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Some bivariate gamma distributions

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Abstract

We introduce two new bivariate gamma distributions based on a characterizing property involving products of gamma and beta random variables. We derive various representations for their joint densities, product moments, conditional densities and conditional moments. Some of these representations involve special functions such as the complementary incomplete gamma and Whittaker functions. We also discuss ways to construct multivariate generalizations.

Keywords: Beta distribution; Bivariate gamma distribution; Complementary incomplete gamma function; Gamma distribution; Whittaker function

1. Introduction

There have only been a few bivariate gamma distributions proposed in the statistics literature; see Chapter 48 in [1] for a good review. These distributions have found useful applications in several areas; for example, in the modeling of rainfall at two nearby rain gauges [2], data obtained from rainmaking experiments [3,4], the dependence between annual streamflow and areal precipitation [5], wind gust data [6] and the dependence between rainfall and runoff [7]. They have also found applications in reliability theory, renewal processes and stochastic routing problems.

The aim of this work is to construct two new bivariate gamma distributions and to study their properties. The basis for their construction is the following characterization of gamma and beta distributions due to Yeo and Milne [8]. We say that a random variable X is beta distributed with shape parameters α and β if its probability density function (pdf) is

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$$

for $0 < x < 1$, $\alpha > 0$ and $\beta > 0$, where

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} \mathrm{d}t$$

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denotes the beta function. Likewise, a random variable X is gamma distributed with shape parameter α and scale parameter β if its pdf is

$$f(x) = \frac{\beta^\alpha x^{\alpha-1} \exp(-\beta x)}{\Gamma(\alpha)}$$

for $x > 0$, $\alpha > 0$ and $\beta > 0$.

Lemma 1 (*Yeo and Milne [8]*). *Suppose U and V are independent, absolutely continuous and non-negative random variables such that U has bounded support. Then for any $a > 0$ and $b > 0$, any two of the following three conditions imply the third:*

(i) *UV is gamma distributed with shape parameter a and scale parameter $1/\mu$, where $0 < \mu < \infty$;*
(ii) *U is beta distributed with shape parameters a and b;*
(iii) *V is gamma distributed with shape parameter $a + b$ and scale parameter $1/\mu$.*

An obvious way to generate a bivariate gamma from this lemma is to consider the joint distribution of $X = UV$ and V. The joint pdf of U and V is

$$f(u, v) = \frac{u^{a-1}(1-u)^{b-1}}{B(a, b)} \frac{v^{a+b-1} \exp(-v/\mu)}{\mu^{a+b}\Gamma(a+b)}$$

and thus the joint pdf of X and V becomes

$$f(x, v) = \frac{x^{a-1}(v-x)^{b-1} \exp(-v/\mu)}{\mu^{a+b}\Gamma(a)\Gamma(b)} \tag{1}$$

for $x \le v$ and $v > 0$. Unfortunately, the pdf (1) corresponds to a known bivariate gamma distribution—McKay's bivariate gamma distribution (see Section 48.2.1 of [1] for details).

Take U, V and W to be independent, absolutely continuous and non-negative random variables. Then two new bivariate gamma distributions can be constructed as follows:

1. Assume that W is beta distributed with shape parameters a and b. Assume further that U and V are gamma distributed with common shape parameter c and scale parameters $1/\mu_1$ and $1/\mu_2$, respectively, where $c = a + b$. Define

$$X = UW, \qquad Y = VW. \tag{2}$$

Then, by Lemma 1, X and Y will be gamma distributed with common shape parameter a and scale parameters $1/\mu_1$ and $1/\mu_2$, respectively. However, they will be correlated so that (X, Y) will have a bivariate gamma distribution over $(0, \infty) \times (0, \infty)$.

2. Assume that U and V are beta distributed with shape parameters (a_1, b_1) and (a_2, b_2), respectively, where $a_1 + b_1 = a_2 + b_2 = c$ (say). Assume further that W is gamma distributed with shape parameter c and scale parameter $1/\mu$. Define

$$X = UW, \qquad Y = VW. \tag{3}$$

Then, by Lemma 1, X and Y will be gamma distributed with common scale parameter $1/\mu$ and shape parameters a_1 and a_2, respectively. However, they will be correlated so that (X, Y) will again have a bivariate gamma distribution over $(0, \infty) \times (0, \infty)$.

In the rest of this work, we derive various representations for the joint densities, product moments, conditional densities and conditional moments associated with (2) and (3). We also discuss ways to construct multivariate generalizations.

