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A Characteristic Free Approach to Invariant Theory

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0. INTRODUCTION

In this paper we treat that portion of classical invariant theory which goes under the name of "first" and "second" fundamental theorem for the classical groups, in a characteristic free way, i.e., where the base ring A is any commutative ring (in particular the integers $\mathbb{Z}$ or an arbitrary field).

The results we obtain are exactly the ones predicted by the classical theory (see [5]), provided we interpret the word invariant to mean formal or absolute ones, they are contained in Theorems 3.1, 3.3, 4.1, 5.6, 6.6.

For instance we have

THEOREM 3.1 [5]. *The ring of polynomial functions over A in the entries of n m-vectors $x_1, \ldots, x_n$ and n m-covectors $\xi_1, \ldots, \xi_n$, left formally invariant under the action of $GL(m, -)$ is generated over A by the scalar products*

$$\langle x_i \mid \xi_j \rangle = \sum_{k=1}^{m} x_{ik}\xi_{jk}.$$

This theorem has an important corollary, due to Schur in the characteristic 0 case.

Let P be a projective module over a commutative ring A, G its group of linear transformations. If we consider $P^{\otimes n}$ with its canonical G action, the symmetric group S_n acts on $P^{\otimes n}$ commuting G; then we have, under a mild condition on A:

THEOREM 4.1.[5]. *The centralizer of G on $P^{\otimes n}$ is spanned by S_n.*

The condition, which is often satisfied, is that a polynomial $f(x) \in A[x]$ of degree n, vanishing identically on A, should be 0.

The fact that Theorem 3.1 implies Theorem 4.1 is essentially trivial and well known (cf. [5] for historical comments and classical references). Certainly Theorems 3.1 and 4.1 are equivalent if one restricts the attention to multilinear invariant. Thus in characteristic 0 the two statements are completely equivalent and in fact there are independent proofs of the two facts.

The classical proofs of invariant theoretic results follow essentially two equivalent paths. Polarization and the Gordan–Capelli expansion (via the theorem of E. Pascal) or double centralizer theorems, owing to the linear reductivity of the group in consideration.

There is, on the other hand, another line of approach based on the standard tableaux of Young which should be traced at least to Hodge [3], to Igusa [4] (who proves the first fundamental theorem of vector invariants in a characteristic free approach), and finally to Doubilet, Rota, and Stein [2], which gives the main technical tool: the straightening formula.

In [2] there is a gap which we fill with Theorem 1.2. The ideal J_d, of Doubilet, Rota, and Stein's Theorem 4, is in fact, as one expects, the ideal of elements vanishing on d-vectors and d-covectors. At the same time we give a different proof of their Theorem 3 which is more suitable for the functional interpretation of the ring R_d.

The line of the proof is the following: We have an algebraic group G acting on an affine variety V with coordinate ring R and we have a subring B of R^G which we want to show equals R^G. First we show that on an open set $U \subseteq V$, where an element $d \in B$ is invertible, the group action is a product action; thus we compute the invariant ring which turns out to be the localized ring $B[1/d]$. Then we have to find a way to cancel d; i.e., if $da \in B$, $a \in R^G$ we must show that $a \in B$. This is accomplished by finding an explicit basis of the ring B and deducing the cancellation result from this. This part is the main contribution of the paper.

1. The Straightening Formula

Let A be a commutative ring with 1. We construct the polynomial ring in the "variables" $\langle x_i \mid \xi_j \rangle$, $i, j = 1,..., k$.

If B is a commutative A algebra, M a module over B, $m_1 ,..., m_k \in M$, $\varphi_1 ,..., \varphi_k \in M^*$, we can evaluate the variables $\langle x_i \mid \xi_j \rangle$ on the "vectors" m_j's and "covectors" φ_j's to obtain the elements $\langle m_i \mid \varphi_j \rangle = \varphi_j(m_i)$. In this fashion every element $f \in R$ becomes a function on $M^k \times M^{*k}$.

Let us denote $I_n = \{f \in R \mid f$ vanishes on B^d, $d \leqslant n$, B any commutative A algebra$\}$. Set $R_n = R/I_n$. It is quite easy to verify that $\bigcap_n I_n = 0$.

The ring R possesses a formal structure which makes it a free algebra in a suitable sense. In fact one can operate formally, in R, substitutions of type $x_i = \sum \lambda_{ij} x_j$, $\xi_h = \sum \mu_{hk} \xi_k$ by just imposing the bilinearity of the symbol $\langle x_i \mid \xi_j \rangle$. Such a substitution, applied to an element $f \in R$, gives rise to a new element in $R \otimes_A A[\lambda_{ij}, \mu_{hk}]$. It is compatible, in an obvious sense, with the evaluation of f as a function, on vectors and covectors.

The monomials in the $\langle x_i \mid \xi_j \rangle$ can be naturally given a multidegree (content [2]) in the x_i's and ξ_j's. It is immediate that the ideals I_n are homogeneous, with respect to such content.

The starting point of our work is the straightening formula [2, Theorems 1.3]. A double tableau (cf. [2]) is an array

$$T = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1m_1} & b_{11} & \cdots & b_{1m_1} \\ a_{21} & \cdots & a_{2m_2} & b_{22} & \cdots & b_{2m_2} \\ \vdots & & & \vdots & & \\ a_{s1} & \cdots & a_{sm_s} & b_{s1} & \cdots & b_{sm_s} \end{array} \right) = (A \mid B),$$

where the a_{ij}'s are indices out of $1, 2,..., k$ and one assumes that

(i) $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$.

Furthermore, if we have

(ii) $a_{ij} < a_{ik}$, $b_{ij} < b_{ik}$ when $k > j$,

(iii) $a_{ij} \leqslant a_{kj}$, $b_{ij} \leqslant b_{kj}$ when $k \geqslant i$,

then we say that T is standard.

One associates [2] to a tableau T an element, still denoted by T, of R,

$$T = \prod_{j=1}^{s} \det |\langle x_{a_{ji}}, \xi_{b_{jk}} \rangle|, \qquad i, k = i,..., m_j .$$

The main result of [2] is

THEOREM 1.1 (Straightening formula [2, Theorems 1, 3]). *The double standard tableau are a basis of R over A.*

We are now ready to start our work. Let $T = (A \mid B)$ be a double tableau. We make, in T, the substitution

$$x_1 = x_1 + \sum_{i=2}^{n} \lambda_i x_i .$$

Due to the linearity of the rows of T the resulting element has the form

$$\sum \lambda_2^{h_2} \lambda_3^{h_3} \cdots \lambda_n^{h_n} P_{h_2 \cdots h_n}$$

where $P_{h_2 \cdots h_n}$ is a sum of tableaux $(A_i \mid B)$. A_i is obtained from A substituting $h_2, h_3, ..., h_n$ entries of x_1, respectively, with $x_2, x_3, ..., x_n$.

