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A solution to a colouring problem of P. Erdős

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1. Introduction

The purpose of this paper is to prove Theorem 1.1 which provides an affirmative solution to a colouring problem posed by P. Erdős at the Julius Petersen Graph Theory Conference held at Hindsgavl, July 1990.

Theorem 1.1. *Let n be a positive integer, and let G be a 4-regular graph on $3n$ vertices. Assume that G has a decomposition into a Hamiltonian circuit and n pairwise vertex disjoint triangles. Then $\chi(G) = 3$.*

A consequence of Theorem 1.1 is that G has independence number n, as conjectured by D.Z. Du and D.F. Hsu in 1986 at MIT. In April 1987 Erdős visited MIT and took an interest in the conjecture. He formulated the above colouring extension, which soon became known as the 'cycle plus triangles'-problem. It was mentioned in [3–5, 8].

Fellows [5] observed that Theorem 1.1 is in fact equivalent to an old conjecture of Schur, that for any partition of the integers $\mathbb{Z}$ into triples, there is another partition S_1, S_2, S_3 of $\mathbb{Z}$ such that each S_i contains a member of each triple, but no consecutive pair of integers.

The proof of Theorem 1.1 given in Section 2 uses a recent result of Alon and Tarsi [1], see Theorem 1.2 in Subsection 1.2. Alon and Tarsi's results provides a colouring criteria for a graph G based on orientations of G.

1.1. Notation

Most of the concepts used in this paper can be found in any book on graph theory, see e.g. [2] or [6].

All (undirected) graphs and digraphs considered are finite and loopless. Multiple edges (arcs) are permitted. A multi(di)graph is a (di)graph with multiple edges (arcs).

A digraph D is a pair (V, A) where $V = V(D)$ is a finite non-empty set of elements called vertices, and $A = A(D)$ is a finite set of ordered pairs of distinct elements of V called arcs.

If $a = (x, y)$ is an arc of D, then we say that a is incident from x and incident to y. For $x \in V(D)$, let $A^+(x:D)$ $(A^-(x:D))$ denote the set of all arcs of D incident from x (incident to x). Moreover, put $A(x:D) := A^+(x:D) \cup A^-(x:D)$. The out-degree (in-degree) of x with respect to D is

$$\mathrm{od}(x:D) := |A^+(x:D)| \qquad (\mathrm{id}(x:D) := |A^-(x:D)|).$$

If D is a digraph, the underlying graph of D is the graph or multigraph obtained from D by replacing each arc by an (undirected) edge joining the same pair of vertices. A digraph D is called an orientation of a graph G iff G is the underlying graph of D. We denote by $a_e \in A(D)$ the arc corresponding to the edge $e \in E(G)$.

For a given digraph D, if we replace every arc $(x, y) \in A(D)$ with (y, x), then we call the new digraph the reverse orientation D^{R} of D. In fact, in many instances we shall need to reverse only certain arcs or arc subsets of D. Thus, in generalizing the above notation we denote by a^{R} the reverse arc of a, i.e., if $a = (x, y)$, then $a^{\mathrm{R}} = (y, x)$, whereas if $A_0 \subseteq A(D)$, then $A_0^{\mathrm{R}} := \{a^R : a \in A_0\}$.

If $A_0 \subseteq A(D)$, then we write $D - A_0$ for the digraph obtained from D by removing the arcs of A_0, i.e. $D - A_0 = (V(D), A(D) - A_0)$. Moreover, we write $(D - A_0) \cup A_0^{\mathrm{R}}$ for the digraph obtained from D by reversing the orientation of the elements of A_0, i.e. $(D - A_0) \cup A_0^{\mathrm{R}} = (V(D), (A(D) - A_0) \cup A_0^{\mathrm{R}})$.

A system of non-empty subdigraphs, $S = \{D_1, \ldots, D_m\}$, $m \geq 1$, in a digraph D is called a decomposition of D iff $A(D) = \bigcup_{i=1}^{m} A(D_i)$ and $A(D_i) \cap A(D_j) = \emptyset$, $1 \leq i < j \leq m$. A decomposition of a graph of G is defined analogously.

