NORTH-HOLLAND

Automorphisms and Derivations of Upper Triangular Matrix Rings

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ABSTRACT

Kezlan proved that for a commutative ring C, every C-automorphism of the ring of upper triangular matrices over C is inner. We generalize this result to rings in which all idempotents are central; moreover we show that for a semiprime ring A and central subring C, every C-automorphism of the ring of upper triangular matrices over C is the composite of an inner automorphism and an automorphism induced from a C-automorphism of A. By the method of proof we re-prove results of S. P. Coelho and C. P. Milies and of Mathis, stating that a derivation of a ring of upper triangular matrices of a C-algebra ($n \times n$ matrices over A) is a sum of an inner derivation and a derivation induced from a C-derivation of A. By an example we show that an extra assumption is needed for proving the above result of automorphisms of upper triangular matrices. Finally we consider automorphisms of subrings of $n \times n$ matrices over a commutative ring C, where entries over the diagonal are from C and below the diagonal are taken from a nil ideal. We prove that all such automorphisms are inner.

1. INTRODUCTION

We consider an (associative) C-algebra, A. By $T_n(A)$ we denote the C-algebra of upper triangular matrices over A. If ψ is a C-automorphism

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of A, then ψ induces a C-automorphism Ψ of $T_n(A)$ simply by $\Psi((a_{ij})_{i,j}) = (\psi(a_{ij}))_{i,j}$.

In case A is prime, it was proved that every automorphism of $T_n(A)$ is of the form

$$\Phi \circ \Psi, (*)$$

where Φ is inner and Ψ is induced from a C-automorphism of A [1, 5].

For C commutative every automorphism of $T_n(C)$ is inner; cf. [6].

We let (e_{ij}) denote the matrix units, and we show that under quite general circumstances every automorphism of $T_n(A)$ can be written as in $(*)$.

If B is a matrix in $\mathrm{Mat}_n(A)$ [in $T_n(A)$], then the derivation $X \to [B, X]$ is denoted d_B, and if d is a C-derivation of A, then $\bar{d}$ denotes the derivation $(a_{ij})_{i,j} \to (d(a_{ij}))_{i,j}$ of $\mathrm{Mat}_n(A)$ [$T_n(A)$, respectively].

For a two-sided ideal I of a ring A, we let $T_n(I, A)$ denote the subring of the rings of $n \times n$ matrices consisting of the matrices having the elements strictly below the diagonal from I, and above and on the diagonal any elements of A.

2. AUTOMORPHISMS OF TRIANGULAR MATRIX RINGS

Barker and Kezlan proved in [1] that a C-automorphism of $T_n(A)$ is of the form $\Phi \circ \Psi$, where Φ is inner and Ψ is induced from a C-automorphism of A, provided every C-endomorphism is an automorphism. In case A is prime the result also holds [5].

Kezlan also proved that every C-automorphism of $T_n(C)$ is inner [6].

All these results are generalized in the following

THEOREM 1. *Let A be a ring and C a central subring of A. Every C-automorphism of $T_n(A)$ can be written*

$$\Phi \circ \Psi,$$

where Ψ is induced by a C-automorphism of A and Φ is inner, provided either

(i) *all idempotents of A are central, or*
(ii) *A is semiprime.*

Proof. The argument follows the lines of [4, Theorem A] and [5, Theorem 1].

As usual we let $(e_{ij})_{1 \le i, j \le n}$ denote the matrix units.

The proof is by induction on n, the size of the upper triangular matrices. $n = 1$ being obvious, we may assume the result for $n - 1$ and suppose σ is a C-automorphism of $T_n(A)$.

We will use without specific reference the fact that an automorphism σ is inner if and only if for an inner automorphism τ, either $\sigma\tau$ or $\tau\sigma$ is inner.

We first assume $\sigma(e_{11}) = e_{11}$ and then reduce the general case to the special one.

Let f denote the idempotent $\sum_{i\geq 2} e_{ii}$. Clearly $\sigma(f) = f$, and thus σ induces an automorphism σ' of $fT_n(A)f$, which is isomorphic to $T_{n-1}(A)$. By the induction hypothesis

$$\sigma' = \Phi' \circ \Psi',$$

where Φ' is conjugation by an invertible matrix D in $T_{n-1}(A)$ and Ψ' is induced from C-automorphism ψ of A.