We will always cancel, from this sum, all the tableaux which have repetitions on one row; and hence are formally zero.

Assume now that A is standard, and A has the form

$$A = \left| \begin{matrix} 1 & 2 & 3 & \cdots & k_1 & \tau_1 & \cdots \\ 1 & 2 & 3 & \cdots & k_2 & \tau_2 & \cdots \\ \vdots & \vdots & \vdots & & & & \\ 1 & 2 & 3 & \cdots & k_s & \tau_s & \\ \tau_{s+1} & & & & & & \end{matrix} \right| ,$$

with $\tau_1 > k_1 + 1$, $\tau_2 > k_2 + 1,..., \tau_{s+1} > 1$, τ_i the first element of the ith row which, lying on the jth column, is different from j.

Due to the standard nature of A we have $k_1 \geqslant k_2 \geqslant k_3 \geqslant \cdots \geqslant k_s$. Upon the substitution $x_1 = x_1 + \sum_{i=2}^{n} \lambda_i x_i$ we analyze, in the resulting expression $\sum \lambda_2^{h_2} \cdots \lambda_n^{h_n} P_{h_2 \cdots h_n}$, the coefficient of the highest monomial in the lexicographic order of the sequences $(h_2, h_3, ..., h_n)$.

This, of course, is obtained by substituting for 1 the maximum number of 2's which do not make the tableau formally zero, then the maximum number of 3's whith the same property and so on.

Thus it is immediate that $h_2 = h_2(A) =$ number of rows starting with $1\ \tau$, $\tau > 2$; $h_3 = h_3(A) =$ number of rows starting with $1\ 2\ \tau$, $\tau > 3$ and so on. The h_2 rows $1\ \tau,\ldots$ follow the h_3 rows $1\ 2\ \tau,\ldots$, etc.

The resulting tableau, which we denote $F(A)$, can be rearranged in order to be, up to sign, of the form

$$F(A) = \left| \begin{array}{cccccc} 1 & 2 & \cdots & n & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & \cdots & n & \cdots \\ 2 & 3 & \cdots & n-1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 3 & \cdots & n-1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 3 & \cdots & k & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right|.$$

The new tableau is still standard. Moreover $F(A)$ uniquely determines A once one knows the numbers $h_2, h_3, \ldots, h_n$.

In fact, to obtain A from $F(A)$, one has to replace in the first h_n rows starting with 2, which necessarily are of the form $2\ 3\ l \cdots n\ l \cdots$, the elements $2\ 3 \cdots n$ with $1\ 2 \cdots n-1$. On the next h_{n-1} rows, which necessarily start with $2\ 3 \cdots n-1$, one replaces $2\ 3 \cdots n-1$ with $1\ 2 \cdots n-2$ and so on.

Suppose we perform the substitution $x_1 = x_1 + \sum_{i=2}^{n} \lambda_i x_i$ into a sum

$$p = \sum c_i T_i, \qquad c_i \neq 0, \qquad T_i = (A_i \mid B_i)$$

of distinct double standard tableaux.

The coefficient of the highest monomial, in the resulting expression, is

$$\bar{p} = \sum_j \epsilon_j c_j (F(A_j) \mid B_j),$$

$\epsilon_j = \pm 1$, and the sum is taken from the double standard tableaux $T_j = (A_j \mid B_j)$ for which the sequence $h_2(A_j)$, $h_3(A_j),\ldots, h_n(A_j)$ is maximal.

Thus, writing $\bar{p} = \sum \tau_j \overline{T}_j$, we have that the τ_j's are not zero and the $\overline{T}_j$'s are distinct.

We are now ready to draw the conclusions. First of all we clarify the structure of R_n (cf. [2]).

THEOREM 1.2. *The ring R_n has a basis over A formed by the double standard tableaux with every row of length $\leqslant n$.*

Remark. This shows in particular the linear independence in R of the double standard tableaux [2, Theorem 3].

Proof. Clearly, from Theorem 1.1, R_n is spanned by these tableaux. We must show that they are linearly independent. Suppose that $p = \sum c_i T_i$ is a linear combination of these tableaux and $p \in I_n$. We make an induction on the size of the T_i's and on the number of variables x_i, ξ_j appearing in p. Let $x_1, \ldots, x_t$; $\xi_i, \ldots, \xi_s$ be the variables appearing. Make the substitution $x_1 = x_1 + \sum_{i=2} \lambda_i x_i$ and then we obtain

$$\bar{p} = \sum \bar{c}_j \bar{T}_j \in I_n, \qquad \bar{T}_j(F(A_j) | B_j).$$

Now, setting $x_1 = 0$, we obtain a nonzero element of I_n depending on the variables $x_2, \ldots, x_t$ only, unless the first row of each $F(A_j)$ is $1\ 2 \cdots t$ exactly. In this case, of course, $t \leqslant n$. We perform now the same operation on the ξ_j's and finally we, either succeed in reducing the number of variables or reduce ourselves to an element $\sum \tilde{c}_j \tilde{T}_j \in I_n$, where the first row of $\tilde{T}_j$ is always

$$d = (1\ 2 \cdots t \mid 1\ 2 \cdots t).$$

Now in this case $\tilde{T}_j = d \cdot Q_j$, Q_j obtained from $\tilde{T}_j$ erasing the first row. Since d is generically invertible $\tilde{p} = \sum \tilde{c}_j Q_j \in I_n$ and we finish by induction on the size of the tableaux.

We come now to our next theorem.

THEOREM 1.3. *Let $p = \sum c_i T_i$, $T_i = (A_i \mid B_i)$ be a linear combination of distinct double standard tableaux with rows of length at most n. If p vanishes when $x_1, x_2, \ldots, x_n$ are computed on linearly dependent vectors then the first row of each A_i is $1\ 2\ 3 \cdots n$ exactly.*

Proof. If one of the variables $x_1, \ldots, x_n$ say x_1, does not appear in all T_i's then, setting $x_1 = 0$ we obtain a contradiction to the linear independence of the tableaux. Split $p = p_1 + p_2$ in two terms. In p_2 we collect all $T_i = (A_i \mid B_i)$ such that A_i has first row $1\ 2 \cdots n$. Clearly p_1 still satisfies the hypothesis of Theorem 1.3 and we must show $p_1 = 0$. Without loss of generality we can thus assume $p = p_1$.

Perform the substitution $x_1 = x_1 + \sum_{i=2}^{n} \lambda_i x_i$ and extract the polynomial $\bar{p} = \sum c_j \bar{T}_j$. $\bar{P}$ still verifies the hypothesis of Theorem 1.3 but now, since no first row of an A_j starts with $1\ 2 \cdots n$ we have canceled the variable x_1 from p. As before we have a contradiction.

We collect one more result which will be useful for the computation of invariants.