For a given graph G, $\chi(G)$ is the minimum number of colours needed to colour the vertices of G such that adjacent vertices have distinct colours. Such a colouring is called proper. $\chi(G)$ is known as the chromatic number of G. $\chi_l(G)$ denotes the minimum number k such that if we give lists of k colours to each vertex of G there exists a proper vertex colouring of G where each vertex is coloured using a colour from its own list no matter what the lists are. This parameter is known as the upper chromatic number or list-chromatic number of G.

1.2. The results of Alon and Tarsi

First some definitions. We define a (not necessarily connected) graph to be Eulerian iff each vertex has even degree. Analogously, a digraph is said to be Eulerian iff each vertex has equal out-degree and in-degree.

Let D be a digraph. A subset $E \subseteq A(D)$ is called an Eulerian arc set in D iff E induces an Eulerian subdigraph of D, i.e. the digraph $(V(D), E)$ is Eulerian. We introduce the following sets.

$$\varepsilon(D) := \{E \subseteq A(D)\colon E \text{ is an Eulerian arc set in } D\},$$

$$\varepsilon_e(D) := \{E \in \varepsilon(D)\colon |E| \text{ is even}\}, \text{ and}$$

$$\varepsilon_0(D) := \{E \in \varepsilon(D)\colon |E| \text{ is odd}\}.$$

Furthermore, associate with these sets the following numbers.

$$e(D) := |\varepsilon(D)|, \qquad ee(D) := |\varepsilon_e(D)|, \quad \text{and} \quad eo(D) := |\varepsilon_0(D)|.$$

With this notation we can state some of Alon and Tarsi's results [1].

Theorem 1.2 (Alon and Tarsi). *Let D be a digraph. For each $x \in V(D)$, let $S(x)$ be a set of* $\mathrm{od}(x : D) + 1$ *distinct integers. Assume furthermore that $ee(D) \neq eo(D)$. Then there exists a proper vertex colouring $c : V(D) \to Z$ such that $c(x) \in S(x)$ for all $x \in V(D)$.*

Theorem 1.3 (Alon and Tarsi). *If G is a graph which has an orientation D satisfying $ee(D) \neq eo(D)$ in which the maximum out-degree is d, then $\chi(G) \leqslant \chi_l(G) \leqslant d + 1$.*

If an orientation D of a graph G is an Eulerian digraph, then we say that D is an Eulerian orientation of G. It follows from well-known results, see e.g. [2] or [6], that a graph G has an Eulerian orientation iff G itself is Eulerian.

Let G be a $2k$-regular graph, and let D be an orientation of G. If the maximum out-degree of D is at most k, then it is easy to see that D is an Eulerian orientation of G. Conversely, if D is an Eulerian orientation of G, then the maximum out-degree of D is k. Therefore, Theorem 1.3 implies the following.

Corollary 1.4. *If G is a $2k$-regular graph which has an Eulerian orientation D satisfying $ee(D) \neq eo(D)$, then $\chi(G) \leqslant \chi_l(G) \leqslant k + 1$.*

Obviously, for the proof of Theorem 1.1 it suffices to show that every graph G satisfying the assumption of Theorem 1.1 has an Eulerian orientation D such that $eo(D) \neq ee(D)$. Before we arrive at the proof of this statement, we investigate some simple properties of Eulerian digraphs and their Eulerian subdigraphs.

1.3. Eulerian subdigraphs of Eulerian digraphs

Let D be a digraph, let $a_1, \ldots, a_p, b_1, \ldots, b_q \in A(D)$, and let $x_1, \ldots, x_r, y_1, \ldots, y_s \in V(D)$. Then $\varepsilon = \varepsilon(D, a_1, \ldots, a_p, \bar{b}_1, \ldots, \bar{b}_q, x_1, \ldots, x_r, \bar{y}_1, \ldots, \bar{y}_s)$ denotes the set of all Eulerian arc sets $E \in \varepsilon(D)$ satisfying: $a_i \in E$ $(i = 1, \ldots, p)$, $b_i \notin E$ $(i = 1, \ldots, q)$, $A(x_i : D) \subseteq E$ $(i = 1, \ldots, r)$, and $A(y_i : D) \cap E = \emptyset$ $(i = 1, \ldots, s)$. Furthermore, put

$$e(D, a_1, \ldots, a_p, \bar{b}_1, \ldots, \bar{b}_q, x_1, \ldots, x_r, \bar{y}_1, \ldots, \bar{y}_s) := |\varepsilon|.$$

The proof of the next lemma, as an easy exercise, is left to the reader.

