# Minimum average congestion of enhanced and augmented hypercubes into complete binary trees 

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#### Abstract

We study the embedding problem of enhanced and augmented hypercubes into complete binary trees. Using tree traversal techniques, we compute the minimum average edge congestion of enhanced and augmented hypercubes into complete binary trees.


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## 1. Introduction and terminologies

Let $G$ and $H$ be finite graphs with $n$ vertices. $V(G)$ and $V(H)$ denote the vertex sets of $G$ and $H$, respectively. $E(G)$ and $E(H)$ denote the edge sets of $G$ and $H$, respectively. An embedding [2] $f$ of $G$ into $H$ is defined as follows:
(i) $f$ is a bijective map from $V(G) \rightarrow V(H)$
(ii) $f$ is a one-to-one map from $E(G)$ to $\left\{P_{f}(f(u), f(v)): P_{f}(f(u), f(v))\right.$ is a path in $H$ between $f(u)$ and $\left.f(v)\right\}$. Some authors use the name labeling instead of embedding. We may use both terminologies here.

The average congestion problem:
The congestion of an embedding $f$ of $G$ into $H$ is the maximum number of edges of $G$ that are embedded on any single edge of $H$. Normally an embedding $f$ of $G$ into $H$ defines a mapping of $V(G)$ into $V(H)$ and does not map the edges of $E(G)$ into $E(H)$. In the congestion problem, we need the information as to how the edges of $E(G)$ are embedded into $E(H)$. For every edge $(u, v)$ of $G$, there are several paths between its images $f(u)$ and $f(v)$ in $H$. Let us assume that the embedding $f$ of $G$ into $H$ defines a unique path between $f(u)$ and $f(v)$ in $H$ for every edge $(u, v)$ of $G$. Let $P_{f}(f(u), f(v))$ denote the unique path between $f(u)$ and $f(v)$ in $H$ for the edge $(u, v)$ of $G$.

The congestion on an edge $e$ of $H$, with respect to an embedding $f$ is defined by

$$
\begin{equation*}
C_{f}(G, H, e)=\left|\left\{(u, v) \in E(G): e \in P_{f}(f(u), f(v))\right\}\right| \tag{1}
\end{equation*}
$$

[^0]and the minimum congestion on $e$ is
\[

$$
\begin{equation*}
C_{\min }(G, H, e)=\min _{f} C_{f}(G, H, e) \tag{2}
\end{equation*}
$$

\]

where the minimum is taken over all embeddings $f$ of $G$ into $H$.
The average congestion of an embedding $f$ of $G$ into $H$ is denoted by

$$
\begin{equation*}
\widetilde{C}_{f}(G, H)=\frac{1}{|E(H)|} \sum_{e \in E(H)} C_{f}(G, H, e) \tag{3}
\end{equation*}
$$

and the minimum average congestion is denoted by

$$
\begin{equation*}
\widetilde{C}(G, H)=\min _{f} \widetilde{C}_{f}(G, H) \tag{4}
\end{equation*}
$$

where the minimum is taken over all embeddings $f$ of $G$ into $H$. The average congestion problem [3] of a graph $G$ into $H$ is to find $\widetilde{C}(G, H)$.

The following lemma provides a method to estimate the sum of congestions of an embedding. It will be used throughout this paper.

Lemma 1 ([23]). Let $f$ be an embedding of a graph $G$ into an arbitrary tree $T$. Let $e \in E(T)$ and $T_{1}$ be a component of $T-e$. Then the congestion $C_{f}(e)$ on $e$ is given by

$$
\begin{equation*}
C_{f}(e)=\sum_{v \in G_{1}} d_{G}(v)-2\left|E\left(G_{1}\right)\right| \tag{5}
\end{equation*}
$$

where $G_{1}$ is the subgraph of $G$ induced by the vertices $\left\{f^{-1}(u): u \in T_{1}\right\}$ and $d_{G}(v)$ denotes the degree of $v$ in $G$.

