ELSEVIER

Annals of Pure and Applied Logic 76 (1995) 1–46

ANNALS OF
PURE AND
APPLIED LOGIC

A method to single out maximal propositional logics with the disjunction property I

Mauro Ferrari,* Pierangelo Miglioli

Università degli Studi di Milano, Dipartimento di Scienze dell'Informazione, Via Comelico 39/41, 20135 Milano, Italy

Received 16 July 1993; revised 4 October 1994; communicated by D. van Dalen

Abstract

This is the first part of a paper concerning intermediate propositional logics with the disjunction property which cannot be properly extended into logics of the same kind, and are therefore called *maximal*. To deal with these logics, we use a method based on the search of suitable *nonstandard logics*, which has an heuristic content and has allowed us to discover a wide family of logics, as well as to get their maximality proofs in a uniform way. The present part illustrates infinitely many maximal logics with the disjunction property extending the well-known logic of Scott, and aims to provide a first picture of the method, sufficient for the reader who wish to achieve an overall understanding of it without entering into the further aspects developed in the second part. From this point of view, the latter will not be self-standing, but will be seen as a prosecution and a complement of the former, with the aim that the material presented in the whole paper can be used as a starting point for a classification of the subject.

1. Introduction

Since in both parts of this paper we will be interested only in intermediate propositional logics, we will use the term "logic" as synonimous with "intermediate propositional logic". Also, we will call *constructive* any logic with the disjunction property.

The study of the maximal constructive logics originated from Lukasiewicz's conjecture of 1952 [11], which pretended that intuitionistic logic was the greatest consistent and constructive propositional system closed under substitution and detachment. However, after the exhibition of the first counterexamples to the conjecture, due to Kreisel and Putnam and to Scott [10], the notion of maximal constructive logic seemed to loose its foundational importance, and the interest of the researchers was

* Corresponding author.

SSDI 0168-0072(94)00053-0

led to other questions. Various nonconstructive logics were discovered together with constructive ones, and the main concern was in developing general tools to study their semantical properties, independently of their being constructive or not.

Yet, the motivation for a systematical analysis of the maximal constructive logics still remains. After all, the notion of constructive logic is important, and one expects that some features involved in this notion can be captured only at the level of maximality.

Thus, after Kirk's discovery that the greatest constructive logic does not exist [9], in the recent years the interest in the maximal constructive logics has considerably increased. In this frame, an investigation has been made on the number of these logics, which has turned out to be infinite [12, 13, 15]. The final result has been independently obtained by Chagrov [2], Galanter [8] and the authors [4], who have shown that the set of the maximal constructive logics has the power of continuum (in [4] a proof is also given that the set of the maximal intermediate predicate logics with the disjunction property and the explicit definability property has the power of continuum).

However, the majority of these researches has been carried out using *indirect methods* [15, 2, 4], i.e., by means of constructive incompatibility proofs. In this sense, two *constructively incompatible logics* are two constructive logics which cannot be simultaneously extended into a single constructive logic; also, by Zorn's lemma every constructive logic can be extended into a maximal one, hence at least two maximal constructive logics exist for any two constructively incompatible logics. In [15] an effort is made of showing how such indirect existence proofs can provide some information on the structure of the set of the maximal logics. But a satisfactory knowledge of this set can be obtained only giving a *direct characterization* of many logics belonging to it. Now, despite the great abundance of maximal constructive logics, the only logic of this kind for which a semantical characterization has been given in literature is, as far as we know, Medvedev's logic MV [6, 12, 14, 15, 3].

In this paper we will present infinitely many *new* maximal constructive logics, giving a Kripke style semantics for each of them. These results will be obtained by extending a method explained in [14], based on a characterization of the maximal constructive logics in terms of maximal *nonstandard* constructive logics. In [14] the method has been applied only to obtain a new proof of the maximality of MV, which was already known to be maximal; moreover, the semantics for MV there considered is close to Medvedev's original one and is not Kripke style. But only in connection with Kripke semantics the method can be extensively applied in all its generality, as we do in the present paper.

Let us better explain, now, the content of the paper and the criteria according to which the material is distributed.

In the first part (i.e., the present one) we provide the general results upon which our method to get maximal constructive logics is based, together with a concrete illustration of it, exhibiting a family of maximal constructive logics. Thus, together with the (standard) notion of logic, in the next section we will introduce, as in [14], the notion of *nonstandard logic*: this is any consistent set of formulas containing intuitionistic logic and closed under detachment and *restricted* substitutions, the latter allowing to replace the variables only with negated formulas. We will be interested in the nonstandard constructive logics (i.e., nonstandard logics with the disjunction property) and in the maximal nonstandard constructive logics. For, as stated in [14], there is a one-to-one correspondence between the maximal nonstandard constructive logics and the maximal standard ones; and the problem of characterizing the latter can be reduced to the apparently simpler problem of characterizing the former. We will recall the main notions and results of [14] about this correspondence, which is the basis of our method. Successively, in Section 3, we will introduce a Kripke style semantics for particular nonstandard logics, including the maximal constructive ones. This semantics is quite similar to the usual one for standard logics, in terms of classes of frames (posets); the only difference is that in the case of nonstandard logics the forcing defined on the posets must satisfy *special constraints* (as compared with the constrained forcing, we can say that the forcing used for the standard logics is free). We will be interested in comparing semantical characterizations of standard and nonstandard logics, in particular in relating the characterizations of the maximal standard constructive logics to the ones of the corresponding maximal nonstandard constructive logics. From this point of view, the examples we will provide in Section 6 (as well as the example presented in the second part) will involve cases where *the same class of posets* simultaneously defines a maximal nonstandard constructive logic and the corresponding maximal standard one, where constrained forcing is used for the first logic and free forcing is used for the second. Our method consists in the search of examples of this kind, and seems to be rather powerful. For, the cases where an *underlying class of posets* defines *both* a standard and a nonstandard constructive logic seem to be the most (if not all). Moreover, to find underlying classes of posets for maximal nonstandard constructive logics in general one has to look for classes of finite posets *which are uniquely determined* (according to some law) *by the final states.*

Since the maximal nonstandard constructive logic associated with any maximal (standard) constructive logic is uniquely determined as *the greatest* nonstandard constructive logic (properly) extending it, a classification of the maximal nonstandard constructive logics indirectly becomes a classification of the maximal standard constructive logics. In this sense, in Section 6 we will introduce also the notion of *constructive pseudologic*, intended as a set of classically valid formulas closed under detachment (but not necessarily under any kind of substitution), and satisfying the disjunction property. This will allow us to divide the family of maximal nonstandard constructive logics presented in that section into two classes: the first class, illustrated by a single nonstandard logic, contains the maximal nonstandard constructive logics which are not maximal constructive pseudologics; the second class, illustrated by infinitely many examples, contains the nonstandard constructive logics which are maximal *also* in the greater context of the constructive pseudologics (the examples of the second class will include the maximal nonstandard constructive logic associated with Medvedev's logic MV, which can be proved to be a maximal constructive pseudologic also starting from the results of [14]). This distinction seems to be of some interest, since it corresponds to important differences in the maximality proofs of the two kinds of nonstandard logics: in the maximality proofs of the logics of the second class we will treat the involved nonstandard constructive logics just as sets of classically valid formulas satisfying the disjunction property; on the other hand, in the maximality proof of the logic of the first class we will have to consider the involved nonstandard constructive logics as sets of classically valid formulas satisfying the disjunction property and *closed with respect to certain substitutions.*

Other important aspects, besides the above quoted ones treated in Sections 2 and 3 and in Section 6 (which is the core of the present part), will be considered in Sections 4 and 5.

Indeed, all the maximal constructive logics we will exhibit in Section 6 are extensions of the well-known logic ST of Scott [10, 1, 16, 15, 4], which is defined by adding to intuitionistic logic the axiom-schema $((\neg\neg A \to A) \to A \vee \neg A) \to \neg A \vee \neg\neg A$, whose instances are obtained by substitution from the formula in one variable $((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$. In Section 5 we will provide a Kripke semantics for ST and will prove the related soundness and completeness theorems. These results, which have been previously stated, without proof, in [15], may be considered interesting by themselves, since they disprove a conjecture of Minari [16], according to which ST has no semantics in terms of Kripke frames. On the other hand, our semantical characterization of ST is the general framework within which the semantical characterizations of the (so to speak) "scottian" maximal constructive logics of Section 6 will be defined, by means of progressive restrictions of the class of frames for ST (in a similar way, in the second part of the paper we will introduce a constructive logic which is constructively incompatible with ST and, in a sense which will be made precise, plays a role "alternative" to the one of ST; we will call AST, i.e. "anti" ST, such a logic and will present an example of "antiscottian" maximal constructive logic extending AST, whose Kripke semantics will be defined within the framework of a Kripke semantics for AST).

The completeness theorem of Section 5, as well as the completeness theorems needed in Section 6 to get the desired maximality results (providing the axiomatizations of the involved maximal nonstandard constructive logics, but not the axiomatizations of the corresponding maximal standard ones), will require an appropriate filtration technique. In Section 4 we will explain Gabbay's technique of the quotient models [5], which will be refined by means of suitable filtration formulas we will call *extensively complete.* The quotient models technique with extensively complete filtration formulas will be sufficient to treat the logics and the nonstandard logics of the present part, while a more sophisticated technique will be introduced in the second part of the paper.

The present part can also be seen as a self-standing paper which should allow the reader to grasp the main lines of our method to single out maximal constructive logics even disregarding the second part. On the other hand, the second part is intended as a prosecution and a complement of the first, presupposing its knowledge.

In the second part we will deal with the above quoted logic AST, defined by adding to intuitionistic logic the axiom-schema $(((\neg\neg A \to A) \to A \vee \neg A) \to \neg A \vee$

$\neg\neg A) \to \neg\neg A \vee (\neg\neg A \to A)$, whose instances are obtained by substitution from the formula in one variable $(((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p) \to \neg\neg p \vee (\neg\neg p \to p)$ (like ST, AST is one of the constructive logics taken into account in [1], whose superintuitionistic axiom-schemes are generated by formulas in one variable). We will introduce a Kripke semantics for AST, and will prove the related soundness and completeness theorems. We will also compare AST and ST on the basis of their fragments in one variable, which will be seen to be the only maximal constructive fragments in one variable (i.e., the fragment in one variable of any constructive logic is contained in ST or in AST). Moreover, we will show that there are maximal constructive logics which neither contain ST nor contain AST (i.e., they are neither "scottian" nor "antiscottian"; these logics, which we will not directly exhibit, cannot have maximal fragments in one variable).

Despite the maximality of its fragment in one variable, the logic AST is not a maximal constructive one (just as it happens for ST). Thus, we will provide a further application of our method and will exhibit an "antiscottian" maximal constructive logic (whose corresponding maximal nonstandard constructive logic will turn out to be also a maximal constructive pseudo logic, in the sense discussed above).

As compared with the analogous results given in the first part of the paper, the completeness theorem of AST and the completeness theorem needed to get the maximality of the "antiscottian" constructive logic (providing an axiomatization of the corresponding maximal nonstandard constructive logic, but not an axiomatization of the considered logic) involve more complex proofs, requiring (as anticipated above) a more sophisticated filtration technique. To this aim, we will introduce the selective models technique, which has been previously presented in [15] as a refinement of the technique of Gabbay and de Jongh [7]; differently from [15], the selective models technique will be applied in connection with special filtration formulas, such as the extensively complete ones used in the first part of the paper.

Finally, we will reconsider the main aspects involved in the semantical characterizations of the maximal constructive logics presented in the two parts of the paper, both to outline possible new directions of investigation and to better understand the heuristic content of the examples already at disposal. Thus, we will put into evidence from an heuristic point of view some hidden differences involved in the Kripke semantics of the maximal constructive logics considered in the paper and in the Kripke semantics of a constructive logic introduced in [12] by L. L. Maksimova, we call the logic of the rhombuses and denote by RH. Such an heuristic comparison will be the starting point of a proof that, indeed, RH is not a maximal constructive logic. This disproves a conjecture of Chagrov and Zacharyashchev [3], which seemed to be highly plausible.

2. Logics and nonstandard logics

The set of the propositional *well formed formulas* (wff) is defined as usual, using the connectives $\neg$, $\vee$, $\wedge$, $\to$. If A is a wff, $\mathcal{V}_A$ will be the set of propositional variables of A.

We say that a wff A is *negated* iff $A = \neg B$ for some wff B. We say that A is *negatively saturated* iff all its variables are within the scope of $\neg$. We say that A is a *Harrop-formula* [18, 14] iff A satisfies the following inductive conditions:

(1) A is atomic or negated;

(2) $A = B \wedge C$, and B and C are Harrop-formulas;

(3) $A = B \rightarrow C$, and C is an Harrop-formula.

A *substitution* will be any function σ associating, with every propositional variable, a wff. To denote the result of the application of the substitution σ to the wff A, we will write $\sigma(A)$ or, more simply, σA. A *restricted substitution* will be any substitution σ_r such that, for every variable p, $\sigma_r(p)$ is a negated formula. A *negatively saturated substitution* will be any substitution σ_{ns} such that, for every variable p, $\sigma_{ns}(p)$ is a negatively saturated formula. Finally, an *Harrop-substitution* will be any substitution σ_h such that, for every variable p, $\sigma_h(p)$ is an Harrop-formula.

INT (respectively, CL) will denote both an arbitrary calculus for intuitionistic propositional logic (respectively, for classical propositional logic) and the set of intuitionistically valid wff's (respectively, the set of classically valid wff's).

As usual, an *intermediate propositional logic* will be any set L of wff's satisfying the following conditions:

(1) L is consistent;

(2) INT $\subseteq L$;

(3) L is closed under detachment;

(4) L is closed under *arbitrary* subsitutions.

Throughout this paper, the term "logic" will mean an intermediate propositional logic. As is well-known, for every logic L, we have INT $\subseteq L \subseteq$ CL. Following tradition, we will define logics as sets of theorems of deductive systems. If $\mathscr{A}$ is a set of axiom-schemes and L is a logic, then the deductive system closed under detachment and arbitrary substitutions (logic) obtained by adding to L the axiom-schemes of $\mathscr{A}$ will be denoted by $L + \mathscr{A}$. If L_1 and L_2 are logics, then $L_1 + L_2$ will be the smallest logic containing both L_1 and L_2.

Following [14], an *intermediate propositional nonstandard logic* will be any set L of wff's satisfying the following conditions:

(1) L is consistent;

(2) INT $\subseteq L$;

(3) L is closed under detachment;

(4) L is closed under *restricted substitutions.*

Throughout this paper, the term "*nonstandard logic*" will mean an intermediate propositional nonstandard logic. Sometimes we will call "standard" a logic, in order to distinguish it from the nonstandard logics which are not logics. Of course, any logic is also a nonstandard logic (but the converse does not hold in general).

As for the logics, the following fact is immediate [14]:

Proposition 1. *If L is a nonstandard logic, then* INT $\subseteq L \subseteq$ CL.

As discussed above for logics, we will define nonstandard logics as sets of theorems of deductive systems; however, we will adopt different notations, to better distinguish the two cases. Thus, if L is a nonstandard logic and $\mathscr{S}$ is a set of formulas such that, for every $A \in \mathscr{S}$ and every restricted substitution σ_r, $\sigma_r(A) \in L \cup \mathscr{S}$, then $L \oplus \mathscr{S}$ will be the deductive system (nonstandard logic) obtained by closing with respect to detachment the set of formulas $L \cup \mathscr{S}$. In this line, if L_1 and L_2 are nonstandard logics, then $L_1 \oplus L_2$ will be the smallest nonstandard logic containing both L_1 and L_2, i.e., the closure with respect to detachment of $L_1 \cup L_2$. More generally, if $\mathscr{S}_1$ and $\mathscr{S}_2$ are two sets of formulas, $\mathscr{S}_1 \oplus \mathscr{S}_2$ will indicate the closure with respect to detachment of $\mathscr{S}_1 \cup \mathscr{S}_2$ ($\mathscr{S}_1 \oplus \mathscr{S}_2$ is not necessarily a nonstandard logic).

As in [14], we define the *extension operator* E and the *standardization operator* S on the set of the nonstandard logics as follows:

- If L is a nonstandard logic, then $E(L) = L \oplus \{\neg\neg p \to p \mid p$ is a propositional variable$\}$.
- If L is a nonstandard logic, then $S(L) = \{A \mid \sigma A \in L$ for every substitution $\sigma\}$.

As in [14], one easily deduces:

Proposition 2. *If L is a nonstandard logic, then $E(L)$ is a nonstandard logic.*

Proposition 3. *If L is a nonstandard logic, then $S(L)$ is a (standard) logic.*

We call *regular* any nonstandard logic L such that $L = E(L)$. As in [14], one easily shows:

Proposition 4. *If L is a regular nonstandard logic, then L is closed under the Harrop-substitutions.*

Of course, since $\neg\neg p \to p$ belongs to any regular nonstandard logic, the only regular nonstandard logic which is also a (standard) logic is CL.

We say that L is a *nonstandard constructive logic* iff L is a nonstandard logic satisfying the *disjunction property*:

(dp) $\quad A \vee B \in L \Rightarrow A \in L$ or $B \in L$.

As a particular case, a *constructive logic* will be any (standard) logic satisfying (dp).

The two following theorems are proved in [14]:

Theorem 1. *If L is a nonstandard constructive logic, then $E(L)$ is a nonstandard constructive logic.*

Theorem 2. *If L is a regular nonstandard constructive logic, then $S(L)$ is a constructive logic.*

The notion of maximality is defined both for the set of constructive logics and for the set of nonstandard constructive logics as follows [14]:

- We say that L is a *maximal constructive logic* iff L is a constructive logic and, for every constructive logic L', if $L \subseteq L'$ then $L = L'$.
- We say that L is a *maximal nonstandard constructive logic* iff L is a nonstandard constructive logic and, for every nonstandard constructive logic L', if $L \subseteq L'$ then $L = L'$.

According to Theorem 1, every maximal nonstandard constructive logic is regular. Hence, since CL is not a constructive logic, we have: *no maximal constructive logic is a maximal nonstandard constructive logic.*

From Theorems 1 and 2 we also deduce that every maximal constructive logic L satisfies the following fixed point equation: $L = S(E(L))$. This suggests the introduction of the following definitions:

- We say that a (standard) logic L is SE-*stable* iff $L = S(E(L))$.
- We say that a (standard) logic L is *neg.sat.-determined* iff L satisfies the following property: if $\sigma_{ns} A \in L$ for every negatively saturated substitution σ_{ns}, then $A \in L$.

The following theorem is proved in [14]:

Theorem 3. *A logic L is* SE-*stable iff it is neg.sat.-determined.*

Incidentally, we remark that not only the maximal constructive logics are SE-stable (i.e., neg.sat.-determined), e.g., INT is such a logic [14]; of course, also the nonconstructive CL is such.

The fundamental role played by the negatively saturated formulas in characterizing the maximal constructive logics is further stressed by the following results, which require the introduction of the *reduction operator* R on the set of nonstandard logics:

- The *negatively saturated part* of the nonstandard logic L will be the set $\mathrm{NS}(L) = \{A \mid A \in L \text{ and } A \text{ is negatively saturated}\}$.
- If L is a nonstandard logic, then $R(L) = \mathrm{INT} \oplus \mathrm{NS}(L)$.

The following facts can be easily shown [14]:

Proposition 5. *If L is a nonstandard logic, then* $R(L)$ *is a (standard) logic.*

Proposition 6. *If L is a nonstandard logic, then:*

(a) $L \subseteq E(R(L)) = E(L)$ *(hence,* $L = E(R(L))$ *for L regular);*

(b) $R(E(L)) = R(L) \subseteq S(L) \subseteq L$;

(c) *if* $L = \mathrm{INT} + \mathscr{A}_{ns}$ *is a (standard) logic and* $\mathscr{A}_{ns}$ *is a set of negatively saturated axiom schemes, then* $L = R(E(L)) = R(L)$.

In [14] the following important fact is proved:

Theorem 4. *If L is a nonstandard constructive logic, then* $R(L)$ *is a constructive logic.*

Now, the relations between the maximal nonstandard constructive logics and the maximal constructive logics are stated by the two following theorems [14]:

Theorem 5. *If L is a maximal nonstandard constructive logic and L′ is a constructive logic such that* $R(L) \subseteq L'$, *then* $L' \subseteq S(L)$ *(hence,* $S(L)$ *is a maximal constructive logic if L is a maximal nonstandard constructive logic).*

Theorem 6. *If L is a maximal constructive logic, then* $E(L)$ *is a maximal nonstandard constructive logic.*

According to the above, there is a one-to-one correspondence between the maximal constructive logics and the maximal nonstandard constructive logics. Also, quoting [14], "it seems to be easier to find maximal nonstandard constructive logics than to find maximal constructive logics directly, because to find a maximal nonstandard constructive logic essentially amounts to finding a maximal set of *special* formulas, namely negatively saturated formulas". In the present paper we will extensively show that the approach suggested by [14] is right, i.e., that the preliminary search of maximal nonstandard constructive logics is a good method to get maximal constructive logics.

In the next two sections we will refine the method on the semantical ground. Now, we conclude this section quoting a result of [14] which will be useful later.

Theorem 7. *If L is a neg.sat.-determined logic and L′ is a logic such that* $R(L) \subseteq L' \subseteq L$ *and* $L \neq L'$, *then L′ is not neg.sat.-determined.*

3. Kripke frames semantics for logics and regular nonstandard logics

We assume the reader to be familiar with the notion of *Kripke model* $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$, where $\boldsymbol{P} = \langle P, \leqslant \rangle$ is a *poset* and $\Vdash$ is the *forcing relation*, defined between elements of P and atomic formulas (propositional variables) and extended in the usual way to arbitrary wff's; we say that $\boldsymbol{K}$ is *built on the poset* $\boldsymbol{P}$, or that $\boldsymbol{P}$ is the *underlying poset of* $\boldsymbol{K}$. A poset $\boldsymbol{P}$ is said to be *principal* iff $\boldsymbol{P}$ has a least element 0 (called the *root* of the poset). Following tradition, the posets will be called also *frames*. We will only consider principal posets and Kripke models built on them. To avoid unnecessary repetitions, *we will consider "poset" ("frame") as synonimous with "principal poset"*. Sometimes we will explicitly indicate the least element 0 of the poset $\boldsymbol{P}$ or of the Kripke model $\boldsymbol{K}$, by writing respectively $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle$ and $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle$. For any element α of a poset $\boldsymbol{P} = \langle P, \leqslant \rangle$, $\boldsymbol{P}_\alpha$, or, if necessary, also $\langle P_\alpha, \leqslant \rangle$, denotes the principal subordering generated by α in $\boldsymbol{P}$, i.e., the restriction of $\langle P, \leqslant \rangle$ to the set $P_\alpha = \{\beta \mid \beta \in P \text{ and } \alpha \leqslant \beta\}$. The poset $\boldsymbol{P}_\alpha$ will be called the *cone of* α *in* $\boldsymbol{P}$.

An element ϕ of $\boldsymbol{P} = \langle P, \leqslant \rangle$ will be called *final* (in $\boldsymbol{P}$) iff, for every $\phi' \in P$, $\phi \leqslant \phi'$ implies $\phi = \phi'$. For any $\boldsymbol{P} = \langle P, \leqslant \rangle$ and any $\alpha \in P$, $\mathrm{Fin}(\alpha)_{\boldsymbol{P}}$ will denote the set

$\{\phi \mid \alpha \leqslant \phi$ and ϕ is a final element in $\boldsymbol{P}\}$. When ambiguities cannot arise, we will write $\mathrm{Fin}(\alpha)$ instead of $\mathrm{Fin}(\alpha)_{\boldsymbol{P}}$. We remark that $\mathrm{Fin}(\alpha)$ is nonempty if $\boldsymbol{P}$ is finite, while this is not necessarily true for an infinite $\boldsymbol{P}$. Given $\boldsymbol{P} = \langle P, \leqslant \rangle$ and a nonfinal $\alpha \in P$, we will also say that α *is prefinal* (in $\boldsymbol{P}$) iff all immediate successors of α in $\boldsymbol{P}$ are final.

If $\mathscr{F}$ is a nonempty class of posets (of frames), $\mathscr{K}(\mathscr{F})$ will denote the set $\{\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle \mid \langle P, \leqslant \rangle \in \mathscr{F}\}$; $\mathscr{K}(\mathscr{F})$ will be said to be the *class of Kripke models built on* $\mathscr{F}$. Also, for every nonempty class $\mathscr{F}$ of posets, $\mathscr{L}(\mathscr{F})$ will indicate the set $\{A \mid \text{for every } \boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle \in \mathscr{K}(\mathscr{F}), 0 \Vdash A\}$.

