Variational iteration method and homotopy perturbation method for nonlinear evolution equations

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Abstract

He’s variational iteration and homotopy perturbation methods are applied to various evolution equations. To assess the accuracy of the solutions, we compare the results with the exact solutions, revealing that both methods are capable of solving effectively a large number of nonlinear differential equations with high accuracy.

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1. Introduction

There are many nonlinear equations, which are quite useful and applicable in engineering and physics such as the well-known KdV equation [1], MKdV equation, BBM equation [2], Burgers equation [2], KdV–KSV equation [2], RLW equation [2] and so on. Since solving these equations needs some nonphysical assumptions, some various approximate methods have recently been developed to solve linear and nonlinear differential equations [3–9]. In this paper, we solve some evolution equations by the homotopy perturbation method (HPM) [10–12] and variational iteration method (VIM) [13–15], which are widely applied to various engineering problems [16–25], and then compare the obtained results with exact solutions.

2. He’s variational iteration method

2.1. The basic idea

To illustrate the basic concepts of the VIM [13,8,9], we consider the following deferential equation:

\[ Lu + Nu = g(x). \]  

(2.1)
where $L$ is a linear operator, $N$ a nonlinear operator, and $g(x)$ a heterogeneous term. According to the VIM, we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda [Lu_n(\tau) + Nu_n(\tau) - g(\tau)] \, d\tau. \tag{2.2}$$

The VIM is a powerful tool to search for approximate solutions for various nonlinear problems [23–25].

2.2. Applications

2.2.1. Example 1

Let us consider the RLW equation which reads

$$u_t - u_{xxx} + \left(\frac{u^2}{2}\right)_x = 0, \quad -\infty < x < +\infty, \quad t > 0, \tag{2.3}$$

with the following initial condition:

$$u(x, 0) = x. \tag{2.4}$$

To solve Eqs. (2.3) and (2.4), using the VIM, we have the correction functional as

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda [u_t - u_{xxx} + uu_x] \, d\tau \tag{2.5}$$

where $uu_x$ indicates the restricted variations; i.e. $\delta (uu_x) = 0$. Making the above correction functional stationary, we can obtain the following stationary conditions:

$$1 + \lambda |_{\tau=1} = 0, \tag{2.6a}$$

$$\lambda' = 0. \tag{2.6b}$$

The Lagrange multiplier can, therefore, be identified as

$$\lambda = -1. \tag{2.7}$$

Substituting Eq. (2.7) into the correction functional equation system (2.5) results in the following iteration formula:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t [u_t - u_{xxx} + uu_x] \, d\tau. \tag{2.8}$$

A higher number of iterations leads to results closer to the exact solution. Using the iteration formula (2.8) and the initial guess $(u_0)$, five iterations are listed as follows.

The first iteration:

$$u_1(x, t) = x(1 - t). \tag{2.9a}$$

The second iteration:

$$u_2(x, t) = x \left(1 - t + t^2 - \frac{t^3}{3}\right). \tag{2.9b}$$

And, finally, the fifth iteration:

$$u_5(x, t) = x \left(1 - t + t^2 - t^3 + t^4 - t^5 + \frac{43}{45} t^6 - \frac{13}{15} t^7 + \frac{943}{1260} t^8 - \frac{3497}{5670} t^9 + \frac{27 523}{56 700} t^{10} \right.$$

$$- \frac{1477}{40 500} t^{11} + \frac{17 779}{68 040} t^{12} - \frac{13 141}{73 710} t^{13} + \frac{10 19}{8820} t^{14} - \frac{63 283}{893 025} t^{15} + \frac{43 363}{105 8400} t^{16}$$

$$- \frac{1080 013}{48 580 560} t^{17} + \frac{2588}{229 635} t^{18} - \frac{162 179}{3054 1455} t^{19} + \frac{16 511}{7144 200} t^{20} - \frac{207 509}{225 042 300} t^{21}$$
Therefore, we have

\[ u(x, t) = u_5(x, t). \] (2.10)

It is obvious that a higher number of iterations makes \( u_n(x, t) \) converge to the exact solution, \( \frac{x}{1 + t} \). See Fig. 1.