Let Ψ be the automorphism of $T_n(A)$ induced by ψ, and let Φ denote conjugation by

$$\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}.$$

$\Phi \circ \Psi$ is a C-automorphism of $T_n(A)$. It fixes every element of $fT_n(A)f$ and e_{11}.

We will prove that such an automorphism is inner, and the first part of the proof will be completed.

Let σ be a C-automorphism of $T_n(A)$ fixing e_{11} and every element of $fT_n(A)f$. Then

$$\sigma(e_{12}) = \sigma(e_{11}e_{12}e_{22}) = e_{11}\sigma(e_{12})e_{22} = xe_{12}$$

for some $x \in A$.

If $\sigma(X) = e_{12}$ for $X \in T_n(A)$, then $\sigma(X) = \sigma(e_{11})e_{12}\sigma(e_{22})$. Applying σ^{-1} to this equation gives that $e_{12} = \sigma(ye_{12})$ for some $y \in A$. Then

$$\sigma(ye_{12}) = \sigma(e_{12}ye_{22}) = xe_{12}ye_{22} = xye_{12} = e_{12}.$$

Hence $xy = 1$, Also,

$$\sigma(yxe_{12}) = \sigma(e_{12}yxe_{22}) = xe_{12}yxe_{22} = xe_{12} = \sigma(e_{12}).$$

Thus $yx = 1$, and x is a unit in A.

Conjugation by

$$\begin{pmatrix} x & 0 \\ 0 & E \end{pmatrix},$$

where E is the identity $(n-1) \times (n-1)$ matrix, fixes every element of $fT_n(A)f$ as well as e_{11}, and sends e_{12} to $xe_{12} = \sigma(e_{12})$. Hence our first reduction is established if we can show that a C-automorphism σ fixing e_{11}, e_{12}, and every element of $fT_n(A)f$ is the identity.

Notice that

$$\begin{aligned} \sigma(e_{1j}) &= \sigma(e_{12}e_{2j}) = e_{12}e_{2j} = e_{1j}, & j \geq 2, \\ \sigma(ae_{1j}) &= \sigma(e_{1j}ae_{jj}) = e_{1j}ae_{jj} = ae_{1j}, & j \geq 2. \end{aligned}$$

Finally, $\sigma(ae_{11}) = e_{11}\sigma(ae_{11})e_{11} = be_{11}$ for some $b \in A$, and

$$ae_{12} = \sigma(ae_{12}) = \sigma(ae_{11}e_{12}) = \sigma(ae_{11})e_{12} = be_{11}e_{12} = be_{12}.$$

Thus $\sigma(ae_{11}) = ae_{11}$, and $\sigma = 1_{T_n(A)}$.

It remains to show that $\sigma(e_{11}) = e_{11}$. Suppoe first $\sigma(e_{11}) = e_{11} + \sum_{j>1} a_{1j}e_{1j}$; then sending

$$Y \quad \text{to} \quad DYD^{-1},$$

where

$$D = \begin{pmatrix} 1 & & -a_{12} & \cdots & -a_{1n} \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & 1 \end{pmatrix}$$

sends e_{11} to $\sigma(e_{11})$. We are then done with the first part of the proof.

Assume now

$$\sigma(e_{11}) = \begin{pmatrix} a & b_- \\ 0 & C \end{pmatrix}.$$

The matrix e_{11} is idempotent, so are a and C, and moreover

$$b = ab_- + b_-C.$$

In particular $ab_-C = 0$.

Conjugation by

$$\begin{pmatrix} 1 & -ab_- + b_-C \\ 0 & E \end{pmatrix}$$

sends

$$\begin{pmatrix} a & b_- \\ 0 & C \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a & 0 \\ 0 & C \end{pmatrix}.$$

Continuing this process, we may assume $\sigma(e_{11})$ is a diagonal matrix

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}.$$

Since e_{11} is idempotent, so is every $d_1, \dots, d_n$.