Lemma 1.5. *Let D be an Eulerian digraph with $m \geq 1$ arcs. Furthermore, define a mapping φ by setting $\varphi(E) := A(D) - E$ for every $E \in \varepsilon(D)$. Then the following statements hold:*

(1.5.1) *The mapping φ is a bijection from $\varepsilon(D)$ onto itself, and $e(D) \equiv 0 \bmod 2$.*

(1.5.2) *If $a_1, \ldots, a_p, b_1, \ldots, b_q \in A(D)$ and $x_1, \ldots, x_r, y_1, \ldots, y_s \in V(D)$, then φ induces a bijection from $\varepsilon(D, a_1, \ldots, a_p, \bar{b}_1, \ldots, \bar{b}_q, x_1, \ldots, x_r, \bar{y}_1, \ldots, \bar{y}_s)$ onto*

$$\varepsilon(D, \bar{a}_1, \ldots, \bar{a}_p, b_1, \ldots, b_q, \bar{x}_1, \ldots, \bar{x}_r, y_1, \ldots, y_s),$$

and

$$e(D, a_1, \ldots, a_p, \bar{b}_1, \ldots, \bar{b}_q, x_1, \ldots, x_r, \bar{y}_1, \ldots, \bar{y}_s) = e(D, \bar{a}_1, \ldots, \bar{a}_p, b_1, \ldots, b_q, \bar{x}_1, \ldots, \bar{x}_r, y_1, \ldots, y_s).$$

(1.5.3) *If m is odd, then φ induces a bijection from $\varepsilon_0(D)$ onto $\varepsilon_e(D)$, and $eo(D) = ee(D)$.*

(1.5.4) *If m is even, then φ induces a bijection from $\varepsilon_0(D)$ onto itself as well as a bijection from $\varepsilon_e(D)$ onto itself, and $eo(D) \equiv ee(D) \equiv 0 \bmod 2$.*

(1.5.5) *If m is even and $e(D) \equiv 2 \bmod 4$, then $eo(D) \neq ee(D)$.*

Statement (1.5.5) and Corollary 1.4 immediately imply the following.

Corollary 1.6. *If G is a $2k$-regular graph $(k \geq 1)$ on p vertices which has an Eulerian orientation D satisfying $e(D) \equiv 2 \bmod 4$, and if pk is even, then $\chi(G) \leq \chi_l(G) \leq k + 1$.*

Lemma 1.7. *Let D be an Eulerian digraph, and let C be a (directed) circuit in D with m arcs. Put $D_1 := (D - A(C)) \cup A(C)^{\mathrm{R}}$. Furthermore, define a mapping φ by setting $\varphi(E) := (E - A(C)) \cup (A(C) - E)^{\mathrm{R}}$ for every $E \in \varepsilon(D)$. Then the following statements hold:*

(1.7.1) *D_1 is an Eulerian digraph.*

(1.7.2) *The mapping φ is a bijection from $\varepsilon(D)$ onto $\varepsilon(D_1)$, and $e(D) = e(D_1)$.*

(1.7.3) *If m is odd, then φ induces a bijection from $\varepsilon_0(D)$ onto $\varepsilon_e(D_1)$ as well as a bijection from $\varepsilon_e(D)$ onto $\varepsilon_0(D_1)$, and $eo(D) - ee(D) = ee(D_1) - eo(D_1)$.*

(1.7.4) *If M is even, then φ induces a bijection from $\varepsilon_0(D)$ onto $\varepsilon_0(D_1)$ as well as a bijection from $\varepsilon_e(D)$ onto $\varepsilon_e(D_1)$, and $eo(D) - ee(D) = eo(D_1) - ee(D_1)$.*

