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Set theoretic complete intersection for curves in a smooth affine algebra

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Abstract

Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k. Let n be an integer such that $2n \geq d + 3$. Let I be an ideal in A of height n and P be a projective A-module of rank n. Suppose $P \oplus A \approx A^{n+1}$ and there is a surjection $\alpha: P \to I$. It is proved in this note that I is a set theoretic complete intersection ideal. As a consequence, a smooth curve in a smooth affine $\mathbb{C}$-algebra with trivial conormal bundle is a set theoretic complete intersection if its corresponding class in the Grothendieck group is torsion.

MSC: primary 13C10, 13C40

Keywords: Set theoretic complete intersection; Euler Class group

1. Introduction

Let A be a commutative Noetherian ring of dimension d ($d \geq 3$). Let J be a local complete intersection ideal in A of height $d - 1$. Then by the well known Ferrand–Szpiro construction [15], there exists a local complete intersection ideal I which is contained in

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J, such that $\sqrt{I} = \sqrt{J}$ and I/I^2 is free A/I-module of rank n. One can ask when such an ideal I is a set theoretic complete intersection. Taking inspiration in part from the results in [5,6,12] and [13], it is conceivable that the property of an ideal being a set theoretic complete intersection is related to its class in the Chow group or Grothendieck group being a torsion element; thus it is natural to ask the following question:

Question 1.1. *Let A be a commutative Noetherian ring of dimension d ($d \geq 3$). Let I be a local complete intersection ideal in A of height $n = d - 1$ such that I/I^2 is a free A/I-module of rank n. Suppose (A/I) is torsion in $K_0(A)$. Is I a set theoretic complete intersection in A?*

When $n \geq 5$ and odd, the above Question has an affirmative answer (for example, see the proof of [17, Theorem 3.2]). For the case when n is even, the author gave an affirmative answer to the above Question too in [17, Theorem 3.6] if A is a polynomial algebra containing $\mathbb{Q}$. This note is another attempt by the author to settle the above Question.

In this note, we shall give an affirmative answer to Question 1.1 in the case when A is a smooth affine $\mathbb{C}$-algebra (see Corollary 2.15).

Theorem 1.2. *Let A be a smooth affine $\mathbb{C}$-algebra of dimension $n + 1$, where $n \geq 4$ and even. Let I be a local complete intersection ideal of height n such that I/I^2 is a free A/I-module of rank n. Suppose (A/I) is torsion in $K_0(A)$. Then I is a set theoretic complete intersection in A.*

All rings in this paper are assumed to be commutative and Noetherian. All modules considered are assumed to be finitely generated. We denote by $K_0(A)$ the Grothendieck group of projective modules over the ring A.

2. Main theorem

Let A be a commutative Noetherian ring of dimension n, and $I \subseteq A$ be a local complete intersection of height r ($r \leq n$). Suppose I/I^2 is A/I-free with base $\bar{f}_1, \ldots, \bar{f}_r$, $f_i \in I$, $\bar{f}_i$ is the class of f_i in I/I^2. Let $J = I^{(r-1)!} + (f_1, \ldots, f_{r-1})$. Then, by a result of Murthy [12, Theorem 2.2], there exists a surjection $P \to J$ with P a projective A-module of rank r, such that $(P) - (A^r) = -(A/I)$ in $K_0(A)$. Therefore to show that Question 1.1 has an affirmative answer in the case when A is a smooth affine $\mathbb{C}$-algebra, it suffices to answer the following much more general question positively:

Question 2.1. *Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k. Let n be an integer such that $2n \geq d + 3$. Let I be an ideal in A of height n and P be a projective A-module of rank n. Suppose $P \oplus A \approx A^{n+1}$ and there is a surjection α: $P \to I$. Is I a set theoretic complete intersection ideal in A?*

Remark 2.2. If n is odd, we have the following proposition:

Proposition 2.3. *Let A be a ring of dimension d ($d \geq 3$) and n be an odd integer such that $2n \geq d + 3$. Let I be a local complete intersection ideal in A of height n and P be a projective A-module of rank n. Suppose $P \oplus A \approx A^{n+1}$ and there is a surjection α: $P \to I$. Then I is a set theoretic complete intersection ideal in A.*