Suppose

$$\sigma\begin{pmatrix} r & s \\ 0 & T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e_{11};$$

then

$$\sigma(e_{11})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\sigma(e_{11}) = \begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying σ^{-1} to these equations gives

$$\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$$

and $s = 0$, and thus

$$\sigma\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \sigma\left(\begin{pmatrix} r & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D = \begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\sigma(e_{1j}) = T_j$. In case A is semiprime, T_j must have 0 diagonal, because $e_{1j} \in \operatorname{rad} T_n(A)$, whence $T_j \in \operatorname{rad} T_n(A)$, which is the ideal of strictly upper triangular matrices.

In the other situation, where idempotents of A are central, then

$$\sigma(e_{11} \cdot e_{1j}) = D \cdot T_j = T_j \quad \text{and} \quad \sigma(e_{1j}e_{11}) = T_j D = 0,$$

and thus for all j, T_j has 0 diagonal.

If $T \neq 0$, then for some j,

$$e_{1j}\begin{pmatrix} r & 0 \\ 0 & T \end{pmatrix} \neq 0;$$

hence $\sigma(e_{1j})e_{11} = T_j e_{11} \neq 0$, which is a contradiction. Therfore $\sigma(re_{11}) = e_{11}$, and thus

$$\sigma((1-r)e_{11}) = \begin{pmatrix} d_1 - 1 & 0 \\ 0 & C_0 \end{pmatrix}.$$

But

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1-r & 0 \\ 0 & 0 \end{pmatrix} = 0;$$

thus $d_1 = 1$, and

$$\sigma\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & C_0 \end{pmatrix},$$

where C_0 is diagonal. Replacing σ^{-1} by τ, we have an automorphism τ with

$$\tau\begin{pmatrix} 1 & 0 \\ 0 & C_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and we also have

$$\tau(e_{11}) = re_{11}.$$

Let $\tau(e_{1j}) = T_j$. We then get $T_j = \tau(e_{11}e_{1j}) = re_{11}T_j$, and since $e_{1j}e_{11} = 0$, $T_j re_{11} = 0$. The two assumptions now imply that T_j has 0 diagonal and nonzero elements only in the first row.

If $C_0 \neq 0$, then

$$e_{1j}\begin{pmatrix} 1 & 0 \\ 0 & C_0 \end{pmatrix} \neq 0 \qquad \text{for some } j.$$

Hence

$$T_j \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0,$$

which is a contradiction. Thus $C = 0$, and the proof is completed by the first part of the argument. ■

EXAMPLE. Let Λ be the ring of 2×2 upper trianguler matrices over a commutative ring C. We have a C-automorphism σ of $T_2(\Lambda)$ defined by

$$\sigma\begin{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} & \begin{pmatrix} d & f \\ 0 & e \end{pmatrix} \\ 0 & \begin{pmatrix} g & i \\ 0 & h \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a & d \\ 0 & g \end{pmatrix} & \begin{pmatrix} c & f \\ 0 & i \end{pmatrix} \\ 0 & \begin{pmatrix} b & e \\ 0 & h \end{pmatrix} \end{pmatrix}.$$

It is straightforward to check that in fact σ is a C-automorphism of order 2.

σ is not of the form described in Theorem 1, because $\sigma(e_{11}) \neq e_{11} + r_2e_{12}$ for all $r_2 \in \Lambda$.

3. DERIVATIONS

In this section we re-prove a result of Mathis [7] on C-derivations on $n \times n$ matrices over a ring A.

We have learned that a similar result for triangular matrices has been obtained by Coelho and Milies. We give a short elementary proof of both results, using the technique of the previous section.

THEOREM 2. (Cf. [2], [4], and [7]). *Let A be a C-algebra . Every C-derivation of* $\text{Mat}_n(A)$ *or* $T_n(A)$ *is the sum of an inner derivation and a derivation induced from a C-derivation of A.*

Proof. The proof is by induction. Let D be a C-derivation of $\text{Mat}_n(A)$ $[T_n(A)]$. The equations

$$D(e_{11}) = D(e_{11}^2) = e_{11}D(e_{11}) + D(e_{11})e_{11}$$

show that $D(e_{11})$ has nonzero elements only in the first row and column [row] and 0 in the (1, 1) position.

Let A_0 denote $D(e_{11})$, where the first row is replaced by minus the first row. d_{A_0} and D take the same value on e_{11}.

