All-derivable points in the algebra of all upper triangular matrices

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Abstract

Let \( \mathcal{T}_n \) be the algebra of all \( n \times n \) upper triangular matrices. We say that an element \( G \in \mathcal{T}_n \) is an all-derivable point of \( \mathcal{T}_n \) if every derivable linear mapping \( \varphi \) at \( G \) (i.e. \( \varphi(ST) = \varphi(S)T + S\varphi(T) \) for any \( S, T \in \mathcal{T}_n \) with \( ST = G \)) is a derivation. In this paper we show that \( G \in \mathcal{T}_n \) is an all derivable point of \( \mathcal{T}_n \) if and only if \( G/0 \).

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1. Introduction and preliminaries

Before proceeding let us fix some notation and symbols in this paper. Let \( H \) and \( K \) be complex and separable Hilbert spaces. \( B(H, K) \) stands for the set of all bounded linear operators from \( H \) into \( K \), and abbreviate \( B(H, H) \) to \( B(H) \). We may regard every element of \( B(M, N) \) as of \( B(H, K) \) for any closed subspace \( M \subseteq H \) and \( N \subseteq K \) naturally. We use the symbols \( x \otimes y \) to denote the rank one operator \( \langle \cdot, y \rangle x \) on \( H \). A nest \( \mathcal{N} \) is a family of subspace totally ordered

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by inclusion on $H$. $\mathcal{N}$ is said to be complete if (i) it contains $[0]$ and $H$, and (ii) given any subfamily $\mathcal{N}_0$ of $\mathcal{N}$, the subspace $\bigvee_{M \in \mathcal{V}_0} M$ and $\bigwedge_{M \in \mathcal{V}_0} M$ are both members of $\mathcal{N}$. If we denotes by $\mathcal{N}$ a complete nest on $H$, then the nest algebra $\alg\mathcal{N}$ is the set of all operators which leave every member of $\mathcal{N}$ invariant. The algebra $\alg\mathcal{N}$ is a Banach algebra. If $N \in \mathcal{N}$, we write $N_\perp$ for $\bigvee\{M \in \mathcal{N}: M \subset N\}$. If $M$ is a closed subspace in $H$, we write $P(M)$ for the orthogonal projection operator from $H$ onto $M$. We write $\mathcal{C}$ for the complex number field.

Let $\mathcal{T}\mathcal{M}_n$ be the algebra of all $n \times n$ upper triangular matrices, and let $L(\mathcal{T}\mathcal{M}_n)$ denote the set of all linear mappings on $\mathcal{T}\mathcal{M}_n$. If $\varphi \in L(\mathcal{T}\mathcal{M}_n)$, we say that $\varphi$ is a derivation if $\varphi(ST) = \varphi(S)T + S\varphi(T)$ for any $S, T \in \mathcal{T}\mathcal{M}_n$. Fix a matrix $G \in \mathcal{T}\mathcal{M}_n$. We say that $\varphi \in L(\mathcal{T}\mathcal{M}_n)$ is a derivable mapping at $G$ if $\varphi(ST) = \varphi(S)T + S\varphi(T)$ for any $S, T \in \mathcal{T}\mathcal{M}_n$ with $ST = G$. It is obvious that the definition of the derivable mapping at $G$ only need to consider the elements $S, T$ which satisfy $ST = G$ in the above equation. So the condition of a derivable mapping at $G$ is stronger than of a derivation. We say that an element $G \in \mathcal{T}\mathcal{M}_n$ is an all-derivable point of $\mathcal{T}\mathcal{M}_n$ if every derivable mapping at $G$ is a derivation. Let $\mathcal{A}$ be an operator subalgebra in $B(H)$, where $H$ is a Hilbert space. Similarly, we say that an element $G \in \mathcal{A}$ is an all-derivable point of $\mathcal{A}$ for the norm-topology (strong operator topology etc.) if every norm-topology (strong operator topology etc.) continuous derivable linear mapping at $G$ is a derivation.

We describe some of the results related to ours. Using the main theorems in [5,6], Jing et al. [7] showed that every derivable mapping $\varphi$ at 0 with $\varphi(I) = 0$ on nest algebras is a derivation. Zhu and Xiong in [15] and Zhu in [18] showed that (1) every norm-continuous generalized derivable mapping at 0 on finite CSL algebras is a generalized derivation; (2) every invertible operator in nest algebras is an all-derivable point for the strong operator topology; and (3) a matrix $G \in \mathcal{T}\mathcal{M}_2$ is an all-derivable point of $\mathcal{T}\mathcal{M}_2$ if and only if $G \neq 0$. Šemrl [10] presented the notion of 2-local derivation and showed that every 2-local derivation on $B(H)$ is a derivation (no linearity is assumed), where $\operatorname{dim} H = \infty$. For other results, see [1–3,8–17].

It is the aim of this paper to show the following two statements: (1) Let $\mathcal{N}$ be a complete nest on $H$ with $\operatorname{dim}(H_\perp) = 1$. We obtain in Section 2 some sufficient conditions that $G \in \alg\mathcal{N}$ is an all-derivable point of $\alg\mathcal{N}$ for the strong operator topology. (2) In Section 3 we prove that $G$ is an all-derivable point of $\mathcal{T}\mathcal{M}_n$ if and only if $G \neq 0$.

### 2. All-derivable points in nest algebras

Let $\mathcal{N}$ be a complete nest on $H$ with $\operatorname{dim}(H_\perp) = 1$. All $2 \times 2$ operator matrices are always represented as relative to the orthogonal decomposition $H = H_\perp \oplus (H_\perp)^\perp$ in this section. We use the symbols $I$ and $I_H$ to denote the unit operator on $H_\perp$ and $H$, respectively. We often use the following two lemmas in the proof of Theorem 2.3.

**Lemma 2.1.** Let $\mathcal{A}$ be an operator algebra on a Hilbert space $H$, and let $K$ be a Banach space. If $\varphi$ is a linear mapping from $\mathcal{A}$ into $K$ such that $\varphi(X) = 0$ for any invertible operator $X \in \mathcal{A}$, then $\varphi \equiv 0$.