As is well-known, *for every nonempty class* $\mathscr{F}$ *of frames,* $\mathscr{L}(\mathscr{F})$ *is a logic* [15,4]. According to this fact, $\mathscr{L}(\mathscr{F})$ will be called the (standard) *logic generated by* $\mathscr{F}$. Also, we will say that a logic L *has a Kripke frames semantics* iff there is a nonempty $\mathscr{F}$ such that $L = \mathscr{L}(\mathscr{F})$. As proved in [17], *there are logics without Kripke frames semantics.*

We say that a Kripke model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ is *regular* iff, for every $\alpha \in P$ and every variable p, the following condition holds:

(∗) if $\alpha \nVdash p$ and $\alpha \nVdash \neg p$, then there is $\beta \in P$ such that $\alpha \leqslant \beta$ and $\beta \nVdash \neg p$ (of course, from $\alpha \nVdash \neg p$ it follows that there is $\gamma \in P$ such that $\alpha \leqslant \gamma$ and $\gamma \Vdash p$).

For any nonempty class $\mathscr{F}$ of posets, $\mathscr{K}_{\text{reg}}(\mathscr{F})$ will denote the set $\{\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle \mid \boldsymbol{K}$ *is regular* and $\langle P, \leqslant \rangle \in \mathscr{F}\}$; $\mathscr{K}_{\text{reg}}(\mathscr{F})$ will be said to be the *class of regular Kripke models built on* $\mathscr{F}$. Also, for every nonempty class $\mathscr{F}$ of posets, $\mathscr{L}_{\text{reg}}(\mathscr{F})$ will indicate the set $\{A \mid \text{for every } \boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle \in \mathscr{K}_{\text{reg}}(\mathscr{F}), 0 \Vdash A\}$.

The regular Kripke models and the related definitions of $\mathscr{K}_{\text{reg}}(\mathscr{F})$ and $\mathscr{L}_{\text{reg}}(\mathscr{F})$ will be used to give a notion of Kripke frame semantics for regular nonstandard logics. To better analyze what is involved in this semantics, we need a definition and a related proposition.

We say that a poset $\boldsymbol{P} = \langle P, \leqslant \rangle$ *has the property of the final elements* iff the following condition holds:

(∗∗) for every $\alpha \in P$, there is $\phi \in P$ such that $\alpha \leqslant \phi$ and ϕ is final.

We also say that a class $\mathscr{F}$ of posets *has the property of the final elements* iff every element of $\mathscr{F}$ has.

As far as only the propositional level is involved, the following proposition holds:

Proposition 7. *For every class* $\mathscr{F}$ *of frames there is a class* $\mathscr{F}'$ *of frames such that* $\mathscr{F}'$ *has the property of the final elements,* $\mathscr{L}(\mathscr{F}) = \mathscr{L}(\mathscr{F}')$ *and* $\mathscr{L}_{\text{reg}}(\mathscr{F}) = \mathscr{L}_{\text{reg}}(\mathscr{F}')$.

Proof (*Outline*). Let v be a finite set of propositional variables, and let α be an element of a Kripke model $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle$. We call *atomic v-forcing of* α (in $\boldsymbol{K}$) the set $f_\alpha^v = (p \mid p \in v \text{ and } \alpha \Vdash p\} \cup \{\neg p \mid p \in v \text{ and } \alpha \Vdash \neg p\}$. We also say that α is v-final (in $\boldsymbol{K}$) iff $|f_\alpha^v| = |v|$, where $| \ldots |$ indicates the cardinality of a set. The following facts can be easily proved:

(i) for every $\alpha \in P$, there is $\beta \in P$ such that $\alpha \leqslant \beta$ and β is v-final;

(ii) if α and β are two v-final elements of P such that $f_\alpha^v = f_\beta^v$, then, for every wff A such that $\mathscr{V}_A \subseteq v$, $\alpha \Vdash A$ iff $\beta \Vdash A$.

Given $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash\rangle$, v and $\alpha \in P$, we set $\mathrm{Fin}(\alpha)^v = \{\beta \in P \mid \alpha \leqslant \beta$ and β is v-final$\}$. Since v is finite, there is only a finite number of distinct atomic v-forcings of elements of P. Let $f_1^v, \ldots, f_h^v$ be all such atomic v-forcings and let us suppose, without loss of generality, that $\{f_1^v, \ldots, f_h^v\} \cap P = \emptyset$. We say that $\{\phi_1, \ldots, \phi_n\} \subseteq \{f_1^v, \ldots, f_h^v)$ is *the complete set of atomic v-forcings for* $\mathrm{Fin}(\alpha)^v$ iff the following conditions hold:

- for every $\gamma \in \mathrm{Fin}(\alpha)^v$, there is i such that $1 \leqslant i \leqslant n$ and $f_\gamma^v = \phi_i$;
- for every i such that $1 \leqslant i \leqslant n$, there is $\gamma \in \mathrm{Fin}(\alpha)^v$ such that $\phi_i = f_\gamma^v$.

Now, for every $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash\rangle$ and every finite set of propositional variables v, we define the *v-pruned model of* $\boldsymbol{K}$, denoted by $\boldsymbol{K}^v = \langle P^v, \leqslant^v, 0^v, \Vdash^v\rangle$, as follows:

- every element of P which is not v-final belongs to P^v; moreover, for every element α of P which is not v-final, if $\{\phi_1, \ldots, \phi_n\}$ is the complete set of atomic v-forcings for $\mathrm{Fin}(\alpha)^v$ then $\{\phi_1, \ldots, \phi_n\} \subseteq P^v$;
- $\alpha \leqslant^v \beta$ iff:
  - α and β are not v-final in $\boldsymbol{K}$ and $\alpha \leqslant \beta$;
  - α is not v-final in $\boldsymbol{K}$ and β belongs to the complete set of atomic v-forcings for $\mathrm{Fin}(\alpha)^v$;
- $0^v = 0$ (if 0 is not v-final);
- for every $\alpha \in P^v \cap P$ and every p, $\alpha \Vdash^v p$ (in $\boldsymbol{K}^v$) iff $\alpha \Vdash p$ (in $\boldsymbol{K}$);
- for every $\phi \in P^v \cap \{f_1^v, \ldots, f_h^v\}$ and every $p \in v$, $\phi \Vdash^v p$ (in $\boldsymbol{K}^v$) iff $p \in \phi$;
- for every propositional variable p which does not belong to v, we set (e.g.) $\phi \Vdash^v p$ (in $\boldsymbol{K}^v$).

We call $\boldsymbol{P}^v = \langle P^v, \leqslant^v, 0^v\rangle$ the *v-pruned poset obtained from* $\boldsymbol{K}$. By (i) and the definition of $\boldsymbol{P}^v$, we have that in $\boldsymbol{P}^v$ every element is followed by a final element. Also, using the above property (ii), one easily proves:

(iii) for every wff A such that $\mathscr{V}_A \subseteq v$ and every $\phi \in P^v \cap \{f_1^v, \ldots, f_h^v\}$, $\phi \Vdash^v A$ (in $\boldsymbol{K}^v$) iff, for every $\alpha \in P$ such that $f_\alpha^v = \phi$, $\alpha \Vdash A$ (in $\boldsymbol{K}$); for every wff A such that $\mathscr{V}_A \subseteq v$ and every $\alpha \in P^v \cap P$, $\alpha \Vdash^v A$ (in $\boldsymbol{K}^v$) iff $\alpha \Vdash A$ (in $\boldsymbol{K}$);

(iv) if $\boldsymbol{K}$ is regular then $\boldsymbol{K}^v$ is.

On the other hand, we immediately get:

(v) if $\boldsymbol{P}^v = \langle P^v, \leqslant^v, 0^v\rangle$ is the v-pruned poset obtained from $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash\rangle$, $\boldsymbol{K}' = \langle P^v, \leqslant^v, 0^v, \Vdash'\rangle$ is a model built on $\boldsymbol{P}^v$, $\boldsymbol{P} = \langle P, \leqslant, 0\rangle$ is the underlying poset of $\boldsymbol{K}$ and v' is any finite set of propositional variables, then there is a model $\boldsymbol{K}'' = \langle P, \leqslant, 0, \Vdash''\rangle$ built on $\boldsymbol{P}$ such that the following conditions hold:

(v_1) for every wff A with $\mathscr{V}_A \subseteq v'$ and every $\phi \in P^v \cap \{f_1^v, \ldots, f_h^v\}$, $\phi \Vdash' A$ (in $\boldsymbol{K}'$) iff, for every α of $\boldsymbol{K}$ such that $f_\alpha^v = \phi$, $\alpha \Vdash'' A$ (in $\boldsymbol{K}''$); for every wff A with $\mathscr{V}_A \subseteq v'$ and every $\alpha \in P^v \cap P$, $\alpha \Vdash' A$ (in $\boldsymbol{K}'$) iff $\alpha \Vdash'' A$ (in $\boldsymbol{K}''$);

(v_2) if $\boldsymbol{K}'$ is regular then $\boldsymbol{K}''$ is.

Now, let $\mathscr{F}' = \{\boldsymbol{P}^v \mid$ there are $\boldsymbol{K} \in \mathscr{K}(\mathscr{F})$ and v such that $\boldsymbol{P}^v$ is the v-pruned poset obtained from $\boldsymbol{K}\}$. Then, properties (iii) and (iv) provide $\mathscr{L}(\mathscr{F}') \subseteq \mathscr{L}(\mathscr{F})$ and $\mathscr{L}_{\mathrm{reg}}(\mathscr{F}') \subseteq \mathscr{L}_{\mathrm{reg}}(\mathscr{F})$; on the other hand, property (v) provides $\mathscr{L}(\mathscr{F}) \subseteq \mathscr{L}(\mathscr{F}')$ and $\mathscr{L}_{\mathrm{reg}}(\mathscr{F}) \subseteq \mathscr{L}_{\mathrm{reg}}(\mathscr{F}')$. □

Remark. Let $\mathscr{F}$ be any class of frames, let Γ be a set of wff's and let A be a wff. We say that A is a *$\mathscr{F}$-consequence of Γ* (respectively, a *regular $\mathscr{F}$-consequence of Γ*), and we write $\Gamma \models_{\mathscr{F}} A$ (respectively, $\Gamma \models_{\text{reg}, \mathscr{F}} A$), iff the following condition holds: for every $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle \in \mathscr{K}(\mathscr{F})$ (respectively, $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle \in \boldsymbol{K}_{\text{reg}}(\mathscr{F})$) and every $\alpha \in P$, if $\alpha \Vdash B$ for every $B \in \Gamma$ then $\alpha \Vdash A$. We have, in particular, that $A \in \mathscr{L}(\mathscr{F})$ iff $\emptyset \models_{\mathscr{F}} A$ (respectively, $A \in \mathscr{L}_{\text{reg}}(\mathscr{F})$ iff $\emptyset \models_{\text{reg}, \mathscr{F}} A$).

Now, if Γ is an infinite set, the construction involved in the proof of the above Proposition 7 cannot be used to guarantee that, for every class $\mathscr{F}$ of posets, there is a class $\mathscr{F}'$ with the property of the final elements such that $\Gamma \models_{\mathscr{F}} A$ iff $\Gamma \models_{\mathscr{F}'} A$ (respectively, $\Gamma \models_{\text{reg}, \mathscr{F}} A$ iff $\Gamma \models_{\text{reg}, \mathscr{F}'} A$).

In the following, we will not be concerned with $\mathscr{F}$-consequence and regular $\mathscr{F}$-consequence, but only with the sets of formulas $\mathscr{L}(\mathscr{F})$ and $\mathscr{L}_{\text{reg}}(\mathscr{F})$. Thus, unless otherwise stated, *henceforth poset and class of posets will be synonimous, respectively, with poset with the property of the final elements and with class of posets with the property of the final elements.* By Proposition 7, this convention will not affect the generality of our treatment.

One easily proves:

Proposition 8. *A model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ is regular iff, for every $\alpha \in P$ and every variable p, $\alpha \Vdash p$ iff, for every $\phi \in \text{Fin}(\alpha)$, $\phi \Vdash p$.*

Proposition 9. *A model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ is regular iff, for every $\alpha \in P$ and every variable p, $\alpha \Vdash \neg\neg p \rightarrow p$.*

The proof of the following proposition is easy and can be carried out using Propositions 8 and 9. We leave it to the reader.

Proposition 10. *For every nonempty class $\mathscr{F}$ of frames, $\mathscr{L}_{\text{reg}}(\mathscr{F})$ is a regular nonstandard logic.*

According to Proposition 10, $\mathscr{L}_{\text{reg}}(\mathscr{F})$ will be called the *regular nonstandard logic generated by $\mathscr{F}$*. Also, we will say that a regular logic L *has a Kripke frames semantics* iff there is a nonempty $\mathscr{F}$ such that $L = \mathscr{L}_{\text{reg}}(\mathscr{F})$.

Now, we compare $\mathscr{L}(\mathscr{F})$ and $\mathscr{L}_{\text{reg}}(\mathscr{F})$, using the operators E, S and R defined in the previous section.

First of all, we make the following remark. Let $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ be any Kripke model and let $\boldsymbol{K}' = \langle P, \leqslant, \Vdash' \rangle$ be obtained from $\boldsymbol{K}$ by (possibly) modifying its forcing in the following way: for every $\alpha \in P$ and every variable p, $\alpha \Vdash' p$ iff, for every $\phi \in \text{Fin}(\alpha)$, $\phi \Vdash p$ (in $\boldsymbol{K}$). Then, $\boldsymbol{K}'$ turns out to be a regular Kripke model, we call the *regular Kripke model associated with $\boldsymbol{K}$* (one has $\boldsymbol{K} = \boldsymbol{K}'$ iff $\boldsymbol{K}$ is regular). One easily shows that, for every Kripke model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$, for every $\alpha \in P$ and every negatively saturated formula A, $\alpha \Vdash A$ (in $\boldsymbol{K}$) iff $\alpha \Vdash' A$ (in the regular Kripke model $\boldsymbol{K}'$ associated with $\boldsymbol{K}$). Hence (since the class of all the regular Kripke models associated with elements of $\mathscr{K}(\mathscr{F})$ coincides with $\mathscr{K}_{\text{reg}}(\mathscr{F})$), one immediately deduces:

Proposition 11. *For every nonempty class $\mathscr{F}$ of frames, $R(\mathscr{L}_{\text{reg}}(\mathscr{F})) \subseteq \mathscr{L}(\mathscr{F})$.*

Using Proposition 11, we can prove:

Theorem 8. *For every nonempty class $\mathscr{F}$ of frames, $\mathscr{L}_{\text{reg}}(\mathscr{F}) = E(\mathscr{L}(\mathscr{F}))$.*

Proof. Since $\mathscr{K}_{\text{reg}}(\mathscr{F}) \subseteq \mathscr{K}(\mathscr{F})$, we have $\mathscr{L}(\mathscr{F}) \subseteq \mathscr{L}_{\text{reg}}(\mathscr{F})$. Also, by Proposition 9, $\neg\neg p \to p \in \mathscr{L}_{\text{reg}}(\mathscr{F})$ for every variable p. It follows that $E(\mathscr{L}(\mathscr{F})) \subseteq \mathscr{L}_{\text{reg}}(\mathscr{F})$.

To prove the converse, let us assume, on the contrary, that $\mathscr{L}_{\text{reg}}(\mathscr{F})$ is not included in $E(\mathscr{L}(\mathscr{F}))$. Then there is a formula $A(p_1, \dots, p_n) \in \mathscr{L}_{\text{reg}}(\mathscr{F})$ such that $A(p_1, \dots, p_n) \notin E(\mathscr{L}(\mathscr{F}))$. Since $\neg\neg p_i \leftrightarrow p_i \in E(\mathscr{L}(\mathscr{F}))$ for $1 \leqslant i \leqslant n$ (by definition of $E(\mathscr{L}(\mathscr{F}))$) and since the replacement holds in $E(\mathscr{L}(\mathscr{F}))$, we must have $A(\neg\neg p_1, \dots, \neg\neg p_n) \notin E(\mathscr{L}(\mathscr{F}))$. A fortiori, $A(\neg\neg p_1, \dots, \neg\neg p_n) \notin \mathscr{L}(\mathscr{F})$. But $A(\neg\neg p_1, \dots, \neg\neg p_n)$ is a negatively saturated formula belonging to $\mathscr{L}_{\text{reg}}(\mathscr{F})$, since $A(p_1, \dots, p_n) \in \mathscr{L}_{\text{reg}}(\mathscr{F})$. It follows that $A(\neg\neg p_1, \dots, \neg\neg p_n) \in R(\mathscr{L}_{\text{reg}}(\mathscr{F}))$, by definition of R. Hence, by Proposition 11, $A(\neg\neg p_1, \dots, \neg\neg p_n) \in \mathscr{L}(\mathscr{F})$, a contradiction. □

Corollary 1. *If $\mathscr{L}(\mathscr{F})$ is a maximal constructive logic, then $\mathscr{L}_{\text{reg}}(\mathscr{F})$ is a maximal nonstandard constructive logic.*

Thus, if L is a maximal constructive logic with Kripke frames semantics, a fortiori the corresponding maximal nonstandard constructive logic *has* a Kripke frames semantics. More generally, it seems to be harder to find regular nonstandard logics without Kripke frames semantics than to find (standard) logics without Kripke frames semantics, in line with the following proposition:

Proposition 12. *Let L be any regular nonstandard logic without Kripke frames semantics and let L' be any (standard) logic such that $R(L) \subseteq L' \subseteq S(L)$. Then L' is without Kripke frames semantics.*

Proof. Since $L = E(L)$, from Proposition 6 we easily deduce $E(R(L)) = E(L') = E(S(L)) = L$. Now, suppose $L' = \mathscr{L}(\mathscr{F})$. It follows, by Theorem 8, $E(L') = \mathscr{L}_{\text{reg}}(\mathscr{F})$, hence $L = \mathscr{L}_{\text{reg}}(\mathscr{F})$. But L is a regular logic without Kripke frames semantics, a contradiction. □

In this paper we will not be concerned, however, in generalizing the results of [17] to regular nonstandard logics. Thus, *we leave open the problem of finding regular nonstandard logics (possibly, maximal nonstandard constructive logics) without Kripke frames semantics.*

We will be interested, on the other hand, in *providing a Kripke frames semantics for the maximal constructive logic $S(L)$, once L has been recognized to be a maximal* *nonstandard constructive logic and to have the form* $L = \mathscr{L}_{\text{reg}}(\mathscr{F})$. Using Theorem 7, we can prove:

Theorem 9. *Let L be a maximal nonstandard constructive logic, let* $L = \mathscr{L}_{\text{reg}}(\mathscr{F})$ *for some nonempty class* $\mathscr{F}$ *of frames, and let* $\mathscr{L}(\mathscr{F})$ *be a neg.sat.-determined logic. Then the corresponding maximal constructive logic* $S(L)$ *coincides with* $\mathscr{L}(\mathscr{F})$.

Proof. Since, by definition of S, $S(L)$ is the greatest (standard) logic contained in L, we have $\mathscr{L}(\mathscr{F}) \subseteq S(L)$. On the other hand, by Proposition 11, one has that $R(\mathscr{L}_{\text{reg}}(\mathscr{F})) = R(L) \subseteq \mathscr{L}(\mathscr{F})$. Also, since $R(L) \subseteq S(L)$ and since $R(L) = R(R(L))$, one has $R(L) = R(R(L)) \subseteq R(S(L)) \subseteq R(L)$, from which it follows $R(L) = R(S(L))$. Thus, we have $R(S(L)) \subseteq \mathscr{L}(\mathscr{F}) \subseteq S(L)$. This implies, by Theorem 7, that $S(L) = \mathscr{L}(\mathscr{F})$. As a matter of fact, $S(L)$ is a maximal constructive logic, hence $S(L)$ is SE-stable, hence, by Theorem 3, $S(L)$ is neg.sat.-determined; thus, by Theorem 7, $\mathscr{L}(\mathscr{F}) \neq S(L)$ implies that $\mathscr{L}(\mathscr{F})$ is not neg.sat.-determined, a contradiction. □

The above Theorem 9 will be one of the key points of our method. We do not know whether there is a maximal nonstandard constructive logic L such that $L = \mathscr{L}_{\text{reg}}(\mathscr{F})$ for some $\mathscr{F}$, but $\mathscr{L}(\mathscr{F}')$ is not neg.sat.-determined for every $\mathscr{F}'$ such that $L = \mathscr{L}_{\text{reg}}(\mathscr{F}')$; in such a case, by the above results, $S(L)$ could not have a Kripke frames semantics.

To conclude this section, we extend a well-known sufficient condition generally used to state the constructiveness of (standard) logics.

- We say that a nonempty class $\mathscr{F}$ of posets has the *strong disjoint embedding property* (which we will indicate by (s.d.e.p.)) iff, for any two posets $\boldsymbol{P} = \langle P, \leqslant \rangle \in \mathscr{F}$ and $\boldsymbol{P}' = \langle P', \leqslant' \rangle \in \mathscr{F}$, there are a poset $\boldsymbol{P}'' = \langle P'', \leqslant'' \rangle \in \mathscr{F}$ and two elements $\alpha \in P''$ and $\beta \in P''$ such that the following conditions are satisfied:
  (1) $\boldsymbol{P}$ is isomorphic to $\boldsymbol{P}''_\alpha$ and $\boldsymbol{P}'$ is isomorphic to $\boldsymbol{P}''_\beta$;
  (2) $P''_\alpha \cap P''_\beta = \emptyset$.

It is well known that if $\mathscr{F}$ is a nonempty class of posets satisfying (s.d.e.p.), then $\mathscr{L}(\mathscr{F})$ is a constructive logic [2–7, 12, 15, 16, 18]. The proof is easy and can be restated, without problems, for the regular logic $\mathscr{L}_{\text{reg}}(\mathscr{F})$. Thus, we have:

Proposition 13. *If* $\mathscr{F}$ *is a nonempty class of frames satisfying* (s.d.e.p.), *then*:
(a) $\mathscr{L}(\mathscr{F})$ *is a constructive logic*;
(b) $\mathscr{L}_{\text{reg}}(\mathscr{F})$ *is a regular nonstandard constructive logic.*

4. The filtration technique of the quotient models

If Γ is any set of wff's and L is *any nonstandard logic*, we say that A is *L-provable from* Γ, and we denote it by $\Gamma \vdash_L A$, iff there are $B_1, \ldots, B_n$ such that $\{B_1, \ldots, B_n\} \subseteq \Gamma$

and $B_1 \wedge \ldots \wedge B_n \to A \in L$. On the other hand, $\Gamma \nvdash_L A$ will mean that $\Gamma \vdash_L A$ does not hold. We introduce the following notions:

- A *saturated set* Γ is any consistent set of wff's closed under INT-provability (i.e., $\Gamma \vdash_{\mathrm{INT}} A$ implies $A \in \Gamma$) and under the disjunction property (i.e., $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$).
- If L is a *nonstandard logic*, $L \subseteq \Gamma$ and Γ is saturated, then Γ is closed under L-provability: in this sense, we say that Γ is *L-saturated.*
- If Γ is a saturated set, by the *canonical model generated by* Γ, in symbols, $\mathscr{C}(\Gamma)$, we mean the Kripke model $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle$ satisfying the following properties:
  (1) $P = \{\Gamma' \mid \Gamma \subseteq \Gamma' \text{ and } \Gamma' \text{ is saturated}\}$;
  (2) for any two $\Gamma', \Gamma'' \in P$, $\Gamma' \leqslant \Gamma''$ iff $\Gamma' \subseteq \Gamma''$;
  (3) $0 = \Gamma$;
  (4) for any $\Gamma' \in P$ and any propositional variable p, $\Gamma' \Vdash p$ iff $p \in \Gamma'$.
- If Γ is L-saturated, *then all elements of* $\mathscr{C}(\Gamma)$ include L; in this case we say that $\mathscr{C}(\Gamma)$ is the *L-canonical model generated by* Γ, and we write $\mathscr{C}_L(\Gamma)$ instead of $\mathscr{C}(\Gamma)$.