The calculations of this work make use of the following special functions: the complementary incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t) dt;$$

and the Whittaker function defined by

$$W_{\lambda,\mu}(a) = \frac{a^{\mu+1/2}\exp(-a/2)}{\Gamma(\mu-\lambda+1/2)} \int_0^\infty t^{\mu-\lambda-1/2}(1+t)^{\mu+\lambda-1/2}\exp(-at)\mathrm{d}t.$$

The properties of these special functions can be found in [9,10].

2. Joint PDFs

Theorem 1 states that the joint pdf of (X, Y) for the first construct can be expressed in terms of the Whittaker function.

Theorem 1. *Under the assumptions of (2), the joint pdf of X and Y is given by*

$$f(x,y) = C\Gamma(b)(xy)^{c-1}\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right)^{\frac{a-1}{2}-c}\exp\left\{-\frac{1}{2}\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right)\right\} W_{c-b+\frac{1-a}{2},c-\frac{a}{2}}\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right) \quad (4)$$

for $x > 0$ and $y > 0$, where the constant C is given by

$$\frac{1}{C} = (\mu_1\mu_2)^c\Gamma(c)\Gamma(a)\Gamma(b).$$

Proof. The joint pdf of U, V and W is

$$f(u,v,w) = C(uv)^{c-1}w^{a-1}(1-w)^{b-1}\exp\left\{-\left(\frac{u}{\mu_1}+\frac{v}{\mu_2}\right)\right\}$$

from which the joint pdf of X, Y and W becomes

$$f(x,y,w) = C(xy)^{c-1}w^{a-2c-1}(1-w)^{b-1}\exp\left\{-\frac{1}{w}\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right)\right\}.$$

Integrating over $0 < w < 1$, one obtains

$$f(x,y) = C(xy)^{c-1}I(x,y), \quad (5)$$

where $I(x, y)$ denotes the integral

$$I(x,y) = \int_0^1 w^{a-2c-1}(1-w)^{b-1}\exp\left\{-\frac{1}{w}\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right)\right\}\mathrm{d}w.$$

Substituting $u = 1/w - 1$ and then using the definition of the Whittaker function, one can write

$$I(x,y) = \Gamma(b)\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right)^{\frac{a-1}{2}-c}\exp\left\{-\frac{1}{2}\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right)\right\} W_{c-b+\frac{1-a}{2},c-\frac{a}{2}}\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right). \quad (6)$$

The result in (4) follows by substituting (6) into (5). □

Corollary 1. *If $b = 1$ then the joint pdf (4) reduces to the simpler form*

$$f(x,y) = C(xy)^{c-1}\left(\frac{x}{\mu_1}+\frac{y}{\mu_2}\right)^{a-2c}\Gamma\left(2c-a, \frac{x}{\mu_1}+\frac{y}{\mu_2}\right).$$

Proof. Immediate from properties of the Whittaker function; see, for example, Sections 9.22 and 9.23 of [10]. □

Theorem 2 states that the joint pdf of (X, Y) for the second construct (3) can be expressed as an infinite sum of the Whittaker functions.

Theorem 2. *Under the assumptions of (3), the joint pdf of X and Y is given by*

$$f(x,y) = C\Gamma(b_1)\Gamma(b_2)\mu^{\frac{b_1+b_2-c+1}{2}} x^{\frac{a_1+b_2-3}{2}} y^{a_2-1} \exp\left(-\frac{x}{2\mu}\right) \times \sum_{j=0}^{\infty} \frac{(-1)^j (\mu x)^{-j/2} y^j}{j!\Gamma(b_2-j)} W_{\frac{b_2-b_1-c-j-1}{2},\frac{b_1+b_2-c-j}{2}}\left(\frac{x}{\mu}\right) \tag{7}$$

for $x \geq y > 0$, where the constant C is given by

$$\frac{1}{C} = \mu^c \Gamma(c) B(a_1,b_1) B(a_2,b_2).$$

The corresponding expression for $0 < x \leq y$ can be obtained from (7) by symmetry, i.e. interchange x with y, a_1 with a_2, and b_1 with b_2.