Proof. The only statement which is not entirely trivial is (1.7.2) (note that (1.7.3) and (1.7.4) follow from (1.7.2)). To prove this statement, let $E \in \varepsilon(D)$. Put $D' := (V(D), E)$ and $D_1' := (V(D), \varphi(E))$. By definition, D' is an Eulerian digraph, and we have to show that D_1' is an Eulerian digraph, too. Let $x \in V(D)$. If $x \notin V(C)$, then we conclude from the definition of $\varphi(E)$ that $\mathrm{od}(x:D_1') = \mathrm{od}(x:D') = \mathrm{id}(x:D') = \mathrm{id}(x:D_1')$. Now assume that $x \in V(C)$. Let $a_1 \in A^+(x:C)$ and $a_2 \in A^-(x:C)$. Clearly, $a_i \in E$ iff $a_i^{\mathrm{R}} \notin \varphi(E)$ $(i = 1, 2)$. This implies immediately that $\mathrm{od}(x:D_1') = \mathrm{id}(x:D_1')$. Therefore, D_1' is an Eulerian digraph. This means that $\varphi(E) \in \varepsilon(D_1)$. Now it is easy to see that φ is a bijection from $\varepsilon(D)$ onto $\varepsilon(D_1)$. This proves (1.7.2.) □

Lemma 1.8. *Let D_1 and D_2 be Eulerian orientations of the Eulerian graph G. Then $D_0 := D_1 - A(D_2) = (D_2 - A(D_1))^{\mathrm{R}}$, and D_0 is an Eulerian digraph. Therefore, also $D_3 := D_1 \cap D_2$ is an Eulerian digraph.*

Proof. By definition, $a_e \in A(D_0)$, $e \in E(G)$, implies $a_e \neq A(D_2)$. Thus $a_e^{\mathrm{R}} \in A(D_2)$ follows of necessity. That is, $D_1 - A(D_2) = (D_2 - A(D_1))^R$. We have for every $x \in V(D)$

$$\mathrm{id}(x:D_1) = \mathrm{id}(x:D_3) + \mathrm{id}(x:D_0) = \mathrm{od}(x:D_1) = \mathrm{od}(x:D_3) + \mathrm{od}(x:D_0),$$

$$\mathrm{id}(x:D_2) = \mathrm{id}(x:D_3) + \mathrm{od}(x:D_0) = \mathrm{od}(x:D_2) = \mathrm{od}(x:D_3) + \mathrm{id}(x:D_0)$$

(note that $\mathrm{id}(x:D_0) = \mathrm{od}(x:D_0^{\mathrm{R}})$). Now, $\mathrm{id}(x:D_1) = \mathrm{id}(x:D_2) = \frac{1}{2}d(x:G)$ implies $\mathrm{id}(x:D_0) = \mathrm{od}(x:D_0)$; therefore, $\mathrm{id}(x:D_3) = \mathrm{od}(x:D_3)$. Whence both D_0 and D_3 are Eulerian digraphs. The lemma follows. □

Since every Eulerian digraph has a decomposition into circuits, Lemma 1.7 and Lemma 1.8 imply the following.

Lemma 1.9. *Let D_1 and D_2 be Eulerian orientations of the Eulerian graph G. Then $e(D_1) = e(D_2)$.*

2. Proof of Theorem 1.1.

Let G be a 4-regular graph satisfying the hypothesis of Theorem 1.1. Since G has a triangle, $\chi(G) \geq 3$ and all we have to show is that $\chi(G) \leq 3$. Because of Corollary 1.6, it suffices to prove that G has an Eulerian orientation D satisfying $e(D) \equiv 2 \bmod 4$. Therefore, Theorem 1.1 is an immediate consequence of

Theorem 2.1. *Let D be an Eulerian digraph. Assume that D has a decomposition into a (directed) Hamiltonian circuit and $n \geq 0$ pairwise vertex disjoint (directed) triangles. Then $e(D) \equiv 2 \bmod 4$.*

Proof (by induction on n). Clearly, if $n=0$, then $e(D)=2$ since $E_1=\emptyset$ and $E_2=A(D)$ are the only Eulerian arc sets in D. Note that D is a circuit in this case.