Since every saturated set can be extended into some maximal consistent one, it is easily seen that the underlying poset of any model $\mathscr{C}_L(\Gamma)$ has the property of the final elements. The following facts are well known for the (standard) logics [18, 15, 4] and hold as well for arbitrary nonstandard logics:

Proposition 14. *If L and Δ are (respectively) a nonstandard logic and a set of wff's such that $\Delta \nvdash_L A$, then there is a L-saturated set Γ such that $\Delta \subseteq \Gamma$ and $A \notin \Gamma$.*

Proposition 15. *If L is a nonstandard logic, Γ is L-saturated, Γ' is an element of $\mathscr{C}_L(\Gamma)$ and B is any wff, then $\Gamma' \Vdash B$ holds in $\mathscr{C}_L(\Gamma)$ iff $B \in \Gamma'$.*

Propositions 9 and 15 immediately yield:

Proposition 16. *If L is a regular nonstandard logic and Γ is L-saturated, then $\mathscr{C}_L(\Gamma)$ is a regular Kripke model.*

Propositions 14 and 15 are the basic tools to prove the completeness of a logic according to the following well-known proof strategy. Suppose L is a (standard) logic syntactically characterized, e.g., as $L = \mathrm{INT} + \mathscr{A}$, where $\mathscr{A}$ is set of axiom-schemes. Suppose also that $\mathscr{F}$ is a nonempty class of posets, intended to provide the Kripke frames semantics of L. Then one has to show that $L = \mathscr{L}(\mathscr{F})$, which splits into the separate proofs that $L \subseteq \mathscr{L}(\mathscr{F})$ (*soundness theorem*) and that $\mathscr{L}(\mathscr{F}) \subseteq L$ (*completeness theorem*). Quite often, the proof of the soundness theorem is direct and easy, so that to state that $L = \mathscr{L}(\mathscr{F})$ amounts to state the completeness theorem. Now, the latter theorem immediately follows from Propositions 14 and 15 if, for every L-saturated set Γ, the underlying poset of the canonical model $\mathscr{C}_L(\Gamma)$ turns out to belong to $\mathscr{F}$.

This use of the canonical models to prove the completeness theorem for a logic is successful in many cases; in these cases, we say that the proof is carried out with the *method of the canonical models*. This method *can be used, as well, to obtain completeness theorems for a regular nonstandard logic L*, in order to show that $L = \mathscr{L}_{\text{reg}}(\mathscr{F})$. For, in this case the problem might be that the models $\mathscr{C}_L(\Gamma)$ are not regular. But this circumstance is excluded by Proposition 16.

Remark. When it can be successfully applied, the method of the canonical models provides more than the completeness theorem for a logic (respectively, for a regular nonstandard logic) L. As a matter of fact, suppose that, for every L-saturated set Γ, the underlying poset of the canonical model $\mathscr{C}_L(\Gamma)$ belongs to some class $\mathscr{F}$ of posets. Then, by Propositions 14 and 15 (respectively, by Propositions 14–16) we get the *strong completeness theorem for L with respect to* $\mathscr{F}$, i.e.: for every set Δ of wff's $\Delta \models_{\mathscr{F}} A$ implies $\Delta \vdash_L A$ (respectively, $\Delta \models_{\text{reg},\mathscr{F}} A$ implies $\Delta \vdash_L A$). As discussed above for the ordinary completeness theorem, usually the strong completeness theorem provides the main part of a proof that, for every Δ, $\Delta \models_{\mathscr{F}} A$ iff $\Delta \vdash_L A$ (respectively, $\Delta \models_{\text{reg},\mathscr{F}} A$ iff $\Delta \vdash_L A$). Indeed, in the most cases the proof of the *strong soundness theorem for L with respect to* $\mathscr{F}$, according to which $\Delta \vdash_L A$ implies $\Delta \models_{\mathscr{F}} A$ for every Δ (respectively, $\Delta \vdash_L A$ implies $\Delta \models_{\text{reg},\mathscr{F}} A$ for every Δ), is as easy and direct as the proof of the related ordinary soundness theorem.

Since the underlying poset of any canonical model $\mathscr{C}_L(\Gamma)$ has the property of the final elements, notice that from a successful application of the method of the canonical models one can always get a strong completeness theorem with respect to some class of posets *with the property of the final elements.*

The method of the canonical models (both for standard logics, using Propositions 14 and 15, and for regular nonstandard logics, using Propositions 14–16) may fail to be successful in various cases, such as the ones with which we will be concerned in both parts of the present paper. To treat these cases, we will make an indirect use of the canonical models, which will be transformed, by means of techniques we are going to explain, into finite models equivalent to them with respect to a set of relevant formulas which can be codified, so to speak, by a "finite amount of information" (from this point of view, the resulting finite models *cannot be used to get strong completeness theorems, generally involving "infinite amounts of information"*; see the remark at the end of this section). These techniques are the so-called filtration techniques [5–7, 16, 15, 4] and can be applied to any model (canonical or not), even if they become interesting only when the starting models have nice properties, as in the case of the canonical models. In this part of the paper we will present and use a construction introduced by Gabbay for the logic KP of Kreisel and Putnam [5, 6], which we call the *quotient models technique*; on the other hand, in the second part we will present and use the *selective models technique*, introduced in [15], which is a variant of the technique of Gabbay and de Jongh [6, 7], enriched by a quotientation of the final states. In both parts we will also use special refinements, introduced in [4], based on careful choices of the formulas defining the filtrations. In this line, notions such as the one of *extensively complete formula* will allow us to considerably strengthen the power of our techniques, as we shall see later.

Now we explain the quotient models technique.

Given any wff H, we let $\mathrm{Sf}(H)$ be the set of subformulas of H, while $\mathrm{Sf}_{\wedge,\to,\neg}(H)$ denotes the infinite set of wff's which can be built starting from the elements of $\mathrm{Sf}(H)$ only using the connectives $\wedge, \to, \neg$. Following [5,6], given a Kripke model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ and $\alpha, \beta \in P$, we set $\alpha \subseteq_H \beta$ iff, for every $H' \in \mathrm{Sf}_{\wedge,\to,\neg}(H)$, if $\alpha \Vdash H'$ then $\beta \Vdash H'$. We also set $\alpha \equiv_H \beta$ iff $\alpha \subseteq_H \beta$ and $\beta \subseteq_H \alpha$. The relation $\equiv_H$ is an equivalence relation. By a result of Diego and Mc Kay quoted in [5,6], there exists only a finite number of intuitionistically non equivalent wff's built up starting from a finite set of propositional variables and using only the connectives $\wedge, \to, \neg$. Hence, as in [5,6], one deduces:

Proposition 17. *The set of equivalence classes of* $\equiv_H$ *on the set of elements of* $\boldsymbol{K}$ *is finite.*

As in [5,6], given $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$, we define the quotient model $\boldsymbol{K}/\equiv_H$ to be the Kripke model $\langle P', \leqslant', \Vdash' \rangle$ with the following properties:

(1) P' is the set of equivalence classes generated by $\equiv_H$ on the set of elements of P;

(2) if $[\alpha]$ and $[\beta]$ are two elements of P' (where $[\gamma]$ is the class of γ), then $[\alpha] \leqslant' [\beta]$ iff $\alpha \subseteq_H \beta$;

(3) for every variable p such that $p \in \mathrm{Sf}_{\wedge,\to,\neg}(H)$, and for every element $[\alpha] \in P'$, $[\alpha] \Vdash' p$ iff $\alpha \Vdash p$ in $\boldsymbol{K}$; for every variable q such that $q \notin \mathrm{Sf}_{\wedge,\to,\neg}(H)$, and for every element $[\alpha] \in P'$, $[\alpha] \Vdash' q$ in $\boldsymbol{K}/\equiv_H$.

The main property of $\boldsymbol{K}/\equiv_H$ is stated in the following proposition, and can be proved by induction on the wff B as in [5,6]:

Proposition 18. *If* $B \in \mathrm{Sf}_{\wedge,\to,\neg}(H)$ *then, for every element* α *of* $\boldsymbol{K}$, $\alpha \Vdash B$ *(in* $\boldsymbol{K}$*) iff* $[\alpha] \Vdash' B$ *(in* $\boldsymbol{K}/\equiv_H$*).*

We also have, as an immediate consequence of Propositions 9 and 16 and of the definitions of $\boldsymbol{K}/\equiv_H$ and of regular Kripke model:

Proposition 19. *If* $\boldsymbol{K}$ *is any regular Kripke model and* H *is any formula, then* $\boldsymbol{K}/\equiv_H$ *is a regular Kripke model.*

Proposition 18 allows us to use the quotient models in completeness proofs of logics. For example, let $\mathscr{F}$ be a class of frames for which one has to prove the completeness theorem of a (standard) logic L, and suppose that $\mathscr{C}_L(\Gamma)/\equiv_H$ turns out to be built on a poset of $\mathscr{F}$ for every Γ and H. Then, by Propositions 14 and 15, for every $A \notin L$ there is a $\mathscr{C}_L(\Gamma)$ whose root does not force A, whence, by Proposition 18, the root of $\mathscr{C}_L(\Gamma)/\equiv_A$ does not force A. Thus, $\mathscr{L}(\mathscr{F}) \subseteq L$, i.e., the completeness theorem of L with respect to $\mathscr{F}$ holds.

The addition of Proposition 19 is needed in order to use the quotient models technique for completeness proofs of regular nonstandard logics.

As said above, a careful choice of the formula H to make the model $\boldsymbol{K}/\equiv_H$ (let us call H the *filtration formula*) may strengthen the quotient models technique. For example, let us consider standard logics (a quite similar discussion can be made for regular nonstandard logics). To prove that $\mathscr{L}(\mathscr{F}) \subseteq L$ (completeness theorem), it suffices to show, by Proposition 18, that $\mathscr{C}_L(\Gamma)/\equiv_H$ is built on a poset belonging to $\mathscr{F}$ *for every wff* H. The proof of the latter fact is the main concern of the usual applications of the quotient models technique. But *the use of arbitrary filtration formulas H is not necessary*. Suppose that there is a class $\mathscr{W}$ of formulas satisfying the following properties:

(a) for every wff H, there is a $W \in \mathscr{W}$ such that H is a subformula of W;

(b) for every $W \in \mathscr{W}$, $\mathscr{C}_L(\Gamma)/\equiv_W$ is built on a poset of $\mathscr{F}$.

Then, Proposition 18 allows *as well* to conclude that $\mathscr{L}(\mathscr{F}) \subseteq L$. Moreover, the choice of $\mathscr{W}$ may be made in such a way that the models $\mathscr{C}_L(\Gamma)/\equiv_W$ have nice properties giving rise to easier completeness proofs.

In this perspective, we are going to define a suitable class $\mathscr{W}$ of formulas, called *extensively complete*.

- Let W be a wff and let $\mathscr{V}_W = \{p_1, \dots, p_n\}$ be the set of propositional variables of W. Let I be any *classical* interpretation of the elements of $\mathscr{V}_W$ and let $\hat{p}_1, \dots, \hat{p}_n$ be wff's so defined, for $1 \leqslant i \leqslant n$: $\hat{p}_i = p_i$ if $I(p_i) = T$ (T the true truth value); $\hat{p}_i = \neg p_i$ if $I(p_i) = F$ (F the false truth value). Let us set $H_I = \hat{p}_1 \wedge \cdots \wedge \hat{p}_n$. Let $\mathscr{I} = \{I_1, \dots, I_k\}$ (with $k \leqslant 2^n$) be any nonempty set of classical interpretations of the elements of $\mathscr{V}_W$, and let $D_{\mathscr{I}} = H_{I_1} \vee \cdots \vee H_{I_k}$. Let $\mathscr{I}_1, \dots, \mathscr{I}_m$ be all the nonempty sets of classical interpretations of the elements of $\mathscr{V}_W$ and let $Z_W = D_{\mathscr{I}_1} \wedge \cdots \wedge D_{\mathscr{I}_m}$. Then, we say that W is a *complete formula* iff Z_W is a subformula of W.
- Given any finite set v of variables, the set of negated formulas containing only variables of v is divided into a finite set of equivalence classes $[\neg B]_v$ by intuitionistic biimplication. By a *v-complete set of negated formulas* we mean any (finite) set $\{\neg C_1, \dots, \neg C_h\}$ satisfying the following conditions:
  (1) for every equivalence class $[\neg B]_v$, there is an i, $1 \leqslant i \leqslant h$, such that $\neg C_i \in [\neg B]_v$;
  (2) for every i,j with $1 \leqslant i,j \leqslant h$ and $i \neq j$, $[\neg C_i]_v \neq [\neg C_j]_v$.
- Let W be a wff and let $\mathscr{V}_W$ be the set of its propositional variables. Let $N = \{\neg C_1, \dots, \neg C_h\}$ be a $\mathscr{V}_W$-complete set of negated formulas and let $N_1, \dots, N_m$ be all the nonempty subsets of N. Let, for every j with $1 \leqslant j \leqslant m$, D_{N_j} be the disjunction of all the formulas of N_j. Finally, let $Z'_W = D_{N_1} \wedge \cdots \wedge D_{N_m}$. We say that W is a *negatively complete formula* if Z'_W is a subformula of W.
- We say that a wff W is an *extensively complete formula* iff W is both a complete formula and a negatively complete formula.

Of course, for every wff H there is a wff W such that H is a subformula of W and W is extensively complete. Thus, we can summarize the above discussion about the possibility of choosing particular (and appropriate) filtration formulas in the following proposition.

Proposition 20. *Let L be a logic (respectively, a regular nonstandard logic) and let $\mathscr{F}$ be a nonempty class of posets such that, for every L-saturated set Γ and every extensively complete formula W, $\mathscr{C}_L(\Gamma)/\equiv_W$ is built on a poset of $\mathscr{F}$. Then $\mathscr{L}(\mathscr{F}) \subseteq L$ (respectively, $\mathscr{L}_{\mathrm{reg}}(\mathscr{F}) \subseteq L$).*

In the following, we will apply the quotient model technique using extensively complete filtration formulas, according to Proposition 20. The extensively complete formulas allow us to simplify many proofs and in some cases are necessary.

Remark. The quotient models technique does not provide strong completeness proofs. Indeed, such a technique allows us to construct *finite models* starting from sets of formulas $\mathrm{Sf}_{\wedge,\to,\neg}(W)$, defined by appropriate formulas W. On the other hand, suppose this technique is applied to prove, for an infinite set Δ of wff's, that (e.g.) $\Delta \models_{\mathscr{F}} A$ implies $\Delta \vdash_L A$. Then the application is successful only if there is some formula Z such that $\Delta \cup \{A\} \subseteq \mathrm{Sf}_{\wedge,\to,\neg}(Z)$; but, being Δ infinite, in general such a Z cannot exist.

5. The logic ST

We now consider the logic ST of Scott, quoted in the Section 1 and syntactically characterized as INT + {(ST)}, where (ST) is the axiom-schema $((\neg\neg A \to A) \to A \vee \neg A) \to \neg A \vee \neg\neg A$. To provide a semantical characterization for ST, we define the class $\mathscr{F}_{\mathrm{ST}}$ of frames as follows.

- Let $\boldsymbol{P} = \langle P, \leqslant \rangle$ be a poset and let ϕ and ψ be two final states of $\boldsymbol{P}$. We say that ϕ and ψ are *prefinally connected in* $\boldsymbol{P}$ iff either $\phi = \psi$ or there is a sequence $\phi_1, \ldots, \phi_n (n > 1)$ of final states of $\boldsymbol{P}$ satisfying the following conditions:
  (1) $\phi_1 = \phi$ and $\phi_n = \psi$;
  (2) for every i, $1 \leqslant i \leqslant n-1$, there is $\alpha \in P$ such that α is prefinal in $\boldsymbol{P}$ and $\{\phi_i, \phi_{i+1}\} \subseteq \mathrm{Fin}(\alpha)$.
- $\mathscr{F}_{\mathrm{ST}}$ will be the class of all finite frames $\boldsymbol{P} = \langle P, \leqslant \rangle$ such that, for every $\alpha \in P$ and for every ϕ and ψ belonging to $\mathrm{Fin}(\alpha)$, ϕ and ψ are prefinally connected in the cone $\boldsymbol{P}_\alpha$ of α in $\boldsymbol{P}$.

First of all, we have:

Proposition 21. ST $\subseteq \mathscr{L}(\mathscr{F}_{\mathrm{ST}})$.

Proof. Let us assume the contrary. Then there is a Kripke model $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle$ together with an instance INST $= ((\neg\neg A \to A) \to A \vee \neg A) \to \neg A \vee \neg\neg A$ of (ST) such that $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}_{\mathrm{ST}}$ and $0 \nVdash$ INST. It follows that there is $\alpha \in P$ such that $\alpha \Vdash (\neg\neg A \to A) \to A \vee \neg A$, and $\alpha \nVdash \neg A$ and $\alpha \nVdash \neg\neg A$. Hence, there are two distinct final states ϕ and ψ of $\boldsymbol{P}$ such that $\{\phi, \psi\} \subseteq \mathrm{Fin}(\alpha)$, $\phi \Vdash A$ and $\psi \Vdash \neg A$. Let $\phi_1, \ldots, \phi_n$ be a sequence of final states of $\boldsymbol{P}$ prefinally connecting ϕ and ψ in $\boldsymbol{P}_\alpha$, with $\phi = \phi_1$ and

$\psi = \phi_n$. We prove by induction on i, $1 \leqslant i \leqslant n$, that $\phi_i \Vdash A$. For, $\phi_1 \Vdash A$. Moreover, let $1 \leqslant j \leqslant n-1$, let β be a prefinal element of $\boldsymbol{P}$ such that $\alpha \leqslant \beta$ and $\mathrm{Fin}(\beta) \supseteq \{\phi_j, \phi_{j+1}\}$, and let (induction hypothesis) $\phi_j \Vdash A$. If $\phi_{j+1} \nVdash A$, then, since β is prefinal, $\beta \Vdash \neg\neg A \to A$. Thus, since $\alpha \Vdash (\neg\neg A \to A) \to A \vee \neg A$ implies $\beta \Vdash (\neg\neg A \to A) \to A \vee \neg A$, one has $\beta \Vdash A \vee \neg A$, which contradicts the fact that $\{\phi_j, \phi_{j+1}\} \subseteq \mathrm{Fin}(\beta)$. It follows that $\phi_{j+1} \Vdash A$, which completes our induction. We therefore have $\phi_n \Vdash A$, i.e., $\psi \Vdash A$, a contradiction. $\square$

We now prove that $\mathscr{L}(\mathscr{F}_{\mathrm{ST}}) \subseteq \mathrm{ST}$ using the quotient models technique with extensively complete filtration formulas. To do so, first of all we remark that $(((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p) \leftrightarrow (((\neg\neg p \to p) \to \neg p \vee \neg\neg p) \to \neg p \vee \neg\neg p) \in \mathrm{INT}$ (the proof is left to the reader as an exercise). Then, if (ST^1) is the axiom-schema $((\neg\neg A \to A) \to \neg A \vee \neg\neg A) \to \neg A \vee \neg\neg A$, we have:

Proposition 22. *The logic* ST *coincides with* $\mathrm{INT} + \{(\mathrm{ST}^1)\}$.

We also need a lemma.

Lemma 1. *Let L be any nonstandard logic such that* $\mathrm{ST} \subseteq L$. *Let Γ be any L-saturated set of formulas and let W be an extensively complete formula. Let $\boldsymbol{P}' = \langle P', \leqslant' \rangle$ be the underlying poset of $\mathscr{C}_L(\Gamma)/\equiv_W$. Let $[\Pi]$ and $[\Phi]$ be two elements of P' such that $[\Phi]$ is final in $\boldsymbol{P}'$ and $[\Phi]$ is an immediate successor of $[\Pi]$ in $\boldsymbol{P}'$. Then $[\Pi]$ is prefinal in $\boldsymbol{P}'$.*

Proof. Let us assume the contrary. Then there are nonfinal immediate successors $[\Sigma_1], \ldots, [\Sigma_k]$ of $[\Pi]$ in $\boldsymbol{P}'$, with $k \geqslant 1$. Let $\{[\Psi_1], \ldots, [\Psi_m]\} = \mathrm{Fin}([\Sigma_1]) \cup \cdots \cup \mathrm{Fin}([\Sigma_k])$. Consider any $[\Psi_i]$ (with $1 \leqslant i \leqslant m$), and let (according to the definition of complete formula, given in the previous section) I_i and H_{I_i} be, respectively, the classical interpretation associated with (the final element) $[\Psi_i]$ and the formula associated with I_i; then, if $[\Delta]$ is any element of P' different from $[\Psi_i]$, we have both $[\Delta] \nVdash' H_{I_i}$ and $[\Psi_i] \Vdash' H_{I_i}$, where $\Vdash'$ is the forcing of $\mathscr{C}_L(\Gamma)/\equiv_W$. Let $A = H_{I_1} \vee \cdots \vee H_{I_m}$; then, for every $[\Theta] \in P'$, $[\Theta] \Vdash' A$ iff $[\Theta] \in \{[\Psi_1], \ldots, [\Psi_m]\}$. Also, since W is an extensively complete formula (hence a complete formula), we can assume, without loss of generality, that $A \in \mathrm{Sf}_{\wedge,\to,\neg}(W)$ (if A is not a subformula of W, then there is a subformula A' of W which is intuitionistically equivalent to A, differing from A only for the ordering of the disjuncts and of the conjuncts within the disjuncts). Finally, since W (being extensively complete) is negatively complete, we can assume, without loss of generality, that $\neg A$, $\neg\neg A$ and $\neg A \vee \neg\neg A$ belong to $\mathrm{Sf}_{\wedge,\to,\neg}(W)$ (e.g., by definition of negatively complete formula, there is a subformula of W which is intuitionistically equivalent to $\neg A \vee \neg\neg A$). It follows that $\neg\neg A \to A$, $(\neg\neg A \to A) \to \neg A \vee \neg\neg A$ and $((\neg\neg A \to A) \to \neg A \vee \neg\neg A) \to \neg A \vee \neg\neg A$ belong to $\mathrm{Sf}_{\wedge,\to,\neg}(W)$. In particular, we get $[\Pi] \Vdash' ((\neg\neg A \to A) \to \neg A \vee \neg\neg A) \to \neg A \vee \neg\neg A$. As a matter of fact, by Proposition 22, all the states of $\mathscr{C}_L(\Gamma)$ contain all the instances of (ST^1); hence $\Pi \Vdash ((\neg\neg A \to A) \to \neg A \vee \neg\neg A) \to \neg A \vee \neg\neg A$, where $\Vdash$ is the forcing of $\mathscr{C}_L(\Gamma)$; hence our assertion, by Proposition 18.

On the other hand, consider any $[\Sigma_i]$, with $1 \leqslant i \leqslant k$. Since $\mathrm{Fin}([\Sigma_i]) \subseteq \{[\Psi_1], \ldots, [\Psi_m]\}$, we have that $[\Sigma_i] \Vdash' \neg\neg A$. Also, if $[\Phi']$ is any final element of $\boldsymbol{P}'$ such that $[\Phi'] \notin \{[\Psi_1], \ldots, [\Psi_m]\}$, then $[\Phi'] \Vdash' \neg A$. It follows that $[\Pi] \Vdash' (\neg\neg A \to A) \to \neg A \vee \neg\neg A$. As a matter of fact, let $[\Pi']$ be any element of P' such that $[\Pi] \leqslant' [\Pi']$ and $[\Pi'] \Vdash \neg\neg A \to A$. Then $[\Pi] \neq [\Pi']$, since $[\Pi] \nVdash' \neg\neg A \to A$ (being, e.g., $[\Pi] \leqslant' [\Sigma_1]$, $[\Sigma_1] \Vdash' \neg\neg A$ and $[\Sigma_1] \nVdash' A$). It follows that either $\mathrm{Fin}([\Pi']) \subseteq \{[\Psi_1], \ldots, [\Psi_m]\}$, which implies $[\Pi'] \Vdash' \neg\neg A$, or $[\Pi']$ is a final element of $\mathscr{C}_L(\Gamma)/\equiv_W$ and $[\Pi'] \notin \{[\Psi_1], \ldots, [\Psi_m]\}$, which implies $[\Pi'] \Vdash' \neg A$; in both cases, $[\Pi'] \Vdash' \neg A \vee \neg\neg A$, which provides our assertion.