Thus to prove the theorem we may assume $D(e_{11}) = 0$. Let $f = \sum_{i\geq 2} e_{ii}$; then $D(f\,\text{Mat}_n(A)f) \subseteq f\,\text{Mat}_n(A)f$. By the induction hypothesis D induces a derivation on $f\,\text{Mat}_n(A)f$ $[fT_n(A)f]$, which is a sum of an inner derivation and a derivation induced from a C-derivation of A.

Expressing the inner derivation and the derivation of A in terms of derivations of $\text{Mat}_n(A)$ $[T_n(A)]$, we see it suffices to show that a derivation D of $\text{Mat}_n(A)$ $[T_n(A)]$ which vanishes on e_{11} and every element of $f\text{Mat}_n(A)f$ [every element of $fT_n(A)f$] is inner. Now

$$D(e_{12}) = D(e_{11}e_{12}e_{22}) = e_{11}D(e_{12})e_{22};$$

hence

$$D(e_{12}) = a_0e_{12}, \qquad a_0 \in A.$$

$D_{a_0e_{11}}$ vanishes on e_{11} and every element of $f\,\text{Mat}_n(A)\,f$ $[fT_n(A)f]$, and $e_{12} \to a_0e_{12}$; hence to show D is inner we might as well assume $D(e_{12}) = 0$. Then

$$D(ae_{1p}) = D(e_{11}e_{12}e_{2p}ae_{pp}) = 0, \qquad p \geq 2.$$

We have $D(ae_{11}) = D(ae_{11} \cdot e_{11}) = D(ae_{11}) \cdot e_{11}$, so $D(ae_{11}) = 0$ if and only if $D(ae_{11}) \cdot e_{12} = 0$, $D(ae_{11})e_{12} = D(ae_{11})e_{12} + ae_{11}D(e_{12}) = D(ae_{11}e_{12}) = D(ae_{12}) = 0$. This completes the proof for derivations of $T_n(A)$.

Finally,

$$D(ae_{p1}) = D(ae_{p1}e_{11}) = D(ae_{p1})e_{11};$$

thus $D(ae_{p1})$ is nonzero if and only if $D(ae_{p1}) \cdot e_{1p} \neq 0$. But

$$D(ae_{p1})e_{1p} = ae_{p1}D(e_{1p}) + D(ae_{p1})e_{1p} = D(ae_{pp}) = 0,$$

and so the theorem is proved. ∎

4. AUTOMORPHISMS OF MORE GENERAL RINGS

For a commutative ring R and a prime P in R, we showed in [4] that an R-automorphism of $T_n(P, R)$ which is the restriction of an inner automorphism of $\mathrm{Mat}_n(R)$ is inner. Moreover, Isaacs has shown [3, Theorem 11] that for every R-automorphism σ of $\mathrm{Mat}_n(R)$, σ^n is inner. In general neither automorphisms of $\mathrm{Mat}_n(R)$ nor those of $T_n(P, R)$ are inner.

In this section we generalize Kezlan's result from [6] by proving that for a nil ideas I, every R-automorphism of $T_n(I, R)$ is inner.

We let I be a nil ideal in the commutative ring R (with an identity). If σ is an R-automorphism of $T_n(I, R)$, and we will show that σ is inner.

If 1 denotes the identity matrix of $T_n(I, R)$, then $\sigma(a \cdot 1) = a1$ for $a \in R$. In particular $I \cdot 1T_n(I, R)$ is an invariant ideal of $T_n(I, R)$. Also notice that $I \cdot T_n(I, R) = T_n(I^2, I)$.

Consequently σ induces an R-automorphism, which we also denote by σ, of the factor ring

$$T_{n,R} = T_n(I, R)/T_n(I^2, I).$$

This is not necessarily a ring of upper triangular matrices over R/I with elements below the diagonal from an ideal, but this ring has a similar sort of structure.

