Asymptotically Linear Elliptic Equations with Resonance at Infinity

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Abstract—Combining the method developed in [1-3] with a modification of Galerkin approximations procedure, we obtain a new multiplicity result for asymptotically linear elliptic equations with resonance at infinity.

Keywords—Morse complex, Even functional, Asymptotically linear equations, Resonance at infinity.

1. INTRODUCTION

As is well known, many physical models lead to studying the following boundary value problem:

\[-\Delta u = p(x, u),\]
\[u \mid \partial \Omega \equiv 0,\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded open set with \(C^2\)-smooth boundary and \(p(x, t) \in C^0(\overline{\Omega} \times \mathbb{R}, \mathbb{R})\). In this paper, under the assumption that \(p\) is odd in \(u\), we study multiple solutions of (1.1) and topological (stability) properties of isolated solutions of (1.1).

Set \(P(x, t) = \int_0^t p(x, s) \, ds\). As is well known (see, for instance, [4]), if \(p\) is of subcritical growth in \(t\) uniformly in \(x\), then the functional

\[f(u) = \int_\Omega \left( \frac{1}{2} \right. \left| \nabla u(x) \right|^2 - P(x, u(x)) \left. \right) \, dx\]

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is well defined on the Sobolev space $W^{1,2}_0(\Omega) = W^{1,2}_0$ and its critical points are weak solutions of (1.1). If, in addition, $p$ is of \textit{superlinear} growth, then one can use the minimax theory (see [4,5]) or classical Morse theory (see [6,7]) to study the problem in question.

From now on, we assume that (1.1) is asymptotically linear, i.e., $p(x,t) = \hat{\lambda} t + \psi(x,t)$, where $\psi(x,t) = o(|t|)$ as $|t| \to \infty$ uniformly in $x \in \Omega$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be eigenvalues of the operator $-\Delta : W^{2,2}_0 \subset L^2(\Omega) \to L^2(\Omega)$. If $\hat{\lambda} \neq \lambda_k$, then the classical methods still can be applied to study (1.2). The situation changes dramatically when $\hat{\lambda} = \lambda_k$ for some $k$, in other words, when (1.1) has the resonance at infinity. As is well known (see, for instance, [7,8]), in this case, the application of classical methods to study functional (1.2) meets serious difficulties because, in general, functional (1.2) does not satisfy the PS-condition on $W^{1,2}_0$.

Observe that the usual way to avoid the “conflict” with the PS-condition in the resonant problem (1.1) is to put certain extra conditions with respect to $\psi$ (cf. [4,7–10], and references therein). The goal of this paper is to attack problem (1.1) in the presence of the $Z_2$-symmetry and resonance at infinity without the extra assumptions on $\psi$. In contrast to the traditional approach, we will investigate (1.2) in the space $L^2(\Omega) = L^2$, on which (1.2) is, evidently, ill-defined. Our strategy is based on an approximation of (1.2) by a nice sequence of finite-dimensional functionals for which the Morse complex arguments can be applied. In particular, we use a modification of classical Galerkin approximations. Observe that the main result of this paper (Theorem 3.1), as well as the Basic Lemma (see Section 4) cannot be obtained in the framework of classical Ljusternik-Schnirelman Theory because the associated functionals do not satisfy the PS-condition in any appropriate sense.

\section{2. PRELIMINARIES}

Let $U$ be an open domain in $\mathbb{R}^n$, $\Phi$ a $C^1$-field on $U$, $f : U \to \mathbb{R}$ a $C^2$-smooth functional, and $K$ a set of critical points of $f$.

**Definition 2.1.** (See [1–3].)

(1) The field $\Phi$ is said to be pseudo-gradient for $f$ if:

(a) $||\Phi(x)|| \leq 2||\nabla f(x)||$ for any $x \in U$, and

(b) $(\Phi(x), \nabla f(x)) > ||\nabla f(x)||^2$ for any noncritical point $x \in U$.