Now, from $[\Pi] \Vdash' ((\neg\neg A \to A) \to \neg A \vee \neg\neg A) \to \neg A \vee \neg\neg A$ and $[\Pi] \Vdash' (\neg\neg A \to A) \to \neg A \vee \neg\neg A$ we get $[\Pi] \Vdash' \neg A \vee \neg\neg A$. This is a contradiction, since, e.g., in $\boldsymbol{P}'$ one has $\{[\Phi], [\Psi_1]\} \subseteq \mathrm{Fin}([\Pi])$ with $[\Phi] \Vdash' \neg A$ and $[\Psi_1] \Vdash' \neg\neg A$. □

Using Lemma 1, we can prove:

Theorem 10. *If L is any nonstandard logic such that* $\mathrm{ST} \subseteq L$, *Γ is any L-saturated set and W is any extensively complete formula, then $\mathscr{C}_L(\Gamma)/\equiv_W$ is built on a poset of $\mathscr{F}_{\mathrm{ST}}$.*

Proof. Assume the contrary. Then, in the underlying poset $\boldsymbol{P}' = \langle P', \leqslant' \rangle$ of $\mathscr{C}_L(\Gamma)/\equiv_W$, there are $[\Delta]$, $[\Phi]$ and $[\Psi]$ such that $\mathrm{Fin}([\Delta]) \supseteq \{[\Phi], [\Psi]\}$, but $[\Phi]$ and $[\Psi]$ are not prefinally connected in the cone $\boldsymbol{P}'_{[\Delta]}$ of $[\Delta]$ in $\boldsymbol{P}'$. Since $\boldsymbol{P}'$ is finite, we can assume, without loss of generality, that $[\Delta]$ satisfies the following further property:

(i) for every immediate successor $[\Sigma]$ of $[\Delta]$ in $\boldsymbol{P}'$, either all the element of $\mathrm{Fin}([\Sigma])$ are prefinally connected with $[\Phi]$ in the cone $\boldsymbol{P}'_{[\Sigma]}$ of $[\Sigma]$ in $\boldsymbol{P}'$, or no element of $\mathrm{Fin}([\Sigma])$ is prefinally connected with $[\Phi]$ in the cone $\boldsymbol{P}'_{[\Sigma]}$ of $[\Sigma]$ in $\boldsymbol{P}'$.

For, if $[\Delta]$ does not satisfy (i), take, in place of $[\Delta]$, some $[\Delta_1]$ which is an immediate successor of $[\Delta]$ and is followed by $[\Phi]$ and a second final element not prefinally connected with $[\Phi]$ in the cone of $[\Delta_1]$ in $\boldsymbol{P}'$; and so on, until a nonfinal state $[\Delta_n]$ is reached which satisfies the properties required for $[\Delta]$.

Now, we will divide the states of $\mathrm{Fin}([\Delta])$ into two classes: the first class, which is nonempty and will be called $\mathscr{I}$, contains all the elements of $\mathrm{Fin}([\Delta])$ which are prefinally connected with $[\Phi]$ in $\boldsymbol{P}'_{[\Delta]}$; the second class, which is nonempty too, contains all the elements of $\mathrm{Fin}([\Delta])$ which are not prefinally connected with $[\Phi]$ in $\boldsymbol{P}'_{[\Delta]}$. Arguing as in the proof of Lemma 1, we can choose a formula A such that:

(ii) $A, \neg A, \neg\neg A$ and $\neg A \vee \neg\neg A$ are intuitionistically equivalent to elements of $\mathrm{Sf}_{\wedge, \to, \neg}(W)$;

(iii) for every $[\Theta] \in P'$, $[\Theta] \Vdash' A$ (in $\mathscr{C}_L(\Gamma)/\equiv_W$) iff $[\Theta] \in \mathscr{I}$.

Now, we remark that, by Lemma 1, there must be an element $[\Pi]$ of $\mathscr{C}_L(\Gamma)/\equiv_W$ such that $[\Delta] \leqslant' [\Pi]$, $[\Pi]$ is not a final element of $\boldsymbol{P}'$ and $\mathrm{Fin}([\Pi]) \subseteq \mathscr{I}$. For, consider any element of $\mathscr{I}$, say $[\Phi]$, and let $[\Pi]$ be an element of the cone $\boldsymbol{P}'_{[\Delta]}$ such that $[\Phi]$ is an immediate successor of $[\Pi]$ in $\boldsymbol{P}'_{[\Delta]}$. Then, by Lemma 1, $[\Pi]$ is a prefinal state of $\boldsymbol{P}'$. Hence, all the elements of $\mathrm{Fin}([\Pi])$ are prefinally connected with $[\Phi]$, i.e., $\mathrm{Fin}([\Pi]) \subseteq \mathscr{I}$ as required.

Now, we have:

(iv) $[\Delta] \Vdash' (\neg\neg A \to A) \to \neg A \vee \neg\neg A$.

To prove (iv), let $[\Delta] \leqslant' [\Delta']$ and let $[\Delta'] \Vdash' \neg\neg A \to A$. Then $[\Delta] \neq [\Delta']$. As a matter of fact, let us consider the element $[\Pi]$ defined above. Since $\mathrm{Fin}([\Pi]) \subseteq \mathscr{I}$, from (iii) we get $[\Pi] \Vdash \neg\neg A$. On the other hand, since $[\Pi]$ is not final in $\boldsymbol{P}'$, (iii) yields also $[\Pi] \nVdash' A$. Being $[\Delta] \leqslant' [\Pi]$, we therefore get $[\Delta] \nVdash' \neg\neg A \to A$, which implies $[\Delta] \neq [\Delta']$, q.e.d.. Thus, there is an immediate successor $[\Sigma]$ of $[\Delta]$ in $\boldsymbol{P}'$ such that $[\Sigma] \leqslant' [\Delta']$; by (i) and (iii), this implies that either $[\Delta'] \Vdash' \neg A$ or $[\Delta'] \Vdash' \neg\neg A$. Since $[\Delta']$ is any element in $\boldsymbol{P}'$ following $[\Delta]$ and forcing $\neg\neg A \to A$ in $\mathscr{C}_L(\Gamma)/\equiv_W$, we have therefore proved (iv).

Now, by (ii), $((\neg\neg A \to A) \to \neg A \vee \neg\neg A) \to \neg A \vee \neg\neg A$ turns out to be intuitionistically equivalent to a formula of $\mathrm{Sf}_{\wedge,\to,\neg}(W)$. Hence, using Propositions 22 and 18, as in the proof of Lemma 1 we get $[\Delta] \Vdash' ((\neg\neg A \to A) \to \neg A \vee \neg\neg A) \to \neg A \vee \neg\neg A$; this implies, by (iv), $[\Delta] \Vdash' \neg A \vee \neg\neg A$. Arguing as in the proof of Lemma 1, the latter fact gives rise to a contradiction. □

Proposition 21 and Theorem 10 immediately yield:

Corollary 2. $\mathrm{ST} = \mathscr{L}(\mathscr{F}_{\mathrm{ST}})$.

As is well-known [10, 1, 16, 15], ST is a constructive logic. A proof of this fact, on the other hand, can be easily obtained also from Proposition 13 and Corollary 2, since a trivial construction shows that $\mathscr{F}_{\mathrm{ST}}$ satisfies (s.d.e.p.). Thus, we have:

Proposition 23. ST *is a constructive logic.*

Remarks. (a) In [16] Minari conjectures that ST has not a Kripke frames semantics. On the other hand, in that paper a class $\mathscr{F}$ of frames is defined such that *if* ST *has a Kripke frames semantics, then* $\mathrm{ST} = \mathscr{L}(\mathscr{F})$. Indeed, it can be proved that the class of all finite posets of $\mathscr{F}$ coincides with $\mathscr{F}_{\mathrm{ST}}$.

(b) Under our notion of poset as a partially ordered set satisfying the property of the final elements, we can prove Proposition 23 also eliminating from the definition of $\mathscr{F}_{\mathrm{ST}}$ the requirement that the involved posets be finite. Thus, if $\mathscr{F}'_{\mathrm{ST}}$ is the resulting class of posets, we obtain also $\mathrm{ST} = \mathscr{L}(\mathscr{F}'_{\mathrm{ST}})$.

(c) Under our notion of poset, the class $\mathscr{F}$ of [16] coincides with the class $\mathscr{F}'_{\mathrm{ST}}$ of the previous remark. On the other hand, if the property of the final elements is not required, one has $\mathscr{F}'_{\mathrm{ST}} \subseteq \mathscr{F}$. However, since the soundness theorem holds for $\mathscr{F}$, one has $\mathrm{ST} = \mathscr{L}(\mathscr{F})$ also in this case.

(d) If H is *any* formula, then we are no longer able to prove that $\mathscr{C}_L(\Gamma)/\equiv_H$ is built on a poset of $\mathscr{F}_{\mathrm{ST}}$; in other words, to prove the completeness theorem for ST by means of the quotient models technique, we need special filtration formulas such as the extensively complete ones. On the other hand, with a proof slightly more complex than the one given in Theorem 10, we can provide the completeness of ST using arbitrary filtration formulas in connection with the selective models technique to be explained in the second part of the paper.

6. Maximal constructive logics extending ST

In this section we give examples of progressive extensions of ST, which are carried out until a constructive logic L is reached such that the corresponding nonstandard regular logic $E(L)$ can be shown to be a maximal nonstandard constructive logic. The progressive extensions of ST are made, on the syntactical ground, by introducing new axiom schemes and give rise, from the semantical point of view, to progressive restrictions of the class $\mathscr{F}_{\mathrm{ST}}$ of frames. From an heuristic point of view, the main interest is involved in the progressive restrictions of the class of frames. The final goal is to reach a logic L characterized by a class of frames $\mathscr{F}$ with a subclass $\mathscr{F}^{\mathrm{c}} \subseteq \mathscr{F}$ of (finite) "canonical frames" such that $\mathscr{L}_{\mathrm{reg}}(\mathscr{F}) = \mathscr{L}_{\mathrm{reg}}(\mathscr{F}^{\mathrm{c}})$ and the elements of $\mathscr{F}^{\mathrm{c}}$ satisfy properties of this kind: if $\boldsymbol{P} \in \mathscr{F}^{\mathrm{c}}$ has an appropriate number of final states and $\boldsymbol{K}$ is a regular Kripke model built on $\boldsymbol{P}$, then the final states of $\boldsymbol{K}$ together with the forcing defined on them uniquely determine $\boldsymbol{K}$.

The first extensions are made by introducing the following axiom schemes:

(EST_n): $((\neg\neg A_1 \to \neg A_2 \vee \neg\neg A_2 \vee \cdots \vee \neg A_n \vee \neg\neg A_n)$

$\to \neg\neg A_1 \vee \neg\neg A_2 \vee \cdots \vee \neg\neg A_n)$

$\to \neg\neg A_1 \vee \neg\neg A_2 \vee \cdots \vee \neg\neg A_n$, for any $n \geqslant 2$

(with $A_1, A_2, \ldots, A_n$ any formulas).

We are interested in the logics ST + $\{(\mathrm{EST}_n)\}$, we call EST_n (n-extended ST). For every $n \geqslant 2$, the class of frames $\mathscr{F}_{\mathrm{EST}_n}$ for the logic EST_n is so defined:

- $\boldsymbol{P} = \langle P, \leqslant \rangle \in \mathscr{F}_{\mathrm{EST}_n}$ iff $\boldsymbol{P}$ is finite and, for every α and distinct $\phi_1, \ldots, \phi_m \in P$ with $2 \leqslant m \leqslant n$, if $\{\phi_1, \ldots, \phi_m\} \subseteq \mathrm{Fin}(\alpha)$ then there is $\beta \in P$ such that $\alpha \leqslant \beta$, β is prefinal and $\{\phi_1, \ldots, \phi_m\} \subseteq \mathrm{Fin}(\beta)$.

One easily sees that $\mathscr{F}_{\mathrm{EST}_n} \subseteq \mathscr{F}_{\mathrm{ST}}$, hence ST $\subseteq \mathscr{L}(\mathscr{F}_{\mathrm{EST}_n})$ (for every $n \geqslant 2$). One also has:

Proposition 24. $\mathrm{EST}_n \subseteq \mathscr{L}(\mathscr{F}_{\mathrm{EST}_n})$.

Proof. The assumption of the contrary yields the existence of wff's $A_1, \ldots, A_n$, a Kripke model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ built on a $\boldsymbol{P} = \langle P, \leqslant \rangle \in \mathscr{F}_{\mathrm{EST}_n}$ and states $\alpha, \phi_1, \phi_2, \ldots, \phi_n \in P$ such that $\alpha \Vdash (\neg\neg A_1 \to \neg A_2 \vee \neg\neg A_2 \vee \cdots \vee \neg A_n \vee \neg\neg A_n) \to \neg\neg A_1 \vee \neg\neg A_2 \vee \cdots \vee \neg\neg A_n$, $\phi_1 \Vdash \neg\neg A_1$, $\phi_2 \Vdash \neg A_2, \ldots, \phi_n \Vdash \neg A_n$ and $\{\phi_1, \phi_2, \ldots, \phi_n\} \subseteq \mathrm{Fin}(\alpha)$ ($\phi_1, \phi_2, \ldots, \phi_n$ are not necessarily distinct). Since $\boldsymbol{P} \in \mathscr{F}_{\mathrm{EST}_n}$, there is $\beta \in P$ such that $\alpha \leqslant \beta$, β is prefinal and $\{\phi_1, \ldots, \phi_n\} \subseteq \mathrm{Fin}(\beta)$. It follows that

$\beta \Vdash (\neg\neg A_1 \to \neg A_2 \vee \neg\neg A_2 \vee \cdots \vee \neg A_n \vee \neg\neg A_n) \to \neg\neg A_1 \vee \neg\neg A_2 \vee \cdots \vee \neg\neg A_n$ and $\beta \nVdash \neg\neg A_1$. Since β is prefinal, the latter fact gives $\beta \Vdash \neg\neg A_1 \to \neg A_2 \vee \neg\neg A_2 \vee \cdots \vee \neg A_n \vee \neg\neg A_n$, which implies, by the above, $\beta \Vdash \neg\neg A_1 \vee \neg\neg A_2 \vee \cdots \vee \neg\neg A_n$, which gives rise to a contradiction. □

We also have the following theorem, whose proof makes an essential use of the fact that $\mathrm{ST} \subseteq \mathrm{EST}_n$:

Theorem 11. *If L is any nonstandard logic such that $\mathrm{EST}_n \subseteq L$ ($n \geqslant 2$), W is any extensively complete formula and Γ is any L-saturated set, then the poset $\boldsymbol{P}' = \langle P', \leqslant' \rangle$ on which $\mathscr{C}_L(\Gamma)/\equiv_W$ is built belongs to $\mathscr{F}_{\mathrm{EST}_n}$.*

Proof. By Theorem 10, we can start from the fact that $\boldsymbol{P}' \in \mathscr{F}_{\mathrm{ST}}$. We prove the theorem by induction on $n \geqslant 2$.

Step: $n = 2$. Suppose that the assertion of the theorem does not hold. Then in $\boldsymbol{P}'$ there are states $[\Delta]$, $[\Phi_1]$ and $[\Phi_2]$ such that $\mathrm{Fin}([\Delta]) \supseteq \{[\Phi_1], [\Phi_2]\}$, but, for every $[\Theta]$ such that $[\Delta] \leqslant' [\Theta]$ and $\mathrm{Fin}([\Theta]) \supseteq \{[\Phi_1], [\Phi_2]\}$, $[\Theta]$ is not prefinal. Since $\boldsymbol{P}'$ is finite, we can furtherly assume, without loss of generality, that $[\Delta]$ satisfies the following property: for every $[\Theta]$ of $\boldsymbol{P}'$ such that $[\Delta] \leqslant' [\Theta]$ and $[\Delta] \neq [\Theta]$, $\mathrm{Fin}([\Theta]) \supseteq \{[\Phi_1], [\Phi_2]\}$ does not hold. As said above, being $\mathrm{ST} \subseteq L$, by Theorem 10 the poset $\boldsymbol{P}'$ turns out to belong to $\mathscr{F}_{\mathrm{ST}}$. It follows, by the previous hypotheses, that in $\boldsymbol{P}'$ there is $[\Phi_3]$ different from $[\Phi_1]$ and $[\Phi_2]$ such that $[\Phi_3] \in \mathrm{Fin}([\Delta])$ (otherwise, $[\Phi_1]$ and $[\Phi_2]$ could not be prefinally connected in the cone $\boldsymbol{P}'_{[\Delta]}$). Also, we can assume, without loss of generality, that in $\boldsymbol{P}'$ there is a prefinal state $[\Pi]$ such that $[\Delta] \leqslant' [\Pi]$ and $\{[\Phi_2], [\Phi_3]\} \subseteq \mathrm{Fin}([\Pi])$ (where, of course, $[\Phi_1] \notin \mathrm{Fin}([\Pi])$).

Starting from these hypotheses and along the usual lines, we can select two wff's B and C of $Sf_{\wedge,\to,\neg}(W)$ such that $[\Phi_1] \nVdash' \neg\neg B$ and $[\Psi] \Vdash' \neg\neg B$ for every $[\Psi] \in \mathrm{Fin}([\Delta])$ different from $[\Phi_1]$, $[\Phi_2] \nVdash' \neg\neg C$ and $[\Psi] \Vdash' \neg\neg C$ for every $[\Psi] \in \mathrm{Fin}([\Delta])$ different from $[\Phi_2]$ (where $\Vdash'$ is the forcing of $\mathscr{C}_L(\Gamma)/\equiv_W$).

With these positions, one has that $[\Pi] \Vdash' \neg\neg B$ and $[\Pi] \nVdash' \neg C \vee \neg\neg C$, which implies $[\Delta] \nVdash' \neg\neg B \to \neg C \vee \neg\neg C$. Thus, any $[\Theta]$ such that $[\Delta] \leqslant' [\Theta]$ and $[\Theta] \Vdash' \neg\neg B \to \neg C \vee \neg\neg C$ must satisfy $[\Theta] \Vdash' \neg\neg B$ or $[\Theta] \Vdash' \neg\neg C$ (by the properties of $[\Delta]$ and the definitions of B and C); it follows that $[\Delta] \Vdash' (\neg\neg B \to \neg C \vee \neg\neg C) \to \neg\neg B \vee \neg\neg C$. On the other hand, since W is extensively complete (hence negatively complete), we can assume, without loss of generality, that $\neg C \vee \neg\neg C$ and $\neg\neg B \vee \neg\neg C$ are subformulas of W, from which $Z = ((\neg\neg B \to \neg C \vee \neg\neg C) \to \neg\neg B \vee \neg\neg C) \to \neg\neg B \vee \neg\neg C \in \mathrm{Sf}_{\wedge,\to,\neg}(W)$; being $\Delta \Vdash Z$ in $\mathscr{C}_L(\Gamma)$, by Proposition 18 we get $[\Delta] \Vdash' Z$. It follows that $[\Delta] \Vdash' \neg\neg B \vee \neg\neg C$, which gives rise to a contradiction.

Step: $n > 2$. Since (EST_n) is an axiom schema of L, one easily shows that also (EST_{n-1}) is an axiom schema of L, i.e., $\mathrm{EST}_{n-1} \subseteq L$. Hence, we assume, as induction hypothesis, that $\boldsymbol{P}'$ belongs to $\mathscr{F}_{\mathrm{EST}_{n-1}}$. On the other hand, suppose that $\boldsymbol{P}' \notin \mathscr{F}_{\mathrm{EST}_n}$. Then in $\boldsymbol{P}'$ there are states $[\Delta], [\Phi_1], \ldots, [\Phi_n]$ such that $\{[\Phi_1], \ldots, [\Phi_n]\} \subseteq \mathrm{Fin}([\Delta])$

and, for every prefinal state $[\Pi]$ for which $[\Delta] \leqslant' [\Pi]$ holds, there is $[\Phi_i]$ with $1 \leqslant i \leqslant n$ such that $[\Phi_i] \notin \mathrm{Fin}([\Pi])$. Since $\boldsymbol{P}'$ is finite, we also assume, without loss of generality, that $[\Delta]$ satisfies the following condition: for every immediate successor $[\Sigma]$ of $[\Delta]$ in $\boldsymbol{P}'$, $\mathrm{Fin}([\Sigma])$ does not include $\{[\Phi_1], \ldots, [\Phi_n]\}$. Since $\boldsymbol{P}' \in \mathscr{F}_{\mathrm{EST}_{n-1}}$, we have, in addition, that in $\boldsymbol{P}'$ there is a prefinal state $[\hat{\Pi}]$ such that $[\Delta] \leqslant [\hat{\Pi}]$ and (e.g.) $\{[\Phi_2], \ldots, [\Phi_n]\} \subseteq \mathrm{Fin}([\hat{\Pi}])$, while, of course, $[\Phi_1] \notin \mathrm{Fin}([\hat{\Pi}])$.

With these assumptions, we can select formulas $B_1, \ldots, B_n$ of $\mathrm{Sf}_{\wedge,\rightarrow,\neg}(W)$ satisfying the following properties:

- for every i, $1 \leqslant i \leqslant n$, $[\Phi_i] \nVdash' \neg\neg B_i$;
- for every i and every $[\Psi] \in \mathrm{Fin}([\Delta])$ such that $[\Psi] \neq [\Phi_i]$, $[\Psi] \Vdash' \neg\neg B_i$.

The above implies that $[\hat{\Pi}] \Vdash' \neg\neg B_1$ and $[\hat{\Pi}] \nVdash' \neg B_2 \vee \neg\neg B_2 \vee \cdots \vee \neg B_n \vee \neg\neg B_n$, which, taking into account the properties of $[\Delta]$ and arguing as in the above proof of the basis, yields $[\Delta] \Vdash' (\neg\neg B_1 \rightarrow \neg B_2 \vee \neg\neg B_2 \vee \cdots \vee \neg B_n \vee \neg\neg B_n) \rightarrow \neg\neg B_1 \vee \neg\neg B_2 \vee \cdots \vee \neg\neg B_n$. On the other hand, since (being W extensively complete) $\neg B_2 \vee \neg\neg B_2 \vee \cdots \vee \neg B_n \vee \neg\neg B_n$ and $\neg\neg B_1 \vee \neg\neg B_2 \vee \cdots \vee \neg\neg B_n$ can be assumed to be subformulas of W, setting $Z = ((\neg\neg B_1 \rightarrow \neg B_2 \vee \neg\neg B_2 \vee \cdots \vee \neg B_n \vee \neg\neg B_n) \rightarrow \neg\neg B_1 \vee \neg\neg B_2 \vee \cdots \vee \neg\neg B_n) \rightarrow \neg\neg B_1 \vee \neg\neg B_2 \vee \cdots \vee \neg\neg B_n$ and arguing as in the proof of the basis, we get $[\Delta] \Vdash' Z$. It follows that $[\Delta] \Vdash' \neg\neg B_1 \vee \neg\neg B_2 \vee \cdots \vee \neg\neg B_n$, which gives rise to a contradiction. □

From Proposition 24 and Theorem 11 we immediately obtain:

Corollary 3. *For every* $n \geqslant 2$, $\mathrm{EST}_n = \mathscr{L}(\mathscr{F}_{\mathrm{EST}_n})$.

We define the logic EST (extended ST) setting $\mathrm{EST} = \mathrm{ST} + \bigcup_{n \geqslant 2}\{(\mathrm{EST}_n)\}$. Since, as seen in the proof of Theorem 11, $\mathrm{EST}_n \subseteq \mathrm{EST}_{n+1}$ for every $n \geqslant 2$, we have:

Proposition 25. $\mathrm{EST}_2 \subseteq \mathrm{EST}_3 \subseteq \cdots \subseteq \mathrm{EST}_n \subseteq \cdots \subseteq \mathrm{EST}$.

To provide the semantics of the logic EST, we define the class $\mathscr{F}_{\mathrm{EST}}$ of frames as follows:

- $\mathscr{F}_{\mathrm{EST}} = \bigcap_{n \geqslant 2} \mathscr{F}_{\mathrm{EST}_n}$.

From Proposition 24 we obtain $\mathrm{EST} \subseteq \mathscr{L}(\mathscr{F}_{\mathrm{EST}})$, while Theorem 11 immediately yields $\mathscr{L}(\mathscr{F}_{\mathrm{EST}}) \subseteq \mathrm{EST}$. Hence:

Corollary 4. $\mathrm{EST} = \mathscr{L}(\mathscr{F}_{\mathrm{EST}})$.