## 2. Overview of the paper

The concept of embedding is widely studied in the literature of fixed interconnection parallel architectures [27]. The average congestion problem of an arbitrary graph on a path is called the linear layout or the linear arrangement problem in the VLSI literature [18]. The average congestion problem is NP-complete for general graphs [14,18,25]. Even though there are numerous results and discussions on the congestion-sum problem, most of them deal with only approximate results [5, 12, 17].

The average congestion problem is studied for binary trees into paths [5,12], hypercubes into grids [3], complete graphs into hypercubes [17]. The bounded cost of dilation and congestion has been estimated for the embedding on binary trees [25]. Most of the works on the average congestion problem are on a particular case in which the host graph is a path, or a cycle [9,18]. There are also other general results on embeddings [1].

The concept of congestion is similar to cutwidth in graph theory [9,24]. There are several results on the congestion problem of various architectures such as complete binary trees into star and pancake graphs [7], trees into hypercubes [21], hypercubes into grids [3], complete binary trees into grids and extended grids [22], ladders and caterpillars into hypercubes [4,8] and generalized wheels into arbitrary trees [23].

The embeddings discussed in this paper produce exact results for the average congestion. We demonstrate that the average congestion problem of folded hypercubes, and enhanced hypercubes into complete binary trees can be constructed in linear time.

## 3. Embedding of enhanced hypercubes into complete binary trees

Hypercube network topology has become the most popular message-passing architecture, and several multicomputer configurations based on this topology have been designed and even marketed. There are different variations of hypercubes, for example, folded hypercubes [27], crossed hypercubes [27], Fibonacci cubes [15], enhanced cubes [26], and augmented cubes [10]. The problem of embedding complete binary trees into folded and enhanced hypercubes has been considered in [11]. In this paper we describe the embedding of enhanced and augmented hypercubes into complete binary trees optimizing the average congestion. We begin with the definition of a hypercube.

Definition 1. For $r \geq 1$, let $Q_{r}$ denote the graph of the $r$-dimensional hypercube. The vertex set $V\left(Q_{r}\right)=\left\{x_{0} x_{1} \ldots x_{r-1}\right.$ : $x_{i}=0$ or $\left.1,0 \leq i \leq r-1\right\}$. Two vertices $x_{0} x_{1} \ldots x_{r-1}$ and $y_{0} y_{1} \ldots y_{r-1}$ are adjacent if and only if they differ exactly in one position.

Definition 2. The enhanced hypercube $Q_{r, k}, 0 \leq k \leq r-1$, is a graph with vertex set $V\left(Q_{r, k}\right)=V\left(Q_{r}\right)$ and edge set $E\left(Q_{r, k}\right)=E\left(Q_{r}\right) \cup\left\{\left(x_{0} x_{1} \ldots x_{k-2} x_{k-1} x_{k} \ldots x_{r-1}, x_{0} x_{1} \ldots x_{k-2} \bar{x}_{k-1} \bar{x}_{k} \ldots \bar{x}_{r-1}\right), x_{i}=0\right.$ or $\left.1,0 \leq i \leq r-1\right\}$. The edges of $Q_{r}$ in $Q_{r, k}$ are hypercube edges and the remaining edges of $Q_{r, k}$ are called complementary edges.


Fig. 1. Folded hypercube $Q_{3,1}$.


Fig. 2. (a) Enhanced hypercube $Q_{3,2}$. (b) Enhanced hypercube $Q_{4,2}$. The dashed edges are complementary edges and the remaining edges are hypercube edges.

Remark 1. The set $\left\{\left(x_{0} x_{1} \ldots x_{k-2} x_{k-1} x_{k} \ldots x_{r-1}, x_{0} x_{1} \ldots x_{k-2} \bar{x}_{k-1} \bar{x}_{k} \ldots \bar{x}_{r-1}\right)\right\}$ is empty when $k=0$. Hence $Q_{r, 0}$ reduces to the $r$-dimensional hypercube. In what follows $Q_{r}$ is denoted by $Q_{r, 0}$.