An element can be written

$$\begin{pmatrix} c_{11} & & c_{1n} \\ a_{21} & \ddots & \\ a_{n1} & a_{n_{n-1}} & c_{nn} \end{pmatrix},$$

where $c_{ij} \in R/I$ and $a_{kl} \in I/I^2$, and multiplication is for instance given by

$$\begin{aligned} a_{k\ell}e_{k\ell}c_{\ell t}e_{\ell t} &= 0, & k > \ell, \quad \ell < t, \text{ and } k \leq t, \\ a_{k\ell}e_{k\ell}c_{\ell t}e_{\ell t} &= a_{k\ell}c_{\ell t}e_{kt}, & k > \ell, \quad \ell \leq t, \text{ and } k > t, \end{aligned}$$

i.e., it is induced by the action on I/I^2 of R/I.

We will prove that σ is inner. Let

$$\sigma(e_{11}) = \begin{pmatrix} a & c_{12} & \cdots & c_{1n} \\ b_{21} & & & \\ \vdots & & D & \\ b_{n1} & & & \end{pmatrix},$$

where D is an $(n-1)\times(n-1)$ matrix. By r_- $(s_|)$ we denote a row (column) matrix of appropriate size. Let

$$\sigma(e_{11}) = \begin{pmatrix} a & c_- \\ b_| & D \end{pmatrix}.$$

By the definition of the multiplication in $T_{n,R}$ we get $c_-b_| = 0$ in R/I.

If we express that $\sigma(e_{11})$ is idempotent, we get

$$\begin{aligned} \begin{pmatrix} a^2 + c_-b_| & ac_- + c_-D \\ b_|a + Db_| & b_|c_- + D^2 \end{pmatrix} &= \begin{pmatrix} a^2 & ac_- + c_-D \\ b_|a + Db_| & b_|c_- + D^2 \end{pmatrix} \\ &= \begin{pmatrix} a & c_- \\ b_| & D \end{pmatrix}. \end{aligned}$$

We have the following equations:

$$\begin{aligned} \begin{pmatrix} 0 & c_- \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b_|c_- + D^2 \end{pmatrix} &= \begin{pmatrix} 0 & c_-D^2 \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & c_- \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} &= \begin{pmatrix} 0 & c_-D \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We denote by U the matrix

$$\begin{pmatrix} 1 & ac_- - c_-D \\ 0 & E \end{pmatrix}.$$

The inner automorphism given by $X \to UXU^{-1}$ sends

$$\sigma(e_{11}) \quad \text{to} \quad \begin{pmatrix} a & -ac_- + c_- - c_- D^2 \\ b_| & \widehat{D}_1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b_| & \widehat{D}_1 \end{pmatrix}.$$

[Also notice that U defines an inner automorphism when considered as a matrix in $T_n(I, R)$.] Thus without loss of generality we may assume

$$\sigma(e_{11}) = \begin{pmatrix} a & 0_- \\ b_| & D \end{pmatrix},$$

where

$$\begin{pmatrix} a^2 & 0_- \\ b_| a + D b_| & D^2 \end{pmatrix} = \sigma(e_{11}).$$

Let

$$U_1 = \begin{pmatrix} 1 & 0 \\ -b_| a + D b_| & E \end{pmatrix};$$

then the inner automorphism given by $X \to U_1 X U_1^{-1}$ sends $\sigma(e_{11})$ to the matrix

$$\begin{pmatrix} a & 0 \\ -b_| + D b_| a + b_| - D b_| a - D^2 b_| & D \end{pmatrix}$$
$$= \begin{pmatrix} a & 0 \\ b_| - b_| a - D b_| & D \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & D \end{pmatrix}.$$

This inner automorphism is also induced from an inner automorphism of $T_n(I, R)$.

We will show that $a = 1$. Assume $a \neq 1$. To prove that a given automorphism of $T_n(I, R)$ is inner, we may without loss of generality assume that the automorphism σ induced on $T_{n,R}$ has

$$\sigma(e_{11}) = \begin{pmatrix} a & 0 \\ 0 & D \end{pmatrix}.$$

We let e be a lifting of a to R modulo the nil ideal I, and replace R by $(1 - e)R$ and I be $(1 - e)I$. Thus we have an automorphism of $T_n(I, R)$ such that the induced automorphism of $T_{n,R}$ has

$$\sigma(e_{11}) = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

We will show this is impossible.