Proof. The joint pdf of U, V and W is

$$f(u,v,w) = Cu^{a_1-1} v^{a_2-1} (1-u)^{b_1-1} (1-v)^{b_2-1} w^{c-1} \exp\left(-\frac{w}{\mu}\right)$$

from which the joint pdf of X, Y and W becomes

$$f(x,y,w) = Cx^{a_1-1} y^{a_2-1} w^{1-c} (w-x)^{b_1-1} (w-y)^{b_2-1} \exp\left(-\frac{w}{\mu}\right) \tag{8}$$

for $w \geq \max(x,y)$. The integration of (8) over $\max(x,y) \leq w < \infty$ is not easy. However, using the series representation

$$(1+z)^{\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)} \frac{z^j}{j!},$$

one can write

$$f(x,y) = C\Gamma(b_2) x^{a_1-1} y^{a_2-1} \sum_{j=0}^{\infty} \frac{(-y)^j I_j(x)}{j!\Gamma(b_2-j)} \tag{9}$$

for $x \geq y > 0$, where $I_j(x)$ denotes the integral

$$I_j(x) = \int_x^{\infty} w^{b_2-c-j} (w-x)^{b_1-1} \exp\left(-\frac{w}{\mu}\right) \mathrm{d}w.$$

Substituting $u = w/x - 1$ and then using the definition of Whittaker function, one can obtain the expression

$$I_j(x) = \Gamma(b_1)\mu^{\frac{b_1+b_2-c-j+1}{2}} x^{\frac{b_1+b_2-c-j-1}{2}} \exp\left(-\frac{x}{2\mu}\right) W_{\frac{b_2-b_1-c-j-1}{2},\frac{b_1+b_2-c-j}{2}}\left(\frac{x}{\mu}\right). \tag{10}$$

The result in (7) follows by substituting (10) into (9). □

Corollary 2. *If $b_1 = 1$ then the joint pdf (7) reduces to the simpler form*

$$f(x,y) = C\Gamma(b_2)\mu^{1-a_2} x^{a_1-1} y^{a_2-1} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(b_2-j)} \left(\frac{y}{\mu}\right)^j \Gamma\left(1+b_2-c-j, \frac{x}{\mu}\right).$$

On the other hand, if $b_2 = 1$ then (7) reduces to

$$f(x,y) = C\Gamma(b_1)\mu^{\frac{b_1-c+2}{2}} W_{\frac{2-b_1-c}{2},\frac{1-a_1}{2}}\left(\frac{x}{\mu}\right) x^{\frac{a_1}{2}-1} y^{a_2-1} \exp\left(-\frac{x}{2\mu}\right).$$

If both $b_1 = 1$ and $b_2 = 1$ then (7) reduces to

$$f(x,y) = C\mu^{2-c} x^{a_1-1} y^{a_2-1} \Gamma\left(2-c, \frac{x}{\mu}\right).$$

Proof. As in Corollary 1. □

3. Product moments

For the two distributions introduced in the previous section, the product moments can expressed in terms of elementary functions, as shown by the following theorems.

Theorem 3. *The product moment of X and Y associated with (4) is given by*

$$E(X^m Y^n) = \frac{\mu_1^m \mu_2^n \Gamma(c+m)\Gamma(c+n)B(a+m+n,b)}{\Gamma(a)\Gamma(b)\Gamma(c)} \tag{11}$$

for $m \geq 1$ and $n \geq 1$. In particular,

$$\mathrm{Cov}(X,Y) = \frac{\mu_1\mu_2 ab}{a+b+1}$$

and

$$\mathrm{Corr}(X,Y) = \frac{\sqrt{ab}}{a+b+1}. \tag{12}$$

Proof. Note from (2) that $E(X^mY^n) = E((UW)^m(VW)^n) = E(U^m)E(V^n)E(W^{m+n})$. So, (11) follows since U and V are gamma random variables and W a beta random variable. □

Theorem 4. *The product moment of X and Y associated with (7) is given by*

$$E(X^m Y^n) = \frac{\mu^{m+n}\Gamma(m+n+c)B(m+a_1,b_1)B(n+a_2,b_2)}{\Gamma(c)B(a_1,b_1)B(a_2,b_2)} \tag{13}$$

for $m \geq 1$ and $n \geq 1$. In particular,

$$\mathrm{Cov}(X,Y) = \frac{\mu^2 a_1 a_2}{c}$$

and

$$\mathrm{Corr}(X,Y) = \frac{\sqrt{a_1a_2}}{c}. \tag{14}$$

Proof. Note from (3) that $E(X^mY^n) = E((UW)^m(VW)^n) = E(U^m)E(V^n)E(W^{m+n})$. So, (11) follows since U and V are beta random variables and W a gamma random variable. □

4. Conditional PDFs and moments

Theorems 5 and 6 derive the conditional distributions corresponding to (4) and (7), respectively.