It is easy to show that any $\mathscr{F}_{\mathrm{EST}_n}$ satisfies (s.d.e.p.) and that the same holds for $\mathscr{F}_{\mathrm{EST}}$. Hence, by Proposition 13, we have:

Proposition 26. *For every* $n \geqslant 2$, EST_n *is a constructive logic*; EST *is a constructive logic.*

Remark. The above results on the logics EST_n and EST have been stated, without proof, in [15] (however, in [15] the intended proofs use arbitrary filtration formulas and are based on the selective models technique we will explain in the second part of the paper).

Later we will consider, for every $n \geqslant 2$, a maximal constructive logic containing EST_n and constructively incompatible with EST_m, for every $m \geqslant 2$ such that $m \neq n$; such a logic, of course, is constructively incompatible also with EST. Now, we want to extend EST into a maximal constructive logic. To do so, we introduce new axiom-schemes to be added to EST.

First of all, we present two following auxiliary schemes:

(DE_2): $A \vee (A \to B \vee \neg B)$;
(FIN_n): $\neg A_1 \vee (\neg A_1 \to \neg A_2) \vee \cdots \vee (\neg A_1 \wedge \cdots \wedge \neg A_{n-1} \to \neg A_n) \vee (\neg A_1 \wedge \cdots \wedge \neg A_{n-1} \to \neg\neg A_n)$, with $n \geqslant 2$.

Let $\mathscr{F}_{\mathrm{DE}_2}$ be the class of all the posets whose root is final or prefinal (i.e., the class of all the posets whose depth is at most 2, 1 being the depth of the final states), and let $\mathscr{F}_{\mathrm{FIN}_n}$ be the class of all the posets having at most n final states. As is known [6, 4] and can be easily shown using the method of the canonical models (filtrations are not necessary), one has:

- $\mathrm{INT} + \{(\mathrm{DE}_2)\} = \mathscr{L}(\mathscr{F}_{\mathrm{DE}_2})$;
- $\mathrm{INT} + \{(\mathrm{FIN}_n)\} = \mathscr{L}(\mathscr{F}_{\mathrm{FIN}_n})$.

Using (DE_2) and (FIN_n), we introduce the following axiom schemes:

(S_3): $(\neg A \to \neg B_1 \vee \neg B_2) \to (\mathrm{DE}_2)' \vee (\mathrm{FIN}_2)' \vee \neg(\neg A \wedge B_1 \wedge \neg B_2) \vee \neg(\neg A \wedge \neg B_1 \wedge B_2)$;
(S_n): $(\neg A \to \neg B_1 \vee \cdots \vee \neg B_{n-1} \vee (\neg B_1 \to \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1}))$
$\to (\mathrm{DE}_2)' \vee (\mathrm{FIN}_{n-1})' \vee \neg(\neg A \wedge B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_{n-1}) \vee \cdots \vee \neg(\neg A \wedge \neg B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_{n-2} \wedge B_{n-1})$,
with $n \geqslant 4$.

In the axiom-schemes (S_3) and (S_n) the subformulas $(\mathrm{DE}_2)'$, $(\mathrm{FIN}_2)'$ and $(\mathrm{FIN}_{n-1})'$ represent *any instances* of the axiom schemes (DE_2), (FIN_2) and (FIN_{n-1}) introduced above.

We also define the following classes of posets:

- $\mathscr{F}_{S_3}$ will be the class of all the finite posets $\boldsymbol{P} = \langle P, \leqslant \rangle$ such that, for every $\alpha, \phi_1, \phi_2 \in P$ with $\phi_1 \neq \phi_2$, the following holds: if α is not prefinal, and $|\mathrm{Fin}(\alpha)| \geqslant 3$, and $\{\phi_1, \phi_2\} \subseteq \mathrm{Fin}(\alpha)$, then there is a (possibly prefinal) $\beta \in P$ such that $\alpha \leqslant \beta$ and $\mathrm{Fin}(\beta) = \{\phi_1, \phi_2\}$.
- For every $n \geqslant 4$, $\mathscr{F}_{S_n}$ will be the class of all the finite posets $\boldsymbol{P} = \langle P, \leqslant \rangle$ such that, for every $\alpha \in P$ and distinct final states $\phi_1, \ldots, \phi_{n-1}$, the following holds: if α is not prefinal, and $|\mathrm{Fin}(\alpha)| \geqslant n$, and $\{\phi_1, \ldots, \phi_{n-1}\} \subseteq \mathrm{Fin}(\alpha)$, then there is $\beta \in P$ such that $\alpha \leqslant \beta$, and β *is not prefinal*, and $\mathrm{Fin}(\beta) = \{\phi_1, \ldots, \phi_{n-1}\}$.

We now introduce the (auxiliary) logic L_{aux} by setting $L_{\mathrm{aux}} = \mathrm{INT} + \bigcup_{n \geqslant 3}\{(S_n)\}$. We also set $\mathscr{F}_{L_{\mathrm{aux}}} = \bigcap_{n \geqslant 3}\mathscr{F}_{S_n}$. We have:

Proposition 27. $L_{\mathrm{aux}} \subseteq \mathscr{L}(\mathscr{F}_{L_{\mathrm{aux}}})$.

Proof. We have to show that every instance of (S_n) is forced in every state of every Kripke model built on a poset of $\mathscr{F}_{L_{\mathrm{aux}}}$, for every $n \geqslant 3$.

To prove the assertion for (S_3), assume the contrary. Then there is a Kripke model $\boldsymbol{K} = \langle P, \leqslant, \Vdash\rangle$ such that $\boldsymbol{P} = \langle P, \leqslant\rangle \in \mathscr{F}_{L_{\mathrm{aux}}}$, a state $\alpha \in P$, formulas A, B_1, B_2, an instance $(\mathrm{DE}_2)'$ of (DE_2) and an instance $(\mathrm{FIN}_2)'$ of (FIN_2) such that $\alpha \Vdash \neg A \to \neg B_1 \vee \neg B_2$, $\alpha \nVdash (\mathrm{DE}_2)'$, $\alpha \nVdash (\mathrm{FIN}_2)'$, $\alpha \nVdash \neg(\neg A \wedge B_1 \wedge \neg B_2)$ and $\alpha \nVdash \neg(\neg A \wedge \neg B_1 \wedge B_2)$. It follows, on the one hand, that α is not prefinal (since $\alpha \nVdash (\mathrm{DE}_2)'$) and $|\mathrm{Fin}(\alpha)| \geqslant 3$ (since $\alpha \nVdash (\mathrm{FIN}_2)'$), on the other hand that there are ϕ_1 and ϕ_2 such that $\{\phi_1, \phi_2\} \subseteq \mathrm{Fin}(\alpha)$, $\phi_1 \Vdash \neg A$, $\phi_1 \Vdash B_1$, $\phi_1 \Vdash \neg B_2$, $\phi_2 \Vdash \neg A$, $\phi_2 \Vdash \neg B_1$ and $\phi_2 \Vdash B_2$ (hence $\phi_1 \neq \phi_2$). By the properties of $\mathscr{F}_{L_{\mathrm{aux}}} \subseteq \mathscr{F}_{S_3}$, there is β such that $\alpha \leqslant \beta$ and $\mathrm{Fin}(\beta) = \{\phi_1, \phi_2\}$. Since $\beta \Vdash \neg A$ and $\beta \nVdash \neg B_1$ and $\beta \nVdash \neg B_2$ easily follows, one has $\beta \nVdash \neg A \to \neg B_1 \vee \neg B_2$, a contradiction.

Now, assume that $n \geqslant 4$ and that some instance of (S_n) is not forced on some state of some Kripke model built on a poset of $\mathscr{F}_{L_{\mathrm{aux}}}$. Then, arguing as above, we have $\boldsymbol{K} = \langle P, \leqslant, \Vdash\rangle$ such that $\boldsymbol{P} = \langle P, \leqslant\rangle \in \mathscr{F}_{L_{\mathrm{aux}}}$, a nonprefinal $\alpha \in P$ such that $|\mathrm{Fin}(\alpha)| \geqslant n$, distinct final states $\phi_1, \ldots, \phi_{n-1}$ such that $\{\phi_1, \ldots, \phi_{n-1}\} \subseteq \mathrm{Fin}(\alpha)$, and wff's $A, B_1, \ldots, B_{n-1}$ such that $\alpha \Vdash \neg A \to \neg B_1 \vee \cdots \vee \neg B_{n-1} \vee (\neg B_1 \to \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1})$, $\phi_i \Vdash \neg A$, $\phi_i \Vdash B_i$ and $\phi_i \nVdash B_j$ for $1 \leqslant i,j \leqslant n-1$ and $i \neq j$. By the properties of $\mathscr{F}_{L_{\mathrm{aux}}} \subseteq \mathscr{F}_{S_n}$, there is $\beta \in P$ such that $\alpha \leqslant \beta$, β is not prefinal and $\mathrm{Fin}(\beta) = \{\phi_1, \ldots, \phi_{n-1}\}$. Of course, by the above, $\beta \Vdash \neg A$ and $\beta \nVdash \neg B_1 \vee \neg B_2 \vee \cdots \vee \neg B_{n-1}$. On the other hand, $|\mathrm{Fin}(\beta)| = n-1 \geqslant 3$, β is not prefinal and $\mathscr{F}_{L_{\mathrm{aux}}} \subseteq \mathscr{F}_{S_{n-1}}$. It follows, by the properties of $\mathscr{F}_{S_{n-1}}$, that there is γ such that $\beta \leqslant \gamma$ and $\mathrm{Fin}(\gamma) = \{\phi_2, \ldots, \phi_{n-1}\}$. Then, $\gamma \Vdash \neg B_1$ and $\gamma \nVdash \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1}$, which implies, by the above, $\beta \nVdash \neg A \to \neg B_1 \vee \cdots \vee \neg B_{n-1} \vee (\neg B_1 \to \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1})$. This gives rise to a contradiction. □

Now we can prove:

Theorem 12. *Let L be any nonstandard logic such that $L_{\mathrm{aux}} \subseteq L$, let Γ be an L-saturated set and let W be any extensively complete formula. Then $\mathscr{C}_L(\Gamma)/\equiv_W$ is built on a poset $\boldsymbol{P}' = \langle P', \leqslant'\rangle$ of $\mathscr{F}_{L_{\mathrm{aux}}}$.*

Proof. We will give only the proof that, for any $n \geqslant 4$, $\mathscr{C}_L(\Gamma)/\equiv_W$ is built on a poset of $\mathscr{F}_{S_n}$. As matter of fact, the proof that $\mathscr{C}_L(\Gamma)/\equiv_W$ is built on a poset of $\mathscr{F}_{S_3}$ is similar, but easier (it depends on the axiom schema (S_3), instead of the axiom schemes (S_n) with $n \geqslant 4$).

Now, let $n \geqslant 4$ and let us assume that $\boldsymbol{P}'$ is not built on a poset of $\mathscr{F}_{S_n}$. Then, there are states $[\Delta], [\Phi_1], \dots, [\Phi_{n-1}]$ of $\boldsymbol{P}'$ such that:

- $[\Delta]$ is not prefinal;
- $|\mathrm{Fin}([\Delta])| \geqslant n$;
- $\mathrm{Fin}([\Delta]) \supseteq \{[\Phi_1], \dots, [\Phi_{n-1}]\}$;
- for every state $[\Theta]$ such that $[\Delta] \leqslant' [\Theta]$ in $\boldsymbol{P}'$ and $\mathrm{Fin}([\Theta]) = \{[\Phi_1], \dots, [\Phi_{n-1}]\}$, $[\Theta]$ is prefinal in $\boldsymbol{P}'$.

By the finiteness of $\boldsymbol{P}'$, we furtherly assume, without loss of generality, that $[\Delta]$ satisfies the following property:

- for every $[\Theta]$ such that $[\Delta] \leqslant' [\Theta]$ and $[\Delta] \neq [\Theta]$, either
  (a) $\mathrm{Fin}([\Theta]) \supseteq \{[\Phi_1], \dots, [\Phi_{n-1}]\}$ and $[\Theta]$ is prefinal in $\boldsymbol{P}'$
  or
  (b) $\mathrm{Fin}([\Theta]) \supseteq \{[\Phi_1], \dots, [\Phi_{n-1}]\}$ does not hold.

Now, as usual, we can choose wff's $A, B_1, \dots, B_{n-1}$ of $\mathrm{Sf}_{\wedge, \rightarrow, \neg}(H)$ such that, for every i with $1 \leqslant i \leqslant n-1$, the following conditions hold, where $\Vdash'$ is the forcing of $\mathscr{C}_L(\Gamma)/\equiv_W$:

- $[\Phi_i] \Vdash' \neg A$;
- $[\Phi_i] \Vdash' B_i$;
- if $[\Psi] \in \mathrm{Fin}([\Delta])$ and $[\Psi] \neq [\Phi_i]$, then $[\Psi] \Vdash' \neg B_i$;
- if $[\Psi] \in \mathrm{Fin}([\Delta])$ and $[\Psi] \notin \{[\Phi_1], \dots, [\Phi_{n-1}]\}$, then $[\Psi] \Vdash' A$.

With these assumptions, we prove that $\Delta \Vdash \neg A \rightarrow \neg B_1 \vee \cdots \vee \neg B_{n-1} \vee (\neg B_1 \rightarrow \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1})$ in $\mathscr{C}_L(\Gamma)$.

To do so, let Δ' be such that $\Delta \leqslant \Delta'$ and $\Delta' \Vdash \neg A$ (in $\mathscr{C}_L(\Gamma)$). Then, from the above, one gets $\Delta \not\equiv_H \Delta'$, since $\Delta \nVdash \neg A$, as a consequence of Proposition 18 and of the fact that in $\boldsymbol{P}'$ there is $[\Psi] \in \mathrm{Fin}([\Delta'])$ such that $[\Psi] \notin \{[\Phi_1\}, \dots, [\Phi_{n-1}]\}$ (we have $|\mathrm{Fin}([\Delta])| \geqslant n$); if follows that $[\Delta] \leqslant' [\Delta']$ and $[\Delta] \neq [\Delta']$. Let $[\Sigma]$ be an immediate successor of $[\Delta]$ in $\boldsymbol{P}'$ such that $[\Sigma] \leqslant' [\Delta']$. We have some cases, depending on whether Condition (a) or Condition (b) is satisfied setting $[\Sigma] = [\Theta]$.

Case 1. $\mathrm{Fin}([\Sigma]) = \{[\Phi_1], \dots, [\Phi_{n-1}]\}$ and $[\Sigma]$ is prefinal.

Here we have that either $[\Delta']$ is a final state of $\boldsymbol{P}'$ or $[\Delta'] = [\Sigma]$. If $[\Delta']$ is final, then $[\Delta'] \in \{[\Phi_1], \dots, [\Phi_{n-1}]\}$ (by definition of A), from which $\Delta' \Vdash \neg B_1 \rightarrow \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1}$ easily follows, using the properties of $B_1, \dots, B_{n-1}$ and Proposition 18. On the other hand, if $[\Delta'] = [\Sigma]$ then we have again $\Delta' \Vdash \neg B_1 \rightarrow \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1}$. As a matter of fact, $\Omega \nVdash \neg B_1$ for every Ω such that $\Delta' \leqslant \Omega$ and $\Delta' \equiv_H \Omega$ (since in $\boldsymbol{P}'$ $\mathrm{Fin}([\Omega]) = \{[\Phi_1], \dots, [\Phi_{n-1}]\}$, from which, $[\Omega] \nVdash' \neg B_1$); moreover, if $\Delta' \leqslant \Omega$ and $\Delta' \not\equiv_H \Omega$ then $[\Omega]$ is final in $\boldsymbol{P}'$, from which, by Proposition 18 and the properties of $B_1, \dots, B_n$, $\Omega \nVdash \neg B_1$ or $\Omega \Vdash \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1}$. Thus, in Case 1 we always have $\Delta' \Vdash \neg B_1 \rightarrow \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1}$.

Case 2. $\mathrm{Fin}([\Sigma])$ properly contains $\{[\Phi_1], \dots, [\Phi_{n-1}]\}$ and $[\Sigma]$ is prefinal.

Here, since $\Sigma \nVdash \neg A$ (following from $[\Sigma] \nVdash' \neg A$) and $\Delta' \Vdash \neg A$, one has that $[\Delta']$ is a final element of $\boldsymbol{P}'$, hence (being $\Delta' \Vdash \neg A$) $[\Delta'] \in \{[\Phi_1], \dots, [\Phi_{n-1}]\}$. It follows, as above, that $\Delta' \Vdash \neg B_1 \rightarrow \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1}$.

Case 3. $\mathrm{Fin}([\Sigma]) \supseteq \{[\Phi_1],\ldots,[\Phi_{n-1}]\}$ does not hold.

Here there is i, $1 \leqslant i \leqslant n-1$, such that $[\Phi_i] \notin \mathrm{Fin}([\Sigma])$. Then, since $\Delta' \Vdash \neg A$, from the properties of A one obtains $\mathrm{Fin}([\Delta']) \subseteq \{[\Phi_1],\ldots,[\Phi_{n-1}]\} - \{[\Phi_i]\}$. It follows $\Delta' \Vdash \neg B_i$, hence $\Delta' \Vdash \neg B_1 \vee \cdots \vee \neg B_{n-1}$.

Combining Cases 1–3 we get the desired proof that $\Delta \Vdash \neg A \rightarrow \neg B_1 \vee \cdots \vee \neg B_{n-1} \vee (\neg B_1 \rightarrow \neg\neg B_2 \vee \cdots \vee \neg\neg B_{n-1})$ in $\mathscr{C}_L(\Gamma)$. Hence, we have $\Delta \Vdash (\mathrm{DE}_2)' \vee (\mathrm{FIN}_{n-1})' \vee \neg(\neg A \wedge B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_{n-1}) \vee \cdots \vee \neg(\neg A \wedge \neg B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_{n-2} \wedge B_{n-1})$ in $\mathscr{C}_L(\Gamma)$, where $(\mathrm{DE}_2)'$ and $(\mathrm{FIN}_{n-1})'$ are arbitrary instances of the axiom-schemes (DE_2) and (FIN_{n-1}) respectively. Since $[\Delta]$ is not prefinal in $\boldsymbol{P}'$, and W is extensively complete, then one can show that Δ is not prefinal in the poset on which $\mathscr{C}_L(\Gamma)$ is built; hence it is not difficult to choose an instance $(\mathrm{DE}_2)''$ of (DE_2) such that $\Delta \nVdash (\mathrm{DE}_2)''$ (we leave the details to the reader). Likewise, since $|\mathrm{Fin}([\Delta])| \geqslant n$ in $\boldsymbol{P}'$, it is not difficult to choose an instance $(\mathrm{FIN}_{n-1})''$ of (FIN_{n-1}) such that $\Delta \nVdash (\mathrm{FIN}_{n-1})''$. It follows that $\Delta \Vdash \neg(\neg A \wedge B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_{n-1}) \vee \cdots \vee \neg(\neg A \wedge \neg B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_{n-2} \wedge B_{n-1})$. From the latter fact and the properties of $[\Delta]$, A, $B_1,\ldots,B_{n-1}$, one easily gets a contradiction. □

Combining Proposition 27 and Theorem 12, we obtain that $L_{\mathrm{aux}} = \mathscr{L}(\mathscr{F}_{L_{\mathrm{aux}}})$. Also, it is easily seen that $\mathscr{F}_{L_{\mathrm{aux}}}$ satisfies (s.d.e.p.), so that L_{aux} is constructive. However, we are not particularly interested in the logic L_{aux} alone. We introduce, on the other hand, the logic CEST (*completed* EST) and the class $\mathscr{F}_{\mathrm{CEST}}$ of frames as follows:
- CEST = EST + L_{aux};
- $\mathscr{F}_{\mathrm{CEST}} = \mathscr{F}_{\mathrm{EST}} \cap \mathscr{F}_{L_{\mathrm{aux}}}$.

Combining Propositions 24 and 27 and Theorems 11 and 12, we obtain that CEST = $\mathscr{L}(\mathscr{F}_{\mathrm{CEST}})$ and, since $\mathscr{F}_{\mathrm{CEST}}$ is easily seen to satisfy (s.d.e.p.), that CEST is a constructive logic. But the logic in which we are really interested is the (nonstandard) regular logic E(CEST). Combining Propositions 19, 24 and 27 and Theorems 8, 11 and 12, we get:

Corollary 5. $E(\mathrm{CEST}) = \mathscr{L}_{\mathrm{reg}}(\mathscr{F}_{\mathrm{CEST}})$.

Combining Propositions 9 and 19 and Theorems 11 and 12, we also have:

Corollary 6. *For every nonstandard logic L such that* $E(\mathrm{CEST}) \subseteq L$, *for every extensively complete formula W and for every L-saturated set* Γ, $\mathscr{C}_L(\Gamma)/\equiv_W$ *is a regular Kripke model built on a poset of* $\mathscr{F}_{\mathrm{CEST}}$.

We now single out a nice subclass of $\mathscr{F}_{\mathrm{CEST}}$ and show that it is sufficient to characterize E(CEST).

First of all, we inductively define the notion of m-CEST-*almost-canonical* state α of a poset $\boldsymbol{P} = \langle P, \leqslant \rangle$, for every $m \geqslant 1$:

(1) If $\alpha \in P$ is a final state of $\boldsymbol{P}$, then α is a 1-CEST-almost-canonical state of $\boldsymbol{P}$.

(2) If $\alpha \in P$ is a prefinal state of $\boldsymbol{P}$ and $|\text{Fin}(\alpha)| = 2$, then α is a 2-CEST-almost-canonical state of $\boldsymbol{P}$.

(3) Let $\alpha \in P$ be such that α is not prefinal and $\text{Fin}(\alpha) = \{\phi_1, \ldots, \phi_m\}$. Let $m > 2$ and let $\mathscr{I}_i = \{\phi_1, \ldots, \phi_m\} - \{\phi_i\}$ for $1 \leqslant i \leqslant m$. We say that α is a m-CEST-almost-canonical state of $\boldsymbol{P}$ iff the following conditions are satisfied:
- α has *exactly* $m + 1$ immediate successors $\beta_1, \ldots, \beta_m, \gamma$ in $\boldsymbol{P}$;
- for every i, $1 \leqslant i \leqslant m$, $\text{Fin}(\beta_i) = \mathscr{I}_i$ and β_i is a $(m-1)$-CEST-almost-canonical state of $\boldsymbol{P}$;
- $\text{Fin}(\gamma) = \{\phi_1, \ldots, \phi_m\}$ and γ is prefinal.

(4) The only m-almost-canonical states of $\boldsymbol{P}$ are the ones satisfying one of the conditions (1)–(3).

Now, let $\boldsymbol{P} = \langle P, \leqslant \rangle$ be a poset and let $\alpha \in P$ be a m-CEST-almost-canonical state of $\boldsymbol{P}$. We say that α is a m-CEST-*canonical* state of $\boldsymbol{P}$ iff the following conditions are satisfied:
- Let $\mathscr{I}$ be any nonempty subset of $\text{Fin}(\alpha)$ and let $|\mathscr{I}| = h \leqslant m$. Then there is a *unique* $\beta \in P$ such that $\alpha \leqslant \beta$, $\text{Fin}(\beta) = \mathscr{I}$ and β is a h-CEST-almost-canonical state of $\boldsymbol{P}$.
- Let $\mathscr{I}$ be any subset of $\text{Fin}(\alpha)$ such that $2 \leqslant |\mathscr{I}| = h \leqslant m$. Then there is a *unique* $\gamma \in P$ such that $\alpha \leqslant \gamma$, $\text{Fin}(\gamma) = \mathscr{I}$ and γ is prefinal in $\boldsymbol{P}$.

Having defined the m-CEST-canonical states, we introduce the class $\mathscr{F}^{c}_{\text{CEST}}$ of posets as follows:

- $\mathscr{F}^{c}_{\text{CEST}} = \{\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \mid \boldsymbol{P}$ is finite, and 0 is prefinal in $\boldsymbol{P}$ or 0 is a m-CEST-canonical state of $\boldsymbol{P}$ for some $m \geqslant 1\}$.