Notice that the enhanced hypercubes $Q_{r, 1}$ are folded hypercubes. The folded hypercube $Q_{r, 1}$ proposed by El-Amawy and Latifi [13] is $(r+1)$-regular, has $2^{r}$ vertices and $(r+1) 2^{r-1}$ edges; $Q_{r, 1}$ has diameter $\lceil r / 2\rceil$ and connectivity $r+1$. The graph shown in Fig. 1 is a 3-dimensional folded hypercube $Q_{3,1}$, where the complementary edges are $(000,111),(001,110),(010,101)$, and $(011,100)$.

The enhanced hypercubes $Q_{r, k}, 1 \leq k \leq r-1$, proposed by Tzeng and Wei [26] are $(r+1)$-regular and have diameter $k+\lceil(r-k) / 2\rceil$. The graphs in Fig. 2(a) and (b) are enhanced hypercubes $Q_{3,2}$ and $Q_{4,2}$, respectively.

A subcube represents a hypercube of lower order which is an induced subgraph of a hypercube of higher order.

### 3.1. Properties of enhanced hypercubes and complete binary trees

A few properties of enhanced hypercubes and complete binary trees are listed below.
Proposition 1. $\left|E\left(Q_{r, k}\right)\right|=(r+1) 2^{r-1}, \quad k \geq 1$.
Proof. The proof follows from the fact that $Q_{r, k}$ is $(r+1)$-regular.
Notation 1. Let $S_{\alpha}$ denote a set of $\alpha$ vertices of a guest graph $G$ and $G\left[S_{\alpha}\right]$ denote the subgraph of $G$ induced by $S_{\alpha}$. Let $\bar{S}_{\alpha}$ represent some $S_{\alpha}$ for which the number of edges $\left|E\left(G\left[S_{\alpha}\right]\right)\right|$ is maximum. This $\bar{S}_{\alpha}$ is called a maximum subgraph of $\alpha$ vertices in $G$.

Lemma 2 ([20]). The cardinality of the edge set of a maximum subgraph of $Q_{r}$ induced by $\alpha$ vertices is given by

$$
\left|E\left(G\left[\bar{S}_{\alpha}\right]\right)\right|= \begin{cases}m 2^{m-1} & \text { if } \alpha=2^{m} \\ m\left(2^{m-1}-1\right) & \text { if } \alpha=2^{m}-1\end{cases}
$$

where $m \leq r$.
Lemma 3. The cardinality of the edge set of a maximum subgraph of $Q_{r, k}$ induced by $\alpha$ vertices is given by

$$
\left|E\left(G\left[\bar{S}_{\alpha}\right]\right)\right|= \begin{cases}(m+1) 2^{m-1} & \text { if } \alpha=2^{m} \\ (m+1)\left(2^{m-1}-1\right) & \text { if } \alpha=2^{m}-1\end{cases}
$$

where $r-k+1 \leq m \leq r$.


Fig. 3. $T^{4}$ with $\Psi\left(e_{1}\right)=8, \Psi\left(e_{2}\right)=1, \Psi\left(e_{3}\right)=3, \Psi\left(e_{4}\right)=1, \Psi\left(e_{7}\right)=1$, and $\Psi\left(e_{8}\right)=1$.
Proof. When $m<r-k+1, Q_{m, k-(r-m)}$ has no complementary edges of $Q_{r, k}$. Thus $Q_{m, k-(r-m)}$, when $m<r-k+1$, is nothing but $Q_{m, 0}$. In other words, $Q_{m, k-(r-m)}$ has complementary edges of $Q_{r, k}$ only when $m \geq r-k+1$.
$\left|E\left(G\left[\bar{S}_{\alpha}\right]\right)\right|=$ largest possible hypercube edges of $S+$ largest possible complementary edges of $S$ where $|S|=\alpha$.
By the definition of enhanced hypercubes, whenever $m \geq r-k+1$, largest possible complementary edges of $S=2^{m-1}$ if $|S|=2^{m}$ and largest possible complementary edges of $S=2^{m}-1$ if $|S|=2^{m}-1$. Thus by using Lemma 2, we write