PROPOSITION 1.4. *Let* $p = \sum c_i T_i$, $T_i = (A_i \mid B_i)$ *be a sum of double standard tableaux with rows of length* $\leqslant n$.

Assume that the variables $x_1, x_2, \ldots, x_n$; $x_{n+1}, \ldots, x_{2n}$; ...; $x_{(s-1)n+1}, \ldots, x_{sn}$ *appear linearly in* p *and that* p *vanishes when one of the* s *groups of* n*-variables* $x_1, \ldots, x_n$; $x_{n+1}, \ldots, x_{2n}$;...; $x_{(s-1)n+1}, \ldots, x_{sn}$ *is computed on linearly dependent vectors. Then each* A_i *has the first* s *rows*

$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \cdots & \cdots & \cdots & \cdots \\ (s-1)n+1 & & \cdots & sn. \end{array}$$

Proof. First of all, setting $x_j = 0$, $j \leqslant sn$ we get a contradiction to the linear independence of the T_j's, unless x_j appears in each T_i.

For $s = 1$, it is contained in Theorem 1.3, assume it true for $s - 1$, then each A_i starts as

$$\begin{array}{ccc} 1 & 2 \cdots & n \\ n+1 & \cdots & 2n \\ \cdots & \cdots & \cdots \\ (s-2)n+1 & \cdots & (s-1)n. \end{array}$$

Since, by assumption, A_i is linear in $1\ 2 \cdots (s-1)n$ and these indices have all been used, the standard nature of A_i forces the next row necessarily to start with $(s-1)n+1$.

For $x_{(s-1)n+1}$, substitute $x_{(s-1)n+1} + \sum_{j=2}^{n} \lambda_j\, x_{(s-1)n+j}$. The coefficient of the highest monomial (which is linear) is obtained by substituting $x_{(s-1)n+j}$ for $x_{(s-1)n+1}$ in all the tableaux A_i whose sth row is of the form $(s-1)n+1$, $(s-1)n+2\ldots$, $(s-1)+j-1$, τ, where $\tau > (s-1)n+j$ or the row ends before τ. j is the minimum index for which such a violation occurs. Now the resulting p does not contain $x_{(s-1)n+1}$ and we have a contradiction, unless all the A_i's have the required form.

2. ABSOLUTE INVARIANTS

We develop here a minimum formal machinary to deal correctly with formal invariants.

Let G be an affine group over A (we assume that 1_G is defined over A). If B is any A algebra, denote by $G(B)$ the group of points of G with coordinates in B. Let A_G denote the coordinate ring of G, thus by definition $G(B) = \text{Maps}\,(A_G, B)$. Now let C be a fixed A algebra. Denote $Sp(C)$ the functor $Sp(C)(B) = \text{Maps}\,(C, B)$.

DEFINITION 2.1. An algebraic action of G on $Sp(C)$ is a natural transformation of functors $G \times Sp(C) \to Sp(C)$ which is a group action in the category of functors.

Since all functors in question are representable, such an action is given by a map

$$\mu\colon C \to C \otimes A_G$$

satisfying the appropriate axioms.

If B is any A-algebra we have an induced action of $G(B)$ on $C \otimes_A B$. Given $\varphi \in G(B)$, φ is a map $\varphi\colon A_G \to B$. The induced automorphism on $C \otimes_A B$ is given by extending linearly to $C \otimes_A B$ the map

$$C \xrightarrow{\mu} C \otimes_A A_G \xrightarrow{1 \otimes \varphi} C \otimes_A B.$$

DEFINITION 2.2. An element $c \in C$ is called an absolute (or formal) invariant if for all A algebras B the element $c \otimes 1 \in C \otimes_A B$ is invariant under the group $G(B)$.

Remark. It is trivial to verify that c is a formal invariant if and only if $\mu(c) = c \otimes 1$ (invariance under the generic element).

We will denote by C^G the subring of formal invariants.

PROPOSITION 2.3. *The absolute invariants of the action of G on itself, $A_G{}^G$, are the "constants" A.*

Proof. Let $c \in A_G{}^G$, i.e., $\mu(c) = c \otimes 1$. Let $\tau\colon A_G \to A_G$ be the inverse map.

The composition

$$\psi\colon A_G \xrightarrow{\mu} A_G \otimes A_G \xrightarrow{1 \otimes \tau} A_G \otimes A_G \xrightarrow{\pi} A_G .$$

$\pi(a \otimes b) = a \cdot b$ is the unit

$$A_G \to A \to A_G .$$

Now $\psi(c) = \pi(1 \otimes \tau)\,\mu(c) = \pi(c \otimes 1) = c \in A.$

COROLLARY 2.4. *Let W be an affine scheme over A with coordinate ring B. Let us endow $G \times W$ with the trivial G action. G acting on the first factor. The absolute invariants of $A_G \otimes_A B$ are $A \otimes_A B$.*

Proof. Trivial by base change.

This corollary can be generalized in a useful way, assume that G acts on an affine scheme U and we have a projection map $U \to W$ compatible with the G action. If $W' \to W$ is faithfully flat and $U \times_W W' \simeq G \times_{\mathrm{Spec} A} W'$, with trivial G action, then the coordinate ring of W is identified with the formal invariants of the G action on U (same proof and descent).

3. THE FIRST FUNDAMENTAL THEOREM

In this paragraph we deduce the main result. Let A be a commutative ring, $x_1, ..., x_m$ be n-vector, $\xi_1, ..., \xi_m$ be n covector variables:

$$x_i = (x_{i1}, x_{i2}, ..., x_{in}); \qquad \xi_j = (\xi_{j1}, \xi_{j2}, ..., \xi_{jn}).$$

Consider the polynomial ring $S = A[x_{ij}, \xi_{st}]$ in the entries. We act on $Sp(S)$ with the general linear group $GL(n, -)$, which we will denote in this section with G, with the standard action.

THEOREM 3.1. *The ring S^G is generated by the elements $\langle x_i \mid \xi_j \rangle$.*

Proof. Let us call $B = A[\langle x_i \mid \xi_j \rangle]$, the ring that we want to show equals S^G.

Since it is clear that, if Theorem 3.1 is true for m variables x_i's it is also true for $m' \leqslant m$ variables; we may assume m very large or even infinite. Now let d be the determinant of the $n \times n$ matrix

$$\| \langle x_i \mid \xi_j \rangle \|, \qquad i, j = 1, ..., n.$$

We have $d = [x_1, x_2, ..., x_n] \cdot [\xi_1, ..., \xi_n]$, where $[x_1, ..., x_n]$ denotes the determinant of the matrix

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix};$$

the proof rests on the following fact, which will be proved in Lemma 3.2. $B[1/d] = S[1/d]^G$. Assuming this fact let $a \in S^G$; since $S^G \subseteq B[1/d]$ there is a power d^k such that $d^k a \in B$.