**Proof.** For arbitrary $X \in \mathcal{A}$, there exists a complex number $\lambda \in \mathcal{C}$ with $\|X\| < \lambda$. So $\lambda$ is in the resolvent set of $X$, i.e. $\lambda I_H + X$ is an invertible operator. It follows from the condition of lemma that $0 = \varphi(\lambda I_H + X) = \lambda \varphi(I_H) + \varphi(X) = \varphi(X)$. Hence $\varphi \equiv 0$. $\Box$

**Lemma 2.2.** Let $\mathcal{N}$ be a complete nest on a complex and separable Hilbert space $H$. If $\varphi : \alg\mathcal{N} \to \alg\mathcal{N}$ be a derivable mapping at 0. Then there exist two operators $C, D \in B(H)$ such that
\[ \varphi(X) = XC + DX \]
for any \( X \in \text{alg} \mathcal{N} \).

**Proof.** Since \( \varphi \) is a derivable mapping at 0 on \( \text{alg} \mathcal{N} \), we know from Theorem 5 in [7] that \( \varphi(ST) = \varphi(S)T + S\varphi(T) - S\varphi(I)T \) for any \( S, T \in \text{alg} \mathcal{N} \). We define a linear mapping \( \psi : \text{alg} \mathcal{N} \rightarrow \text{alg} \mathcal{N} \) with
\[
\psi(T) = \varphi(T) - T\varphi(I) \quad \forall T \in \text{alg} \mathcal{N}.
\]
It is easy to verify that \( \psi \) is a derivation on \( \text{alg} \mathcal{N} \). By Theorem 19.7 in [4], \( \psi \) is an inner derivation, i.e., there exist an operator \( D_0 \in B(H) \) such that \( \psi(T) = TD_0 - D_0T \) for any \( T \in \text{alg} \mathcal{N} \). Furthermore \( \varphi(T) = \psi(T) + T\varphi(I) = TD_0 - D_0T + T\varphi(I) \) for any \( T \in \text{alg} \mathcal{N} \). It is obvious that \( C = \varphi(I) + D_0 \) and \( D = -D_0 \) are desired in the lemma. \( \square \)

**Theorem 2.3.** Let \( \mathcal{N} \) be a complete nest on \( H \) with \( \dim(H_\perp) = 1 \), and let \( P = P(H_\perp) , Q = P((H_\perp)^\perp) , \mathcal{A} = P_{\text{alg} \mathcal{N}}P \). Then the following statements hold.

1. If \( F \) is an all-derivable point of \( \mathcal{A} \) for the strong operator topology and \( 0 \neq F_0 \in B((H_\perp)^\perp, (H_\perp)^\perp) \), then \( G = \begin{bmatrix} F & D \\ 0 & R_0 \end{bmatrix} \) is an all-derivable point of \( \text{alg} \mathcal{N} \) for the strong operator topology.
2. If \( F \) is an all-derivable point of \( \mathcal{A} \) for the strong operator topology and \( D \in B(H_\perp^\perp, H_\perp^\perp) \), then \( G = \begin{bmatrix} F & D \\ 0 & R_0 \end{bmatrix} \) is an all-derivable point of \( \text{alg} \mathcal{N} \) for the strong operator topology.
3. If \( 0 \neq F_0 \in B((H_\perp)^\perp, (H_\perp)^\perp) \), then \( G = \begin{bmatrix} 0 & 0 \\ 0 & R_0 \end{bmatrix} \) is an all-derivable point of \( \text{alg} \mathcal{N} \) for the strong operator topology.
4. If \( 0 \neq D \in B(H_\perp^\perp, H_\perp^\perp) \), then \( G = \begin{bmatrix} 0 & D \\ 0 & R_0 \end{bmatrix} \) is an all-derivable point of \( \text{alg} \mathcal{N} \) for the strong operator topology.

**Proof.** We write \( H_0 = (H_\perp)^\perp = \text{span}(g) \), where \( g \in (H_\perp)^\perp \) with \( \| g \| = 1 \). Obviously, \( \mathcal{A} \) is a nest algebra in \( B(H_\perp) \). Simultaneously, we may regard \( \mathcal{A} \) as a subalgebra of \( \text{alg} \mathcal{N} \), naturally. For arbitrary \( S \in \text{alg} \mathcal{N} \), then \( S \) can be represented as a \( 2 \times 2 \) operator matrix relative to the orthogonal decomposition \( H = H_\perp \oplus H_0 \) as follows:
\[
S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix},
\]
where \( X = PSP, Y = PSQ = y \otimes g \) for some \( y \in H_\perp, Z = QS = zg \otimes g \) for some \( z \in \mathcal{C} \).
Let \( \varphi : \text{alg} \mathcal{N} \rightarrow \text{alg} \mathcal{N} \) be a strongly operator topology continuous linear mapping. Then there exist \( A(X) \in \mathcal{A}, a(X) \in H_\perp \) and \( \alpha(X) \in \mathcal{C} \) such that
\[
\varphi \left( \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \right) = P\varphi(PSP)P + P\varphi(PSP)Q + Q\varphi(PSP)Q = A(X) + a(X) \otimes g + \alpha(X)g \otimes g
\]
and there exist \( B(y) \in \mathcal{A}, b(y) \in H_\perp \) and \( \beta(y) \in \mathcal{C} \) such that
\[ \varphi \left( \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \right) = P \varphi(PSQ)P + P \varphi(PSQ)Q + Q \varphi(PSQ)Q \]

\[ = B(y) + b(y) \otimes g + \beta(y) g \otimes g \]

\[ = \begin{bmatrix} B(y) & b(y) \otimes g \\ 0 & \beta(y) g \otimes g \end{bmatrix}. \]

Simultaneously, there exist \( C_0 \in \mathcal{A}, c_0 \in H_\gamma \) and \( \gamma_0 \in \mathcal{C} \) such that

\[ \varphi \left( \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \right) = \varphi(z(g \otimes g)) \]

\[ = z[P \varphi(g \otimes g)P + P \varphi(g \otimes g)Q + Q \varphi(g \otimes g)Q] \]

\[ = z \begin{bmatrix} C_0 + c_0 \otimes g + \alpha g \otimes g \\ 0 & \gamma_0 g \otimes g \end{bmatrix}. \]

Since \( \varphi \) is a strongly operator topology continuous linear mapping on \( \text{alg}_N \), \( A(\cdot) \), \( a(\cdot) \) and \( \alpha(\cdot) \) are strongly operator topology continuous linear mappings from \( B(H_\gamma) \) into \( B(H_\gamma) \), \( H_\gamma \) and \( \mathcal{C} \), respectively, and \( B(\cdot), b(\cdot) \) and \( \beta(\cdot) \) are continuous linear mappings from \( H_\gamma \) into \( B(H_\gamma) \), \( H_\gamma \) and \( \mathcal{C} \), respectively.