Let $n \geqslant 1$. Suppose Theorem 2.1 is true for every Eulerian digraph which has a decomposition into a Hamiltonian circuit and $k<n$ pairwise vertex disjoint triangles. Let D be an Eulerian digraph which has a decomposition into a Hamiltonian circuit, say C, and n pairwise vertex disjoint triangles. Let T be one of the triangles of this decomposition. Denote by x_1, x_2 and x_3 the three vertices of T, where $a_1:=(x_3, x_2)$, $a_2:=(x_1, x_3)$, and $a_3:=(x_2, x_1)$ are the arcs of T.

Put

$$e^*(D):=e(D, \bar{a}_1, a_2, a_3)+e(D, a_1, \bar{a}_2, \bar{a}_3)+e(D, a_1, \bar{a}_2, a_3)$$
$$+e(D, \bar{a}_1, a_2, \bar{a}_3)+e(D, a_1, a_2, \bar{a}_3)+e(D, \bar{a}_1, \bar{a}_2, a_3).$$

Then, clearly,

$$e(D)=e(D, a_1, a_2, a_3)+e(D, \bar{a}_1, \bar{a}_2, \bar{a}_3)+e^*(D). \tag{P1}$$

Let $D':=D-A(T)$. Since $A(T)$ is an Eulerian arc set in D, it is easy to see that $e(D')=e(D, \bar{a}_1, \bar{a}_2, \bar{a}_3)=e(D, a_1, a_2, a_3)$. From the induction hypothesis it follows that $e(D') \equiv 2 \bmod 4$, hence $e(D, a_1, a_2, a_3) \equiv e(D, \bar{a}_1, \bar{a}_2, \bar{a}_3) \equiv 2 \bmod 4$. This and (P1) imply

$$e(D) \equiv e^*(D) \bmod 4. \tag{P2}$$

Consequently, all we have to show is that $e^*(D) \equiv 2 \bmod 4$. To do this, we shall construct three digraphs D_1', D_2', and D_3' from D to which the induction hypothesis can be applied.

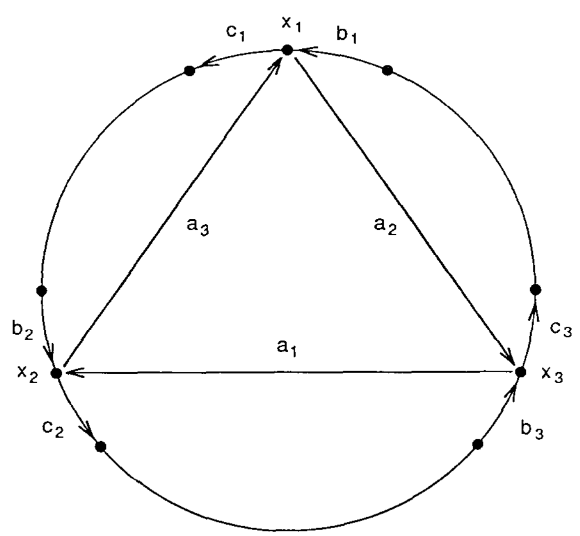

Fig. 1.

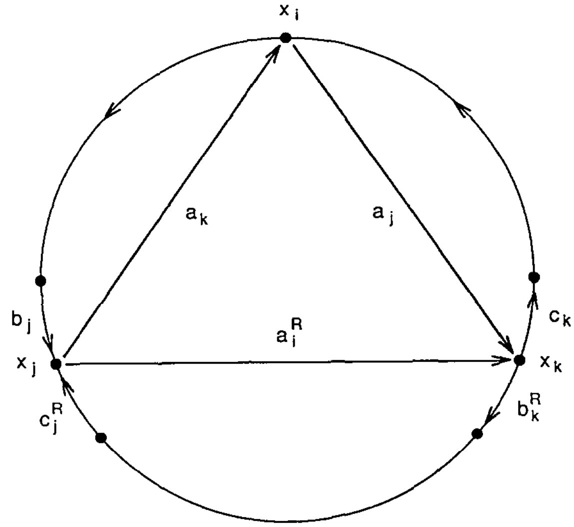

Fig. 2.