(2) Let $\Phi$ be a pseudo-gradient field for $f$. Then $(\Phi, f)$ is called a regular pair if there exist open bounded sets $Q_0, Q_1$ such that:

(a) $K \subset Q_0 \subset \text{cl}(Q_1) \subset U$, and

(b) if the endpoints of an integral curve $\mu$ of $\Phi$ belong to $Q_0$, then $\mu \subset Q_1$.

(3) A homotopy of pairs $\{\Phi_\lambda, f_\lambda\}, \lambda \in [0,1]$, defined on $U$ is said to be regular if every pair $(\Phi_\lambda, f_\lambda)$ is regular with $Q_0$ and $Q_1$ independent of $\lambda$.

Let $(\Phi, f)$ be a regular pair with $\Phi$ of the Morse-Smale type. In a standard way (see, for instance, [1–3,11]), one can associate with $(\Phi, f)$ the Morse complex $C_*(\Phi, f)$ (over $\mathbb{Z}_2$). Denote the corresponding homology groups by $H_*(\Phi)$.

**Theorem 2.2.** (See [1–3].)

(i) Let $\{\Phi_\lambda, f_\lambda\}, \lambda \in [0,1]$, be a regular homotopy of pairs with $\Phi_0$ and $\Phi_1$ of the Morse-Smale type. Then $H_*(\Phi_0) = H_*(\Phi_1)$.

(ii) Let $(\Phi, f)$ be a regular pair. There exists a regular homotopy $\{\Phi_\lambda, f_\lambda\}, \lambda \in [0,1]$, such that $(\Phi_0, f_0) = (\Phi, f)$ and $\Phi_1$ is of the Morse-Smale type.

In particular, by Theorem 2.2, the homology groups are well defined for an arbitrary regular pair $(\Phi, f)$ (in general, with $\Phi$ not of the Morse-Smale type).

**Definition 2.3.** Let $V$ be a vector field on $H$. Take a ball $B(x,r)$ of radius $r$ centered at $x$, and denote by $L(x,V,r)$ the minimal Lipschitz constant for $V$ on $B(x,r)$. Then the real number
lim_{r \to 0} L(x, V, r) is called the local Lipschitz constant at x and denoted by \( L(x) = L(x, V) \). Finally, if \( E \) is a linear subspace of \( H \), the number \( L_{E}(x, V) = L(x, V | \{x + E\}) \) is said to be a local Lipschitz constant of \( V \) at \( x \) along \( E \).

Observe that if \( V \) is differentiable then \( L(x, V) = \| V'(x) \| \).

3. STATEMENT OF THE MAIN RESULT

Consider (1.1) with \( p(x, t) = \lambda_k t + \psi(x, t) \), and assume that the nonlinearity \( \psi \) satisfies the following conditions.

- (ψ₁) The function \( \psi(x, t) \) satisfies the Lipschitz condition in both arguments with constant \( l \).
- (ψ₂) \( \psi \) has the zero derivative with respect to \( t \) at infinity, in the sense that the local Lipschitz constants in \( t \) go to zero as \( |t| \to \infty \), uniformly in \( x \).
- (ψ₃) \( \psi(x, t) = \nu t + \psi_0(x, t) \), where \( \nu > 0 \) and the local Lipschitz constants of \( \psi_0(x, t) \) in \( t \) go to zero as \( |t| \to 0 \), uniformly in \( x \).
- (ψ₄) \( \psi(x, -t) = -\psi(x, t) \).

Set \( H = L_2 \) and \( \varphi(u) = \int_{\Omega} \Psi(x, u(x)) \, dx \), where \( \Psi(x, t) = \int_{0}^{t} \psi(x, s) \, ds \). Denote by \( E_n \) the sum of \( -\Delta \)-eigenspaces corresponding to eigenvalues \( \leq \lambda_n \). Define a sequence of functionals \( \{f_n : E_n \to \mathbb{R}\} \) by setting \( f_n = f | E_n \), where \( f \) is defined by (1.2) and considered as a functional on \( H \) (in general, ill-defined).