Theorem 5. *For the pdf (4), the conditional pdf of Y given $X = x$ is given by*

$$f(y \mid x) = C\mu_1^a \Gamma(a)\Gamma(b) x^b y^{c-1} \left(\frac{x}{\mu_1} + \frac{y}{\mu_2}\right)^{\frac{a-1}{2}-c} \exp\left\{\frac{1}{2}\left(\frac{x}{\mu_1} - \frac{y}{\mu_2}\right)\right\} W_{c-b+\frac{1-a}{2}, c-\frac{a}{2}}\left(\frac{x}{\mu_1} + \frac{y}{\mu_2}\right).$$

The corresponding pdf of X given $Y = y$ is obtained by interchanging x with y and μ_1 with μ_2.

Proof. Immediate from (4) and the fact that X and Y are gamma distributed with the common shape parameter a and scale parameters $1/\mu_1$ and $1/\mu_2$, respectively. □

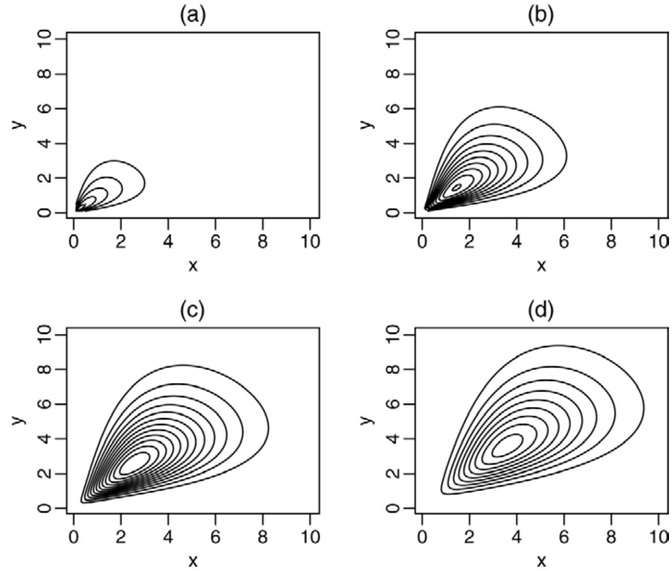

Fig. 1. Contours of the joint pdf (4) with $(a, b) = (2, 2)$ for (a), $(a, b) = (3, 3)$ for (b), $(a, b) = (4, 4)$ for (c) and $(a, b) = (5, 5)$ for (d).

Theorem 6. *For the pdf (7), the conditional pdf of Y given $X = x$ is given by*

$$f(y \mid x) = C\Gamma(a_1)\Gamma(b_1)\Gamma(b_2)\mu^{\frac{a_1+b_2+1}{2}} x^{\frac{b_2-1-a_1}{2}} y^{a_2-1} \exp\left(\frac{x}{2\mu}\right) \times \sum_{j=0}^{\infty} \frac{(-1)^j (\mu x)^{-j/2} y^j}{j!\Gamma(b_2-j)} W_{\frac{b_2-b_1-c-j-1}{2}, \frac{b_1+b_2-c-j}{2}}\left(\frac{x}{\mu}\right) \tag{15}$$

for $x \geq y > 0$, and by

$$f(y \mid x) = C\mu^{a_1}\Gamma(a_1)\Gamma(b_1)\Gamma(b_2)\mu^{\frac{b_1+b_2-c+1}{2}} y^{\frac{a_2+b_1-3}{2}} \exp\left(\frac{x}{\mu} - \frac{y}{2\mu}\right) \times \sum_{j=0}^{\infty} \frac{(-1)^j (\mu y)^{-j/2} x^j}{j!\Gamma(b_1-j)} W_{\frac{b_2-b_1-c-j-1}{2}, \frac{b_1+b_2-c-j}{2}}\left(\frac{y}{\mu}\right) \tag{16}$$

for $y \geq x > 0$. The corresponding pdf of X given $Y = y$ is obtained by interchanging x with y, a_1 with a_2 and b_1 with b_2.

Proof. Immediate from (7) and the fact that X and Y are gamma distributed with common scale parameter $1/\mu$ and shape parameters a_1 and a_2, respectively. □

The conditional moments of the form $E(X^m \mid Y = y)$ and $E(Y^n \mid X = x)$ for (4) and (7) appear mathematically intractable to obtain.

It is of interest to know how the McKay's bivariate gamma distribution given by (1) compares to the two new models given by (4) and (7). It is known that the correlation coefficient corresponding to (1) is $\sqrt{a/(a+b)}$. Comparing with the correlation coefficient in (12), we note that $\sqrt{a/(a+b)} \geq \sqrt{ab}/(a+b+1)$ for all $a > 0$ and $b > 0$. This implies that the correlation structure exhibited by (4) is always weaker than that of the McKay's bivariate gamma distribution. The correlation coefficient in (14) depends on the three parameters (a_1, a_2, c) and so the correlation structure of (7) is more flexible than that of McKay's bivariate gamma distribution (this means that the amount of correlation exhibited can be stronger, weaker or about the same).