Proof. Since n is odd, P has a free summand of rank 1 by Bass [1], say $P \approx Q \oplus A$. Let x be the image of (0, 1) under α and J be the image of Q under α; then $I = (J, x)$. By some suitable elementary transformations on P, we may assume $\text{ht}J = n-1$. $2n \geq d+3$ implies $\text{rank}(Q/JQ) \geq \dim(A/J) + 1$. By Bass cancellation [2], Q/JQ is a free A/J-module of rank $n - 1$. By [11, Lemma 1], I is generated by n elements. In particular, I is a set theoretic complete intersection ideal in A. The proof of the proposition is complete. □

Therefore, if n is odd, Question 2.1 has an affirmative answer.

In order to give a complete answer to the Question 2.1, we need a few lemmas.
First, let us restate a lemma of Van der Kallen [7, Lemma 4.9]:

Lemma 2.4. *Let A be a commutative ring. Let $[a_0, a_1, \dots, a_n]$ be a unimodular row over A and P be the cokernel of the natural map:*

$$A \xrightarrow{(a_0^2, a_1, \dots, a_n)} A^{n+1}.$$

Then the projective A-module P has a free summand of rank 1.

Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k. Let n be an integer such that $2n \geq d + 3$. Let $[a_0, a_1, \dots, a_n] \in Um_{n+1}(A)$, and $P = A^{n+1} / \sum_{0 \leq i \leq n} a_i e_i$. The Euler Class group of A is defined by Bhatwadekar and Sridharan in [3], and is denoted by $E^n(A)$. And they also attached to P an element $e([a_0, a_1, \dots, a_n])$ in $E^n(A)$.

The following is an easy corollary to the above lemma:

Corollary 2.5. *Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k. Let n be an integer such that $2n \geq d + 3$. Suppose $[a_0, a_1, \dots, a_n] \in Um_{n+1}(A)$. Then $e([a_0^2, a_1, \dots, a_n]) = 0$ in $E^n(A)$.*

Proof. By [3, Theorem 5.4] and the above lemma, we are done. □

Let B be a ring and M be a finitely generated B-module. We use the following convention throughout the rest of this note for simplicity:

Convention. Let $m_1, \dots, m_n \in M$. We say a map: $B^n \to M$ is given by $(m_1, \dots, m_n)$ to mean a B-module homomorphism: $B^n \to M$ defined by sending e_i to m_i for $i = 1, \dots, n$, where $(e_1, \dots, e_n)$ is a standard basis of B^n.

Lemma 2.6. *Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k and n be an even integer such that $2n \geq d + 3$. Let $I = (f_1, \dots, f_n)$ be an ideal of height n and $u \in A$ such that $1 - uv \in I$. Let $u^2\omega$ be the surjection: $(A/I)^n \to I/I^2$ given by $(\overline{u^2 f_1}, \overline{f_2}, \dots, \overline{f_n})$, where bar denotes reduction modulo I. Then $(I, u^2\omega) = 0$ in $E^n(A)$.*

Proof. Applying Lemma 5.6 in [3], we have $e([v^2, f_1, \dots, f_n]) = (I, u^2\omega)$ in $E^n(A)$. By Corollary 2.5, we are done. □

The proof of the following lemma is analogous to the proof of Lemma 5.4 in [4].

Lemma 2.7. *Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k and n be an even integer such that $2n \geq d+3$. Let I be an ideal of height n and $u \in A$ such that $1-uv \in I$. Suppose $I = (f_1, f_2, \ldots, f_n) + I^2$. Let ω be the surjection: $(A/I)^n \to I/I^2$ given by $(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_n})$ and let $u^2\omega$ be the surjection: $(A/I)^n \to I/I^2$ given by $(\overline{u^2 f_1}, \overline{f_2}, \ldots, \overline{f_n})$. Then $(I, \omega) = (I, u^2\omega)$ in $E^n(A)$.*