### 2.2.2. Example 2

We consider the following equation:

\[ u_t + u_x = 2u_{xxt}, \quad x \in \mathbb{R}, t > 0, \] (2.11)

\[ u(x, 0) = \exp(-x). \] (2.12)

To solve Eqs. (2.11) and (2.12) using the VIM, we have the correction functional as

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda [u_t + u_x - 2u_{xxt}] \, d\tau. \] (2.13)

Making the above correction functional stationary, we obtain the following stationary conditions:

\[ 1 + \lambda |_{\tau=1} = 0, \] (2.14a)

\[ \lambda' = 0. \] (2.14b)

The Lagrange multiplier can therefore be identified as

\[ \lambda = -1. \] (2.15)

Substituting Eq. (2.15) into the correction functional Eq. (2.13) results in the following iteration formula:

\[ u_{n+1}(x, t) = u_n(x, t) - \int_0^t \{u_t + u_x - 2u_{xxt}\} \, d\tau. \] (2.16)

Using the iteration formula (2.16) and the initial guess \( (u_0) \), six iterations are made as follows.
The first iteration
\[ u_1(x, t) = e^{-x}(1 + t). \]  
(2.17a)

The second iteration
\[ u_2(x, t) = \frac{e^{-x}}{2}(t^2 + 6t + 2). \]  
(2.17b)

And, finally, the sixth iteration reads
\[ u_6(x, t) = \frac{e^{-x}}{720}(t^6 + 66t^5 + 1470t^4 + 13320t^3 + 46440t^2 + 45360t + 720). \]  
(2.17c)

Therefore, we have
\[ u(x, t) = u_6(x, t) = \frac{e^{-x}}{720}(t^6 + 66t^5 + 1470t^4 + 13320t^3 + 46440t^2 + 45360t + 720). \]  
(2.18)

See Fig. 2.

2.2.3. Example 3

We consider the following example:
\[ u_t + u_{xxxx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \]  
\[ u(x, 0) = \sin(x). \]  
(2.19)

(2.20)

The correction functional is then in the form
\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \{ u_t + u_{xxxx} \} \, dt \]  
(2.21)

where \( \lambda = -1 \), so that Eq. (2.21) changes to
\[ u_{n+1}(x, t) = u_n(x, t) - \int_0^t \{ u_t + u_{xxxx} \} \, dt. \]  
(2.22)

Using the iteration formula (2.22) and the initial condition as \( u_0 \), four iterations are made and results are as follows:
\[ u_1(x, t) = (1 - t) \sin(x). \]  
(2.23)
Fig. 3. The surface shows the approximate solution $u(x, t)$ for Eqs. (2.19) and (2.20).

\[ u_2(x, t) = \left( \frac{t^2}{2} - t + 1 \right) \sin(x). \]  
\[ u_3(x, t) = \left( \frac{-t^3}{6} + \frac{t^2}{2} - t + 1 \right) \sin(x). \]  
\[ u_4(x, t) = (t^4 - 4t^3 + 12t^2 - 24t + 24) \frac{\sin(x)}{24} \]  
and therefore,

\[ u(x, t) = u_4(x, t) = (t^4 - 4t^3 + 12t^2 - 24t + 24) \frac{\sin(x)}{24}. \]

One can now try to make a higher number of iterations which results in the convergence of $u_n(x, t)$ to the exact solution, $e^{-t} \sin(x)$. See Fig. 3.

3. He’s homotopy perturbation method

3.1. The basic idea

To illustrate the basic ideas of this method [11,8,9], we consider the following nonlinear differential equation:

\[ A(u) - f(r) = 0, \quad r \in \Omega. \]  
(3.1)

The boundary condition is

\[ B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \]  
(3.2)

where $A$ is a general differential operator, $B$ a boundary operator, $f(r)$ a known analytical function and $\Gamma$ is the boundary of the domain $\Omega$.