Denote by \( \overline{N} \) and \( r \) the numbers of the eigenvalues of \( -\Delta \) in the intervals \( (-\infty, \lambda_k] \) and \( (\lambda_k, \lambda_k + \nu) \) (taking into account their multiplicities), respectively.

**Theorem 3.1.** Under the Assumptions (ψ₁)-(ψ₄), assume that \( q = r - 1 > 0 \). Then problem (1.1) has at least \( q \) distinct pairs of classical solutions of class \( C^{2,\alpha} \) with any \( \alpha < 1 \). Moreover, if the set of classical solutions of (1.1) is discrete (in the \( L_2 \)-metric), then there exist \( q \) pairs of classical solutions, say, \( u_1, \ldots, u_q \), of problem (1.1) and \( L_2 \)-neighborhoods \( W_i \) of \( u_i \) such that for \( n \) large enough, the homology group \( H_{N+1}(\nabla f_n | W_i \cap E_n) \) is well-defined and nontrivial (\( i = 1, \ldots, q \)).

**Remark 3.2.** Assume that \( \psi \in C^1(\Omega \times \mathbb{R}, \mathbb{R}) \). Then Condition (ψ₂) means that \( \psi'_i(x, t) \to 0 \) as \( t \to \infty \) uniformly in \( x \); Condition (ψ₃) means that \( \psi'_i(x, 0) = \nu \).

4. BASIC LEMMA

Let \( H \) be a separable Hilbert space and let \( F : \text{dom} F \subset H \to H \) be a linear, self-adjoint operator densely defined on \( H \) and having a discrete spectrum only. Denote by \( N(F) \), the dimension of the negative space in the spectral decomposition for \( F \), and set \( \overline{N}(F) = N(F) + \text{dim ker } F \).

In this section, we consider an asymptotically quadratic functional \( f(x) = (1/2)(Ax, x) + \varphi(x) \), in general, ill-defined on \( H \). Here \( A : \text{dom} A \subset H \to H \) is assumed to be a linear, closed, self-adjoint (in general, unbounded) operator densely defined on \( H \), \( \text{dim ker } A < \infty \); in addition, \( \varphi \in C^1(H, \mathbb{R}) \).

To study \( f \) in a neighborhood of zero, we will use another representation of \( f \) in the form \( f(x) = (1/2)(Bx, x) + \varphi_0(x) \), where \( B : \text{dom} B \subset H \to H \) is a linear self-adjoint operator densely defined on \( H \), \( \text{dim ker } B < \infty \), \( \text{dom } B = \text{dom } A \); in addition, \( \varphi_0 \in C^1(H, \mathbb{R}) \). In this representation, the first summand is assumed to be a principal part of \( f \) around zero.

Set \( \tilde{E}_n = \text{ker } B + \tilde{E}_n \), where \( \tilde{E}_n \) is the sum of \( A \)-eigenspaces corresponding to all the eigenvalues \( \lambda \) with \( |\lambda| \leq n \). Define a sequence of functionals \( \{f_n : E_n \to \mathbb{R}\} \) by setting \( f_n = f | \tilde{E}_n \).

We will assume the following conditions.

- (ψ₁) The operator \( A \) has a discrete spectrum only, and any eigenvalue is of finite multiplicity.
- (ψ₂) The functional \( \varphi \) is even; the operator \( \nabla \varphi \) is bounded on any ball, satisfies the Lipschitz condition with the Lipschitz constant \( L \) outside a ball and \( \nabla \varphi(x)/\|x\| \to 0 \) as
Moreover, for any finite-dimensional subspace $E \subset H$ and any sequence $\{x_n\}$ with $\|x_n\| \to \infty$ and $\rho(x_n, \ker A)/\|x_n\| \to 0$, one has $L_E(x_n, \nabla \varphi) \to 0$ (here $\rho$ stands for the metric induced by the norm in $H$).