If we replace

$$\begin{pmatrix} a & c_- \\ b_| & D \end{pmatrix} \text{ by } \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$$

in the above, it follows that there is a ring R, a nil ideal I, and an automorphism of $T_n(I,R)$ such that the induced automorphism σ of $T_{n,R}$ has

$$\sigma(e_{11}) = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & L \end{pmatrix},$$

where 1 is in the jth row, and L is a matrix of appropriate size. Let f be such that $\sigma(f) = e_{jj}$; then $\sigma(f)\sigma(e_{11}) = \sigma(e_{11})\sigma(f) = \sigma(f)$ and hence $fe_{11} = e_{11}f = f$, so $f = \widehat{e}e_{11}$, where $\widehat{e}$ is idempotent. Let e be an idempotent in R that is a lifting of $\widehat{e}$. Thus we have a commutative ring (eR), a nil ideal I, and an automorphism σ of $T_n(I,R)$ such that the induced automorphism $\widehat{\sigma}$ of $T_{n,R}$ has $\widehat{\sigma}(e_{11}) = e_{jj}$. We have the following relation:

$$\widehat{\sigma}^{-1}(e_{11}) \cdot e_{11} = e_{11}\widehat{\sigma}^{-1}(e_{11}) = 0.$$

Hence $\widehat{\sigma}^{-1}(e_{1j}) = \widehat{\sigma}^{-1}(e_{11})\widehat{\sigma}^{-1}(e_{ij})e_{11}$, so

$$\widehat{\sigma}^{-1}(e_{1j}) = \begin{pmatrix} 0 & 0 \\ b_| & 0 \end{pmatrix}, \qquad \text{or} \quad \widehat{\sigma}\begin{pmatrix} 0 & 0 \\ b_| & 0 \end{pmatrix} = e_{1j},$$

for some $b_| = \binom{b_1}{b_n}$. This means that

$$\sigma\begin{pmatrix} 0 & 0 \\ b_1 & \\ \vdots & 0 \\ b_n & \end{pmatrix} = e_{1j} + A,$$

where $A = (a_{ij}),\ a_{ij} \in I$.

Let $\mathfrak{A}$ be the ideal generated by the b_j's and the a_{ij}'s. $\mathfrak{A}$ is nilpotent, so there exists a k such that $\mathfrak{A}^{k-1} \neq 0$ and $\mathfrak{A}^k = 0$. Thus

$$\mathfrak{A}^{k-1} \cdot \sigma\begin{pmatrix} 0 & 0 \\ b_| & 0 \end{pmatrix} = \sigma(0) = \mathfrak{A}^{k-1}e_{1j} \neq 0,$$

a contradiction. Thus we have proved that if σ is an automorphism of $T_{n,R}$ and

$$\sigma(e_{11}) = \begin{pmatrix} a & 0 \\ 0 & D \end{pmatrix},$$

then $a = 1$.

We will now prove that $D = 0$. To do so notice that

$$e_{11} = e_{11}\sigma(e_{11}) = \sigma(e_{11})e_{11};$$

hence

$$\sigma^{-1}(e_{11}) = \sigma^{-1}(e_{11})e_{11} = e_{11}\sigma^{-1}(e_{11}),$$

so

$$\sigma^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & 0 \end{pmatrix}$$

for some a_0.

If $a_0 \neq 1$, then $\sigma^{-1}((1 - a_0)e_{11}) = 0$, which is a contradiction, so

$$\sigma^{-1}(e_{11}) = e_{11} \text{ and } \sigma(e_{11}) = e_{11}.$$

If we lift back to $T_n(I, R)$, it follows that up to an inner automorphism

$$\sigma(e_{11}) = e_{11} + (a_{ij})_{1 \leq i, j \leq n}, \qquad a_{ij} \in I.$$

(In fact $i > j$, $a_{ij} \in I^2$.) We write the matrix $(a_{ij})_{i,j}$ as follows:

$$(a_{i,j})_{i,j} = \begin{pmatrix} a & b_- \\ c_| & C \end{pmatrix}.$$

The following equations express that $\sigma(e_{11})$ is idempotent:

$$a + a^2 + b_- c_| = 0, \tag{1}$$
$$c_| a + C c_| = 0, \tag{2}$$
$$a b_- + b_- C = 0, \tag{3}$$
$$c_| b_- + C^2 = C. \tag{4}$$

Conjugation by $\left(\begin{smallmatrix} 1 & b_- \\ 0 & E \end{smallmatrix}\right)$ maps $e_{11} + (a_{i,j})_{i,j}$ to

$$\begin{pmatrix} 1 - a^2 & a^2 b_- + b_- C \\ c_| & C^2 \end{pmatrix},$$

as is easily seen by a direct calculation using the relations (1)–(4).