Eq. (3.1) can, therefore, be rewritten as

\[ L(u) + N(u) - f(r) = 0. \]  
(3.3)

According to the HPM, we construct a homotopy as follows

\[ H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad P \in [0, 1], r \in \Omega. \]  
(3.4)
Obviously, we have
\begin{align}
H(v, 0) &= L(v) - l(u_0) = 0, \\
H(v, 1) &= A(v) - f(r) = 0.
\end{align}

(3.5) (3.6)

According to HPM, we can use the embedding parameter \( p \) as an expanding parameter [12], and assume that the solution of Eq. (3.4) can be written as a power series in \( p \):

\[ v = v_0 + p v_1 + p^2 v_2 + \cdots. \]

Setting \( p = 1 \) results in the approximate solution of Eq. (3.4):

\[ u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \]

(3.7) (3.8)

The combination of the perturbation method and the homotopy method is called HPM, which eliminates the limitations of the traditional perturbation methods, although it can take full advantage of the traditional perturbation techniques.

### 3.2. The applications

#### 3.2.1. Example 1

Similar to the first example solved previously, the equation is

\[ u_t - u_{xxt} + \left(\frac{u^2}{2}\right)_x = 0, \quad x \in \mathbb{R}, \; t > 0, \]

with the initial condition

\[ u(x, 0) = x. \]

(3.9) (3.10)

Substituting Eq. (3.9) into (3.4), we have an equation system including \( n + 1 \) equations to be simultaneously solved; \( n \) is the order of \( p \) in Eq. (3.7). Assuming \( n = 5 \) the system is as follows:

\[
\begin{align}
0 &- x = 0, \quad u_0(x, 0) = x, \\
0 &+ u_0 u_0_x - u_0_{xx} + x = 0, \quad u_1(x, 0) = 0, \\
0 &+ u_1 u_0 + u_0 u_{1x} - u_{1xx} = 0, \quad u_2(x, 0) = 0, \\
0 &- u_2_{xx} + u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} = 0, \quad u_3(x, 0) = 0, \\
0 &- u_3_{xx} + u_0 u_{3x} + u_0 u_{3x} + u_2 u_{1x} + u_1 u_{2x} = 0, \quad u_4(x, t) = 0, \\
0 &- u_4_{xx} + u_3 u_{1x} + u_3 u_{3x} + u_2 u_{2x} + u_0 u_{4x} = 0, \quad u_5(x, t) = 0.
\end{align}
\]

(3.11)

One can now obtain a solution for equation system (3.11) in the form

\[
\begin{align}
u_0(x, t) &= x(1 + t), \\
u_1(x, t) &= -xt \left(2 + t + \frac{t^2}{3}\right). \\
u_2(x, t) &= \frac{2}{15} x^2 t^2 (t^3 + 5t^2 + 15t + 15). \\
u_3(x, t) &= \frac{-xt^3}{315} (17t^4 + 119t^3 + 483t^2 + 945t + 840). \\
u_4(x, t) &= \frac{2}{2835} x^4 t^4 (31t^5 + 279t^4 + 1422t^3 + 4032t^2 + 6615t + 4725). \\
u_5(x, t) &= \frac{-2}{155925} x^5 (691t^6 + 7601t^5 + 46530t^4 + 170280t^3 + 396495t^2 + 537075t + 332640).
\end{align}
\]

(3.12a) (3.12b) (3.12c) (3.12d) (3.12e) (3.12f)
Having \( u_i, i = 0, 1, \ldots, 5 \), the solution \( u(x, t) \) is as follows:

\[
 u(x, t) = \sum_{i=0}^{5} u_i(x, t) = x \left( 1 - t + t^2 - t^3 + t^4 - t^5 - \frac{199}{45} t^6 - \frac{1303}{315} t^7 - \frac{626}{315} t^8 
 - \frac{326}{567} t^9 - \frac{1382}{14175} t^{10} - \frac{1382}{155925} t^{11} \right). 
\]  

(3.13)

Again, we observe that when \( n \to \infty \), \( u(x, t) \to \frac{x}{1 + t} \). See Fig. 4.