For $i = 1, 2, 3$, let x_i^+ be the successor of x_i on C, and let x_i^- be the predecessor of x_i on C. Put $b_i := (x_i^-, x_i)$ and $c_i := (x_i, x_i^+)$. Because of Lemma 1.7, we may assume that C is directed from x_1 to x_2, x_2 to x_3, and x_3 to x_1 (see Fig. 1).

Furthermore, let C_i denote the unique (directed) circuit satisfying $a_i \in A(C_i) \subseteq A(C) \cup \{a_i\}$. Then C_1, C_2 and C_3 are edge disjoint.

In what follows, let (i, j, k) be one of the triples $(1, 2, 3)$, $(2, 3, 1)$, or $(3, 1, 2)$.

First, we define D_i to be the digraph obtained from D by reversing the orientation of C_i, i.e. $D_i := (D - A(C_i)) \cup A(C_i)^{\mathrm{R}}$ (see Fig. 2).

Next, we construct a graph D_i' from D_i by splitting away the arcs b_j, a_k, a_j and b_k^{R} (see Fig. 3).

Let x_i', x_j', and x_k' denote the three new vertices which arise by splitting each x_l into two 2-valent vertices, $l = i, j, k$. The digraph D_i' arises from D_i by putting $b_j := (x_j^-, x_j')$, $a_k := (x_j', x_i')$, $a_j := (x_i', x_k')$, and $b_k^{\mathrm{R}} := (x_k', x_k^-)$, and by leaving all other incidences unaltered.

Clearly, D_i' is an Eulerian digraph which has a decomposition into a Hamiltonian circuit and $(n-1)$ pairwise vertex disjoint triangles. Thus, by virtue of the induction hypothesis, we obtain

$$e(D_1') \equiv e(D_2') \equiv e(D_3') \equiv 2 \bmod 4. \tag{P3}$$

In particular, this implies

$$e(D_1') + e(D_2') + e(D_3') \equiv 2 \bmod 4. \tag{P4}$$

Obviously, in every Eulerian arc set $E \in \varepsilon(D_i')$ either both arcs a_j and a_k are contained in E or both arcs are not contained in E. This implies that, for

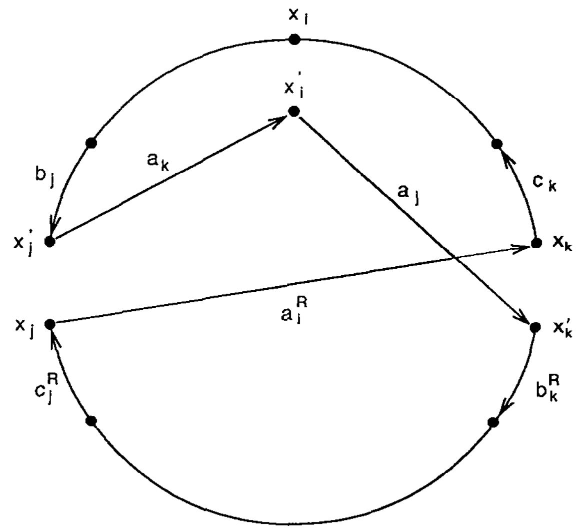

Fig. 3.

$(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$,

$$e(D_i') = e(D_i', a_i^{\mathrm{R}}, a_j, a_k) + e(D_i', \bar{a}_i^{\mathrm{R}}, \bar{a}_j, \bar{a}_k) + e(D_i', \bar{a}_i^{\mathrm{R}}, a_j, a_k) + e(D_i', a_i^{\mathrm{R}}, \bar{a}_j, \bar{a}_k). \tag{P5}$$