Figs. 1 and 2 illustrate the correlation structures of the joint pdfs (4) and (7) for selected values of their parameters. If $a = b$ then the correlation coefficient $a/(2a+1)$ in (12) is an increasing function of a. This is supported by the joint contours in Fig. 1. If $a_1 > a_2$ (respectively, $a_1 < a_2$) then the correlation between X and Y is skewed to the left

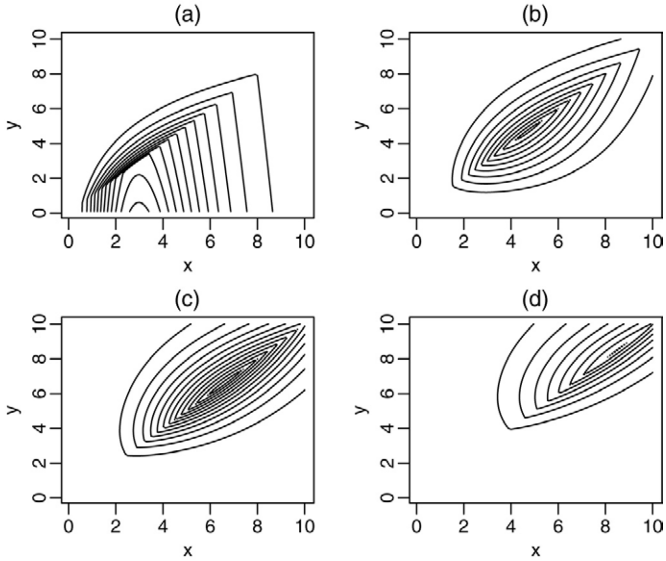

Fig. 2. Contours of the joint pdf (7) with $(a_1, a_2, c) = (4, 1, 8)$ for (a), $(a_1, a_2, c) = (4, 3, 8)$ for (b), $(a_1, a_2, c) = (4, 5, 8)$ for (c) and $(a_1, a_2, c) = (4, 7, 8)$ for (d).

(respectively, to the right) as shown in Fig. 2. It is also evident from Fig. 2 that larger values of a_1 or a_2 correspond to stronger correlation between X and Y.

5. Multivariate case

It is natural to ask how the pdfs (1), (4) and (7) can be generalized to the multivariate case. Lemma 1 can be applied in several ways to generate multivariate gamma distributions. Some of these are:

1. Assume that W is a beta distributed random variable with shape parameters a and b. Assume further that U_j, $j = 1, 2, \ldots, p$, are gamma distributed independent random variables (and independent of W) with common shape parameter c and scale parameters $1/\mu_j$, $j = 1, 2, \ldots, p$, where $c = a + b$.
2. Assume that U_j, $j = 1, 2, \ldots, p$, are beta distributed random variables with shape parameters (a_j, b_j), $j = 1, 2, \ldots, p$, where $a_j + b_j = c$ (say) for $j = 1, 2, \ldots, p$. Assume further that W is a gamma distributed random variable (independent of U_j, $j = 1, 2, \ldots, p$) with shape parameter c and scale parameter $1/\mu$.

In both these cases, by Lemma 1, $(U_1 W, U_2 W, \ldots, U_p W)$ will have a p-variate gamma distribution over $(0, \infty)^p$.

The following generalization of Lemma 1 provided by Yeo and Milne [8] provides other ways to generate multivariate gammas.

Lemma 2 (*Yeo and Milne [8]*). *Suppose for a fixed integer $p \geq 2$ that $X_1, X_2, \ldots, X_p$ are independent and identically distributed (iid) non-negative random variables which are independent of another non-negative random variable X with bounded support, and that*

$$Y = X(X_1 + X_2 + \cdots + X_p).$$

Then the two following conditions are equivalent.

(i) *Y has the same distribution as each of $X_1, X_2, \ldots, X_p$ and belongs to the class of distributions whose characteristic function is of the form*

$$\phi(t) = 1 - A|t|\{1 + o(t)\}$$

as $t \to 0$, where A is a real constant.

(ii) *X is beta distributed with shape parameters 1 and $p - 1$.*

One can generate several multivariate gammas by taking $X_1, X_2, \ldots, X_p$ to be iid gamma distributed. This investigation will be the subject of a future paper.

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