Fig. 1 illustrates the five immediate successors 15, 16, 17, 18 and 19 of the root 20 of the (uniquely determined up to isomorphisms) nonprefinal element $\boldsymbol{P}$ of $\mathscr{F}^{c}_{\text{CEST}}$ with the four final states 1, 2, 3 and 4. Notice that $\boldsymbol{P}$ has 20 states, among which 19 is the

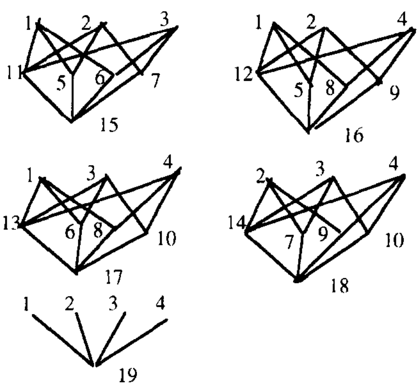

Fig. 1.

prefinal immediate successor of the root and 15, 16, 17 and 18 are the four 3-CEST-canonical states of $\boldsymbol{P}$. Notice also that the cones of 15 and 16 have in common the 2-CEST-complete (and prefinal) state 5, the cones of 15 and 17 have in common the 2-CEST-complete (and prefinal) state 6, and so on.

One immediately has:

Proposition 28. $\mathscr{F}^{c}_{\text{CEST}} \subseteq \mathscr{F}_{\text{CEST}}$.

To show that $E(\text{CEST}) = \mathscr{L}_{\text{reg}}(\mathscr{F}^{c}_{\text{CEST}})$, we recall the well known notion of open epimorphism [16] between two posets (another name for this notion, more frequent in literature, is *p-morphism*):

- Let $\boldsymbol{P} = \langle P, \leqslant \rangle$ and $\boldsymbol{P}' = \langle P', \leqslant' \rangle$ be two posets, and let $\mathscr{E}: P \to P'$ be a surjective application of P on P'. We say that $\mathscr{E}$ is an *open epimorphism of* $\boldsymbol{P}$ *in* $\boldsymbol{P}'$ iff the following conditions are satisfied:
  - if $\alpha \leqslant \beta$ then $\mathscr{E}(\alpha) \leqslant' \mathscr{E}(\beta)$;
  - if $\mathscr{E}(\alpha) \leqslant' \beta'$ then there is $\beta \in P$ such that $\alpha \leqslant \beta$ and $\mathscr{E}(\beta) = \beta'$

The following fact is well known [16] and can be easily proved by induction on A:

Proposition 29. *Let v be any nonempty set of propositional variables, and let $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ and $\boldsymbol{K}' = \langle P', \leqslant', \Vdash' \rangle$ be two Kripke models. Let $\mathscr{E}$ be an open epimorphism of the poset $\boldsymbol{P} = \langle P, \leqslant \rangle$ in the poset $\boldsymbol{P}' = \langle P', \leqslant' \rangle$ satisfying the following property: for every $\alpha \in P$ and every $p \in v$, $\alpha \Vdash p$ iff $\mathscr{E}(\alpha) \Vdash' p$. Then, for every $\alpha \in P$ and every wff A containing only variables of v, $\alpha \Vdash A$ iff $\mathscr{E}(\alpha) \Vdash' A$.*

Let $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle$ and $\boldsymbol{K}' = \langle P', \leqslant', 0', \Vdash' \rangle$ be two Kripke models such that:
- $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}_{\text{CEST}}$ and $\boldsymbol{P}' = \langle P', \leqslant', 0' \rangle \in \mathscr{F}^{c}_{\text{CEST}}$;
- $|\text{Fin}(0)_{\boldsymbol{P}}| = |\text{Fin}(0')_{\boldsymbol{P}'}|$;
- either both 0 and 0′ are prefinal states (of $\boldsymbol{P}$ and $\boldsymbol{P}'$, respectively), or both 0 and 0′ are not prefinal states (of $\boldsymbol{P}$ and $\boldsymbol{P}'$ respectively).

Also, let v be any nonempty set of propositional variables. We say that the application $\mathscr{E}: P \to P'$ is a v-CEST-*canonical application of* $\boldsymbol{K}$ *in* $\boldsymbol{K}'$ iff $\mathscr{E}$ satisfies the following conditions:

(i) Let ϕ be a final state of $\boldsymbol{P}$. Then $\mathscr{E}(\phi)$ is a final state of $\boldsymbol{P}'$.

(ii) If ϕ_1 and ϕ_2 are two different final states of $\boldsymbol{P}$, then $\mathscr{E}(\phi_1) \neq \mathscr{E}(\phi_2)$.

(iii) If $\alpha \in P$ and $\text{Fin}(\alpha)_{\boldsymbol{P}} = \{\phi\}$ for some (single) final state ϕ of $\boldsymbol{P}$, then $\mathscr{E}(\alpha) = \mathscr{E}(\phi)$.

(iv) Let $\alpha \in P$ and let $\text{Fin}(\alpha)_{\boldsymbol{P}} = \{\phi_1, \phi_2\}$, with $\phi_1 \neq \phi_2$. Let α' be the unique prefinal state of $\boldsymbol{P}'$ such that $\text{Fin}(\alpha')_{\boldsymbol{P}'} = \{\mathscr{E}(\phi_1), \mathscr{E}(\phi_2)\}$. Then $\mathscr{E}(\alpha) = \alpha'$.

(v) Let α be a prefinal state of $\boldsymbol{P}$, and let $\text{Fin}(\alpha)_{\boldsymbol{P}} = \{\phi_1, \dots, \phi_n\}$, with $n \geqslant 3$. Let α' be the unique element of $\boldsymbol{P}'$ such that α' is prefinal and $\text{Fin}(\alpha')_{\boldsymbol{P}'} = \{\mathscr{E}(\phi_1), \dots, \mathscr{E}(\phi_n)\}$. Then $\mathscr{E}(\alpha) = \alpha'$.

(vi) Let α be a nonprefinal element of $\boldsymbol{P}$ such that $\mathrm{Fin}(\alpha)_{\boldsymbol{P}} = \{\phi_1, \ldots, \phi_n\}$ and $n \geqslant 3$. Let α' be the unique n-CEST-canonical state of $\boldsymbol{P}'$ such that $\mathrm{Fin}(\alpha')_{\boldsymbol{P}'} = \{\mathscr{E}(\phi_1), \ldots, \mathscr{E}(\phi_n)\}$. Then $\mathscr{E}(\alpha) = \alpha'$.

(vii) For every $\alpha \in P$ and every $p \in v$, $\alpha \Vdash p$ (in $\boldsymbol{K}$) iff $\mathscr{E}(\alpha) \Vdash' p$ (in $\boldsymbol{K}'$).

Starting from the definition of $\mathscr{F}_{\mathrm{CEST}}$ and $\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}}$, one easily states the two following propositions, whose proofs are left to the reader.

Proposition 30. *Let $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ be a regular Kripke model such that $\boldsymbol{P} = \langle P, \leqslant \rangle \in \mathscr{F}_{\mathrm{CEST}}$, and let v be any nonempty set of propositional variables. Then there is a Kripke model $\boldsymbol{K}' = \langle P', \leqslant', \Vdash' \rangle$, together with an application $\mathscr{E}: P \to P'$, such that:*

(1) *$\boldsymbol{P}' = \langle P', \leqslant' \rangle \in \mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}}$ and $\boldsymbol{P}'$ is uniquely determined up to isomorphisms;*

(2) *$\mathscr{E}$ is a v-CEST-canonical application of $\boldsymbol{K}$ in $\boldsymbol{K}'$.*

Proposition 31. *Let $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ and $\boldsymbol{K}' = \langle P', \leqslant', \Vdash' \rangle$ be two Kripke models such that $\boldsymbol{P} = \langle P, \leqslant \rangle \in \mathscr{F}_{\mathrm{CEST}}$ and $\boldsymbol{P}' = \langle P', \leqslant' \rangle \in \mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}}$. Let v be a nonempty set of propositional variables and let $\mathscr{E}$ be a v-CEST-canonical application of $\boldsymbol{K}$ in $\boldsymbol{K}'$. Then $\mathscr{E}$ is an open epimorphism of $\boldsymbol{P}$ in $\boldsymbol{P}'$ such that, for every $\alpha \in P$ and every $p \in v$, $\alpha \Vdash p$ (in $\boldsymbol{K}$) iff $\mathscr{E}(\alpha) \Vdash' p$ (in $\boldsymbol{K}'$).*

Now we can prove:

Theorem 13. $E(\mathrm{CEST}) = \mathscr{L}_{\mathrm{reg}}(\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}})$.

Proof. From Proposition 28 we immediately obtain $E(\mathrm{CEST}) = \mathscr{L}_{\mathrm{reg}}(\mathscr{F}_{\mathrm{CEST}}) \subseteq \mathscr{L}_{\mathrm{reg}}(\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}})$.

On the other hand, let $A \notin \mathscr{L}_{\mathrm{reg}}(\mathscr{F}_{\mathrm{CEST}})$ and let v be the set of all the variables of A. Let $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle$ be a *regular* Kripke model such that $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}_{\mathrm{CEST}}$ and $0 \nVdash A$, according to the above. Let $\boldsymbol{K}' = \langle P', \leqslant', 0', \Vdash' \rangle$ and $\mathscr{E}$ be, respectively, a *regular* Kripke model such that $\boldsymbol{P}' = \langle P', \leqslant', 0' \rangle \in \mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}}$ and a v-CEST-canonical application of $\boldsymbol{K}$ in $\boldsymbol{K}'$. Note that $\boldsymbol{K}'$ and $\mathscr{E}$ exist by Proposition 30 and that $\boldsymbol{K}'$ is regular (the forcing of the regular Kripke models is uniquely determined by the forcing on the final states).

Now, by Proposition 31, $\mathscr{E}$ is an open epimorphism of $\boldsymbol{P}$ in $\boldsymbol{P}'$ such that $\alpha \Vdash p$ iff $\mathscr{E}(\alpha) \Vdash' p$, for every $\alpha \in P$ and every $p \in v$. Since $0' = \mathscr{E}(0)$ and $0 \nVdash A$, by Proposition 29 we get $0' \nVdash' A$, which implies, being $\boldsymbol{K}'$ regular, that $A \notin \mathscr{L}_{\mathrm{reg}}(\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}})$. □

Remarks. (a) Starting from the fact that $\mathscr{C}_L(\Gamma)/\equiv_W$ is a regular Kripke model built on a poset of $\mathscr{F}_{\mathrm{CEST}}$, for $E(\mathrm{CEST}) \subseteq L$, Γ a L-saturated set and W an extensively complete formula, one can directly show: *if L is a nonstandard logic such that $E(\mathrm{CEST}) \subseteq L$, Γ is a L-saturated set and W is an extensively complete formula, then $\mathscr{C}_L(\Gamma)/\equiv_W$ is built on a poset of $\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}}$.*

(b) Even if $\mathscr{L}_{\text{reg}}(\mathscr{F}^{c}_{\text{CEST}}) = \mathscr{L}_{\text{reg}}(\mathscr{F}_{\text{CEST}})$, we have not $\mathscr{L}(\mathscr{F}^{c}_{\text{CEST}}) = \mathscr{L}(\mathscr{F}_{\text{CEST}}) =$ CEST, as we will furtherly discuss later.

Since $\mathscr{F}^{c}_{\text{CEST}}$ (as well as $\mathscr{F}_{\text{CEST}}$) satisfies (s.d.e.p.), we have:

Proposition 32. *E(CEST) is a nonstandard constructive logic.*

Now we show:

Theorem 14. *E(CEST) is a maximal nonstandard constructive logic.*

Proof. Let L be any nonstandard constructive logic such that $E(\text{CEST}) \subseteq L$. Suppose that $E(\text{CEST}) \neq L$. Then there is a formula $A \in L$ such that $A \notin E(\text{CEST})$. Let $\{p_1, \dots, p_m\}$ be the set of all the propositional variables of A, and let $\{p_1, \dots, p_n\} = \{p_1, \dots, p_m\}$ if $m \geqslant 2$ and $\{p_1, \dots, p_n\} = \{p_1, p_2\}$ if $m = 1$. Let W be any extensively complete formula containing the subformula $A \wedge (p_1 \vee (p_1 \rightarrow p_2 \vee \neg p_2))$ and such that $\mathscr{V}_W = \{p_1, \dots, p_n\}$. Since E(CEST) is a constructive logic, E(CEST) itself is a E(CEST)-saturated set of formulas. As a consequence, the canonical model $\mathscr{C}_{E(\text{CEST})}(E(\text{CEST}))$ is defined. Likewise, since L is constructive, L is a L-saturated set of formulas and the canonical model $\mathscr{C}_L(L)$ is defined (we also have, being $E(\text{CEST}) \subseteq L$, that L is a E(CEST)-saturated set, so that $\mathscr{C}_L(L)$ coincides with $\mathscr{C}_{E(\text{CEST})}(L)$ and becomes a submodel of $\mathscr{C}_{E(\text{CEST})}(E(\text{CEST}))$).

Now, let us consider the two models $\mathscr{C}_{E(\text{CEST})}(E(\text{CEST}))/\equiv_W$ and $\mathscr{C}_L(L)/\equiv_W$. Let $\boldsymbol{K}_1 = \langle P_1, \leqslant_1, 0_1, \Vdash_1 \rangle = \mathscr{C}_{E(\text{CEST})}(E(\text{CEST}))/\equiv_W$ and $\boldsymbol{K}_2 = \langle P_2, \leqslant_2, 0_2, \Vdash_2 \rangle = \mathscr{C}_L(L)/\equiv_W$. Then, first of all we have that the set of final states of $\boldsymbol{K}_1$ and the set of final states of $\boldsymbol{K}_2$ coincide and correspond to the 2^n classical interpretations of the variables $p_1, \dots, p_n$. For, every such a final state defines (in the obvious way) a classical interpretation of $p_1, \dots, p_n$. On the other hand, suppose that there is a classical interpretation I of $p_1, \dots, p_n$ such that the corresponding final state (i.e., a state forcing p_i if $I(p_i) = T$, and forcing $\neg p_i$ if $I(p_i) = F$, for $1 \leqslant i \leqslant n$) is not a state of $\boldsymbol{K}_j$, with $j = 1$ or $j = 2$. If H_I is the formula $\hat{p}_1 \wedge \cdots \wedge \hat{p}_n$ with $\hat{p}_i = p_i$ if $I(p_i) = T$, and $\hat{p}_i = \neg p_i$ if $I(p_i) = F$ (for $1 \leqslant i \leqslant n$), we therefore have that all the final states of $\boldsymbol{K}_j$ force $\neg H_I$. It follows that the root of $\boldsymbol{K}_j$ forces $\neg H_I$. Since $\neg H_I \in \text{Sf}_{\wedge, \rightarrow, \neg}(W)$, the latter fact implies that the root of $\mathscr{C}_{L_j}(L_j)$ forces $\neg H_I$, L_j being E(CEST) if $j = 1$ and being L if $j = 2$. But $\neg H_I$ is not a classical tautology, while L_j, being a nonstandard logic, contains only classical tautologies. A contradiction.

Now, we show that, for $j = 1$ or $j = 2$, the root 0_j of the poset $\boldsymbol{P}_j$ on which $\boldsymbol{K}_j$ is built is not prefinal. For otherwise, $0_j \Vdash_j p_1 \vee (p_1 \rightarrow p_2 \vee \neg p_2)$ holds in $\boldsymbol{K}_j$, since $p_1 \vee (p_1 \rightarrow p_2 \vee \neg p_2)$ is an instance of the axiom-schema (DE_2). Since $p_1 \vee (p_1 \rightarrow p_2 \vee \neg p_2) \in \text{Sf}_{\wedge, \rightarrow, \neg}(W)$ by definition of our W, we obtain $p_1 \vee (p_1 \rightarrow p_2 \vee \neg p_2) \in L_j$, which implies (since p_1 is not a classical tautology) that $p_1 \rightarrow p_2 \vee \neg p_2 \in L_j$. Since L_j *is closed under arbitrary restricted substitutions*, we get, e.g., $\neg\neg(B \rightarrow B) \rightarrow \neg p_2 \vee \neg\neg p_2 \in L_j$. Hence, being $\neg\neg(B \rightarrow B) \in L_j$,

$\neg p_2 \vee \neg\neg p_2 \in L_j$ by modus ponens. Since neither $\neg p_2$ nor $\neg\neg p_2$ is a classical tautology, and L_j is a nonstandard constructive logic, this gives rise to a contradiction.

We have therefore proved that $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ have the same $2^n = k$ final states with the same forcing, and that their roots are not prefinal states. Thus, by Proposition 30, we can take a regular Kripke model $\hat{\boldsymbol{K}} = \langle \hat{P}, \leqslant^{\wedge}, \hat{0}, \Vdash^{\wedge} \rangle$ such that $\hat{\boldsymbol{P}} = \langle P, \leqslant^{\wedge}, \hat{0} \rangle \in \mathscr{F}^c_{\text{CEST}}$ and $\hat{0}$ is a k-CEST-canonical state of $\hat{\boldsymbol{P}}$, and two applications $\mathscr{E}_1 : P_1 \to \hat{P}$ and $\mathscr{E}_2 : P_2 \to \hat{P}$ such that:

- $\mathscr{E}_1$ is a $\{p_1, \ldots, p_n\}$-CEST- canonical application $\boldsymbol{K}_1$ in $\hat{\boldsymbol{K}}$;
- $\mathscr{E}_2$ is a $\{p_1, \ldots, p_n\}$-CEST-canonical application of $\boldsymbol{K}_2$ in $\hat{\boldsymbol{K}}$.

Since $A \in \mathrm{Sf}_{\wedge, \to, \neg}(W)$ and $A \notin E(\text{CEST})$, we have $0_1 \nVdash_1 A$ (in $\boldsymbol{K}_1$), which implies, by Propositions 29–31, $\hat{0} \nVdash^{\wedge} A$ (in $\hat{\boldsymbol{K}}$). The latter fact, still by Propositions 29–31, implies $0_2 \nVdash_2 A$ (in $\boldsymbol{K}_2$). Hence, being $A \in \mathrm{Sf}_{\wedge, \to, \neg}(W)$, $A \notin L$, a contradiction. $\square$

Remark. In the proof of Theorem 14 an essential use has been made of the fact that the logic $L \supseteq E(\text{CEST})$ *is closed under restricted substitutions.* On this aspect we will come back later.

From Theorems 5 and 14 we immediately deduce:

Corollary 7. *$S(E(\text{CEST}))$ is a maximal constructive logic.*

We are interested in giving a semantical characterization of the logic $S(E(\text{CEST}))$, i.e., in showing that this logic coincides with $\mathscr{L}(\mathscr{F}^c_{\text{CEST}})$. To do so, in order to apply Theorem 9, we prove the two following propositions.

Proposition 33. *Let $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}^c_{\text{CEST}}$ and let $\alpha \in P$ be a prefinal state of $\boldsymbol{P}$. Then there is a set $\mathscr{V}_\alpha$ of propositional variables, together with a negatively saturated formula H_α containing only variables of $\mathscr{V}_\alpha$, and a forcing $\Vdash'$ on the final states of $\boldsymbol{P}$ for the variables of $\mathscr{V}_\alpha$ such that the following holds: for every Kripke model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ such that $\boldsymbol{K}$ is built on $\boldsymbol{P}$ and the forcing $\Vdash$ coincides with $\Vdash'$ on the final states of $\boldsymbol{P}$ for the variables of $\mathscr{V}_\alpha$, $\alpha \Vdash H_\alpha$ and, for every $\beta \in P$ such that β is not final and $\alpha \neq \beta$, $\beta \nVdash H_\alpha$.*

Proof. Let $\mathrm{Fin}(\alpha) = \{\phi_1, \ldots, \phi_n\}$ (with $n \geqslant 2$) and let $\psi_1, \ldots, \psi_h$ be all the final states of $\boldsymbol{P}$ (where, of course, $\{\phi_1, \ldots, \phi_n\} \subseteq \{\psi_1, \ldots, \psi_h\}$). We select the distinct variables $p_1, \ldots, p_n, q_1, \ldots, q_h, r_1, \ldots, r_n$, and set $\mathscr{V}_\alpha = \{p_1, \ldots, p_n, q_1, \ldots, q_h, r_1, \ldots, r_n\}$. Now, let H^1_α, H^2_α and H_α be the wff's so defined:

- $H^1_\alpha = (\neg p_1 \to \neg q_1 \vee \cdots \vee \neg q_h) \wedge \cdots \wedge (\neg p_n \to \neg q_1 \vee \cdots \vee \neg q_h)$;
- $H^2_\alpha = \neg r_1 \vee (\neg r_1 \to \neg r_2) \vee \cdots \vee (\neg r_1 \wedge \cdots \wedge \neg r_{n-1} \to \neg r_n) \vee (\neg r_1 \wedge \cdots \wedge \neg r_{n-1} \to \neg\neg r_n)$ (H^2_α is an instance of the axiom schema (FIN_n));
- $H_\alpha = H^1_\alpha \wedge H^2_\alpha$.

We define $\Vdash'$ to be any forcing on the final states of $\boldsymbol{P}$ satisfying the following conditions:

- for every j, $1 \leqslant j \leqslant n$, $\phi_j \Vdash' p_j$, and, for every k with $1 \leqslant k \leqslant n$ and $k \neq j$, $\phi_k \Vdash' \neg p_j$;

- $\phi_1 \Vdash' r_1$, $\phi_2 \Vdash' \neg r_1 \wedge r_2, \dots, \phi_n \Vdash' \neg r_1 \wedge \cdots \wedge \neg r_{n-1} \wedge r_n$;
- for every j, $1 \leqslant j \leqslant h$, $\psi_j \Vdash' \neg q_j$, and, for every k with $1 \leqslant k \leqslant h$ and $k \neq j$, $\psi_k \Vdash' q_j$;
- for every $\psi \in \{\psi_1, \dots, \psi_h\} - \{\phi_1, \dots, \phi_n\}$, $\psi \Vdash' \neg r_1 \wedge \cdots \wedge \neg r_n$, and, for every j with $1 \leqslant j \leqslant n$, $\psi \Vdash' \neg p_j$.

Now, let $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ be any Kripke model built on $\boldsymbol{P}$ such that $\Vdash$ coincides with $\Vdash'$ on the final states of $\boldsymbol{P}$ for the variables of $\mathscr{V}_\alpha$. With these positions, it is easy to show that $\alpha \Vdash H_\alpha$ in $\boldsymbol{K}$.

On the other hand, let $\beta \in P$ such that β is not final and $\beta \neq \alpha$. If β is prefinal, then, by the properties of $\mathscr{F}^c_{\text{CEST}}$, $\text{Fin}(\beta) \neq \text{Fin}(\alpha)$. In this case, if $\text{Fin}(\alpha) \subseteq \text{Fin}(\beta)$, then there is $\psi \in \text{Fin}(\beta)$ such that $\psi \notin \text{Fin}(\alpha)$, which implies $\beta \nVdash H^2_\alpha$, hence $\beta \nVdash H_\alpha$. If β is prefinal and $\text{Fin}(\alpha) \subseteq \text{Fin}(\beta)$ does not hold, then there is ϕ_i with $1 \leqslant i \leqslant n$ such that $\phi_i \notin \text{Fin}(\beta)$; it follows that $\beta \Vdash \neg p_i$, from which, since $\beta \nVdash \neg q_1 \vee \cdots \vee \neg q_h$ (as a consequence of the fact that $|\text{Fin}(\beta)| \geqslant 2$), one gets $\beta \nVdash \neg p_i \to \neg q_1 \vee \cdots \vee \neg q_h$, which implies $\beta \nVdash H^1_\alpha$, which implies $\beta \nVdash H_\alpha$. If β is not prefinal, since β is not final too, we have $|\text{Fin}(\beta)| \geqslant 3$. Here, if $\text{Fin}(\beta) \neq \text{Fin}(\alpha)$ and $\text{Fin}(\alpha) \subseteq \text{Fin}(\beta)$, then, as before, one gets $\beta \nVdash H^2_\alpha$, hence $\beta \nVdash H_\alpha$; also, if $\text{Fin}(\alpha) \subseteq \text{Fin}(\beta)$ does not hold, then, as before, $\beta \nVdash \neg p_i \to \neg q_1 \vee \cdots \vee \neg q_h$ for some i with $1 \leqslant i \leqslant n$, hence $\beta \nVdash H^1_\alpha$, hence $\beta \nVdash H_\alpha$. It remains to consider the case where β is not prefinal and $\text{Fin}(\beta) = \text{Fin}(\alpha)$.