Proof. If $(I, \omega) = 0$ in $E^n(A)$, then we are done by Lemma 2.6. So we may assume $(I, \omega) \neq 0$; then by corollary 2.4 in [3] we can find an ideal J of height n such that $I + J = A$, $J \cap I = (f_1, \ldots, f_n)$ and $J = (f_1, \ldots, f_n) + J^2$. Let ω_J be the surjection: $(A/J)^n \to J/J^2$ given by $(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_n})$; then $(I, \omega) + (J, \omega_J) = 0$ in $E^n(A)$. Since $I + J = A$, we can write $1 - u = x + y$ for some $x \in I$, $y \in J$. Let $b = 1 - y$; then $b = 1$ modulo J and $b = u$ modulo I. By Lemma 2.6 above and Theorem 4.2 in [3], we see that there exists a surjection ϕ: $A^n \to I \cap J$, such that $\phi \otimes A/I$ is given by $(\overline{u^2 f_1}, \overline{f_2}, \ldots, \overline{f_n})$ and $\phi \otimes A/J$ is given by $(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_n})$. From the surjection ϕ, we have $(I, u^2\omega) + (J, \omega_J) = 0$ in $E^n(A)$. Combining the relation $(I, \omega) + (J, \omega_J) = 0$, we have $(I, \omega) = (I, u^2\omega)$ in $E^n(A)$. □

Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k and n be an integer such that $2n \geq d + 3$. By a theorem of Van der Kallen [8, Theorem 4.1], the universal weak Mennicke symbol

$$wms : Um_{n+1}(A)/E_{n+1}(A) \to WMS_n(A)$$

is a bijection with an abelian target, which provides $Um_{n+1}(A)/E_{n+1}(A)$ with the desired structure of an abelian group. In [3, Theorem 5.7], Bhatwadekar and Sridharan showed that the natural map

$$e : Um_{n+1}(A)/E_{n+1}(A) \to E^n(A)$$

is a group homomorphism, where the group structure of $Um_{n+1}(A)/E_{n+1}(A)$ is the one defined above by Van der Kallen.

Let $[a_0, a_1, \ldots, a_n]$ be a unimodular row over A and let

$$P = A^{n+1} / \sum_{0 \leq i \leq n} a_i e_i,$$

where $(e_0, \ldots, e_n)$ is a standard basis of A^{n+1}. Let p_i denote the image of e_i in P; then $P = \sum_{0 \leq i \leq n} A p_i$ and $\sum_{0 \leq i \leq n} a_i p_i = 0$. Suppose there is a surjection $\alpha : P \to I$, where I is an ideal of height n. Let f_i be the image of p_i under the surjection α. Since $n + 1 \geq \dim(A/I) + 2$, $[\overline{a_0}, \overline{a_1}, \ldots, \overline{a_n}]$ is completable to an elementary matrix in $E_{n+1}(A/I)$. So we may assume $[a_0, a_1, \ldots, a_n] \equiv [1, 0, \ldots, 0]$ modulo I. $2n \geq d + 3$ implies P/IP is a free A/I-module of rank n by Bass cancellation. Since $\sum_{0 \leq i \leq n} a_i p_i = 0$, we can write $I = (f_1, \ldots, f_n) + I^2$. So if we let $\omega : (A/I)^n \to I/I^2$ denote the surjection given by $(\overline{f_1}, \ldots, \overline{f_{n-1}}, \overline{f_n})$, then $e([a_0, a_1, \ldots, a_n]) = (I, \omega)$ in $E^n(A)$. Let $-\omega : (A/I)^n \to I/I^2$ denote the surjection given by $(\overline{f_1}, \ldots, \overline{f_{n-1}}, -\overline{f_n})$. Then we have the following proposition:

Proposition 2.8. *Let A, P, I, α, ω and $-\omega$ be as above. Then $(I, \omega) + (I, -\omega) = 0$ in $E^n(A)$.*