$$
\left|E\left(G\left[\bar{S}_{\alpha}\right]\right)\right|= \begin{cases}m 2^{m-1}+2^{m-1} & \text { if } \alpha=2^{m} \\ m\left(2^{m-1}-1\right)+\left(2^{m-1}-1\right) & \text { if } \alpha=2^{m}-1\end{cases}
$$

when $m \geq r-k+1$. Thus

$$
\left|E\left(G\left[\bar{S}_{\alpha}\right]\right)\right|= \begin{cases}(m+1) 2^{m-1} & \text { if } \alpha=2^{m} \\ (m+1)\left(2^{m-1}-1\right) & \text { if } \alpha=2^{m}-1\end{cases}
$$

when $r-k+1 \leq m \leq r$.

Notation 2. Let $T^{r}$ denote a complete binary tree rooted at $w$, with $2^{r}$ nodes. An edge $e=u v$ of $T^{r}$ with $d(w, v)=$ $d(w, u)+1$, is called $e_{\alpha}$ if the subtree $T_{e_{\alpha}}$ of $T^{r}$ rooted at $v$ has $\alpha$ vertices. Let $\Psi\left(e_{\alpha}\right)$ denote the number of $e_{\alpha}$ 's in $T^{r}$. In other words, $\Psi\left(e_{\alpha}\right)$ denotes the number of subtrees of $T^{r}$ with $\alpha$ vertices. See Fig. 3.

Proposition 2 ([19]). The number of subtrees of $T^{r}$ with $\alpha$ vertices is given by

$$
\Psi\left(e_{\alpha}\right)=\left\{\begin{array}{lll}
1 & \text { if } \alpha=2^{m}, & m=1,2, \ldots, r-1 \\
2^{r-m}-1 & \text { if } \alpha=2^{m}-1, & m=2,3, \ldots, r-1 \\
2^{r-1} & \text { if } \alpha=1 . &
\end{array}\right.
$$

### 3.2. Inorder embedding of enhanced hypercube into complete binary tree

We apply the well-known inorder traversal to construct an optimal embedding of $Q_{r, k}$ into a complete binary tree $T^{r}$ with $2^{r}$ nodes. Inorder traversal on a tree is a widely known technique. This traversal is used to read the labels of the tree and output the inorder listing of the labels. Here we use this technique to assign labels $\left\{0,1, \ldots, 2^{r}-1\right\}$ to the nodes of the tree $T^{r}$.

### 3.2.1. The inorder embedding

The inorder embedding I of $Q_{r, k}$ into a complete binary tree $T^{r}$ assigns labels to the nodes of $T^{r}$ by inorder traversal as described in Fig. 4.

Theorem 1. The inorder embedding I of $Q_{r, k}$ into complete binary tree $T^{r}$ yields minimum average congestion $\widetilde{C}\left(Q_{r, k}, T^{r}\right)$.
Proof. It is enough to prove that the edge congestion on each edge of $T^{r}$ is minimum. The nodes of $T^{r}$ are labeled using inorder traversal. Let $v_{i}^{l}, i=0,1,2, \ldots, 2^{l}-1$ be the vertices of $T^{r}$ at level $l, 0 \leq l \leq r$. Let $T\left(v_{i}^{l}\right)$ denote the subtree of $T^{r}$ rooted at $v_{i}^{l}$.

We claim that the subcube of $Q_{r, k}$ induced by the labels of $T\left(v_{i}^{l}\right)$ is isomorphic to the subcube of $Q_{r, k}$ induced by the labels of $T\left(v_{0}^{l}\right) \backslash\{0\}$. A binary representation of the label of $v$ belonging to the subtree $T\left(v_{i}^{l}\right)$ is of the form $\alpha_{0} \alpha_{1} \ldots \alpha_{l-1} x_{l} x_{l+1} \ldots x_{r-1}$, where $\alpha_{0} \alpha_{1} \ldots \alpha_{l-1}$ is the binary representation of $i$. In other words, if $\alpha_{0} \alpha_{1} \ldots \alpha_{l-1}$ is the binary representation of $i$, then a


Fig. 4. Inorder embedding of $Q_{4,3}$ into $T^{4}$.