For arbitrary \( S = PSP + PSQ + QSQ = \begin{bmatrix} P^2 & PSQ \\ 0 & QSQ \end{bmatrix} \in \text{alg}_N \) and \( T = PT P + PT Q + QT Q = \begin{bmatrix} PT P & PTQ \\ 0 & QTQ \end{bmatrix} \in \text{alg}_N \), we always write \( X = PSP, Y = PSQ = y \otimes g, Z = QSQ = zg \otimes g, U = PT P, V = PT Q = v \otimes g, W = QT Q = wg \otimes g \) in the rest of the section.

The proof of the statement (1). Let \( F \) be an all derivable point of \( \mathcal{A} \) for the strong operator topology. Let \( \varphi \) be a derivable mapping at \( G = \begin{bmatrix} F & D \\ 0 & g \otimes g \end{bmatrix} \) on \( \text{alg}_N \) for the strong operator topology. It is easy to verifies that \( A(\cdot) \) is a derivable mapping at \( F(= PG P) \) on \( \mathcal{A} \) for the strong operator topology. Since \( F \) is an all derivable point of \( \mathcal{A} \) for the strong operator topology, it follows that \( A(\cdot) \) is a derivation on \( \mathcal{A} \). Note that \( \mathcal{A} \) is a nest algebra. It follows from Theorem 19.7 in [4] that \( A(\cdot) \) is an inner derivation. Hence there exists an operator \( A \in B(H_\gamma) \) such that

\[ A(X) =XA - AX \quad \forall X \in \mathcal{A}. \]

Without loss of generality, we may assume that \( F_0 = g \otimes g \) and \( D = d \otimes g \). If \( ST = G = \begin{bmatrix} F & d \otimes g \\ 0 & g \otimes g \end{bmatrix} \), then we have

\[ \begin{bmatrix} F & d \otimes g \\ 0 & g \otimes g \end{bmatrix} = G = ST \]

\[ = \begin{bmatrix} X & y \otimes g \\ 0 & zg \otimes g \end{bmatrix} \begin{bmatrix} U & v \otimes g \\ 0 & wg \otimes g \end{bmatrix} \]

\[ = \begin{bmatrix} XU & (Xv + wy) \otimes g \\ 0 & zwg \otimes g \end{bmatrix}, \]

i.e. \( XU = F, Xv + wy = d, zw = 1 \). Since \( \varphi \) is a derivable mapping at \( G \), we have \( \varphi(G) = \varphi(S)T + S \varphi(T) \). Furthermore the following matrix equation holds:

\[ \begin{bmatrix} FA - AF + B(d) + C_0 & (\alpha(F) + b(d) + c_0) \otimes g \\ 0 & (\alpha(F) + b(d) + \gamma_0)g \otimes g \end{bmatrix} = \varphi \left( \begin{bmatrix} F & d \otimes g \\ 0 & g \otimes g \end{bmatrix} \right) \]

\[ = \varphi(G) = \varphi(S)T + S \varphi(T) \]
\[ \varphi \left( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right) \left[ \begin{array}{cc} U & V \\ 0 & W \end{array} \right] + \left[ \begin{array}{cc} X & Y \\ 0 & Z \end{array} \right] \varphi \left( \begin{bmatrix} U \\ 0 \end{bmatrix} \right) = \left[ \begin{array}{cc} XA - AX & a(X) \otimes g \\ 0 & \alpha(X)g \otimes g \end{array} \right] + \left[ \begin{array}{cc} b(y) & 0 \\ 0 & \beta(y)g \otimes g \end{array} \right] \\
+z \left[ \begin{array}{cc} C_0 & 0 \\ 0 & \gamma_0g \otimes g \end{array} \right] \left[ \begin{array}{cc} U & v \otimes g \\ 0 & wg \otimes g \end{array} \right] + \left[ \begin{array}{cc} X & y \otimes g \\ 0 & zg \otimes g \end{array} \right] \left[ \begin{array}{cc} UA - AU & a(U) \otimes g \\ 0 & \alpha(U)g \otimes g \end{array} \right] + \left[ \begin{array}{cc} B(v) & b(v) \otimes g \\ 0 & \beta(v)g \otimes g \end{array} \right] + w \left[ \begin{array}{cc} C_0 & 0 \\ 0 & \gamma_0g \otimes g \end{array} \right] = \left[ \begin{array}{cc} XUA - AXU & (XA - AX)v + wa(X) + B(y)v \\ +B(y)U + zC_0U & +wb(y) + zC_0v + zwc_0 \\ +XB(v) & +Xa(U) + \alpha(U)y + Xb(v) \\ +wXC_0 & +\beta(v)y + wXc_0 + w\gamma_0y \otimes g \\ 0 & [w(\alpha(X) + \beta(y)) + 2z\gamma_0y + z(\alpha(U) + \beta(v))]g \otimes g \end{array} \right] \]

for any \( XU = F, Xv + wy = d, zw = 1. \) It follows from the above matrix equation that the following three equations hold:

1. \( C_0 + B(d) = B(y)U + zC_0U + XB(v) + wXC_0, \)  \( (1) \)

2. \( a(F) + b(d) + c_0 = [(XA - AX) + B(y) + zC_0]v + w(a(X) + b(y)) + wz_0 + X(a(U) + b(v) + wc_0) + (\alpha(U) + \beta(v) + w\gamma_0)y, \)

3. \( \alpha(F) + \beta(d) + \gamma_0 = w(\alpha(X) + \beta(y)) + z(\alpha(U) + \beta(v)) + 2zw_0 \)  \( (3) \)

for any \( X, U \in B(H_-), v, y \in H_- \) and \( w, z \in \mathcal{C} \) with \( XU = F, Xv + wy = d, zw = 1. \)