Clearly it is sufficient to prove that, if $df \in B$ then $f \in B$ (for $f \in S$). Let us write $df = \sum c_i T_i$, where the T_i's are double standard tableaux with rows of length $\leqslant n$. Since, setting $x_1, \ldots, x_n$ or $\xi_1, \ldots, \xi_n$ linearly dependent, one has $d = 0$, we are in the position to apply Theorem 1.3 and we deduce that each T_i has the form

$$\left(\begin{array}{cccc|cccc} x_1 & x_2 & \cdots & x_n & \xi_1 & \xi_2 & \cdots & \xi_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right).$$

Thus $T_i = dT_i'$ and $f = \sum c_i T_i' \in B$.

We now prove the fact used before.

LEMMA 3.2. $B[1/d] = S[1/d]^G$.

Proof. We use the results of Section 2.

$S[1/d]$ is the coordinate ring of the open set W of an $2nm$-dimensional affine space where d is invertible. The action of G on such affine space restricts to W and we claim that $W = G \times V$ is a product. In fact let V' be the subspace, where $x_i = e_i$, $i = 1, \ldots, n$, and e_i represent the vectors of the canonical basis. Let $V = V' \cap W$ (we will identify V with $1 \times V$). V is the open set of V', where $[\xi_1, \xi_2, \ldots, \xi_n]$ is invertible. If C is any A algebra and $u \in W(C)$, there is a unique element $g \in GL(n, C)$ and $u' \in V(C)$ such that $u = g \cdot u'$. In fact one has to set

$$g = \begin{pmatrix} u_{11} & \cdots & u_{n1} \\ u_{12} & \cdots & u_{n2} \\ \vdots & \cdots & \vdots \\ u_{1n} & \cdots & u_{nn} \end{pmatrix}, \qquad u' = g^{-1} \cdot u.$$

This establishes the isomorphism $W \sim G \times V$ of schemes. By Corollary 2.3 the ring of G invariants is identified with the coordinate ring D of V. D is generated by the elements x_{ij}, $i = n + 1, \ldots, m$, $j = 1, \ldots, n$; ξ_{st}, $s = 1, \ldots, m$; $t = 1, \ldots, n$; and $[\xi_1, \xi_2, \ldots, \xi_n]^{-1}$. We have thus to identify the corresponding invariants on W. By construction $f \in D$ induces the invariant $\bar{f}$ on W by setting $\bar{f}(u) = f(u')$, $u = g \cdot u'$.

If $u = (x_1, \ldots, x_m, \xi_1, \ldots, \xi_m)$ we have

$$\bar{x}_{ij}(u) = x_{ij}(u') = x_{ij}(g^{-1}u) = \langle g^{-1}x_i, e^j \rangle.$$

Since $g^{-1}x_i = e_i$, $i = 1,..., n$, we have

$$x_i = \sum_{j=1}^{n} \langle g^{-1}x_i , e^j\rangle ge_j = \sum \langle g^{-1}x_i , e^j\rangle x_j ;$$

hence

$$\langle g^{-1}x_i , e^j\rangle = \frac{[x_1 ,..., x_{j-1} , x_i , x_{j+1} ,..., x_n]}{[x_1 , x_2 ,..., x_n]}$$
$$= \frac{[\xi_1 ,..., \xi_n][x_1 \cdots x_i \cdots x_n]}{d} \in B[1/d].$$

Similarly

$$\xi_{st}(u) = \xi_{st}(u') = \xi_{st}(g^{-1}u) = \langle e_t , g^t\xi_s\rangle = \langle ge_t , \xi_s\rangle = \langle x_t , \xi_s\rangle$$

and finally

$$\overline{[\xi_1 ,..., \xi_n]^{-1}} = 1/d.$$

We come now to the invariants under the special linear group $Sl(n, —)$.

THEOREM 3.3. *The invariant ring S^{Sl} is generated over A by the elements*

$$\langle x_i \mid \xi_j\rangle, \qquad [x_{i_1} ,..., x_{i_n}], \qquad [\xi_{j_1} ,..., \xi_{j_n}].$$

Proof. Let $f \in S^{Sl}$. We may assume f homogeneous in the variables x_i's and ξ_j's. Let us call t_1, t_2 the total degree of f in the x_i's and ξ_j's, respectively.

If B is any A algebra and $g \in GL(n, B)$ we claim that

$$f^g(x, \xi) = f(g^{-1}x_1 ,..., g^{-1}\xi_1 ,..., g^{-1}\xi_s) = \det(g)^d f(x, \xi),$$

where $d \cdot n = t_2 - t_1$. In fact, let $\alpha = \det(g)$ and consider the ring $B' = B[x]/x^n - \alpha$. In B' the element g can be written $g = \bar{g} \cdot \beta I$; $\bar{g} \in Sl(n, B')$, β a scalar (e.g., $\beta = \bar{x}$) with $\beta^n = \alpha$. Thus $f^g(x, \xi) = f^{\beta I}(x, \xi) = f(\beta^{-1}x, \beta\xi) = \beta^{t_2-t_1}f(x, \xi)$. But since $1, \beta,..., \beta^{n-1}$ are a basis of B' over B and $f(x, \xi), \beta^{t_2-t_1}f(x, \xi) \in B$ we must have $t_2 - t_1 = d \cdot n$.

Let us treat the case $d \geqslant 0$, the other being similar. We introduce $d \cdot n$ new vector variables $y_1 ,..., y_n ; y_{n+1} ,..., y_{2n} \cdots y_{(d-1)n+1} ,..., y_{dn}$. We make the convention that the x's follow the y's in the lexicographic order. Consider the element

$$h = f \cdot [y_1 \cdots y_n][y_{n+1} \cdots y_{2n}] \cdots [y_{(d-1)n+1} \cdots y_{dn}].$$

Clearly $h \in S^G$ and so $h = \sum c_i T_i$, T_i double standard tableaux. We are now in a position to apply Proposition 1.4 and deduce that each T_i has the form

$$\left(\begin{array}{cccc|cc} 1 & 2 & \cdots & n & \cdots & \cdots \\ n+1 & & \cdots & 2n & \cdots & \cdots \\ & \cdots & & & & \\ (d-1)n+1 & & \cdots & dn & & \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \end{array}\right).$$

If we specialize the y_j's to make $[y_{ln+1}, \ldots, y_{(l+1)n}] = 1$, $l = 0, 1, \ldots, d-1$ we obtain the result.

Remarks. (i) In case $s = 0$ we have back the theorem of Igusa [4] (cf. also [2]).

(ii) We have in fact explicit bases for the rings S^{Sl}, S^G over A.

One can deduce from these bases the second fundamental theorem as in [5]. One has immediately.