Proof. We first show there is a unimodular row over A which represents $(I, -\omega)$ in $E^n(A)$. Notice that the $\sum_{0\le i\le n} a_i p_i = 0$ implies $\sum_{0\le i\le n} a_i f_i = 0$. Let ϕ be the surjection: $A^{n+1} \to I$ given by $(f_0, \dots, f_{n-1}, -f_n)$. Let Q be the projective A-module of rank n defined by

$$Q = A^{n+1}/(a_0e_0 + \cdots + a_{n-1}e_{n-1} - a_ne_n).$$

Since $[a_0, a_1, \dots, a_n]$ is a unimodular row over A, there exist b_i's $\in A$ such that $a_0b_0 + a_1b_1 + \cdots + a_nb_n = 1$. Let $q_n = e_n - (-b_n)(a_0e_0 + \cdots + a_{n-1}e_{n-1} - a_ne_n)$ and $q_i = e_i - (b_i)(a_0e_0 + \cdots + a_{n-1}e_{n-1} - a_ne_n)$ for $i = 0, \dots, n-1$; then $Q = \sum_{0\le i\le n} Aq_i$, $\phi(q_n) = -f_n$ and $\phi(q_i) = f_i$ for $i = 0, \dots, n-1$. Thus the restriction of ϕ to Q gives us a surjection: $Q \to I$; call it β. Then it is rather obvious that $e([a_0, \dots, a_{n-1}, -a_n]) = (I, -\omega)$ in $E^n(A)$ via the surjection β and the projective A-module Q.

Next, we show the image of $[a_0, \dots, a_{n-1}, a_n] * [a_0, \dots, a_{n-1}, -a_n]$ under the group homomorphism e is zero, where $*$ is the group operation on

$$Um_{n+1}(A)/E_{n+1}(A)$$

which is defined in [8] by Van der Kallen. Applying [8, Lemma 3.5(i)], we have

$$[a_0, \dots, a_{n-1}, a_n] * [a_0, \dots, a_{n-1}, -b_n] = 0$$

in $Um_{n+1}(A)/E_{n+1}(A)$. Also by [8, Lemma 3.5(v)],

$$[a_0, \dots, a_{n-1}, -a_n] * [a_0, \dots, a_{n-1}, b_n^2] = [a_0, \dots, a_{n-1}, -a_nb_n^2].$$

But $[a_0, \dots, a_{n-1}, -a_nb_n^2] = [a_0, \dots, a_{n-1}, -b_n]$ in $Um_{n+1}(A)/E_{n+1}(A)$. Taking the image in $E^n(A)$ under the group homomorphism e and applying Corollary 2.5, it follows that $(I, \omega) + (I, -\omega) + 0 = 0$ in $E^n(A)$. The proof of the proposition is complete. □

From Lemma 2.7 and Proposition 2.8, we have the following interesting corollary:

Corollary 2.9. *Let A be a smooth affine $\mathbb{C}$-algebra of dimension $n + 1$ ($n \ge 4$, even). Let $[a_0, \dots, a_n] \in Um_{n+1}(A)$. Then $e([a_0, \dots, a_n])$ is 2-torsion in $E^n(A)$.*

Proof. Let P denote the projective A-module defined by $[a_0, \dots, a_n]$. Choose a general section of the dual of P, say α; then α gives us a surjection: $P \to I$, where I is a local complete intersection ideal of height n. From this surjection, we can write $e([a_0, \dots, a_n]) = (I, \omega)$ in $E^n(A)$, where ω is some surjection: $A/I^n \to I/I^2$. By Lemma 2.7, we have $(I, \omega) = (I, -\omega)$ since -1 is a square in A/I. By Proposition 2.8, we have $2e([a_0, \dots, a_n]) = 0$ in $E^n(A)$. The proof of the corollary is complete. □

Lemma 2.10. *Let A be a commutative Noetherian ring containing a field k. Let I be a proper ideal of height n which is a local complete intersection in A, such that I/I^2 is a free A/I-module of rank n. Then there exists a regular sequence $f_1, \dots, f_n$ in A and $s_1 \in I^2$ such that*

(1) *$I = (f_1, \dots, f_n, s_1)$, $s_1(1 - s_1) \in (f_1, \dots, f_n)$, $I = (f_1, \dots, f_n) + I^2$, and*
(2) *$\{f_1, \dots, f_{n-1}, f_n - s_1^2\}$ is a regular sequence in A.*