If we take \( X = F, U = I, y = d, v = 0 \) and \( z = w = 1 \) in Eq. (1), then \( FC_0 = 0. \) If we take \( X = F, U = I, y = -d, v = 0 \) and \( z = w = -1 \) in Eq. (1), then \( C_0 + B(d) = 0. \) If we take \( X = I, U = F, y = d, v = 0 \) and \( z = w = 1 \) in Eq. (1) and notice that \( C_0 + B(d) = 0, \) then \( C_0 = 0. \) It follows that:

\( B(y)U + XB(v) = 0. \)

For arbitrary \( y \in H_-, \) if we take \( X = \xi^{-1}I(0 \neq \xi \in \mathcal{C}), U = \xi F, v = y + \xi d, w = -\xi^{-1} \) and \( z = -\xi \) in the above equation, then \( \xi B(y)F + \xi^{-1}B(y + \xi d) = 0. \) Thus we have

\( \xi^2 B(y)F + B(y + \xi d) = 0. \)

Let \( \xi \to 0 \) in the above equation, we obtain \( B(y) = 0. \)

If we take \( X = \xi I(0 \neq \xi \in \mathcal{C}), U = \xi^{-1}F, v = y + \xi^{-1}d, z = -\xi^{-1} \) and \( w = -\xi \) in Eq. (3), then

\( \alpha(F) + \beta(d) + \gamma_0 = -\xi(\xi\alpha(I) + \beta(y)) - \xi^{-1}(\xi^{-1}\alpha(F) + \beta(y + \xi^{-1}d)) + 2\gamma_0. \)
If we multiply the above equation by $\xi^2$ and let $\xi \to 0$, we obtain $\alpha(F) + \beta(d) = 0$. Thus we have

$$\gamma_0 = -\xi(\xi\alpha(I) + \beta(y)) - \xi^{-1}\beta(y) + 2\gamma_0.$$  

If we multiply the above equation by $\xi$ and let $\xi \to 0$, we get that $\beta(y) = 0$. Furthermore $\gamma_0 = 0$ and $\alpha(I) = 0$. Hence the following equation holds by Eq. (3)

$$0 = w\alpha(X) + z\alpha(U)$$

for any $X, U \in B(H)$, $v, y \in H$ and $w, z \in \mathbb{C}$ with $XU = F, Xv + wy = d, zw = 1$. For arbitrary invertible operator $X \in \mathcal{A}$, if we take $U = X^{-1}F, y = -\xi Xv + \xi d, z = \xi$ and $w = \xi^{-1}$ in the above equation, then we get

$$\xi^{-1}\alpha(X) + \xi\alpha(X^{-1}F) = 0.$$  

If we multiply the above equation by $1/w$ and let $w \to \infty$, then $a(F) = -Xc_0$ for any invertible operator $X \in \mathcal{A}$. By Lemma 2.1, $a(X) = -Xc_0$ for any $X \in \mathcal{A}$. In particular, $a(I) + c_0 = 0$.

In a word, we have

$$\varphi \left( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right) = \varphi \left( \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Z \end{bmatrix} \right)$$

$$= \begin{bmatrix} XA - AX & -Xc_0 \otimes g \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b(y) \otimes g \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & zc_0 \otimes g \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} XA - AX & (-Xc_0 + b(y) + zc_0) \otimes g \end{bmatrix},$$

for any $S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \text{alg} \cdot \mathcal{N}$.

We now claim that $\varphi$ is a derivable mapping at $\tilde{G} = \begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix}$. In fact, for arbitrary $S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$, $T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \in \text{alg} \cdot \mathcal{V}$ with $ST = \tilde{G}$, then we have

$$\begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix} = \tilde{G} = ST$$

$$= \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$$

$$= \begin{bmatrix} X & y \otimes g \\ 0 & zg \otimes g \end{bmatrix} \begin{bmatrix} U & v \otimes g \\ 0 & wg \otimes g \end{bmatrix}$$

$$= \begin{bmatrix} XU & (Xv + wy) \otimes g \\ 0 & zwg \otimes g \end{bmatrix}.$$
i.e. $XU = I$, $Xv + wy = d$, $zw = 1$. For $S, T$ with $ST = \tilde{G}$ we have

$$\varphi(\tilde{G}) = \varphi\left(\begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix}\right) = \begin{bmatrix} 0 & (a(I) + b(d) + c_0) \otimes g \\ 0 & b(d) \otimes g \end{bmatrix}. \tag{5}$$

Thus we have

$$\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} = G.$$ Notice that $\varphi$ is a derivable mapping at $G$. Thus we have

$$\varphi\left(\begin{bmatrix} XY \\ 0 \end{bmatrix}\right) \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} + \begin{bmatrix} XY \\ 0 \end{bmatrix} \varphi\left(\begin{bmatrix} UF & V \\ 0 & W \end{bmatrix}\right) = \begin{bmatrix} FA - AF & (F - c_0 + b(d) + c_0) \otimes g \\ 0 & 0 \end{bmatrix}, \tag{6}$$

Let us calculate

$$\varphi\left(\begin{bmatrix} XY \\ 0 \end{bmatrix}\right) \begin{bmatrix} U(I - F) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} XY \\ 0 \end{bmatrix} \varphi\left(\begin{bmatrix} U(I - F) & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} FA - AF & (F - I)c_0 \otimes g \\ 0 & 0 \end{bmatrix}. \tag{7}$$

We then have by Eqs. (5)–(7) that

$$\varphi(ST) + S\varphi(T) = \varphi\left(\begin{bmatrix} XY \\ 0 \end{bmatrix}\right) \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} + \begin{bmatrix} XY \\ 0 \end{bmatrix} \varphi\left(\begin{bmatrix} UF & V \\ 0 & W \end{bmatrix}\right) + \varphi\left(\begin{bmatrix} XY \\ 0 \end{bmatrix}\right) \begin{bmatrix} U(I - F) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} XY \\ 0 \end{bmatrix} \varphi\left(\begin{bmatrix} U(I - F) & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b(d) \otimes g \\ 0 & 0 \end{bmatrix} = \varphi(\tilde{G}). \tag{8}$$
Thus \( \varphi \) is a derivable mapping at the invertible operator \( \tilde{G} \) for the strong operator topology. It follows from Theorem 2.3 in [18] that \( \varphi \) is a derivation. Hence \( G \) is an all-derivable point of \( \text{alg} \mathcal{V} \) for the strong operator topology.