THEOREM 3.4. *The ideal of relations among the* $\langle x_i \mid \xi_j \rangle$ *is generated by the* $n+1 \times n+1$ *determinants.*

Similarly for the Sl invariants.

4. THE SYMMETRIC GROUP

In this paragraph we draw the corollaries of Theorem 3.1 for the symmetric group. As pointed out in the Introduction this is essentially a standard fact, known in the classical literature (cf. [5]).

We assume that A is a commutative ring and m is an integer such that, if $f(x) \in A[x]$ is a polynomial of degree m and $f(x)$ vanishes on A, then f is identically 0. Then we have

THEOREM 4.1. *Let P be a projective module over A.* $\mathrm{End}_{GL(P)}(P^{\otimes m})$ *is spanned over A by the symmetric group.*

Proof. By localizing one can assume that P is free. If $rk\ P$ is infinite, or just $\geqslant m$, we can find independent vectors $e_1, e_2, \ldots, e_m$ in the basis and $u = e_1 \otimes e_2 \otimes \cdots e_{m-1} \otimes e_m \in P^{\otimes m}$. If $\varphi \in \mathrm{End}_{GL(P)}(P^{\otimes m})$ it is easy to see that

$$\varphi\mu = \sum \alpha_\sigma e_{\sigma^{-1}(1)} \otimes e_{\sigma^{-1}(2)} \otimes \cdots \otimes e_{\sigma^{-1}(m)}$$

and there is a $\psi = \sum \alpha_\sigma \sigma$ coinciding with φ on u.

But then it is easy to prove that u is a linear generator of $P^{\otimes m}$ over $Gl(P)$ and the claim follows. Thus the relevant case is $P = A^n$ and in fact $n < m$.

In this case $GL(P) = GL(n, A)$ and

$$\mathrm{End}_{GL(n,A)}(P^{\otimes m}) \simeq \mathrm{End}_A(P^{\otimes m})^{GL(n,A)} \simeq (P^{\otimes m} \otimes P^{*\otimes m})^{*GL(n,A)}.$$

The identification $\mathrm{End}(W) \simeq (W \otimes W^*)^*$ is given by the usual pairing

$$(A, u \otimes \psi) = \psi(Au).$$

Thus $\mathrm{End}_{GL(n,A)}(P^{\otimes m})$ is identified with the multilinear invariants of m vectors and m covectors. At this point we have to make only two remarks to finish.

First of all the monomial

$$\langle x_1 \mid \xi_{\sigma(1)}\rangle\langle x_2 \mid \xi_{\sigma(2)}\rangle \cdots \langle x_n \mid \xi_{\sigma(n)}\rangle$$

corresponds to the permutation $\sigma \in \mathrm{End}(V^{\otimes m})$, since

$$\begin{aligned}
&(\sigma, x_1 \otimes \cdots \otimes x_m \otimes \xi, \otimes \cdots \otimes \xi_m) \\
&\quad = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m(x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}) \\
&\quad = \Pi\langle x_{\sigma^{-1}(i)} \mid \xi_i\rangle = \Pi\langle x_i \mid \xi_{\sigma(i)}\rangle.
\end{aligned}$$

Second, if $\varphi \in (P^{\otimes m} \otimes P^{*\otimes m})^{*sl(n,a)}$, φ is an absolute invariant; for then from Theorem 3.3 and the fact that φ has the same degree in the x_i's and ξ_j's the claim will follow. The fact that φ is an absolute invariant follows from our assumptions on A. In our assumptions, A local, $Sl(n, A)$ is generated by the elementary matrices $I + \alpha e_{ij}$. It is sufficient to show that, if $a \in \mathrm{End}(P^{\otimes m})$ and a commutes with $I + \alpha e_{ij}$, $\alpha \in A$ then a commutes with $I + \beta e_{ij}$, $\beta \in B$ any A algebra. Now the commutator $[a, I + xe_{ij}]$ is a polynomial in x of degree m and so the hypotheses on A imply the claim.

The explicit structure of the invariant ring implies also the structure of the Kernel of the map, from the group algebra of S_m to $\mathrm{End}(P^{\otimes m})$.

THEOREM 4.2. *If P is a free A module of rank n the kernel of the canonical map* $\lambda\colon A[S_m] \to \operatorname{End}(P^{\otimes m})$ *is*

(i) 0 *if* $n \geqslant m$;

(ii) *generated by* $\pi = \sum \epsilon_\sigma \sigma$, σ *running on all the permutations of the first* $n+1$ *indices, if* $m \geqslant n+1$.

Proof. By the proof of Theorem 4.1 we identify $A[S_m]$ with the space spanned by the double standard tableaux of content $(1, 1,\ldots, 1 \mid 1, 1,\ldots, 1)$. The kernel of λ is given by the tableaux with the first row of length $\geqslant n+1$. It is sufficient to show that each such tableau lies in the ideal generated by π. If

$$T = \begin{pmatrix} i_1 & \cdots & i_k & \big| & j_1 & \cdots & j_k \\ & \vdots & & & & \vdots & \\ & \cdots & & \big| & & \cdots & \end{pmatrix}, \qquad k \geqslant n+1,$$

T is a sum of Terms:

$$u = (i_1 \cdots i_k \mid j_1 \cdots j_k)(s_1 \mid t_1) \cdots (s_r \mid t_r)$$

we claim that $u = (i_1 ,\ldots, i_k \mid j_1 ,\ldots, j_k)(s_1 \mid t_1) \cdots (s_m \mid t_2)$ is a consequence of π.

In fact the monomial $(a_1 \mid b_1)(a_2 \mid b_2) \cdots (a_m \mid b_m)$ corresponds to the permutation $\sigma\colon a_i \to b_i$; thus

$$(a_1 \mid \tau(b_1))(a_2 \mid \tau(b_2)) \cdots (a_n \mid \tau(b_n))$$

corresponds to $\tau\sigma$ and

$$(\tau(a_1) \mid b_1)(\tau(a_2) \mid b_2) \cdots (\tau(a_n) \mid b_n)$$

to $\sigma\tau^{-1}$. Thence u corresponds to $\tau_1 \bar{u} \tau_2$ for some permutations τ_1, τ_2, where $\bar{u} = (1\,2 \cdots k \mid 1\,2 \cdots k)(k+1 \mid k+1) \cdots (m \mid m)$.

Now clearly $\bar{u}$ is the antisymmetrizer of the first k letters and if $k \geqslant n+1$ this is a consequence of π.

5. The Orthogonal Group

First of all we want to deduce the straightening formula for scalar products.

Consider a ring A and m n-vector variables u_i, $u = (u_{i1} ,\ldots, u_{in})$. Define $(u_i , u_j) = \sum_{k=1}^{n} u_{ik} u_{jk}$ and $R = A[(u_i , u_j)]$. We will find, as in Theorem 1.1, a basis of R over A.