Proof. As in the proof of Lemma 2.3 in [10], we can find a regular sequence $f_1, \dots, f_n$ in A such that $I = (f_1, \dots, f_n) + I^2$. By [11, Lemma 1], there exists $s \in I$ such that $s(1-s) \in (f_1, \dots, f_n)$ and $I = (f_1, \dots, f_n, s)$. Since $s(1-s) \in (f_1, \dots, f_n)$, we may further assume that $s \in I^2$. Notice that we can change s to $\prod_{i=1}^m (s - b_i f_n)$ for any positive integer m and $b_i \in I$. If $p_1, \dots, p_t$ are the maximal elements in $\mathrm{Ass}(A/(f_1, \dots, f_{n-1}))$, then $f_n \notin p_1, \dots, p_t$.

If $s \in p_1, \dots, p_t$, then $f_n - s^2 \notin p_1, \dots, p_t$, and we are through.

If, say for example, $s \notin p_1$, but $s + bf_n \in p_1$ for some $b \in I$, we replace s by $s(s + bf_n)$ and assume $s \in p_1$. Repeating this procedure (that is, replacing s by $\prod_{i=1}^m (s - b_i f_n)$) and reordering p_i where $i \in \{1, \dots, t\}$ if necessary, we may assume that $s \in p_1, \dots, p_r$, $s - bf_n \notin p_k$ for $k > r$ and any $b \in I$.

Since $s \in p_1, \dots, p_r$, $f_n - s^2 \notin p_1, \dots, p_r$. If $f_n - s^2 \notin p_{r+1}, \dots, p_t$, then we are done. So by reordering $p_{r+1}, \dots, p_t$, we may assume $f_n - s^2 \notin p_{r+1}, \dots, p_{r+l}$ and $f_n - s^2 \in p_{r+l+1}, \dots, p_t$. Let $\lambda \in I \cap (\cap_{i=1}^{r+l} p_i) \setminus \cup_{j=r+l+1}^t p_j$ (such a λ does exist), and $s_1 = s + \lambda f_n$. Then $f_n - s_1^2 = f_n - s^2 - \lambda f_n(2s + \lambda f_n)$, and $f_n - s_1^2 \notin p_1, \dots, p_{r+l}$ by our choice of λ.

Now we claim that $f_n - s_1^2 \notin p_{r+l+1}, \dots, p_t$. If $f_n - s_1^2 \in p_j$ for some $j \in \{r+l+1, \dots, t\}$, then $2s + \lambda f_n \in p_j$. Notice that since A is a commutative ring containing a field k, either 2 is invertible in A or 2 is zero in A. If 2 is zero in A, then $\lambda f_n \in p_j$, which is impossible. If 2 is invertible in A, then $s + (1/2)\lambda f_n \in p_j$, which contradicts that $s - bf_n \notin p_k$ for $k > r$ and any $b \in I$. So the claim follows.

Therefore $f_n - s_1^2$ is a nonzero divisor in $A/(f_1, \dots, f_{n-1})$. By our choice of s_1, we have that $I = (f_1, \dots, f_n, s_1)$, $s_1(1 - s_1) \in (f_1, \dots, f_n)$, $s_1 \in I^2$, $I = (f_1, \dots, f_n) + I^2$, and $\{f_1, \dots, f_{n-1}, f_n - s_1^2\}$ is a regular sequence in I. $\square$

The proof of the following lemma is a generalization of [16, Proposition 4.3] which is inspired by the statements of Bhatwadekar, Das, and Mandal in [9, Lemma 6.1 and Proposition 6.2].