Fig. 5. The binary labels of $T^{6}$ are the vertices of $Q_{6, k}$ assigned by inorder embedding. $T\left(v_{0}^{2}\right) \backslash\{0\}$ and $T\left(v_{2}^{2}\right)$ are marked in circles. The labels of $T\left(v_{0}^{2}\right)$ are $\{0,1, \ldots, 15\}$. The labels of $T\left(v_{0}^{2}\right) \backslash\{0\}$ are $\{1,2, \ldots, 15\}$. The subcube of $Q_{6, k}$ induced by the labels of $T\left(v_{0}^{2}\right) \backslash\{0\}$ is isomorphic to the subcube of $Q_{6, k}$ induced by the labels of $T\left(v_{2}^{2}\right)$.
sequence of the form $\alpha_{0} \alpha_{1} \ldots \alpha_{l-1} x_{l} x_{l+1} \ldots x_{r-1}$ is a label of some vertex of $T\left(v_{i}^{l}\right)$. There are $2^{r-l}$ vertices of $T\left(v_{i}^{l}\right)$ and each is represented by $x_{l} x_{l+1} \ldots x_{r-1}$ preceded by $\alpha_{0} \alpha_{1} \ldots \alpha_{l-1}$. Let $a, b \in T\left(v_{i}^{l}\right)$ whose labels are

$$
\begin{aligned}
& a=\alpha_{0} \alpha_{1} \ldots \alpha_{l-1} x_{l} x_{l+1} \ldots x_{r-1} \\
& b=\alpha_{0} \alpha_{1} \ldots \alpha_{l-1} y_{l} y_{l+1} \ldots y_{r-1}
\end{aligned}
$$

where $\alpha_{0} \alpha_{1} \ldots \alpha_{l-1}$ is the binary representation of $i$. Then consider the corresponding vertices $u, v \in T\left(v_{0}^{l}\right) \backslash\{0\}$ where

$$
u=00 \ldots 0 x_{l} x_{l+1} \ldots x_{r-1}, \quad \text { and } \quad v=00 \ldots 0 y_{l} y_{l+1} \ldots y_{r-1}
$$

Thus, $(a, b)$ is an edge of the subcube of $Q_{r, k}$ induced by the labels of $T\left(v_{i}^{l}\right)$ if and only if $(u, v)$ is an edge of the subcube of $Q_{r, k}$ induced by the labels of $T\left(v_{0}^{l}\right) \backslash\{0\}$. See Fig. 5. Therefore, the subcube of $Q_{r, k}$ induced by the labels of $T\left(v_{i}^{l}\right)$ is isomorphic to the subcube of $Q_{r, k}$ induced by the labels of $T\left(v_{0}^{l}\right) \backslash\{0\}$. The subcube induced by the labels of $T\left(v_{0}^{l}\right)$ is $Q_{r-l, 0}$ which is a maximum subgraph on $2^{r-l}$ vertices of $Q_{r, k}$. In the same way the subcube induced by the labels of $T\left(v_{0}^{l}\right) \backslash\{0\}$ is a maximum subgraph on $2^{r-l}-1$ vertices of $Q_{r, k}[6,16]$. Hence the subcube induced by the labels of $T\left(v_{i}^{l}\right)$ is a maximum subgraph on $2^{r-l}-1$ vertices of $Q_{r, k}$.

When an edge $e$ is deleted from $T^{r}$, one of the two components is rooted at some $v_{j}^{l}$. See Fig. 6. In other words, one of the two components is $T\left(v_{j}^{l}\right)$. Since the subcube induced by the labels of $T\left(v_{j}^{l}\right)$ is maximum, the edge congestion on $e$ of $T^{r}$ is minimum by Lemma 1 . Since the edge congestion on each edge of $T^{r}$ is minimum, the average congestion is also minimum.
3.2.2. Estimation of $\widetilde{C}\left(Q_{r, k}, T^{r}\right)$

We now estimate the average congestion of an enhanced cube on a complete binary tree.