Consider a double tableau

$$T = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1m_1} & b_{11} & \cdots & b_{1m_1} \\ & \cdots & & & \cdots & \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \end{array}\right)$$

and associate to T the element in R product of the determinants

$$(a_{i1}, \ldots, a_{in_i} \mid b_{i1}, \ldots, b_{in_i}) = \det \|(u_{a_{ih}}, u_{b_{ik}})\|.$$

Let us form from T the single tableau

$$T' = \left|\begin{array}{ccc} a_{11} & \cdots & a_{1m_1} \\ b_{11} & \cdots & b_{1m_1} \\ a_{21} & \cdots & a_{2m_2} \\ b_{21} & \cdots & b_{2m_2} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \end{array}\right|$$

with $2k$ rows of length $m_1, m_1, m_2, m_2, \ldots, m_k, m_k$. We will say that T is doubly standard, briefly a d-tableau, if T' is standard.

THEOREM 5.1. *The d-tableaux, with rows of length $\leqslant n$, are a basis of R over A.*

The proof will be based on two lemmas. Given

$$T = (u_1 \cdots u_{n-\lambda} x_1 \cdots x_\lambda \mid x_{\lambda+1} \cdots x_{n+1} v_1 \cdots v_{\lambda-1})$$

and a permutation σ of $x_1, x_2, \ldots, x_{n+1}$ the element

$$\epsilon_\sigma(u_1 \cdots u_{n-\lambda} x_{\sigma(1)} \cdots x_{\sigma(\lambda)} \mid x_{\sigma(\lambda+1)} \cdots x_{\sigma(n+1)}\, v_1 \cdots v_{\lambda-1})$$

depends only on the class of σ modulo the subgroup $S_\lambda \times S_{n+1-\lambda}$ fixing the set $\{x_1, x_2, \ldots, x_\lambda\}$. Let us indicate

$$\dot{T} = \sum_\sigma \epsilon_\sigma(u_1 \cdots u_{n-\lambda} x_{\sigma(1)} \cdots x_{\sigma(1)} \mid x_{\sigma(\lambda+1)} \cdots x_{\sigma(n+1)} v_1 \cdots v_{\lambda-1})$$

σ running over a set of representatives in the lateral classes of $S_\lambda \times S_{n+1-\lambda}$ in S_{n+1}. $\dot{T}$ is the signed sum of T and other terms, in each of which at least one of $x_{\lambda+1}, \ldots, x_{n+1}$ is passed to the left part of the row.

LEMMA 5.2. $\dot{T} = 0$.

Proof. Since this is a formal identity we may assume $A = Q$, the rational numbers. We make induction on λ. For $\lambda = 1$ we must prove $(u_1 \cdots u_{n-1}\, x_1 \mid x_2 \cdots x_{n+1}) = \sum_{\rho=2}^{n+1} (u_1 \cdots u_{n-1} x_\rho \mid x_2 \cdots x_{\rho-1} x_1 x_{\rho+1} \cdots x_{n+1})$.
Expand the determinants with respect to the last row:

$$\begin{aligned}
&\sum_{\rho=2}^{n+1} (u_1 \cdots u_{n-1} x_\rho \mid x_2 \cdots x_{\rho-1} x_1 x_{\rho+1} \cdots x_{n+1}) \\
&= \sum_{\rho=2}^{n+1} \Bigg(\sum_{i=1}^{\rho-1} (-1)^{i+n-1} (x_\rho\,, x_i)(u_1 \cdots u_{n-1} \mid x_2 \cdots \check{x}_i \cdots x_{\rho-1} x_1 x_{\rho+1} \cdots x_{n+1}) \\
&\quad + (-1)^{\rho+n-1} (x_\rho\,, x_1)(u_1 \cdots u_{n-1} \mid x_2 \cdots \check{x}_\rho \cdots x_{n+1}) \\
&\quad + \sum_{j=\rho+1}^{n+1} (-1)^{j+n-1} (x_\rho\,, x_j)(u_1 \cdots u_{n-1} \mid x_2 \cdots x_{\rho-1} x_1 x_{\rho+1} \cdots \check{x}_j \cdots x_{n+1}) \Bigg) \\
&= \sum_{h<k} (-1)^{h+n-1} (x_k\,, x_h)(u_1 \cdots u_{n-1} \mid x_2 \cdots \check{x}_h \cdots x_{k-1} x_1 x_{k+1} \cdots x_{n+1}) \\
&\quad + \sum_{h<k} (-1)^{k+n-1} (x_h\,, x_k)(u_1 \cdots u_{n-1} \mid x_2 \cdots x_{h-1} x_1 x_{h+1} \cdots \check{x}_k \cdots x_{n+1}) \\
&\quad + (u_1 \cdots u_{n-1} x_1 \mid x_2 x_3 \cdots x_{n+1}).
\end{aligned}$$

Now $(x_h\,, x_k) = (x_k\,, x_h)$ and $(u_1 \cdots u_{n-1} \mid x_2 \cdots \check{x}_h \cdots\ x_{k-1} x_1 x_{k+1} \cdots x_{n+1}) = (-1)^{k-h+1} (u_1 \cdots u_{n-1} \mid x_2 \cdots x_{h-1} x_1 x_{h+1} \cdots \check{x}_k \cdots x_{n+1})$ and $(-1)^{k-h+1} \cdot (-1)^{h+n+1} + (-1)^{k+n-1} = 0$.
Thus the claim follows.

Now suppose the lemma proved for $\lambda - 1$. Since $\lambda!(n + 1 - \lambda)!\, \dot{T} = \sum_{\sigma \in S_{n+1}} \epsilon_\sigma (u_1 \cdots u_{n-\lambda} x_{\sigma(1)} \cdots x_{\sigma(\lambda)} \mid x_{\sigma(\lambda+1)} \cdots x_{\sigma(n+1)} v_1 \cdots v_{\lambda-1})$ it is sufficient to show that this second expression is zero.