(f3) The operator $\nabla \varphi_0$ satisfies the Lipschitz condition in a neighborhood of zero and $\nabla \varphi_0(x)/\|x\| \to 0$ as $\|x\| \to 0$. Moreover, for any finite-dimensional subspace $E \subset H$ and any sequence $\{x_n\}$ with $\|x_n\| \to 0$ and $\rho(x_n, \ker B)/\|x_n\| \to 0$, one has $L_E(x_n, \nabla \varphi_0) \to 0$.

(f4) $N(A) < \infty$. We are interested in solutions of the equation

$$Ax + \nabla \varphi(x) = 0. \quad (4.1)$$

Using the methods developed in [1–3], one can prove the following lemma.

Basic Lemma. Under the Assumptions (f1)–(f4), suppose that $q = N(B) - N(A) - 1 > 0$. Then equation (4.1) has at least $q$ distinct pairs of solutions. If, in addition, equation (4.1) has the discrete set of solutions, then there exist $q$ distinct pairs, say, $\pm x_1, \ldots, \pm x_q$ of solutions of (4.1) and neighborhoods $W_i$ of $x_i$, such that for $n$ large enough, the homology group $H_{N+i}(\nabla f_n | W_i \cap E_n)$ is well defined and nontrivial ($i = 1, \ldots, q; N = N(A)$).

5. PROOF OF THEOREM 3.1

We prove Theorem 3.2 using the Basic Lemma. To take advantage of the Basic Lemma, we must verify Conditions (f1)–(f4).

Set $A u = -\Delta u - \lambda_k u$ and $B u = -\Delta u - (\lambda_k + \nu) u$. By [12, Corollaries 8.2, 8.8, and 8.12], $W_{0,2}^2$ is closed with respect to the norm $\|u\|_A = \|u\|_{L_2} + \|Au\|_{L_2}$. This observation together with (f3) and the standard Laplacian arguments yield (f1) and (f4).

Below we shall verify Condition (f2). The verification of (f3) is similar and we omit it.

By Condition (f4), the functional $\varphi$ is even. By Condition (f1), the operator $\nabla \varphi$ is bounded on any bounded subset of $L_2$ and, moreover, satisfies the Lipschitz condition on the whole space $H$. Let us show that $\nabla \varphi(u)/\|u\| \to 0$ (in the $L_2$-metric) as $\|u\| \to \infty$. Indeed, by Conditions (f1) and (f2), one has $|\varphi(x,t)| = o(|t|)$ as $|t| \to \infty$ uniformly in $x$. Therefore, for any $\varepsilon > 0$, we have $|\varphi(x,t)| \leq \varepsilon |t| + C\varepsilon$, and hence,

$$\left\|\nabla \varphi(u)\right\|^2_{L_2} = \frac{1}{\Omega} \iint_{\Omega} \nabla^2(x,u) \, dx \leq 2\varepsilon^2 \iint_{\Omega} u^2 \, dx + 2C^2\varepsilon \cdot \text{mes}(\Omega) = 2\varepsilon^2 \left\|u\right\|^2_{L_2} + 2C^2\varepsilon \cdot \text{mes}(\Omega).$$

Since $\varepsilon$ is arbitrarily small, the result follows.

It remains to check the last (and the most important) part of Condition (f2), which says that for any finite-dimensional subspace $E \subset L_2$ and any sequence $\{u_n\} \subset L_2$ with $\|u_n\|_{L_2} \to \infty$ and $\rho(u_n, \ker A)/\|u_n\| \to 0$ in the $L_2$-metric), one has $L_E(u_n, \nabla \varphi) \to 0$. To this end, we need the following proposition.

Proposition 5.1. Let $\{u_n\} \subset L_2$ be a sequence such that for any $M > 0$, one has $\text{mes}\{x \in \Omega \mid |u_n(x)| < M\} \to 0$. Then $L_E(u_n, \nabla \varphi) \to 0$, for any finite-dimensional subspace $E \subset L_2$.