In the latter case, being $|\text{Fin}(\beta)| \geqslant 3$, by the properties of $\mathscr{F}^c_{\text{CEST}}$ there is $\gamma \in P$ such that $\beta \leqslant \gamma$ and $\text{Fin}(\gamma) = \{\phi_1, \dots, \phi_{n-1}\}$; it follows that $\gamma \nVdash \neg p_n \to \neg q_1 \vee \cdots \vee \neg q_h$, hence $\beta \nVdash H^1_\alpha$, hence $\beta \nVdash H_\alpha$. □

Proposition 34. *Let $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}^c_{\text{CEST}}$ and let $\alpha \in P$. Then there is a set $\mathscr{V}_\alpha$ of propositional variables, together with a negatively saturated formula H_α containing only variables of $\mathscr{V}_\alpha$, and a forcing $\Vdash'$ on the final states of $\boldsymbol{P}$ for the variables of $\mathscr{V}_\alpha$ such that the following holds: for every Kripke model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ such that $\boldsymbol{K}$ is built on $\boldsymbol{P}$ and the forcing $\Vdash$ coincides with $\Vdash'$ on the final states of $\boldsymbol{P}$ for the variables of $\mathscr{V}_\alpha$, $\alpha \Vdash H_\alpha$ and, for every $\beta \in P$ such that $\alpha \leqslant \beta$ does not hold, $\beta \nVdash H_\alpha$.*

Proof. If α is prefinal, then the assertion follows from Proposition 33.

If α is not prefinal, then α is m-CEST-canonical for $m = 1$ or $m \geqslant 3$. Here we set $\mathscr{V}_\alpha = \{p\}$ and $H_\alpha = \neg\neg p$, while $\Vdash'$ will be any forcing such that $\phi \Vdash' p$ iff $\phi \in \text{Fin}(\alpha)$. If $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$ is any Kripke model built on $\boldsymbol{P}$ whose forcing $\Vdash$ coincides with $\Vdash'$ on the final states of $\boldsymbol{P}$ with respect to p, we trivially have $\alpha \Vdash H_\alpha$. On the other hand, if $\beta \in P$ and $\alpha \leqslant \beta$ does not hold then $\alpha \nVdash H_\alpha$, since we cannot have $\text{Fin}(\beta) \subseteq \text{Fin}(\alpha)$. As a matter of fact, if $\text{Fin}(\beta) \subseteq \text{Fin}(\alpha)$ and $\alpha \leqslant \beta$ does not hold, then, by the properties of $\mathscr{F}^c_{\text{CEST}}$, α is prefinal, a contradiction. □

From Proposition 34 we obtain:

Theorem 15. *$\mathscr{L}(\mathscr{F}^c_{\text{CEST}})$ is neg.sat.-determined.*

Proof. Assume the contrary. Then there is a wff A such that, for every negatively saturated substitution σ_{ns}, $\sigma_{ns}A \in \mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}})$, while $A \notin \mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}})$. It follows that there is a Kripke model $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash\rangle$ such that $\boldsymbol{P} = \langle P, \leqslant, 0\rangle \in \mathscr{F}^c_{\mathrm{CEST}}$ and $0 \nVdash A$.

Let $\mathscr{V}_A$ be the set of propositional variables of A, let $p \in \mathscr{V}_A$ and let $\alpha \in P$. We say that α *is initial for* p iff $\alpha \Vdash p$ and, for every $\beta \in P$ such that $\beta \leqslant \alpha$ and $\beta \neq \alpha$, $\beta \nVdash p$.

Now, for any $p \in \mathscr{V}_A$ such that $\alpha \Vdash p$ for some $\alpha \in P$, let $\alpha^p_1, \ldots, \alpha^p_m$ $(m \geqslant 1)$ be all the states of $\boldsymbol{P}$ which are initial for p. Let, for $1 \leqslant i \leqslant m$, H^p_i be a formula satisfying Proposition 34, replacing in that proposition the state α with α^p_i and the formula H_α with H^p_i. We define the formula H_p as follows:

– $H_p = H^p_1 \vee \cdots \vee H^p_m$.

On the other hand, if $p \in \mathscr{V}_A$ and there is no $\alpha \in P$ such that $\alpha \Vdash p$, we define H_p as follows:

– $H_p = \neg\neg p$.

Notice that, in any case, H_p is a negatively saturated formula. We can assume, without loss of generality, that, if $p \neq q$ and $\{p, q\} \subseteq \mathscr{V}_A$, then the variables occurring in H_p are different from the variables occurring in H_q, and that, if $H_p = H^p_1 \vee \cdots \vee H^p_m$, then the variables occurring in H^p_i are different from the variables of H^p_j for $1 \leqslant i, j \leqslant m$ and $i \neq j$.

With these assumptions, let $\boldsymbol{K}' = \langle P, \leqslant, 0, \Vdash'\rangle$ be a Kripke model built on $\boldsymbol{P}$ in such a way that the following property holds:

(i) for every $p \in \mathscr{V}_A$ and every $\alpha \in P$, $\alpha \Vdash' H_p$ iff there is α^p_i such that α^p_i is initial for p and $\alpha^p_i \leqslant \alpha$.

Notice that such a forcing $\Vdash'$ can be defined, as a consequence of Proposition 34. Thus, let $\hat{\sigma}_{ns}$ be a negatively saturated substitution so defined on the set of variables $\mathscr{V}_A$:

– for every $p \in \mathscr{V}_A$, $\hat{\sigma}_{ns}(p) = H_p$.

Using (i), with an easy induction on the complexity of A we can prove:

(ii) for every $\alpha \in P$, $\alpha \Vdash A$ (in $\boldsymbol{K}$) iff $\alpha \Vdash' \hat{\sigma}_{ns}A$ (in $\boldsymbol{K}'$).

It follows that $0 \nVdash' \hat{\sigma}_{ns}A$, hence $\hat{\sigma}_{ns}A \notin \mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}})$, a contradiction. □

Finally, combining Corollary 7 and Theorems 13 and 15 with the key Theorem 9, we obtain:

Corollary 8. $S(E(\mathrm{CEST})) = \mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}})$.

Thus, our method has succeeded in providing a Kripke frames semantics for the maximal constructive logic $S(E(\mathrm{CEST}))$.

As said in a previous remark, we can find formulas of $\mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}})$ which do not belong to $\mathscr{L}(\mathscr{F}_{\mathrm{CEST}}) = \mathrm{CEST}$. As an example, let us consider the axiom-schema (SCHEMA) so defined:

$$\text{(SCHEMA)}: (A \to \neg F) \wedge ((B \wedge C) \vee (B \wedge D) \vee (C \wedge D) \to F \vee \neg F)$$
$$\wedge ((A \to B \vee C \vee D \vee E)$$

$$\rightarrow B \vee C \vee D \vee E) \wedge ((B \rightarrow A \vee C \vee D \vee E \vee (E \rightarrow \neg F))$$
$$\rightarrow A \vee C \vee D \vee E) \wedge ((C \rightarrow A \vee B \vee D \vee E) \rightarrow A \vee B \vee D \vee E)$$
$$\wedge ((D \rightarrow A \vee B \vee C \vee E) \rightarrow A \vee B \vee C \vee E)$$
$$\wedge ((E \rightarrow A \vee B \vee C \vee D) \rightarrow A \vee B \vee C \vee D) \rightarrow A \vee B \vee C \vee D \vee E,$$

where A, B, C, D, E and F are any formulas.

Then every instance of (SCHEMA) belongs to $\mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}})$. On the other hand, if A, B, C, D, E and F coincide with the propositional variables p. q, r, s, t and u, respectively, then the corresponding instance of (SCHEMA) does not belong to CEST.

To show that (SCHEMA) belongs to $\mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}})$, assume the contrary. Then there is a Kripke model $\boldsymbol{K} = \langle P, \leqslant, \Vdash \rangle$, together with $\alpha \in P$, such that $\boldsymbol{P} = \langle P, \leqslant \rangle \in \mathscr{F}^c_{\mathrm{CEST}}$, α forces in $\boldsymbol{K}$ the antecedent of (SCHEMA) and $\alpha \nVdash A \vee B \vee C \vee D \vee E$. Since (being (SCHEMA) a classical tautology) every element of $\mathrm{Fin}(\alpha)$ forces (SCHEMA) and since $\boldsymbol{P}$ is a finite poset, without loss of generality we can suppose that all the immediate successors of α force A, or B, or C, or D, or E.

Now, since α forces the antecedent of (SCHEMA), in particular we have that $\alpha \Vdash (B \rightarrow A \vee C \vee D \vee E \vee (E \rightarrow \neg F)) \rightarrow A \vee C \vee D \vee E$. Since $\alpha \nVdash A \vee C \vee D \vee E$, it follows that $\alpha \nVdash B \rightarrow A \vee C \vee D \vee E \vee (E \rightarrow \neg F)$, which implies the existence of $\beta \geqslant \alpha$ such that $\alpha \Vdash B$, $\beta \nVdash A$, $\beta \nVdash C$, $\beta \nVdash D$, $\beta \nVdash E$ and $\beta \nVdash E \rightarrow \neg F$. Thus, there is an immediate successor β_1 of α such that $\beta_1 \Vdash B$, $\beta_1 \nVdash A$, $\beta_1 \nVdash C$, $\beta_1 \nVdash D$, $\beta_1 \nVdash E$ and $\beta_1 \nVdash E \rightarrow \neg F$. From $\beta_1 \nVdash E \rightarrow \neg F$ we also get the existence of $\gamma \in P$ and $\phi \in \mathrm{Fin}(\gamma)$ such that $\beta_1 \leqslant \gamma$, $\gamma \Vdash E$ and $\phi \Vdash F$; it follows that $\beta_1 \neq \gamma$, hence β_1 is not final and there is $\phi \in \mathrm{Fin}(\beta_1)$ such that $\phi \Vdash F$.

Likewise, taking into account (in the antecedent of (SCHEMA)) the formulas $(C \rightarrow A \vee B \vee D \vee E) \rightarrow A \vee B \vee D \vee E$, $(D \rightarrow A \vee B \vee C \vee E) \rightarrow A \vee B \vee C \vee E$, $(E \rightarrow A \vee B \vee C \vee D) \rightarrow A \vee B \vee C \vee D$ and $(A \rightarrow B \vee C \vee D \vee E) \rightarrow B \vee C \vee D \vee E$, we can show the existence of immediate successors β_2, β_3, β_4 and β_5 of α such that: $\beta_2 \Vdash C$, while $\beta_2 \nVdash A$, $\beta_2 \nVdash B$, $\beta_2 \nVdash D$, $\beta_2 \nVdash E$; $\beta_3 \Vdash D$, while $\beta_3 \nVdash A$, $\beta_3 \nVdash B$, $\beta_3 \nVdash C$, $\beta_3 \nVdash E$; $\beta_4 \Vdash E$, while $\beta_4 \nVdash A$, $\beta_4 \nVdash B$, $\beta_4 \nVdash C$, $\beta_4 \nVdash D$; $\beta_5 \Vdash A$, while $\beta_5 \nVdash B$, $\beta_5 \nVdash C$, $\beta_5 \nVdash D$, $\beta_5 \nVdash E$.

We have therefore proved that there are at least five *distinct* immediate successors of α, namely, β_1, β_2, β_3, β_4 and β_5. Since $\boldsymbol{P} \in \mathscr{F}^c_{\mathrm{CEST}}$, we get $|\mathrm{Fin}(\alpha)| \geqslant 4$; moreover, since β_1 is not final, α is not prefinal. Also, $\alpha \Vdash A \rightarrow \neg F$ (since α forces the antecendet of (SCHEMA)); then, all the elements of $\mathrm{Fin}(\beta_5)$ force $\neg F$, while $\phi \in \mathrm{Fin}(\beta_1)$ and $\phi \Vdash F$. It follows that not all the elements of $\mathrm{Fin}(\alpha)$ are elements of $\mathrm{Fin}(\beta_5)$, hence (being $\boldsymbol{P} \in \mathscr{F}^c_{\mathrm{CEST}}$ and $|\mathrm{Fin}(\alpha)| \geqslant 4$), β_5 cannot be prefinal; furthermore (still by the properties of the elements of $\mathscr{F}^c_{\mathrm{CEST}}$), for every immediate successor δ of α with $\delta \neq \beta_5$, $\phi \in \mathrm{Fin}(\delta)$. Since $|\mathrm{Fin}(\alpha)| \geqslant 4$, $|\mathrm{Fin}(\delta)| \cap |\mathrm{Fin}(\beta_5)| \geqslant 2$ for every immediate successor δ of α such that $\delta \neq \beta_5$; hence, for every immediate successor δ of α such that $\delta \neq \beta_5$, there are at least two elements of $\mathrm{Fin}(\delta)$ forcing $\neg F$. Finally, by the properties of the elements of $\mathscr{F}^c_{\mathrm{CEST}}$, α has exactly one prefinal immediate successor; hence, either both β_1 and β_2 are not prefinal, or both β_1 and β_3 are not prefinal, or both β_2 and β_3 are not prefinal. Suppose, without loss of generality, that β_1 and β_2 are not prefinal.

Now, since β_1 and β_2 are not prefinal, by the above discussion and the properties of the elements of $\mathscr{F}^c_{\mathrm{CEST}}$, there is $\varepsilon \in P$ such that $\beta_1 \leqslant \varepsilon$, $\beta_2 \leqslant \varepsilon$, $\phi \in \mathrm{Fin}(\varepsilon)$, $|\mathrm{Fin}(\varepsilon) \cap \mathrm{Fin}(\beta_5)| \geqslant 1$; it follows that $\varepsilon \Vdash B \wedge C$, but $\varepsilon \nVdash F \vee \neg F$, hence $\varepsilon \nVdash (B \wedge C) \vee (B \wedge D) \vee (C \wedge D) \to F \vee \neg F$, the latter formula being one of the conjuncts of the antecedent of (SCHEMA). This gives rise to a contradiction.

To prove that if A, B, C, D, E and F coincide with the propositional variables p, q, r, s, t and u, respectively, then the corresponding instance of (SCHEMA) does not belong to CEST $= \mathscr{L}(\mathscr{F}_{\mathrm{CEST}})$, consider the Kripke model of Fig. 2.

It is readily seen that the Kripke model of Fig. 2 is built on a poset of $\mathscr{F}_{\mathrm{CEST}}$ and that

$$\begin{aligned} 0 \nVdash\, & (p \to \neg u) \wedge ((q \wedge r) \vee (q \wedge s) \vee (r \wedge s) \to u \vee \neg u) \\ & \wedge ((p \to q \vee r \vee s \vee t) \to q \vee r \vee s \vee t) \\ & \wedge ((q \to p \vee r \vee s \vee t \vee (t \to \neg u)) \to p \vee r \vee s \vee t) \\ & \wedge ((r \to p \vee q \vee s \vee t) \to p \vee q \vee s \vee t) \wedge ((s \to p \vee q \vee r \vee t) \to p \vee q \vee r \vee t) \\ & \wedge ((t \to p \vee q \vee r \vee s) \to p \vee q \vee r \vee s) \to p \vee q \vee r \vee s \vee t. \end{aligned}$$

We point out that $\mathscr{F}^c_{\mathrm{CEST}}$ (differently from $\mathscr{F}_{\mathrm{CEST}}$) is not closed under open epimorphisms, which implies that $\mathrm{Ep}(\mathscr{F}^c_{\mathrm{CEST}})$ properly contains $\mathscr{F}^c_{\mathrm{CEST}}$, where, for every nonempty class $\mathscr{F}$ of posets, by $\mathrm{Ep}(\mathscr{F})$ we denote $\{\boldsymbol{P}/\text{there is } \boldsymbol{P}' \in \mathscr{F} \text{ together with an open epimorphism } \mathscr{E} \text{ of } \boldsymbol{P}' \text{ in } \boldsymbol{P}\}$. Since, for every $\mathscr{F}$, one has $\mathscr{L}(\mathrm{Ep}(\mathscr{F})) = \mathscr{L}(\mathscr{F})$ (as one easily proves starting from Proposition 29, and as it is well known), it follows that $\mathscr{F}^c_{\mathrm{CEST}}$ is not the greatest class $\hat{\mathscr{F}}$ of frames such that $\mathscr{L}(\hat{\mathscr{F}}) = S(E(\mathrm{CEST}))$. Such a $\hat{\mathscr{F}}$ (which is easily proved to exist) stands between $\mathscr{F}^c_{\mathrm{CEST}}$ and $\mathscr{F}_{\mathrm{CEST}}$ (the latter being the greatest $\mathscr{F}$ such that CEST $= \mathscr{L}(\mathscr{F})$), but we do not know a characterization of $\hat{\mathscr{F}}$.

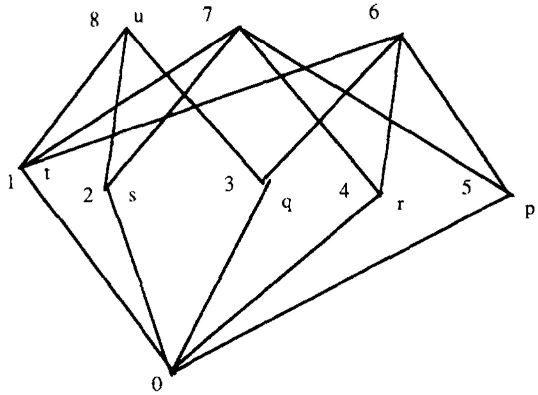

Fig. 2.

From this point of view, consider the application of filtration methods such as the ones used in this paper. Let L be a logic syntactically characterized by a set of axiom-schemes and let $\mathscr{F}$ be a class of posets for which $L = \mathscr{L}(\mathscr{F})$ is to be proved; then, generally a completeness proof by filtration techniques amounts to prove that, for every $\boldsymbol{P} \notin \mathscr{F}$, there is a Kripke model built on $\boldsymbol{P}$ whose root does not force some instance of some axiom-schema of L. In other words (instead of interpreting such a proof as a proof that a contradiction arises from the assumption that the filtration model obtained from the canonical model is built on a poset which does not belong to $\mathscr{F}$), we can look at a completeness proof as the *construction of a counter-model for some formula of L* (a counter-model whose underlying poset does not belong to $\mathscr{F}$). As a consequence, generally a completeness proof *is successful only if $\mathscr{F}$ is the greatest class of frames for L*. Thus, *without a characterization of the greatest class $\hat{\mathscr{F}}$ of frames for* $S(E(\text{CEST}))$, *no attempt of giving a recursive axiomatization of* $S(E(\text{CEST}))$ *seems to be promising.*

A similar discussion could be made also for nonstandard logics. However, in this case particular properties of the involved models (due to the presence of the axioms $\neg\neg p \to p$) may help in simplifying the structure of the frames, i.e., in restricting (under *particular forcings*) classes of frames into narrower classes (of "canonical" frames). In this sense, the above proof that $E(\text{CEST}) = \mathscr{L}_{\text{reg}}(\mathscr{F}^{\text{c}}_{\text{CEST}})$, where $\mathscr{F}^{\text{c}}_{\text{CEST}}$ is not the greatest class of frames for $E(\text{CEST})$ (according to the conventions made for the nonstandard logics), has not been carried out using only filtration techniques (an essential role has been played by the properties of the regular Kripke models, which cannot be used in completeness proofs of standard logics).

We end this illustration of the maximal constructive logic $S(E(\text{CEST}))$ by putting into evidence two important properties of the related maximal nonstandard constructive logic $E(\text{CEST})$.

First of all, for every finite nonempty set of variables $v = \{p_1, \ldots, p_n\}$, let $\boldsymbol{K}_v = \langle P, \leqslant, 0, \Vdash \rangle$ be a regular Kripke model so defined:

- 0 is a 2^n-CEST-canonical state of $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle$;
- for every classical interpretation I of the variables $p_1, \ldots, p_n$, there is a final state ϕ of $\boldsymbol{P}$ such that the forcing $\Vdash$ on ϕ for the variables $p_1, \ldots, p_n$ coincides with I.

For the Kripke model $\boldsymbol{K}_v$ one can easily prove: *if A is any formula containing only variables of v, then $0 \Vdash A$ in $\boldsymbol{K}_v$ iff $A \in E(\text{CEST})$*. Thus, one has:

Theorem 16. *For every finite nonempty set v of propositional variables, $E(\text{CEST})$-biimplication divides the set of formulas containing only variables of v into a finite set of equivalence classes.*

For the second property of $E(\text{CEST})$, we recall that, to prove that any constructive nonstandard logic $L \supseteq E(\text{CEST})$ coincides with $E(\text{CEST})$, *an essential use has been made of the fact that L is closed under restricted substitutions*, as we have remarked after the proof of Theorem 14. It may be interesting to ask what happens when a $L \subseteq \text{CL}$ and closed under modus ponens looses the closure under restricted substitutions, still preserving constructiveness. In this line, we introduce the notions of pseudologic, constructive pseudologic and maximal constructive pseudologic.

- We say that a set L of wff's is a *pseudologic* iff L satisfies the following properties:
  (i) $\mathrm{INT} \subseteq L$ and $L \subseteq \mathrm{CL}$;
  (ii) L is closed under detachment.
- We say that L is a *constructive pseudologic* iff L is a pseudologic and satisfies the disjunction property.
- We say that L is a *maximal constructive pseudologic* iff L is a constructive pseudologic and, for every constructive pseudologic L', if $L \subseteq L'$ then $L = L'$.

Of course, any nonstandard logic is pseudologic, and any nonstandard constructive logic is a constructive pseudologic. However, the converses do not hold, hence a maximal nonstandard constructive logic is not necessarily a maximal constructive pseudologic. According to the treatment of [14], it turns out that the maximal nonstandard constructive logic $E(\mathrm{MV})$ is also a maximal constructive pseudologic. On the other hand, we are going to show that $E(\mathrm{CEST})$ is not a maximal constructive pseudologic. To do so, the following definitions are needed.

- We say that a wff A is *classically well founded* iff $A \in \mathrm{CL}$ and one of the following conditions holds:
  (1) $A = \neg B$;
  (2) $A = B \wedge C$, and A and B are classically well founded;
  (3) $A = B \vee C$, and A is classically well founded or B is classically well founded;
  (4) $A = B \to C$, and if A is classically well founded then B is classically well founded.
- WFCL will be the set of all the classically well founded formulas.

One has that WFCL is closed under substitutions σ of variables with variables or negated variables such that the following condition is satisfied: if $p \neq q$, then the variable occurring in $\sigma(p)$ is different from the variable occurring in $\sigma(q)$. However, WFCL is not closed under substitutions of variables with arbitrary negated formulas. Hence:

Proposition 35. WFCL *is not a nonstandard logic.*

On the other hand, one easily proves:

Proposition 36. WFCL *is a constructive pseudologic.*

Finally, we have:

Proposition 37. WFCL *is a maximal constructive pseudologic.*

Proof. One has to show that if $\mathrm{WFCL} \subseteq L$ and L is a constructive pseudologic, then, for every wff A, $A \in \mathrm{WFCL}$ iff $A \in L$. The proof of this fact is an easy induction on the complexity of A. □

For every finite nonempty set v of propositional variables, let $\boldsymbol{K}'_v = \langle P', \leqslant', 0', \Vdash' \rangle$ be a regular Kripke model so defined:
- the root $0'$ of $\boldsymbol{P}' = \langle P', \leqslant', 0' \rangle$ is a prefinal state of $\boldsymbol{P}'$;
- $|\mathrm{Fin}(0')| = 2^n$, where n is the cardinality of v;
- for every classical interpretation I of the variables of v, there is a final state ϕ of $\boldsymbol{P}'$ such that the forcing $\Vdash'$ of $\boldsymbol{K}'_v$ on ϕ coincides with I for the variables of v.

With an easy induction on the complexity of A one shows: *if A contains only variables of v, then $A \in$ WFCL iff $0' \Vdash' A$ in the (prefinal) Kripke model $\boldsymbol{K}'_v = \langle P', \leqslant', 0', \Vdash' \rangle$ defined above.* We recall, on the other hand, the definition of the (nonprefinal) model $\boldsymbol{K}_v = \langle P, \leqslant, 0, \Vdash \rangle$ immediately before Theorem 16. We have that the root of the poset $\boldsymbol{P}$ on which $\boldsymbol{K}_v$ is built is a 2^n-CEST-canonical state of $\boldsymbol{P}$ (n the cardinality of the nonempty set v); moreover, we can assume (without loss of generality) that $\boldsymbol{K}_v$ has the same final states as $\boldsymbol{K}'_v$, with the same forcing for the variables of v; finally, if A contains only variables of v, then $A \in E(\mathrm{CEST})$ iff $0 \Vdash A$ in $\boldsymbol{K}_v$.