The proof of the statement (2). If \( ST = G = \begin{bmatrix} F & D \\ 0 & 0 \end{bmatrix} \), then the following three matrix equations hold by imitating the proof of the statement (1):

\[
B(d) = B(y)U + zC_0 + Xb(v) + wXc_0,
\]

(9)

\[
a(F) + b(d) = [(A - A) + B(y) + zC_0]v
+ w(a(X) + b(y)) + wzc_0
+ X(a(U) + b(v) + wc_0)
+ (a(U) + b(U) + w\gamma_0)y,
\]

(10)

\[
\alpha(U) + \beta(v) + w\gamma_0.
\]

(11)

for any \( X, U \in B(H), v, y \in H \) and \( w, z \in \mathbb{C} \) with \( Xu = F, Xv + wv = d, zw = 0 \).

If we take \( X = I, U = F, y = 0, v = 0, z = 0 \) and \( w = 1 \) in Eq. (9), then \( C_0 = 0 \). Furthermore

\[
B(d) = B(y)U + Xb(v).
\]

If we take \( X = \frac{1}{2} F, U = 2I, y = d, v = 0, z = 0 \) and \( w = 1 \) in the above equation, then \( B(d) = 2B(d) \). So \( B(d) = 0 \). It follows that:

\[
B(y)U + Xb(v) = 0.
\]

For arbitrary \( 0 \neq y \in H, \) if we take \( X = I, U = F, v = d \) and \( z = w = 0 \) in the above equation, then we have \( B(y)F = 0 \) for any \( y \in H \). On the other hand, for arbitrary \( v \in H \), if we take \( X = I, U = F, y = -v + d, z = 0 \) and \( w = 1 \) in the above equation, then \( 0 = B(-v + d)F + B(v) \). Furthermore \( B(v)(I - F) = 0 \). Notice that \( B(v)F = 0 \). So \( B(v) = 0 \) for any \( v \in H \).

If we take \( X = I, U = F, v = d \) and \( z = w = 0 \) in Eq. (11), then \( a(F) + \beta(d) = 0 \). Furthermore we have

\[
w(a(X) + \beta(y)) + z(\alpha(U) + \beta(v)) = 0.
\]

For arbitrary invertible operator \( X \in \mathcal{A} \), if we take \( U = X^{-1} F, y = d, v = 0, z = 0 \) and \( w = 1 \) in the above equation, then

\[
\alpha(U) + \beta(d) = 0.
\]

Thus \( 2\alpha(U) + \beta(d) = \alpha(2X) + \beta(d) = 0 \). Hence \( \alpha(U) = 0 \) for any invertible operator \( X \in B(H) \). It follows from Lemma 2.1 that \( \alpha(U) = 0 \) for any \( X \in B(H) \). Thus we have

\[
w\beta(y) + z\beta(v) = 0.
\]

For arbitrary \( y \in H \), if we take \( X = I, U = F, v = d - y, z = 0 \) and \( w = 1 \) in the above equation, then \( \beta(y) = 0 \) for any \( y \in H \).

If we take \( X = I, U = F, y = 0, v = d \) and \( z = 0 \) in Eq. (10), then \( a(I) + c_0 = 0 \). For arbitrary \( v \in H \), if we take \( X = I, U = F, y = d - v, w = 1 \) and \( z = 0 \) in Eq. (10), then \( \gamma_0(d - v) = 0 \). So \( \gamma_0 = 0 \). For arbitrary invertible operator \( X \in \mathcal{A} \) and complex number \( \lambda \in \mathbb{C} \), we may find a vector \( v \in H \) such that \( Xv = d \), if we take \( U = X^{-1} F, y = 0, z = 0 \) and \( w = \lambda \) in Eq. (10), then we have

\[
a(F) + b(d) = (X - A) + \lambda^{-1}a(X) + Xa(X^{-1} F) + Xb(v) + \lambda^{-1} Xc_0.
\]
If we multiply the above equation by \( \lambda \) and let \( \lambda \to 0 \), then \( a(X) = -Xc_0 \) for any invertible operator \( X \in \mathcal{A} \). It follows from Lemma 2.1 that \( a(X) = -Xc_0 \) for any \( X \in \mathcal{A} \).

In a word, we have

\[
\varphi \left( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right) = \varphi \left( \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \right)
= \begin{bmatrix} XA - AX & -Xc_0 \otimes g \\ 0 & 0 \end{bmatrix}
+ \begin{bmatrix} 0 & b(y) \otimes g \\ 0 & 0 \end{bmatrix}
+ \begin{bmatrix} 0 & zc_0 \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} XA - AX & (-Xc_0 + b(y) + zc_0) \otimes g, \\ 0 & 0 \end{bmatrix}.
\]

By Theorem 2.3 in [18], we only need to prove that \( \varphi \) is a derivable mapping at \( \tilde{G} = \begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix} \).

In fact, for arbitrary \( S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}, T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \in \text{alg} \cdot \mathcal{V} \) with \( ST = \tilde{G} \), then we have

\[
\begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix} = \tilde{G} = ST
= \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}
= \begin{bmatrix} X & y \otimes g \\ 0 & zg \otimes g \end{bmatrix} \begin{bmatrix} U & v \otimes g \\ 0 & wg \otimes g \end{bmatrix}
= \begin{bmatrix} XU & (Xv + wy) \otimes g \\ 0 & zwg \otimes g \end{bmatrix},
\]
i.e. \( XU = I, Xv + wy = d, zw = 1 \). Thus we have by (12)

\[
\varphi(\tilde{G}) = \varphi \left( \begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix} \right)
= \begin{bmatrix} 0 & (-c_0 + b(d) + c_0) \otimes g \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} 0 & b(d) \otimes g \\ 0 & 0 \end{bmatrix}.
\]

(13)

Since

\[
\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} = \begin{bmatrix} F & XV + YW \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} F & D \\ 0 & 0 \end{bmatrix} = G
\]

and \( \varphi \) is a derivable mapping at \( G \), we have

\[
\varphi \left( \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} + \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \varphi \left( \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} \right)
= \varphi \left( \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} \right)
= \varphi(G) = \varphi \left( \begin{bmatrix} F & d \otimes g \\ 0 & 0 \end{bmatrix} \right)
= \begin{bmatrix} FA - AF & (-Fc_0 + b(d)) \otimes g \\ 0 & 0 \end{bmatrix}.
\]