Apply the result for $\lambda = 1$ and get

$$\begin{aligned}
&\sum_{\sigma \in S_{n+1}} \epsilon_\sigma (u_1 \cdots u_{n-\lambda} x_{\sigma(1)} \cdots x_{\sigma(\lambda)} \mid x_{\sigma(\lambda+1)} \cdots x_{\sigma(n+1)} v_1 \cdots v_{\lambda-1}) \\
&= \sum_{\sigma \in S_{n+1}} \epsilon_\sigma \Bigg(\sum_{\rho=\lambda+1}^{n+1} (u_1 \cdots u_{n-\lambda} x_{\sigma(1)} \cdots x_{\sigma(\lambda-1)} x_{\sigma(\rho)} \mid x_{\sigma(\lambda+1)} \cdots x_{\sigma(\rho-1)} x_{\sigma(\lambda)} \\
&\quad + x_{\sigma(\rho+1)} \cdots x_{\sigma(n+1)} v_1 \cdots v_{\lambda-1}) \\
&\quad + \sum_{\tau=1}^{\lambda-1} (u_1 \cdots u_{n-\lambda} x_{\sigma(1)} \cdots x_{\sigma(\lambda-1)} v_\tau \mid x_{\sigma(\lambda+1)} \cdots x_{\sigma(n+1)} v_1 \cdots v_{\tau-1} x_{\sigma(\lambda)} v_{\tau+1} \cdots v_{\lambda-1}) \Bigg)
\end{aligned}$$

$$= -(n+1-\lambda) \sum_{\sigma \in S_{n+1}} \epsilon_\sigma (u_1 \cdots u_{n-\lambda} x_{\sigma(1)} \cdots x_{\sigma(\lambda)} \mid x_{\sigma(\lambda+1)} \cdots x_{\sigma(n+1)} v_1 \cdots v_{\lambda-1})$$

$$+ \sum_{\tau=1}^{\lambda-1} (-1)^{n+\tau-1} \sum_{\sigma \in S_{n+1}} \epsilon_\sigma (u_1 \cdots u_{n-\lambda} v_\tau x_{\sigma(1)} \cdots x_{\sigma(\lambda-1)} \mid x_{\sigma(\lambda)} \cdots$$

$$x_{\sigma(n+1)} v_1 \cdots \check{v}_\tau \cdots v_{\lambda-1})$$

$$= -(n+1-\lambda) \sum_{\sigma \in S_{n+1}} \epsilon_\sigma (u_1 \cdots u_{n-\lambda} x_{\sigma(1)} \cdots x_{\sigma(\lambda)} \mid x_{\sigma(\lambda+1)} \cdots x_{\sigma(n+1)} v_1 \cdots v_{\lambda-1})$$

by induction on λ. This clearly yields the result.

Now we prove half of Theorem 5.1.

LEMMA 5.3. *The d-tableaux span R over A.*

Proof. We proceed as in [2] using their straightening algorithm plus our Lemma 5.2. A monomial $(u_{i_1}, u_{j_1})(u_{i_2}, u_{j_2}) \cdots (u_{i_k}, u_{j_k})$ can be displayed by the double tableau

$$T = \begin{pmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_k & j_k \end{pmatrix} \qquad \text{or} \qquad T' = \begin{bmatrix} i_1 \\ j_1 \\ i_2 \\ j_2 \\ \vdots \\ i_k \\ j_k \end{bmatrix}.$$

We have to straighten T'. Now, using the procedure of [2], one can straighten T as a double tableau. Looking at a single row of T,

$$C = (i_1 i_2 \cdots i_h \mid j_1 \cdots j_h) \qquad \text{displayed as } \begin{pmatrix} i_1 & \cdots & i_h \\ j_1 & \cdots & j_h \end{pmatrix}$$

one may have a violation of standardness in a position ; we apply Lemma 5.2 to C and see that C is a linear combination of tableaux of the same shape and with the first row higher lexicographically than $i_1 \cdots i_h$. This shows that $T = (A_i \mid B_i)$ is a linear combination of $T' = (A_i' \mid B_i')$ with A_i' higher than A_i. Applying the two straightening laws one always increases A_i as much as possible, then B_i and A_i simultaneously, then A_i again and so on. Clearly one must stop at some point, when all the tableaux are d-standard.

We now complete the theorem.

LEMMA 5.4. *The d-tableaux with rows of length $\leqslant n$ are linearly independent.*

Proof. We proceed as in Section 1. Writing $x_1 = x_1 + \sum_{i=2}^{t} \lambda_i x_i$ we eliminate x_1 unless $t \leqslant n$ and the first row is always $1\ 2 \cdots t$. Now the variables are exactly $1\ 2 \cdots t$ and therefore must all appear also on the second row. Therefore we can take out the factor $(1\ 2 \cdots t \mid 1\ 2 \cdots t)$ and conclude by induction.

We need to extend this theorem to the ring

$$R' = A[(u_i, u_j), [u_{i_1}, \ldots, u_{i_n}]].$$

We can easily see that R' also has a standard basis over A given by the elements

(1) d-tableaux;

(2) products $[u_{i_1}, \ldots, u_{i_n}]T$, T a tableau such that the unique tableau

$$\begin{pmatrix} u_{i_1} & \cdots & u_{i_n} \\ & T' & \end{pmatrix}$$

is standard.

The proof is similar to the previous ones and it will be omitted.

We will call these last tableaux odd d-tableaux and those previously considered, even d-tableaux.

We are now ready to prove the first fundamental theorem for $O(n, -)$. (Note: In what follows A is assumed to be a field of characteristic $\neq 2$. For the general case see the note added in proof.)

Let us consider the space of vectors $(u_1, u_2, \ldots, u_m)$ and assume, as usual, $m \geqslant n$. Let $d = \det |(u_i, u_j)|$, $i, j = 1, \ldots, n$. Let W be the open set where d is invertible, we still propose to show that, if R is the ring $A[(u_h, u_k)]$, S the polynomial ring in the entries of the vectors u_i's and $S^{O(n,-)}$ is the invariant ring; then:

LEMMA 5.5. $R[1/d] = S^{O(n,-)}[1/d]$.

Proof. We show that W is locally isomorphic to a product $G \times V$, not in a Zarisky sense, but in the faithfully flat topology (cf. comments after Corollary 2.4). We follow [1]. The space V is the space $U_1 \times U_2$, where U_1 is the space of invertible $n \times n$ symmetric matrices and U_2 is the space of vectors $u'_{n+1}, \ldots, u_m'$. In fact let us restrict our attention to the first n vectors $u_1, \ldots, u_n$; we are essentially studying the quotient of $GL(n, -)$ under the subgroup $O(n, -)$. The way to study this quotient is to act with $GL(n, -)$ on invertible symmetric matrices, via the action XAX^t, and notice that $O(n, -)$ is the stabilizer of I and $GL(n, -)$ acts transitively. Finally one shows that the principal fibration

$$O(n, —) \to GL(n, —) \to \mathrm{Sym}$$

has a local section in the faithfully flat topology. This finally proves that the invariant ring of $O(n, -)$, acting on $GL(n, -)$, is identified with the coordinate ring of Sym. The only assertion that needs a comment is the existence of a local section. One has to be able to extract the square root of a symmetric matrix, i.e., given $A \in \mathrm{Sym}$ find $B \in GL(n, -)$ with $B\,B^t = A$. This of course is accomplished by diagonalizing first the quadratic form (Ax, x) and then extracting the roots of the diagonal elements. For the diagonalization procedure one has the canonical construction of the basis $f_1 ,..., f_n , f_i = u_i - \sum_{j<i} ((Au_i , f_j)/(Af_j , f_j)) f_j$. This gives rise to a basis at least on an open Zarisky neighborhood, with coordinate ring $\bar{R}$, of the matrix I. Now let $\alpha_i = (Af_i , f_i)$; the extension

$$\bar{R}' = \bar{R}(\alpha_i^{1/2}) = \bar{R}[x_1 , x_2 ,..., x_n]/(x_i^2 - \alpha_i)$$

is faithfully flat and on Spec $\bar{R}'$ one has the canonical section.