Next, we prove that

$$e(D_i', a_i^{\mathrm{R}}, a_j, a_k) = e(D, \bar{a}_i, a_j, a_k). \tag{P6}$$

Put $\varepsilon' := \varepsilon(D_i', a_i^{\mathrm{R}}, a_j, a_k)$ and $\varepsilon := \varepsilon(D, \bar{a}_i, a_j, a_k)$. It suffices to show that there exists a bijection from ε onto ε'. First, we conclude that

$$\varepsilon' = \varepsilon(D_i', c_j^{\mathrm{R}}, a_i^{\mathrm{R}}, c_k, b_j, a_k, a_j, b_k^{\mathrm{R}}) = \varepsilon(D_i, c_j^{\mathrm{R}}, a_i^{\mathrm{R}}, c_k, b_j, a_k, a_j, b_k^{\mathrm{R}}),$$

and

$$\varepsilon = \varepsilon(D, \bar{c}_j, \bar{a}_i, \bar{b}_k, b_j, a_k, a_j, c_k).$$

Next, we define a mapping φ by $\varphi(E) := (E - A(C_i)) \cup (A(C_i) - E)^{\mathrm{R}}$ for every $E \in \varepsilon(D)$. Then (see Lemma 1.7) φ is a bijection from $\varepsilon(D)$ onto $\varepsilon(D_i)$. From the definition of φ it follows that $\varphi(E) \in \varepsilon'$ for every $E \in \varepsilon$, hence φ induces a bijection from ε onto ε'. This proves (P6).

Lemma 1.5, (1.5.2), and (P6) imply

$$e(D_i', \bar{a}_i^{\mathrm{R}}, \bar{a}_j, \bar{a}_k) = e(D, a_i, \bar{a}_j, \bar{a}_k). \tag{P7}$$

Next, we prove

$$e(D_i', a_i^{\mathrm{R}}, \bar{a}_j, \bar{a}_k) = e(D', \bar{x}_j, x_k) = e(D', x_i, \bar{x}_j, x_k) + e(D', \bar{x}_i, \bar{x}_j, x_k). \tag{P8}$$

Put $\varepsilon' := \varepsilon(D_i', a_i^{\mathrm{R}}, \bar{a}_j, \bar{a}_k)$ and $\varepsilon := \varepsilon(D', \bar{x}_j, x_k)$. First, we conclude that

$$\varepsilon' = \varepsilon(D_i', c_j^{\mathrm{R}}, a_i^{\mathrm{R}}, c_k, \bar{b}_j, \bar{a}_j, \bar{a}_k, \bar{b}_k^{\mathrm{R}}) = \varepsilon(D_i, c_j^{\mathrm{R}}, a_i^{\mathrm{R}}, c_k, \bar{b}_j, \bar{a}_j, \bar{a}_k, \bar{b}_k^{\mathrm{R}}),$$

and

$$\varepsilon = \varepsilon(D', \bar{b}_j, \bar{c}_j, b_k, c_k) = \varepsilon(D, \bar{a}_i, \bar{a}_j, \bar{a}_k, \bar{b}_j, \bar{c}_j, b_k, c_k).$$

In the same way as in the proof of (P6) it now follows from Lemma 1.7, that there exists a bijection from ε onto ε', hence $e(D_i', a_i^{\mathrm{R}}, \bar{a}_j, \bar{a}_k) = e(D', \bar{x}_j, x_k)$. Since x_i is a vertex of degree 2 in D', the second equation holds. This proves (P8).

Finally, using Lemma 1.5, (1.5.2), and (P8) we obtain

$$e(D_i', \bar{a}_i^R, a_j, a_k) = e(D', x_j, \bar{x}_k) = e(D', x_i, x_j, \bar{x}_k) + e(D', \bar{x}_i, x_j, \bar{x}_k). \quad \text{(P9)}$$

Put

$$m := e(D_1') + e(D_2') + e(D_3'),$$

and

$$\begin{aligned} m' := {} & e(D', x_1, \bar{x}_2, x_3) + e(D', \bar{x}_1, \bar{x}_2, x_3) + e(D', x_1, x_2, \bar{x}_3) \\ & + e(D', \bar{x}_1, x_2, \bar{x}_3) + e(D', x_1, x_2, \bar{x}_3) + e(D', x_1, \bar{x}_2, \bar{x}_3) \\ & + e(D', \bar{x}_1, x_2, x_3) + e(D', \bar{x}_1, \bar{x}_2, x_3) + e(D', \bar{x}_1, x_2, x_3) \\ & + e(D', \bar{x}_1, x_2, \bar{x}_3) + e(D', x_1, \bar{x}_2, x_3) + e(D', x_1, \bar{x}_2, \bar{x}_3). \end{aligned}$$