(14)
Let us calculate
\[
\varphi \left( \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \right) \left[ \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \varphi \left( \begin{bmatrix} UF & V \\ 0 & W \end{bmatrix} \right) \right] = \begin{bmatrix} 0 & (wzc_0) \otimes g \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_0 \otimes g \\ 0 & 0 \end{bmatrix} ,
\]

(15)

and
\[
\varphi \left( \begin{bmatrix} X & Y \\ Z \end{bmatrix} \right) \left[ U(I - F) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \varphi \left( \begin{bmatrix} U(I - F) & 0 \\ 0 & 0 \end{bmatrix} \right) \right] = \begin{bmatrix} AF - FA & -(I - F)c_0 \otimes g \\ 0 & 0 \end{bmatrix} .
\]

(16)

It follows from Eqs. (13)–(16) that
\[
\varphi(S)T + S\varphi(T) = \begin{bmatrix} 0 & b(d) \otimes g \\ 0 & 0 \end{bmatrix} = \varphi(\tilde{G}).
\]

Thus \(\varphi\) is a derivable mapping at \(\tilde{G}\) for the strong operator topology. It follows from Theorem 2.3 in [18] that \(\varphi\) is a derivation. Hence \(G\) is an all-derivable point of \(\text{alg} N\) for the strong operator topology.

The proof of the statement (3). Without loss of generality, we may assume that \(F_0 = g \otimes g\). If \(\varphi\) is a derivable mapping at \(G = \begin{bmatrix} 0 & 0 \\ 0 & g \otimes g \end{bmatrix} \) on \(\text{alg} N\) for the strong operator topology, it is easy to verifies that \(A(\cdot)\) is a derivable mapping at 0 on \(\mathcal{A}\) for the strong operator topology. By Lemma 2.2, there exist two operators \(C, E \in B(H_-)\) such that \(A(X) = XC + EX\) for any \(X \in \mathcal{A}\).

If \(ST = G = \begin{bmatrix} 0 & 0 \\ 0 & g \otimes g \end{bmatrix} \), then the following three equations hold by imitating the proof of the statement (1)
\[
C_0 = X(C + E)U + B(y)U + zC_0U + XB(v) + wXC_0 ,
\]

(17)
\[
c_0 = [XC + EX + B(y) + zC_0]v + w(a(X) + b(y)) + wzC_0 + X(a(U) + b(v) + wc_0) + (a(U) + \beta(v) + w\gamma_0)y ,
\]

(18)
\[
\gamma_0 = w(a(X) + \beta(y)) + z(\alpha(U) + \beta(v)) + 2zw\gamma_0
\]

(19)
for any \(X, U \in B(H_-)\), \(v, y \in H_-\) and \(w, z \in \mathbb{C}\) with \(XU = 0, XV + wY = 0, zw = 1\).

If we take \(X = U = 0, y = v = 0\) and \(z = w = 1\) in Eq. (17), then \(C_0 = 0\). Thus we have
\[
0 = X(C + E)U + B(y)U + XB(v) .
\]
For arbitrary \( y \in H_\sim \), if we take \( X = I, U = 0, v = y \) and \( z = w = -1 \) in the above equation, then \( B(y) = 0 \) for any \( y \in H_\sim \).

If we take \( X = U = 0, y = v = 0 \) and \( z = w = 1 \) in Eq. (19), then \( \gamma_0 = 0 \). Thus we have

\[
0 = w(\alpha(X) + \beta(y)) + z(\alpha(U) + \beta(v)).
\]

For arbitrary \( X \in \mathcal{A}, \) if we take \( U = 0, y = v = 0 \) and \( z = w = 1 \) in the above equation, then \( B(y) = 0 \) for any \( y \in H_\sim \).

If we take \( X = I, U = 0, v = 0 \) and \( z = w = 1 \) in Eq. (19), then \( \gamma_0 = 0 \). Thus we have

\[
0 = w\beta(y) + z\beta(v).
\]

For arbitrary \( v \in H_\sim, \) if we take \( X = U = 0, y = v = 0 \) and \( z = w = 1 \) in Eq. (18), then \( \alpha(X) = 0 \) for any \( X \in \mathcal{A} \).

Thus we have

\[
0 = XU (Xv + wy) \otimes g + zwg \otimes g.
\]

(20)

We now claim that \( \varphi \) is a derivation on \( \text{alg}. \mathcal{N} \). We only need to show that \( \varphi \) is a derivable mapping at \( IH \). In fact, for arbitrary \( S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \) and \( T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \) in \( \text{alg}. \mathcal{N} \) with \( ST = IH \), we have

\[
\begin{bmatrix}
I & 0 \\
0 & g \otimes g
\end{bmatrix} = \begin{bmatrix} XU & (Xv + wy) \otimes g \\
0 & zwg \otimes g
\end{bmatrix}.
\]

That is \( XU = I, Xv + wy = 0 \) and \( zw = 1 \). Since \( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & R_0 \end{bmatrix} \), and \( \varphi \) is a derivable mapping at \( G \), thus we have

\[
\varphi \left( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right) \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} + \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \varphi \left( \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \right)
= \varphi(G) = \varphi \left( \begin{bmatrix} 0 & 0 \\ 0 & g \otimes g \end{bmatrix} \right)
= \begin{bmatrix} 0 & 0 \\ 0 & c_0 \otimes g \end{bmatrix}.
\]

On the other hand, we calculate

\[
\begin{align*}
\varphi \left( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right) & = \begin{bmatrix} XC - CX & (-Xc_0 + b(y) + zc_0) \otimes g \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} UC - CU & -Uc_0 \otimes g \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} 0 & -c_0 \otimes g \\ 0 & 0 \end{bmatrix}.
\end{align*}
\]
Notice that $\varphi(I_H) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$. It follows from the above two equations that $\varphi(I_H) = \varphi(S)T + S\varphi(T)$. So $\varphi$ is derivable mapping at $I_H$ and it follows from Theorem 2.3 in [18] that $\varphi$ is a derivation. Hence $G$ is an all-derivable point of $\text{alg} \cdot N$ for the strong operator topology.

(4) If $\varphi$ is a derivable mapping at $G = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$ on $\text{alg} \cdot N$ for the strong operator topology ($D \neq 0$). It is easy to verify that $A(\cdot)$ a derivable mapping at 0 on $\mathcal{A}$ for the strong operator topology. By Lemma 2.2, there exist two operators $C, E \in B(H_-)$ such that $A(X) = XC + EX$ for any $X \in \mathcal{A}$.