Lemma 2.11. *Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k and n be an even integer such that $2n \geq d + 3$. Let I be an ideal of height n such that $I = (f_1, \dots, f_{n-1}, f_n) + I^2$, where $f_1, \dots, f_{n-1}, f_n$ form a regular sequence in A. Let ω_I be the surjection: $(A/I)^n \to I/I^2$ given by $(\overline{f_1}, \dots, \overline{f_{n-1}}, \overline{f_n})$ and $-\omega_I$ be the surjection: $(A/I)^n \to I/I^2$ given by $(\overline{f_1}, \dots, \overline{f_{n-1}}, -\overline{f_n})$. Define $J = I^{(2)} = (f_1, \dots, f_{n-1}) + I^2$. Then there exists a surjection ω: $(A/J)^n \to J/J^2$, such that $(I^{(2)}, \omega) = (I, \omega_I) + (I, -\omega_I)$ in $E^n(A)$.*

Proof. Applying Lemma 2.10, we can find $s \in A$ such that the image of $f_n - s^2$ in $A/(f_1, \dots, f_{n-1})$ is a nonzero divisor, $I = (f_1, \dots, f_n, s)$ and $s(1-s) \in (f_1, \dots, f_n)$. Let $K_1 = (f_1, \dots, f_n, 1-s)$; then $K_1 \cap I = (f_1, \dots, f_n)$. Since $\{f_1, \dots, f_{n-1}, f_n - s^2\}$ is a regular sequence and $I = (f_1, \dots, f_{n-1}, f_n - s^2) + I^2$, we can write $(f_1, \dots, f_{n-1}, f_n - s^2) = I \cap K_2$ for some K_2 in A, which is comaximal with I. If $K_1 = A$, or $K_2 = A$, then the conclusion of the lemma clearly holds. So we may assume that K_1, K_2 are ideals of height n. Let $g = f_n - s^2$; then $gA + K_1 = A$, and hence I, K_1, K_2 are pairwise comaximal. It is clear that $g = -s^2$ is a unit modulo K_1 and $f_n = s^2$ is a unit modulo K_2. Since $I^{(2)} = (f_1, \dots, f_{n-1}) + I^2$, $I^{(2)} \cap K_1 \cap K_2 = (f_1, \dots, f_{n-1}, gf_n)$. So we have the surjective homomorphisms

$$A^n \xrightarrow{\alpha} I \cap K_1$$
$$A^n \xrightarrow{\alpha'} I \cap K_1$$

given by $(f_1, \ldots, f_{n-1}, f_n)$ and $(f_1, \ldots, f_{n-1}, -f_n)$ respectively. Then $\omega_I = \alpha \otimes A/I$, $-\omega_I = \alpha' \otimes A/I$. Let $\omega_{K_1} = \alpha \otimes A/K_1$ and $-\omega_{K_1} = \alpha' \otimes A/K_1$. Then we have $(I, -\omega_I) + (K_1, -\omega_{K_1}) = 0$ in $E^n(A)$.

$I^{(2)} \cap K_1 \cap K_2 = (f_1, \ldots, f_{n-1}, gf_n)$. So we also have two natural surjective homomorphisms

$$A^n \xrightarrow{\beta} I \cap K_2$$
$$A^n \xrightarrow{\gamma} I^{(2)} \cap K_1 \cap K_2$$

given by $(f_1, \ldots, f_{n-1}, g)$ and $(f_1, \ldots, f_{n-1}, gf_n)$ respectively.

Let $\omega_{K_2} = \beta \otimes A/K_2$, and $\omega = \gamma \otimes A/I^{(2)}$. Since $g = -s^2$ is a unit modulo K_1 and $f_n = s^2$ is a unit modulo K_2, from the surjections β and γ and Lemma 2.7, we have: $\beta \otimes A/I = \omega_I, \gamma \otimes A/K_1 = -\bar{s}^2\omega_{K_1}, \gamma \otimes A/K_2 = \bar{s}^2\omega_{K_2}$ and the following relations in $E^n(A)$: $(I, \omega_I) + (K_2, \omega_{K_2}) = 0$, and $(I^{(2)}, \omega) + (K_1, -\bar{s}^2\omega_{K_1}) + (K_2, \bar{s}^2\omega_{K_2}) = (I^{(2)}, \omega) + (K_1, -\omega_{K_1}) + (K_2, \omega_{K_2}) = 0$. Hence $(I, \omega_I) + (I, -\omega_I) = (I, \omega_I) + (I, -\omega_I) + (I^{(2)}, \omega) + (K_1, -\omega_{K_1}) + (K_2, \omega_{K_2}) = (I, \omega_I) + (K_2, \omega_{K_2}) + (I^{(2)}, \omega) + (I, -\omega_I) + (K_1, -\omega_{K_1}) = 0 + (I^{(2)}, \omega) + 0 = (I^{(2)}, \omega)$. Thus $(I^{(2)}, \omega) = (I, \omega_I) + (I, -\omega_I)$ in $E^n(A)$. $\square$