3.2.2. Example 2

As in the second example solved previously, we consider the following equation:

\[
 u_t + u_x = 2u_{xxt}, \quad x \in \mathbb{R}, \quad t > 0, 
\]  

(3.14)

\[
 u(x, 0) = \exp(-x). 
\]  

(3.15)

By a similar manipulation, we have

\[
 \begin{align*}
 u_0 - e^{-x} &= 0, \quad u_0(x, 0) = e^{-x}. \\
 u_1 + u_0 - 2u_{0,xx} + e^{-x} &= 0, \quad u_1(x, 0) = 0. \\
 u_2 + u_1 - 2u_{1,xx} &= 0, \quad u_2(x, 0) = 0. \\
 u_3 + u_2 - 2u_{2,xx} &= 0, \quad u_3(x, 0) = 0. \\
 u_4 + u_3 - 2u_{3,xx} &= 0, \quad u_4(x, 0) = 0. \\
 u_5 + u_4 - 2u_{4,xx} &= 0, \quad u_5(x, 0) = 0.
\end{align*}
\]  

(3.16)

One can now obtain a solution for the equation system above in the form

\[
 \begin{align*}
 u_0(x, t) &= e^{-x}(t + 1). \\
 u_1(x, t) &= \frac{te^{-x}}{2}(t + 4). \\
 u_2(x, t) &= \frac{te^{-x}}{6}(t^2 + 12t + 24). \\
 u_3(x, t) &= \frac{te^{-x}}{24}(t^3 + 24t^2 + 144t + 192). \\
 u_4(x, t) &= \frac{te^{-x}}{120}(t^4 + 40t^3 + 480t^2 + 1920t + 1920).
\end{align*}
\]  

(3.17)
Fig. 5. The surface shows the approximate solution $u(x, t)$ for Eqs. (3.14) and (3.15).

$$u_5(x, t) = \frac{t e^{-x}}{720} (t^5 + 60t^4 + 1200t^3 + 9600t^2 + 28 800t + 23 040). \quad (3.17f)$$

Having $u_i, i = 0, 1, \ldots, 5$, the solution $u(x, t)$ is finally

$$u(x, t) = \sum_{i=0}^{5} u_i(x, t) = \frac{e^{-x}}{720} (t^6 + 66t^5 + 1470t^4 + 13 320t^3 + 46 440t^2 + 45 360t + 720). \quad (3.18)$$

See Fig. 5.

3.2.3. Example 3

We consider Eqs. (2.19) and (2.20) and solve them through HPM. Assuming $n = 3$, we have the system equation as follows:

$$\begin{cases}
    u_0 - \sin(x) = 0, & u_0(x, 0) = \sin(x), \\
    u_1 + u_{0xxx} + \sin(x) = 0, & u_1(x, 0) = 0, \\
    u_2 + u_{1xxx} = 0, & u_2(x, 0) = 0, \\
    u_3 + u_{2xxx} = 0, & u_3(x, 0) = 0.
\end{cases} \quad (3.19)$$

Solving system equation (3.19) results in

$$u_0(x, t) = (t + 1) \sin(x). \quad (3.20a)$$

$$u_1(x, t) = -(t + 4) t \frac{\sin(x)}{2}. \quad (3.20b)$$

$$u_2(x, t) = (t + 6) t^2 \frac{\sin(x)}{6}. \quad (3.20c)$$

$$u_3(x, t) = -(t + 8) t^3 \frac{\sin(x)}{24}. \quad (3.20d)$$

$$u(x, t) = \sum_{i=0}^{3} u_i(x, t) = (-t^4 - 4t^3 + 12t^2 - 24t + 24) \frac{\sin(x)}{24}. \quad (3.21)$$

Higher iterations lead to the convergence of $u_n(x, t)$ to the exact solution, $e^{-t} \sin(x)$. See Fig. 6.
4. Conclusions

In this paper, the authors have intended to show that the two methods, HPM and VIM, are considerably capable of solving a wide range of linear and nonlinear equations, especially those with high nonlinearity in engineering and physics problems. The examples given in this paper reveal that both methods are very effective and have high accuracy.

VIM and HPM do not need small parameters, the limitations and non-physical assumptions required in classical perturbation methods are eliminated, furthermore, VIM and HPM can overcome the difficulties arising in the calculation of Adomian polynomials. They do not require linearization; both methods are very promising tools for nonlinear equations. Therefore, both methods will find applications in various fields.

References