If $ST = G$, then the following three equations hold by imitating the proof of the statement (1)

$$B(d) = X(C + E)U + B(y)U + zC_0U + XB(v) + wXC_0,$$

$$b(d) = [XC + EX + B(y) + zC_0]v$$

$$+ w(a(X) + b(y)) + wzC_0$$

$$+ X(a(U) + b(v) + wC_0)$$

$$+ (\alpha(U) + \beta(v) + w\gamma_0)y,$$

$$\beta(d) = w(\alpha(X) + \beta(y)) + z(\alpha(U) + \beta(v)) + 2zw\gamma_0$$

for any $X, U \in B(H_-), v, y \in H_-$ and $w, z \in \mathbb{C}$ with $XU = 0, Xv + wy = d, zw = 0$.

If we take $X = I, U = 0, y = 0, v = d, z = 0$ and $w = 1$ in Eq. (21), then $C_0 = 0$. Thus we have

$$B(d) = X(C + E)U + B(y)U + XB(v).$$

For arbitrary $v \in H_-$, if we take $X = I, U = 0, y = d - v$ and $z = 0$ and $w = 1$ in the above equation, then $B(d) = B(v)$ for any $v \in H_-$. Thus $B(2v) = B(d)$. Hence $B(v) = 0$ for any $v \in H_-$. 

For arbitrary invertible operator $X \in \mathcal{A}$, we may find a vector $v \in H_-$ such that $Xv = d$. If we take $U = 0, y = 0, z = 0$ and $w = 1$ in Eq. (23), then $\alpha(X) = \beta(d)$ for any invertible operator $X \in \mathcal{A}$. So $2\alpha(X) = \alpha(2X) = \beta(d)$. Hence $\alpha(X) = 0$. It follows from Lemma 2.1 that $\alpha(X) = 0$ for any $X \in \mathcal{A}$. Thus we have

$$\beta(d) = w\beta(y) + z\beta(v).$$

For arbitrary $y \in H_-$, if we take $X = I, U = 0, y = d - v, z = 0$ and $w = 1$ in the above equation, then $\beta(y) = \beta(d)$. So $\beta(y) = 0$ for any $y \in H_-$. 

For arbitrary $X \in \mathcal{A}$, if we take $U = 0, y = d, v = 0, z = 0$ and $w = 1$ in Eq. (22), then we have

$$a(X) = Xc_0 + \gamma_0d = 0$$

for any $X \in B(H_-)$. Furthermore we have

$$2a(X) + 2Xc_0 + \gamma_0d = a(2X) + 2Xc_0 + \gamma_0d = 0.$$

It follows from the above two equations that $a(X) = -Xc_0$ and $\gamma_0 = 0$. In particular, $a(I) + c_0 = 0$. For arbitrary $v \in H_-$, if we take $X = I, U = 0, y = d - v$ and $z = 0$ and $w = 1$ in Eq. (22), then $(C + E)v = 0$ for any $v \in H_-$. Hence $C = -E$. 

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In a word, we have
\[
\varphi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix}\right)
\]
\[
= \begin{bmatrix} XC - CX & -Xc_0 \otimes g \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b(y) \otimes g \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & zc_0 \\ 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} XC - CX & (Xc_0 + b(y) + zc_0) \otimes g \\ 0 & 0 \end{bmatrix}.
\] (24)

We now claim that \(\varphi\) is a derivable mapping at \(\tilde{G} = \begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix}\). In fact, for arbitrary \(S = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\) and \(T = \begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix}\) in \(\text{alg} \mathcal{N}\) with \(ST = \tilde{G}\), we have
\[
\begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix} = \begin{bmatrix} XU & (Xv + wy) \otimes g \\ 0 & zwg \otimes g \end{bmatrix}.
\]
That is \(XU = I, Xv + wy = d,\) and \(zw = 1\). Notice that \(G = \begin{bmatrix} 0 & d \otimes g \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}\), and \(\varphi\) is a derivable mapping at \(G\). So we have
\[
\varphi\left(\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} + \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}\varphi\left(\begin{bmatrix} 0 & V \\ 0 & W \end{bmatrix}\right)
\]
\[
= \varphi(G) = \varphi\left(\begin{bmatrix} 0 & d \otimes g \\ 0 & 0 \end{bmatrix}\right)
\]
\[
= \begin{bmatrix} 0 & b(d) \otimes g \\ 0 & 0 \end{bmatrix}.
\]

Let us calculate
\[
\varphi\left(\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}\varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)
\]
\[
= \begin{bmatrix} XC - CX & (Xc_0 + b(y)) \otimes g \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}
\]
\[
+ \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} UC - CU & -Uc_0 \otimes g \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -c_0 \otimes g \\ 0 & 0 \end{bmatrix}.
\]
and
\[
\varphi\left(\begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix}\right)\begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix}\varphi\left(\begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix}\right)
\]
\[
= \begin{bmatrix} 0 & zc_0 \otimes g \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix}
\]
\[
+ \begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix} \begin{bmatrix} UC - CU & (Uc_0 + b(v) + wc_0) \otimes g \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_0 \otimes g \\ 0 & 0 \end{bmatrix}.
\]

We have by Eq. (24) \(\varphi(I_H) = 0\). Thus we obtain
\[
\varphi(\tilde{G}) = \varphi\left(I_H + \begin{bmatrix} 0 & d \otimes g \\ 0 & 0 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 0 & d \otimes g \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b(d) \otimes g \\ 0 & 0 \end{bmatrix}.
\]
It follows from the above four equations that \( \varphi(\tilde{G}) = \varphi(S)T + S\varphi(T) \). So \( \varphi \) is a derivable mapping at \( \tilde{G} \) for the strong operator topology. Note that \( \tilde{G} \) is an invertible operator. It follows from Theorem 2.3 in [18] that \( \varphi \) is a derivation. Hence \( G \) is an all-derivable point of \( \text{alg.} \mathcal{N} \) for the strong operator topology. □

3. All-derivable points in the algebra of \( n \times n \) upper triangular matrices

In this section, we always write \( \mathcal{M}_n \) for the algebra of all \( n \times n \) upper triangular matrices. We use the symbols \( H \) and \( \{e_i : i = 1, 2, \ldots, n\} \) to denote the Euclidean \( n \)-dimensional space and its normal orthogonal basis, respectively. We may regard an \( n \times n \) matrix as an operator on Euclidean \( n \)-dimensional space \( H \), naturally. Thus \( \mathcal{M}_n \) is a nest algebra associated with \( \mathcal{N} \), where \( \mathcal{N} = \{N_i : 0 \leq i \leq n\} \) and \( N_i = \text{span}\{e_j : 0 \leq j \leq i\} \), \((e_0 = 0)\).