Combining (P5)–(P9), we have $m = e^*(D) + m'$. Using Lemma 1.5, we conclude that

$$\begin{aligned} m' &= 2(e(D', x_1, x_2, \bar{x}_3) + e(D', \bar{x}_1, \bar{x}_2, x_3) + e(D', \bar{x}_1, x_2, \bar{x}_3) \\ &\quad + e(D', x_1, \bar{x}_2, x_3) + e(D', \bar{x}_1, x_2, x_3) + e(D', x_1, \bar{x}_2, \bar{x}_3)) \\ &= 4(e(D', x_1, x_2, \bar{x}_3) + e(D', x_1, \bar{x}_2, x_3) + e(D', \bar{x}_1, x_2, x_3)). \end{aligned}$$

Therefore, $m' \equiv 0 \bmod 4$, hence $m \equiv e^*(D) \bmod 4$. Then, by (P4), it follows that $e^*(D) \equiv 2 \bmod 4$. Now, (P2) implies $e(D) \equiv 2 \bmod 4$. □

3. Final remarks

Discussing the above with colleagues in Grenoble in October 1991, the first author was informed of the following interesting facts.

(1) Take an Eulerian orientation D of the line graph of a plane cubic 2-connected graph G. Then $|ee(D) - eo(D)|$ is the number of Tait colorings of G (F. Jaeger). This follows directly from [7] (Proposition 1, Lemma 3, Proposition 4). However, it is not true in general that $e(D) \equiv 2 \bmod 4$ in this case; as F. Jaeger pointed out, the cube yields $e(D) \equiv 0 \bmod 4$.

(2) One can use the theory of nowhere-zero $\mathbb{Z}_3$-flows (in particular, the flow polynomial) to show that if the number of nowhere-zero $\mathbb{Z}_3$-flows equals $2 \bmod 4$ for an arbitrary graph G, then G is 3-colorable. In the case where G is 4-regular, this is equivalent to the implication that G is 3-colorable if $e(D) \equiv 2 \bmod 4$; in fact, such a flow in an eulerian orientation D of G defines arc-disjoint Eulerian subdigraphs D_i which are induced by the arcs labeled i, $i = 1, 2$. That is, one need

not necessarily rely on the result of Alon and Tarsi (Theorem 1.2) to prove Theorem 1.1 (F. Jaeger).

(3) M. Tarsi showed that F. Jaeger's observation (2) can be deduced from the Alon–Tarsi approach to colorability (see [1]). However, the application of Theorem 1.2 and Theorem 2.1 gives a somewhat stronger result than just 3-colorability, namely 3-choosability (see Theorems 1.2 and 1.3, and [1]).

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References

[1] N. Alon and M. Tarsi, Colourings and orientations of graphs, Combinatorica, to appear.
[2] L.W. Beineke and R.J. Wilson, Selected Topics in Graph Theory 2 (Academic Press, New York, 1983).
[3] D.Z. Du, D.F. Hsu and F.K. Hwang, Hamiltonian property of consecutive-d digraphs, Comput. Math. Appl., to appear.
[4] D.Z. Du and D.F. Hsu, On a combinatorial problem related to network design and optimization, manuscript.
[5] M.R. Fellows, Transversals of vertex partitions of graphs, SIAM J. Discrete Math. 3 (1990) 206–215.
[6] H. Fleischner, Eulerian Graphs and Related Topics, Part 1, Vol. 1, Ann. Discrete Math. 45; Vol. 2, Ann. Discrete Math. 50 (North-Holland, Amsterdam 1990/91).
[7] F. Jaeger, On the Penrose number of cubic diagrams, Discrete Math. 74 (1989) 85–97.
[8] D.B. West, Open problems, SIAM Discrete Math. Newsletter 1 (3) (1991) 9–12.