Fig. 6. The labels of the binary tree.
Theorem 2. The minimum average congestion of an enhanced hypercube $Q_{r, k}$ into a complete binary tree $T^{r}$ is

$$
\widetilde{C}\left(Q_{r, k}, T^{r}\right)=\frac{1}{2^{r}-1}\left\{\begin{array}{c}
(r+1) 2^{r-1}+2 r+\sum_{m=2}^{r-k}\left[(r-m+1) 2^{r}+\left(2^{r-m}-1\right)(2 m-r-1)\right] \\
+\sum_{m=r-k+1}^{r-1}\left[(r-m) 2^{r}+\left(2^{r-m}-1\right)(2 m-r+1)\right]
\end{array}\right\}
$$

Proof. By Theorem 1, $\widetilde{C}\left(Q_{r, k}, T^{r}\right)=C_{I}\left(Q_{r, k}, T^{r}\right)$, where $I$ is the inorder embedding of $Q_{r, k}$ into $T^{r}$. As described in Notation 2 , let $e_{\alpha}$ be an edge of $T^{r}$ such that the subtree of $T^{r}$ rooted at $e$ has $\alpha$ vertices. It is enough to prove that the edge congestion on $e_{\alpha}$ is

$$
C_{I}\left(e_{\alpha}\right)= \begin{cases}(r-m+1) 2^{m} & \text { if } m<r-k+1 \\ (r-m) 2^{m} & \text { if } r-k+1 \leq m \leq r-1\end{cases}
$$

when $\alpha=2^{m}$, and

$$
C_{I}\left(e_{\alpha}\right)= \begin{cases}(r-m+1) 2^{m}+(2 m-r-1) & \text { if } m<r-k+1 \\ (r-m)\left(2^{m}-1\right)+m+1 & \text { if } r-k+1 \leq m \leq r-1\end{cases}
$$

when $\alpha=2^{m}-1$.
As described in Notation 1 , let $\bar{S}_{\alpha}$ be a maximum subgraph of $Q_{r, k}$ induced by $\alpha$ vertices. We first consider the claim for $\alpha=2^{m}$.
Case $(m<r-k+1)$ : As is discussed in the proof of Lemma $3, \bar{S}_{\alpha}$ is isomorphic to $Q_{m, 0}$. Let $f$ be an arbitrary embedding of $Q_{r, k}$ into $T^{r}$. By Lemmas 1 and 2 , if $\alpha=2^{m}$, then

$$
\begin{aligned}
C_{I}\left(e_{\alpha}\right) & =(r+1) 2^{m}-2\left(m 2^{m-1}\right) \\
& =(r-m+1) 2^{m}
\end{aligned}
$$

$\operatorname{Case}(r-k+1 \leq m \leq r-1):$ By Lemmas 1 and 3,

$$
\begin{aligned}
C_{I}\left(e_{\alpha}\right) & =(r+1) 2^{m}-2(m+1) 2^{m-1} \\
& =(r-m) 2^{m}
\end{aligned}
$$

The case when $\alpha=2^{m}-1$ can be proved similarly. Now the theorem follows by Proposition 2 .
Theorem 3. The average congestion problem of enhanced hypercubes into complete binary trees is solvable in linear time.

## 4. Embedding of augmented hypercubes into complete binary trees

Following the recursive definition of the hypercube $Q_{r, 0}$, the augmented cubes $\mathrm{AQ}^{r}$ have been introduced in [10]. Augmented cubes are vertex symmetric, $(2 r-1)$-regular, and $(2 r-1)$-connected with diameter $\lceil r / 2\rceil$.