Lemma 3.1. Let \( \mathcal{N} \) be a complete nest on \( H \), and let \( J \) be an invertible operator in \( B(H) \). Then \( G \in \text{alg.} \mathcal{N} \) is an all-derivable point of \( \text{alg.} \mathcal{N} \) if and only if \( JGJ^{-1} \) is an all-derivable point of the operator algebra \( J\text{alg.} \mathcal{N}J^{-1} \).

**Proof.** \( \Rightarrow \) Suppose that \( G \) is an all-derivable point of \( \text{alg.} \mathcal{N} \). Let \( \varphi \) be a derivable mapping at \( JGJ^{-1} \) on \( \text{alg.} \mathcal{N}J^{-1} \). Define a mapping \( \psi : \text{alg.} \mathcal{N} \to \text{alg.} \mathcal{N} \) as follows \( \psi(S) = J^{-1}\varphi(JSJ^{-1})J \). It is easy to check that \( \psi(ST) = \psi(S)T + S\psi(T) \). That is \( \psi \) is a derivable mapping at \( G \) of \( \text{alg.} \mathcal{N} \). Since \( G \) is an all-derivable point of \( \text{alg.} \mathcal{N} \), \( \psi \) is a derivation and therefore by Theorem 19.7 in [4] an inner derivation. Thus there exists an operator \( A \in B(H) \) such that \( \psi(S) = SA - AS \) for any \( S \) in \( \text{alg.} \mathcal{N} \). It is easy to check that \( \varphi(JSJ^{-1}) = J^{-1}\psi(S)J = (J^{-1}SJ)(J^{-1}AJ) - (J^{-1}AJ)(J^{-1}SJ) \) for any \( JSJ^{-1} \in J\text{alg.} \mathcal{N}J^{-1} \). So \( \varphi \) is an inner derivation. Hence \( JGJ^{-1} \) is an all-derivable point of \( J\text{alg.} \mathcal{N}J^{-1} \).

\( \Leftarrow \) If \( JGJ^{-1} \) is an all-derivable point of \( J\mathcal{N}J^{-1} \), then \( J^{-1}(JGJ^{-1})J \) is an all-derivable point of \( J^{-1}(J\text{alg.} \mathcal{N}J^{-1})J \) by the result of the above paragraph, i.e. \( G \) is an all-derivable point of \( \text{alg.} \mathcal{N} \). □

Theorem 3.2. \( G \in \mathcal{M}_n \) is an all derivable point in \( \mathcal{M}_n \) if and only if \( G \neq 0 \).

**Proof.** Let \( e_i (i = 1, 2, \ldots, n) \) be a normal orthogonal basis on an \( n \)-dimensional Euclidean space \( H \). We may regard \( \mathcal{M}_n \) as a nest algebra associated with \( \mathcal{N} \), where \( \mathcal{N} = \{N_k : 0 \leq k \leq n\} \) and \( N_k = \{e_i : 0 \leq i \leq k\} \). Thus \( H_- = N_{n-1} \cdot \text{dim}(H_-) = \text{dim}(N_{n-1}) = 1 \).

Suppose that \( G \neq 0 \). We claim that \( G \) is an all-derivable point of \( \mathcal{M}_n \).

When \( n = 2 \), we know by Example 3.3 in [18] that \( G \) is an all derivable point of \( \mathcal{M}_2 \). If we assume that the statement holds when \( n - 1 \), we only need to show that the statement is true when \( n \).

Suppose that \( 0 \neq G \in \mathcal{M}_n \). Thus we only need to prove that \( G \) is an all derivable point of \( \mathcal{M}_n \). If we write

\[
G = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{bmatrix},
\]

we divide the proof of the statement into the following two cases.
Case 1. Suppose that $a_{nn} = 0$. then $G$ can be represented as a $2 \times 2$ block matrix relative to the orthogonal decomposition $H = N_{n-1} \oplus \text{span}\{e_n\}$ as follows $G = \begin{bmatrix} F & 0 \\ 0 & D \end{bmatrix}$. If $F \neq 0$, by hypothesis, $F$ is an all-derivable point of $P(N_{n-1}) \mathcal{M}_n P(N_{n-1}) = \mathcal{M}_{n-1}$. It follows from the statement (2) in Theorem 2.3 that $G$ is an all-derivable point of $\mathcal{M}_n$. If $F = 0$, then $D \neq 0$. It follows from the statement (4) in Theorem 2.3 that $G$ is an all-derivable point of $\mathcal{M}_n$.

Case 2. Suppose that $a_{nn} \neq 0$. If $F \neq 0$, by hypothesis, $F$ is an all derivable point of $\mathcal{M}_{n-1}$. It follows from the statement (1) in Theorem 2.3 that $G$ is an all-derivable point of $\mathcal{M}_n$. If $F = 0$, then there exists upper triangular invertible matrix $J$ such that $J G J^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & a_{nn} \end{bmatrix}$. It follows from the statement (3) in Theorem 2.3 that $J G J^{-1}$ is an all-derivable point of $\mathcal{M}_n$. Notice that $J \mathcal{M}_n J^{-1} = \mathcal{M}_n$. Hence $J G J^{-1}$ is an all-derivable point of $J \mathcal{M}_n J^{-1}$. It follows from Theorem 3.1 that $G$ is an all-derivable point of $\mathcal{M}_n$.

Finally, we only need to show that $0 \in \mathcal{M}_n$ is not an all-derivable point of $\mathcal{M}_n$. In fact, if we define a linear mapping from $\mathcal{M}_n$ into itself as follows $\varphi(S) = S$ for any $S \in \mathcal{M}_n$, then $\varphi$ is a derivable mapping at 0, but $\varphi$ is not a derivation. □

References