Proof of Proposition 5.1. For any $M > 0$, denote by $\delta_M$, the supremum of the Lipschitz constants of $\psi$ in $t$ taken over $t$ with $|t| > M$. By Condition (f2), one has $\delta_M \to 0$ as $M \to \infty$.

To continue the proof, we need certain auxiliary inequalities.

Fix a finite-dimensional subspace $E \subset H$. Further, take $u \in L_2$ and fix $M > 0$, $r \geq 0$. Set

$$X = X(u, M, r) = \{x \in \Omega \mid |u(x) + v(x)| \geq M \forall v \in E, \|v\| \leq \tau\},$$

$$Y = Y(u, M, r) = \overline{N} \setminus X.$$
Take some $v_1, v_2 \in E$ with $\|v_i\| \leq r$, $v_1 \neq v_2$, and set $v_0 = (v_1 - v_2)/\|v_1 - v_2\|$. From the definition of $\delta_M$, it follows:

$$\int_X |\psi(x, u(x) + v_1(x)) - \psi(x, u(x) + v_2(x))|^2 \, dx \leq \delta_M^2 \cdot \|v_1 - v_2\|_{L^2}^2. \quad (5.1)$$

By Condition $(\psi_1)$,

$$\int_Y |\psi(x, u(x) + v_1(x)) - \psi(x, u(x) + v_2(x))|^2 \, dx \leq l^2 \cdot \|v_0\| \cdot \|Y\|_{L^2}^2 \cdot \|v_1 - v_2\|_{L^2}^2, \quad (5.2)$$

where $l$ is the Lipschitz constant from Condition $(\psi_1)$. From (5.1),(5.2), one gets

$$\|\nabla \varphi(u + v_1) - \nabla \varphi(u + v_2)\|_{L^2} \leq (\delta_M + l \cdot \|v_0\| \cdot \|Y\|_{L^2}) \cdot \|v_1 - v_2\|_{L^2}. \quad (5.3)$$

Let us complete the proof of Proposition 5.1. The standard measure argument yields, for any $n \in \mathbb{N}$, there exists $r_n > 0$ such that

$$\text{mes} \{Y(u_n, M, r_n)\} \leq 2 \cdot \text{mes} \{Y(u_n, 2M, 0)\}. \quad (5.4)$$

Fix a sequence $\{r_n > 0\}$ satisfying (5.4) and set $Y_n = Y(u_n, M, r_n)$. By assumptions of Proposition 5.1, $\text{mes} \{Y(u_n, 2M, 0)\} \to 0$, hence (see (5.4)), $\text{mes} (Y_n) \to 0$. By construction, $v_0$ from (5.3) runs through the unit sphere $S(E)$ in the finite-dimensional space $E$. Since $S(E)$ is compact, $\|v_0\|_{L^2} \to 0$ uniformly in $v_0$. Finally, since $\delta_M \to 0$ as $M \to \infty$, the result follows from (5.3).

To take advantage of Proposition 5.1, we need to show that for any sequence $\{u_n\} \subset L_2$ with $\|u\|_{L^2} \to \infty$ and $\rho(u_n, \ker A)/\|u_n\|_{L^2} \to 0$ (in the $L_2$-metrics), the assumption of Proposition 5.1 holds. This easily follows from the two facts:

(a) any function $u$ satisfying the equation $-\Delta u = \lambda u$ is analytic (see [13]);
(b) for any nonzero analytic function, the pre-image of zero has the Lebesgue measure equals zero (see, for instance, [14, Theorem 2]).

Thus, by the Basic Lemma, problem (1.1) has at least $q$ distinct pairs of solutions with the corresponding homological information in the case of a discrete set of solutions. Now applying $(\psi_1)$ and the standard regularity technique (see, for instance, [15]), one can easily show that each obtained solution is of class $C^{2,\alpha}$ with any $\alpha < 1$.

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