Definition 3. Let $r \geq 1$ be an integer. The augmented cube $\mathrm{AQ}^{r}$ of dimension $r$ has $2^{r}$ vertices, each labeled by an $r$-bit binary string $a_{0} a_{1} a_{2} \ldots a_{r-1}$. We define $\mathrm{AQ}^{1}=K_{2}$. For $r \geq 2, \mathrm{AQ}^{r}$ is obtained by taking two copies of the augmented cube $\mathrm{AQ}^{r-1}$, denoted by $\mathrm{AQ}_{0}^{r-1}$ and $\mathrm{AQ}_{1}^{r-1}$, and adding $2 \times 2^{r-1}$ edges between the two as follows: Let $V\left(\mathrm{AQ}_{0}^{r-1}\right)=\left\{0 a_{1} a_{2} \ldots a_{r-1}\right.$ : $a_{i}=0$ or 1$\}$ and $V\left(\mathrm{AQ}_{1}^{r-1}\right)=\left\{1 b_{1} b_{2} \ldots b_{r-1}: b_{i}=0\right.$ or 1$\}$. A vertex $u=0 a_{1} a_{2} \ldots a_{r-1}$ of $\mathrm{AQ}_{0}^{r-1}$ is joined to a vertex $v=1 b_{1} b_{2} \ldots b_{r-1}$ of $\mathrm{AQ}_{1}^{r-1}$ if and only if for every $i, 1 \leq i \leq r-1$, either $(1) a_{i}=b_{i}$; in this case, $(u, v)$ is called a hypercube edge, or (2) $a_{i}=\bar{b}_{i}$; in this case, $(u, v)$ is called a complement edge.

The inorder embedding $I$ of $\mathrm{AQ}^{r}$ into a complete binary tree $T^{r}$ assigns labels to the nodes of $T^{r}$ by inorder traversal. The proof of the following theorem is similar to that of Theorem 1.

Theorem 4. The inorder embedding I of $\mathrm{AQ}^{r}$ into complete binary tree $T^{r}$ yields minimum average congestion $\widetilde{C}\left(\mathrm{AQ}^{r}, T^{r}\right)$.
Theorem 5. The minimum average congestion of an augmented cube $\mathrm{AQ}^{r}$ into a complete binary tree $T^{r}$ is

$$
\widetilde{C}\left(\mathrm{AQ}^{r}, T^{r}\right)=\frac{1}{2^{r}-1}\left\{\sum_{m=1}^{r-1}(r-m) 2^{m+1}+\sum_{m=2}^{r-1}\left[(r-m)\left(2^{m+1}-2\right)+(2 m-1)\right]\left(2^{r-m}-1\right)+(2 r-1) 2^{r-1}\right\}
$$

Proof. By Theorem $4, \widetilde{C}\left(\mathrm{AQ}^{r}, T^{r}\right)=C_{I}\left(\mathrm{AQ}^{r}, T^{r}\right)$, where $I$ is the inorder embedding of $\mathrm{AQ}^{r}$ into $T^{r}$. The cardinality of the edge set of a maximum subgraph of $\mathrm{AQ}^{r}$ induced by $\alpha$ vertices is given by

$$
\left|E\left(G\left[\bar{S}_{\alpha}\right]\right)\right|= \begin{cases}(2 m-1) 2^{m-1} & \text { if } \alpha=2^{m} \\ (2 m-1)\left(2^{m-1}-1\right) & \text { if } \alpha=2^{m}-1\end{cases}
$$

where $m \leq r$. It is easy to verify that

$$
C_{I}\left(e_{\alpha}\right)= \begin{cases}(r-m) 2^{m+1} & \text { if } \alpha=2^{m} \\ (r-m)\left(2^{m+1}-2\right)+(2 m-1) & \text { if } \alpha=2^{m}-1\end{cases}
$$

for $1 \leq m \leq r-1, \alpha \leq 2^{r-1}$. The rest of the proof is similar to the previous section.
Theorem 6. The average congestion problem of augmented hypercubes into complete binary trees is solvable in linear time.

## 5. Conclusion

We obtain the minimum average congestion of embeddings of enhanced and augmented hypercubes into complete binary trees. It is also an interesting research topic to verify whether this technique can be employed to solve the average congestion problem for architectures such as butterfly, torus, star, and pancake. This paper discusses hypercubes, folded hypercubes, enhanced hypercubes and augmented hypercubes. However, there are two more families of hypercubes such as crossed cubes and Fibonacci cubes which are not discussed here. It is interesting to find a methodology to solve the average congestion problem for these two classes.

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