We immediately obtain (up to isomorphisms) that the root $0'$ of $\boldsymbol{K}'_v$ is a prefinal state of $\boldsymbol{K}_v$ (recall the definition of 2^n-CEST-canonical state), hence that $\boldsymbol{K}'_v$ is a submodel of $\boldsymbol{K}_v$. We therefore have:

Theorem 17. $E(\mathrm{CEST}) \subseteq \mathrm{WFCL}$.

We can also show that any nonstandard constructive logic containing the nonstandard constructive logic $E(\mathrm{EST})$ (which is a proper sublogic of $E(\mathrm{CEST})$), is contained in WFCL. Moreover, we can find infinitely many constructive logics extending EST, which are pairwise constructively incompatible. To give an example, let, for $m \geqslant 2$, $L_m = \mathrm{EST} + \{(\mathrm{AS}^1_m), (\mathrm{AS}^2_{m+1})\}$, where (AS^1_m) and (AS^2_{m+1}) are the axiom-schemes so defined:

- (AS^1_m): $(\neg A \to \neg B_1 \vee \cdots \vee \neg B_m) \to (\mathrm{DE}_2)' \vee (\neg A \to \neg B_1) \vee \cdots \vee (\neg A \to \neg B_m)$, where $(\mathrm{DE}_2)'$ represents an arbitrary instance of the axiom schema (DE_2);
- (AS^2_{m+1}): $((\mathrm{DE}_2)' \to A_1 \vee \cdots \vee A_m \vee A_{m+1} \vee A_{m+2} \vee A_{m+3}) \wedge (\bigwedge_{i=1,\ldots,m+3}((A_i \to \bigvee_{i \neq j} A_j) \to \bigvee_{i \neq j} A_j)) \to A_1 \vee \cdots \vee A_m \vee A_{m+1} \vee A_{m+2} \vee A_{m+3}$, where $(\mathrm{DE}_2)'$ still represents any instance of the axiom schema (DE_2) (remark that (AS^2_{m+1}) is a weakening of Gabbay's and de Jongh's axiom-schema (D_{m+1}) [7, 6, 15]).

To characterize the logic L_m, let $\mathscr{F}_{L_m}$ be the class of frames so defined:
- $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}_{L_m}$ iff $\boldsymbol{P} \in \mathscr{F}_{\mathrm{EST}}$ and the two following additional conditions (the first related to the axiom-schema (AS^1_m) and the second related to the axiom-schema (AS^2_{m+1})) are satisfied:
  (i) for every $\alpha, \phi_1, \ldots, \phi_k \in P$ such that $2 \leqslant k \leqslant m$, α is not prefinal, $\phi_1, \ldots, \phi_k$ are distinct and $\{\phi_1, \ldots, \phi_k\} \subseteq \mathrm{Fin}(\alpha)$, there is $\beta \in P$ such that $\alpha \leqslant \beta$ and $\mathrm{Fin}(\beta) = \{\phi_1, \ldots, \phi_k\}$;
  (ii) for every $\alpha \in P$ such that α is not prefinal, α has at most $m + 2$ immediate successors.

To prove the completeness theorem of the logic L_m, the quotient models technique, even with extensively complete filtration formulas, does not work. However, one can successfully reach the goal using the selective models technique to be explained in the second part of the paper (even with arbitrary filtration formulas). From this we prove, on the one hand, that, for every $m \geqslant 2$, $L_m = \mathscr{L}(\mathscr{F}_{L_m})$ and L_m is constructive, since $\mathscr{F}_{L_m}$ satisfies (s.d.e.p.) (the proof of the latter fact is not trivial, but can be done as in similar cases illustrated in [15]). On the other hand, combining the properties of $\mathscr{F}_{L_m}$ and $\mathscr{F}_{L_n}$ (following the lines of similar cases illustrated in [15]), we obtain that $L_m + L_n = \mathscr{L}(\mathscr{F}_{L_m} \cap \mathscr{F}_{L_n})$ cannot be extended into a constructive logic for m and n such that $m \geqslant 2$, $n \geqslant 2$ and $m \neq n$.

Thus, from the previous discussion and Zorn's lemma, we get:

Theorem 18. *There are infinitely many maximal nonstandard constructive logics contained in* WFCL.

We now want to characterize, in terms of Kripke frames semantics, maximal constructive logics extending ST and contained in maximal nonstandard constructive logics which are, at the same time, maximal constructive pseudologics. To do so, we introduce, for every $n \geqslant 2$ and $m \geqslant 2$, the following axiom-schema:

$(T_{n,m})$: $(\neg A \to \neg B_1 \vee \cdots \vee \neg B_m) \to (\mathrm{FIN}_n)' \vee (\neg A \to \neg B_1) \vee \cdots \vee (\neg A \to \neg B_m)$,
where $(\mathrm{FIN}_n)'$ represents any instance of the axiom schema (FIN_n).

Using the axiom-schemes $(T_{n,m})$, the logics EST_n and L_{aux} previously defined, we can define the logic CEST_n (completed EST_n) as follows:

– $\mathrm{CEST}_n = \mathrm{EST}_n + L_{\mathrm{aux}} + \bigcup_{m \geqslant 2}\{(T_{n,m})\}$.

The class $\mathscr{F}_{\mathrm{CEST}_n}$ for the logic CEST_n is defined in the following way, using the classes $\mathscr{F}_{\mathrm{EST}_n}$ and $\mathscr{F}_{L_{\mathrm{aux}}}$:

– $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}_{\mathrm{CEST}_n}$ iff $\boldsymbol{P} \in \mathscr{F}_{\mathrm{EST}_n} \cap \mathscr{F}_{L_{\mathrm{aux}}}$, and the following property is satisfied:

(i) for every $\alpha \in P$ such that $|\mathrm{Fin}(\alpha)| > n$, every $m \geqslant 2$, every distinct $\phi_1, \ldots, \phi_m$ such that $\{\phi_1, \ldots, \phi_m\} \subseteq \mathrm{Fin}(\alpha)$, there is $\beta \in P$ such that $\alpha \leqslant \beta$ and $\mathrm{Fin}(\beta) = \{\phi_1, \ldots, \phi_m\}$.

Since every Kripke model built on a poset satisfying condition (i) of the above definition of $\mathscr{F}_{\mathrm{CEST}_n}$ is easily seen to satisfy every axiom schema $(T_{n,m})$ (where n is fixed, and $m \geqslant 2$ is any), combining Propositions 24 and 27 with this fact, we obtain:

Proposition 38. $\mathrm{CEST}_n \subseteq \mathscr{L}(\mathscr{F}_{\mathrm{CEST}_n})$.

Without any additional difficulty with respect to the previous examples of application of the quotient models technique, starting from Theorems 11 and 12 and using the axiom-schemes $(T_{n,m})$, one can prove the following theorem:

Theorem 19. *Let $n \geqslant 2$, let L be any nonstandard logic such that* $\mathrm{CEST}_n \subseteq L$, *let Γ be any L-saturated set and let W be any extensively complete formula. Then $\mathscr{C}_L(\Gamma)/\equiv_W$ is built on a poset of $\mathscr{F}_{\mathrm{CEST}_n}$.*

Remark. One obtains an equivalent characterization of CEST_2 by setting $\mathrm{CEST}_2 = \mathrm{EST}_2 + \bigcup_{m \geqslant 2}\{(T_{2,m})\}$, i.e., the axiom schemas of the logic L_{aux} are not necessary. Moreover, for every $n > 2$ one can set $\mathrm{CEST}_n = \mathrm{EST}_n + \bigcup_{3 \leqslant i \leqslant n+1}\{(S_i)\} + \bigcup_{m \geqslant 2}\{(T_{n,m})\}$, i.e., all the axiom schemas (S_j) of the logic L_{aux} such that $j > n + 1$ are not necessary.

From Propositions 19 and 38 and Theorems 8 and 19, we get:

Corollary 9. $E(\mathrm{CEST}_n) = \mathscr{L}_{\mathrm{reg}}(\mathscr{F}_{\mathrm{CEST}_n})$.

Combining Propositions 9 and 19 and Theorem 19, we get:

Corollary 10. *For every nonstandard logic $L \supseteq E(\mathrm{CEST}_n)$, every L-saturated set Γ and every extensively complete formula W, $\mathscr{C}_L(\Gamma)/\equiv_W$ is a regular Kripke model built on a poset of $\mathscr{F}_{\mathrm{CEST}_n}$.*

Now, recalling the notion of m-CEST-almost-canonical state introduced above, we inductively define the notion of m-CEST_n-*almost-canonical* state α of a poset $\boldsymbol{P} = \langle P, \leqslant \rangle$, for every $m \geqslant 1$ and every $n \geqslant 2$:

(1) If $|\mathrm{Fin}(\alpha)| = m$, and $m \leqslant n$, and α is a m-CEST-almost-canonical state of $\boldsymbol{P}$, then α is a m-CEST_n-almost-canonical state of $\boldsymbol{P}$.

(2) Let $\alpha \in P$ be such that $\mathrm{Fin}(\alpha) = \{\phi_1, \ldots, \phi_m\}$ and $m > n$. We say that α is a m-CEST_n-almost-canonical state of $\boldsymbol{P}$ iff the following conditions are satisfied:
- α has *exactly m* immediate successors $\beta_1, \ldots, \beta_m$ in $\boldsymbol{P}$;
- for every i, $1 \leqslant i \leqslant m$, $\mathrm{Fin}(\beta_i) = \{\phi_1, \ldots, \phi_m\} - \{\phi_i\}$ and β_i is a $(m-1)$-CEST_n-almost-canonical state of $\boldsymbol{P}$.

Having defined the m-CEST_n-almost-canonical states, given a poset $\boldsymbol{P} = \langle P, \leqslant \rangle$ and $\alpha \in P$, we say that α is a m-CEST_n-*canonical* state of $\boldsymbol{P}$ iff α is a m-CEST_n-almost-canonical state of $\boldsymbol{P}$ ($m \geqslant 1$ and $n \geqslant 2$) and the following conditions are satisfied:
- Let $\mathscr{I}$ be any nonempty subset of $\mathrm{Fin}(\alpha)$ and let $|\mathscr{I}| = h \leqslant m$. Then there is a *unique* $\beta \in P$ such that $\alpha \leqslant \beta$, $\mathscr{I} = \mathrm{Fin}(\beta)$ and β is a h-CEST_n-almost-canonical state of $\boldsymbol{P}$.
- Let $\mathscr{I}$ be any nonempty subset of $\mathrm{Fin}(\alpha)$ such that $2 \leqslant |\mathscr{I}| = h \leqslant n$. Then there is a *unique* $\gamma \in P$ such that $\alpha \leqslant \gamma$, $\mathscr{I} = \mathrm{Fin}(\gamma)$ and γ is prefinal in $\boldsymbol{P}$.

At this point, we introduce the class $\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}_n}$ of posets as follows, for every $n \geqslant 2$:
- $\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}_n} = \{\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \mid \boldsymbol{P}$ is finite, and 0 is prefinal in $\boldsymbol{P}$ with $|\mathrm{Fin}(0)| \leqslant n$ or 0 is a m-CEST_n-canonical state of $\boldsymbol{P}$ for some $m\}$.

To give examples of elements of $\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}_n}$ for some values of n, consider the poset $\boldsymbol{P}$ related to the previous Fig. 1 ($\boldsymbol{P}$ has twenty states and belongs to $\mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}}$); then, for every $n \geqslant 4$, $\boldsymbol{P} \in \mathscr{F}^{\mathrm{c}}_{\mathrm{CEST}_n}$. For a more interesting example, let $\boldsymbol{P}'$ be the poset with 19 states whose root is immediately followed by the four nonprefinal elements of Fig. 1

(i.e., $\boldsymbol{P}'$ differs from $\boldsymbol{P}$ for the lack of the prefinal state with the four final states 1, 2, 3 and 4): then $\boldsymbol{P}' \in \mathscr{F}^{c}_{\mathrm{CEST}_3}$.

Let $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle$ and $\boldsymbol{K}' = \langle P', \leqslant', 0', \Vdash' \rangle$ be two Kripke models such that:

- $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}_{\mathrm{CEST}_n}$ and $\boldsymbol{P}' = \langle P', \leqslant', 0' \rangle \in \mathscr{F}^{c}_{\mathrm{CEST}_n}$;
- $|\mathrm{Fin}(0)_{\boldsymbol{P}}| = |\mathrm{Fin}(0')_{\boldsymbol{P}'}|$;
- both 0 and 0′ are prefinal states (of $\boldsymbol{P}$ and $\boldsymbol{P}'$, respectively), or both 0 and 0′ are not prefinal states.

Let v be any nonempty set of propositional variables. Then we define the notion of *v-CEST$_n$-canonical application $\mathscr{E}$ of $\boldsymbol{K}$ in $\boldsymbol{K}'$* just as made in the previous definition of v-CEST-canonical application. In other words, a v-CEST$_n$-canonical application of $\boldsymbol{K}$ in $\boldsymbol{K}'$ will be a surjective application $\mathscr{E}: P \to P'$ which is injective on the set of final states of $\boldsymbol{P}$, sends any prefinal state α of $\boldsymbol{P}$ into the unique prefinal state α' of $\boldsymbol{P}'$ whose final states are the $\mathscr{E}$-images of the final states of α, any nonprefinal state β of $\boldsymbol{P}$ into the unique m-CEST$_n$-canonical state β' of $\boldsymbol{P}'$ whose final states are the $\mathscr{E}$-images of the final states of β, and preserves the forcing of the variables of v.

As stated for the v-CEST-canonical applications, one proves that, for every *regular* $\boldsymbol{K} = \langle P, \leqslant, 0, \Vdash \rangle$ with $\boldsymbol{P} = \langle P, \leqslant, 0 \rangle \in \mathscr{F}_{\mathrm{CEST}_n}$, there are $\boldsymbol{K}' = \langle P', \leqslant', 0', \Vdash' \rangle$ and $\mathscr{E}: P \to P'$ such that $\boldsymbol{P}' = \langle P', \leqslant', 0' \rangle \in \mathscr{F}^{c}_{\mathrm{CEST}_n}$, $\boldsymbol{P}'$ is uniquely determined up to isomorphisms and $\mathscr{E}$ is a v-CEST$_n$-canonical application of $\boldsymbol{K}$ in $\boldsymbol{K}'$. Also, one has that a v-CEST$_n$-canonical application of $\boldsymbol{K}$ in $\boldsymbol{K}'$ is an open epimorphism of $\boldsymbol{P}$ in $\boldsymbol{P}'$. Thus, since $\mathscr{F}^{c}_{\mathrm{CEST}_n} \subseteq \mathscr{F}_{\mathrm{CEST}_n}$ trivially holds, along the lines of the proof of Theorem 13 we can prove:

Theorem 20. *For every $n \geqslant 2$, $E(\mathrm{CEST}_n) = \mathscr{L}_{\mathrm{reg}}(\mathscr{F}^{c}_{\mathrm{CEST}_n})$.*

Remark. For a nonstandard logic $L \supseteq E(\mathrm{CEST}_n)$, a L-saturated set Γ and an extensively complete formula W, one can directly show that $\mathscr{C}_L(\Gamma)/\equiv_W$ is a regular Kripke model built on a poset of $\mathscr{F}^{c}_{\mathrm{CEST}_n}$.

Since $\mathscr{F}^{c}_{\mathrm{CEST}_n}$ (as well as $\mathscr{F}_{\mathrm{CEST}_n}$) is easily seen to satisfy (s.d.e.p.), we can state:

Proposition 39. *$E(\mathrm{CEST}_n)$ is a nonstandard constructive logic.*

Now, differently from the case of $E(\mathrm{CEST})$, which is a maximal nonstandard constructive logic but is not a maximal constructive pseudologic, we can prove:

Theorem 21. *For every $n \geqslant 2$, $E(\mathrm{CEST}_n)$ is simultaneously a maximal nonstandard constructive logic and a maximal constructive pseudologic.*

Proof. Let $n \geqslant 2$. It suffices to show that, for every constructive pseudologic $L \supseteq E(\mathrm{CEST}_n)$, $L = E(\mathrm{CEST}_n)$.

Assume the contrary. Then there is a wff A such that $A \in L$ but $A \notin E(\mathrm{CEST}_n)$. We can take a formula W such that:

- A is a subformula of W;
- W is extensively complete;
- the cardinality k of the set $\mathscr{V}_W$ of the propositional variables of W is such that $n < 2^k$.

Now, we remark that both $E(\mathrm{CEST}_n)$ and L, being constructive pseudologics, are $E(\mathrm{CEST}_n)$-saturated sets. Thus, we can consider the two canonical models $\mathscr{C}_{E(\mathrm{CEST}_n)}(E(\mathrm{CEST}_n))$ and $\mathscr{C}_{E(\mathrm{CEST}_n)}(L)$, the latter being a submodel of the former.

Let $L_1 = E(\mathrm{CEST}_n)$ and $L_2 = L$. Consider, for $1 \leqslant i \leqslant 2$, the model $\mathscr{C}_{E(\mathrm{CEST}_n)}(L_i)/\equiv_W$, which is regular by Corollary 10. Since the cardinality of $\mathscr{V}_W$ is k, arguing as in the proof of Theorem 14 we can show that the number of final states of $\mathscr{C}_{E(\mathrm{CEST}_n)}(L_i)/\equiv_W$ is 2^k. By Corollary 10, the poset $\boldsymbol{P}_i = \langle P_i, \leqslant_i, 0_i \rangle$ on which $\mathscr{C}_{E(\mathrm{CEST}_n)}(L_i)/\equiv_W$ is built belongs to $\mathscr{F}_{\mathrm{CEST}_n}$; hence, since $n < 2^k$, *the root* 0_i *of* $\boldsymbol{P}_i$ *is not a prefinal state of* $\boldsymbol{P}_i$.

Let $\boldsymbol{K}_1 = \langle P_1, \leqslant_1, 0_1, \Vdash_1 \rangle$ and $\boldsymbol{K}_2 = \langle P_2, \leqslant_2, 0_2, \Vdash_2 \rangle$ be, respectively, $\mathscr{C}_{E(\mathrm{CEST}_n)}(L_1)/\equiv_W$ and $\mathscr{C}_{E(\mathrm{CEST}_n)}(L_2)/\equiv_W$. Since $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ have the same 2^k final states with the same forcing, and since their roots are not prefinal states, arguing as in the proof of Theorem 14 we obtain that $0_1 \Vdash_1 A$ (in $\boldsymbol{K}_1$) iff $0_2 \Vdash_2 A$ (in $\boldsymbol{K}_2$). This implies that $A \in L_1$ iff $A \in L_2$. Hence, being $L_1 = E(\mathrm{CEST}_n)$, $L_2 = L$ and $A \in L$, we get $A \in E(\mathrm{CEST}_n)$, a contradiction. □

To complete the picture of the nonstandard logics $E(\mathrm{CEST}_n)$, with a proof quite similar to the one of Theorem 16, we get:

Theorem 22. *For every* $n \geqslant 2$ *and for every finite nonempty set* v *of propositional variables,* $E(\mathrm{CEST}_n)$*-biimplication divides the set of formulas containing only variables of* v *into a finite set of equivalence classes.*

Passing from the nonstandard to the standard logics, from Theorems 5 and 21 we immediately deduce:

Corollary 11. *For every* $n \geqslant 2$, $S(E(\mathrm{CEST}_n))$ *is a maximal constructive logic.*

To show that $S(E(\mathrm{CEST}_n))$ coincides with $\mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}_n})$, with a proof quite similar to the one given above for Theorem 15, we get:

Theorem 23. *For every* $n \geqslant 2$, $\mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}_n})$ *is neg.sat.-determined.*

Corollary 12. *For every* $n \geqslant 2$, $S(E(\mathrm{CEST}_n)) = \mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}_n})$.

Remarks. (a) The logic $S(E(\mathrm{CEST}_2))$ turns out to coincide with Medvedev's logic MV. The proof is almost immediate, looking at the class of posets $\mathscr{F}^c_{\mathrm{CEST}_2}$ and at the Kripke frames semantics of MV given, e.g., in [6, 12, 14, 15].

(b) The proof of the maximality of the logics $\mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}_n})$, for every $n \geqslant 2$, provides a *direct proof, without applications of Zorn's lemma, i.e., of the axiom of choice, of the existence of infinitely many maximal constructive logics.* Proofs such as the ones given in [2, 4], which state the existence of uncountably many maximal constructive logics, give a stronger result, but use Zorn's lemma.

(c) For every $n \geqslant 2$, the logic $\mathrm{CEST}_n = \mathscr{L}(\mathscr{F}_{\mathrm{CEST}_n})$ does not coincide with the logic $S(E(\mathrm{CEST}_n)) = \mathscr{L}(\mathscr{F}^c_{\mathrm{CEST}_n})$. As a matter of fact, the axiom-schema (SCHEMA) considered above (which belongs to $S(E(\mathrm{CEST}))$ but not to CEST) can be shown to belong also to $S(E(\mathrm{CEST}_n))$ for every $n \geqslant 2$, while the instance of its with $A = p$, $B = q$, $C = r$, $D = s$, $E = t$ and $F = u$ does not belong to CEST_n for every $n \geqslant 2$ (the proofs of these facts are quite similar to the ones given for $S(E(\mathrm{CEST}))$ and CEST)). Thus, for the problem of axiomatizing the logic $S(E(\mathrm{CEST}_n))$ for any $n \geqslant 2$, we can repeat what has been previously said for $S(E(\mathrm{CEST}))$.

References

[1] J.G. Anderson, Superconstructive propositional calculi with extra schemes containing one variable, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 18 (1972) 113–130.

[2] A.V. Chagrov, The cardinality of the set of maximal intermediate logics with the disjunction property is of continuum, Matematicheskie Zametki 51 (2) (1992) 117–123 (in Russian).

[3] A.V. Chagrov and M.V. Zacharyashchev, The disjunction property of intermediate propositional logics, Studia Logica 50(2) (1991) 189–216.

[4] M. Ferrari and P. Miglioli, Counting the maximal intermediate constructive logics, J. Symbolic Logic 58 (1993) 1365–1401.

[5] D.M. Gabbay, The decidability of the Kreisel–Putnam system, J. Symbolic Logic 35 (1970) 431–437.

[6] D.M. Gabbay, Semantical investigations in Heyting's intuitionistic logic (Reidel, Dordrecht, 1981).

[7] D.M. Gabbay and D.H.J. de Jongh, A sequence of decidable finitely axiomatizable intermediate logics with the disjunction property, J. Symbolic Logic 39 (1974) 67–78.

[8] G.I. Galanter, A continuum of intermediate logics which are maximal among the logics having the intuitionistic disjunctionless fragment, Proc. 10th USSR Conf. for Mathematical Logic (Alma Ata, 1990) 41 (in Russian).

[9] R.E. Kirk, A result on propositional logics having the disjunction property, Notre Dame J. Formal Logic 23(1) (1982) 71–74.

[10] G. Kreisel, H. Putnam, Eine Unableitbarkeitsbeweismethode für den intuitionistischen Aussagenkalkül, Archiv für Mathematische Logik und Grundlagenforschung 3 (1957) 74–78.

[11] J. Lukasiewicz, On the intuitionistic theory of deduction, Indagationes Mathematicae 14 (1952) 202–212.

[12] L.L. Maksimova, On maximal intermediate logics with the disjunction property, Studia Logica 45 (1986) 69–75.

[13] L.L. Maksimova, The number of maximal intermediate logics with the disjunction property, Proc. 7th All-Union Conf. for Mathematical Logic (Novosibirsk, 1984) (in Russian).

[14] P. Miglioli, U. Moscato, M. Ornaghi, S. Quazza and G. Usberti, Some results on intermediate constructive logics, Notre Dame J. Formal Logic 30 (1989) 543–562.

[15] P. Miglioli, An infinite class of maximal intermediate propositional logics with the disjunction property, Archive Math. Logic 31 (1992) 415–432.

[16] P. Minari, On the extension of intuitionistic propositional logic with Kreisel–Putnam's and Scott's schemas, Studia Logica 45 (1986) 455–468.

[17] V.B. Shehtman, On incomplete propositional logics, Doklady AN SSSR 235 (1977) 524–525.

[18] C.A. Smorinski, Applications of Kripke models, in: A.S. Troelstra, ed., Metamathematical Investigation of Intuitionistic Arithmetic and Analysis, Lecture Notes in Mathematics, Vol. 334 (Springer, Berlin, New York, 1973) 324–391.