Now we are ready to state our main theorem which gives an affirmative answer to Question 2.1:

Theorem 2.12. *Let A be a regular ring of dimension d ($d \geq 3$) containing an infinite field k. Let n be an even integer such that $2n \geq d + 3$. Let I be an ideal in A of height n and P be a projective A-module of rank n. Suppose $P \oplus A \approx A^{n+1}$ and there is a surjection α: $P \to I$. Then I is a set theoretic complete intersection ideal in A.*

Proof. Let ω and $-\omega$ be as in Proposition 2.8. Then by Lemma 2.11, there exists a surjection ω': $(A/J)^n \to J/J^2$, where $J = I^{(2)}$, such that $(I^{(2)}, \omega') = (I, \omega_I) + (I, -\omega_I)$ in $E^n(A)$. Using Proposition 2.8, we have $(I^{(2)}, \omega') = 0$ in $E^n(A)$. By Theorem 4.2 in [3], $I^{(2)}$ is generated by n elements and hence $I^{(2)}$ is a complete intersection ideal in A. Therefore, I is a set theoretic complete intersection ideal in A. $\square$

Corollary 2.13. *Let A be a smooth affine $\mathbb{C}$-algebra of dimension $n + 1$, where $n \geq 4$ and even. Let I be an ideal of height n. Suppose I is the image of a stably free A-module of rank n. Then I is a set theoretic complete intersection ideal in A.*

Proof. Applying the cancellation theorem of Suslin [14, Theorem 1] and using Theorem 2.12, we see that I is a set theoretic complete intersection ideal in A. The proof of the corollary is complete. $\square$

Remark 2.14. For the case when $n = 3$ in Corollary 2.13, let α be the surjection: $P \to I$, where P is a stably free module of rank 3. By the cancellation theorem of Suslin [14, Theorem 1], we have $P \oplus A \approx A^4$. By Bass [1], P has a unimodular element. Then using the same arguments as in Proposition 2.3 except for applying the cancellation theorem of Suslin instead of applying Bass cancellation, we can conclude that I is a complete intersection ideal in A. Therefore, Corollary 2.13 also holds when $n = 3$.

The following corollary gives a positive answer to the Question 1.1 in the case when A is a smooth affine $\mathbb{C}$-algebra.

Corollary 2.15. *Let A be a smooth affine $\mathbb{C}$-algebra of dimension $n+1$, where $n \geq 4$ and even. Let I be a local complete intersection ideal of height n such that I/I^2 is a free A/I-module of rank n. Suppose (A/I) is torsion in $K_0(A)$. Then I is a set theoretic complete intersection in A.*

Proof. Let m be the integer such that $m(A/I) = 0$ in $K_0(A)$. Write $I = (f_1, \ldots, f_{n-1}, f_n) + I^2$, where $f_1, \ldots, f_{n-1}, f_n$ form a regular sequence. Let $J = (f_1, \ldots, f_{n-1}) + I^m$. By a result of Mandal [10, Lemma 2.3], $(A/J) = m(A/I)$ in $K_0(A)$ and hence $(A/J) = 0$ in $K_0(A)$. By a result of Murthy [12, Theorem 2.2], we see that K is the image of a stably free A-module of rank n, where $K = (f_1, \ldots, f_{n-1}) + J^{(n-1)!}$. By Corollary 2.13, we conclude that K is a set theoretic complete intersection in A and hence so is I. □

Remark 2.16. As in Remark 2.14, Corollary 2.15 also holds when $n = 3$.

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