By the fact that both a and C are nilpotents, we see that there is an inner automorphism τ such that

$$\tau\sigma(e_{11}) = \begin{pmatrix} 1 & \widetilde{b}_- \\ c_| & 0 \end{pmatrix}.$$

Notice that (1)–(4) in this case are

$$\widetilde{b}_- c_| = 0, \qquad (1')$$
$$c_| \widetilde{b}_- = 0. \qquad (4')$$

So

$$\begin{pmatrix} 1 & \widetilde{b}_- \\ -c_| & E \end{pmatrix}\begin{pmatrix} 1 & -\widetilde{b}_- \\ c_| & E \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix},$$

so we have an inner automorphism sending a matrix X to

$$\begin{pmatrix} 1 & b_-^{\sim} \\ -c_| & E \end{pmatrix} X \begin{pmatrix} 1 & -b_-^{\sim} \\ c_| & E \end{pmatrix}.$$

This automorphism maps

$$\begin{pmatrix} 1 & b_-^{\sim} \\ c_| & 0 \end{pmatrix} \quad \text{to} \quad e_{11}.$$

Thus in order to show that automorphisms of $T_n(I,R)$ are inner, one can without loss of generality assume that e_{11} is fixed.

The proof is now by induction on n. Let $f = \sum_{i\geq 2} e_{ii}$, $fT_n(I,R)f \cong T_{n-1}(I,R)$, and by the induction hypothesis we may as well assume every element of $fT_n(I,R)f$ is fixed.

The argument from the beginning of the proof of Theorem 1 shows that there is an inner automorphism τ fixing e_{11} and every element of $fT_n(I,R)f$ and such that $\sigma(e_{12}) = \tau(e_{12})$. Hence we may assume that $\sigma(e_{12}) = e_{12}$ as well.

For $a \in R$,

$$\sigma(ae_{1p}) = \sigma(e_{12}ae_{2p}) = e_{12}ae_{2p} = ae_{1p}, \qquad p \geq 2,$$
$$\sigma(ae_{11}) \in e_{11}T_n(I,R)e_{11}.$$

Hence

$$\sigma(ae_{11}) = ae_{11} \quad \text{if and only if} \quad \sigma(ae_{11})\sigma(e_{12}) = ae_{11}e_{12},$$

which clearly holds. For $a \in I$,

$$\sigma(ae_{p1}) = \sigma(e_{pp}ae_{p1}e_{11}) = e_{pp}\sigma(ae_{p1})e_{11},$$

so

$$\sigma(ae_{p1}) \in Ie_{p1},$$

and

$$\sigma(ae_{p1}) = ae_{p1} \quad \text{if and only if} \quad \sigma(ae_{p1})e_{1p} = ae_{pp},$$

which clearly holds.

We have now shown

THEOREM 3. *Let R be a commutative ring with an identity element, and I a nil ideal. Every R-automorphism of $T_n(I, R)$ is inner.*

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REFERENCES

1 G. P. Barker and T. P. Kezlan, Automorphisms of algebras of upper triangular matrices, *Arch. Math.* 55:38–43 (1990).
2 S. P. Coelho and C. P. Milies, Derivations of upper triangular matrix rings, preprint.
3 I. M. Isaacs, Automorphisms of matrix algebras over commutative rings, *Linear Algebra Appl.* 31:215–231 (1980).
4 S. Jøndrup, Automorphisms of upper triangular matrix rings, *Arch. Math.* (*Basel*) 49:497–502 (1987).
5 S. Jøndrup, The group of automorphisms of certain subalgebras of matrix algebras, *J. Algebra* 141(1):106–114 (1991).
6 T. P. Kezlan, A note on algebra automorphisms of triangular matrices over commutative rings, *Linear Algebra Appl.* 135:181–184 (1990).
7 P. Mathis, Differential polynomial rings and Mortia equivalence, *Comm. Algebra* 10:2001–2017 (1982).

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