To complete the proof of Lemma 5.5 one has to interpret the coordinate ring of V as invariant functions on W. If $u \in W$, $u = (u_1 , u_2 ,..., u_n , u_{n+1} ,..., u_m)$ projects to the point u' of V, $u' = (A, u'_{n+1} ,..., u_m')$, where $A = gg^t$, $u_j' = gu_j'$ and g is the invertible matrix with rows $u_1 , u_2 ,..., u_n$. Now the coordinates of A are just the scalar products (u_i , u_j). The coordinates of u_j' are (u_i , u_j), $i = 1,..., n$. The determinant of A, which is the final element to add inverted, is just d and the lemma is completely proved.

We come to the main theorem:

THEOREM 5.6. (i) *The ring* $S^{O(n,-)}$ *is* $R = A[(u_i , u_j)]$.

(ii) *the ring* $S^{SO(n,-)}$ *is* $R' = A[(u_i , u_j), [u_{i_1} ,..., u_{i_n}]$.

Proof. (ii) First of all replace in Lemma 5.5 $O(n, -)$ with $SO(n, -)$ and see that we still have a principal fibration. The quotient V' covers V with fiber $Z/(2)$, the element $\Delta = [u_1 ,..., u_n]$ is the extra invariant corresponding to the generator η of the coordinate ring $A[\eta]$, $\eta^2 = 1$, of the group $Z/(2)$. Thus, as in Lemma 5.5, $S^{SO(n,-)}$ is contained in $R'[1/\Delta]$. Now let $f \in S^{SO(n,-)}$ and assume $\Delta f \in R'$; we have to show $f \in R'$, and this will prove (ii). Now since $\Delta f = \sum c_i T_i$, T_i are tableaux (of the type explained after Lemma 5.4.). $\sum c_i T_i$ vanishes when $u_1 ,..., u_n$ are dependent and it is sufficient to show that the first row of each T has the form $1, 2, \ldots, n$. This is essentially the usual argument. Substitute $u_1 + \sum_{i=2}^{n} \lambda_i u_i$ for u_1 and see that either each T_i has the required form, or there is a $\bar{p} = \sum \bar{c}_i \bar{T}_i$, not containing u_1, and vanishing when $u_1, \ldots, u_n$ are dependent; this contradicts Lemma 5.4 and so (ii) is proved.

(i) Now let $\sum c_i T_i \in S^{O(n,-)}$. If we apply the element

$$\begin{pmatrix} \alpha & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{pmatrix},$$

with $\alpha^2 = 1$ (formally), to $\sum c_i T_i$ we have on the one hand $\sum c_i T_i$, by invariance, and on the other hand $\sum_{i=1}^{m_1} c_i T_i + \alpha \sum_{i=1}^{m_2} c_j T_j$, where the T_j are odd d-tableaux of type $[u_{i_1}, \ldots, u_{i_n}]\, T_j' \cdot T_i$, T_j' are even d-tableaux. By the linear independence of d-tableaux over any commutative ring, in particular over $A[\alpha]$, we have $c_j = 0, j = 1, \ldots, m_2$, and the theorem is completed.

As for the second fundamental theorem we have

THEOREM 5.7. *The ideal of relations among the* (u_i, u_j') *is generated by the* $n + 1 \times n + 1$ *minors of the symmetric matrix* $|(u_i, u_j)|$.

Proof. Trivial by Theorem 5.1.

6. THE SYMPLECTIC GROUP

This case is fairly similar in spirit to the case of the orthogonal group, and hence we shall give only the main variations of the argument. We consider k vectors $x_1, \ldots, x_k$ from a $2n$-dimensional vector space with a symplectic form. Consider the skew products $\langle x_i, x_j \rangle$ which we shall display in a skew symmetric matrix

$$Z = |\langle x_i, x_j \rangle|, \qquad i, j = 1, \ldots, k.$$

If $i_1, i_2, \ldots, i_{2h}$ are $2h$ indices out of $1, \ldots, k$ we shall indicate by $[i_1, \ldots, i_{2h}]$ the Pfaffian on the skew matrix obtained from Z taking the rows and columns of indices $i_1, \ldots, i_{2h}$.

We notice that, with these notations, $[i, j] = \langle x_i, x_j \rangle$. Our basic combinatorial lemmas are the following.

LEMMA 6.1. $[a_1, \ldots, a_n][b_1, \ldots, b_m] - \sum_{h=1}^{n} [a_1, \ldots, a_{h-1}, b_1, a_{h+1}, \ldots, a_n]$ $[a_h, b_2, \ldots, b_m] = \sum_{k=2}^{m} (-1)^{k-1}[b_2, \ldots, \check{b}_k, \ldots, b_m][b_k, b_1, a_1, \ldots, a_n]$.

Proof. By standard properties of Pfaffians:

$$[a_1,\dots,a_n][b_1,\dots,b_m] - \sum_{h=1}^{n} [a_1,\dots,a_{h-1}, b_1, a_{h+1},\dots,a_n][a_h, b_2,\dots,b_m]$$

$$= [a_1,\dots,a_n]\left(\sum_{k=2}^{m} (-1)^k [b_1, b_k][b_2,\dots,\check{b}_k,\dots,b_m]\right)$$

$$- \sum_{h=1}^{n} \left([a_1,\dots,a_{h-1}, b_1, a_{h+1},\dots,a_n] \sum_{k=2}^{m} (-1)^k [a_h, b_k][b_2,\dots,\check{b}_k,\dots,b_m]\right)$$

$$= \sum_{k=2}^{m} (-1)^k [b_2,\dots,\check{b}_k,\dots,b_m](-[b_k, b_i][a_1,\dots,a_n] + (-1)^{h-1}[b_k, a_h]$$

$$[b_1, a_1,\dots,a_{h-1}, a_{h+1},\dots,a_n]) = \sum_{k=2}^{m} (-1)^{k-1}[b_2,\dots,\check{b}_k,\dots,b_m]$$

$$\times [b_k, b_1, a_1,\dots,a_n].$$

LEMMA 6.2. $\sum_{\sigma\in S_{\lambda+i+1}} \epsilon_\sigma [a_1,\dots,a_i, x_{\sigma(1)},\dots,x_{\sigma(\lambda)}][x_{\sigma(\lambda+1)},\dots,x_{\sigma(\lambda+i+1)}, v_1,\dots,v_t] = R$, *where R is a linear combination, with rational coefficients, of products $[i_1,\dots,i_n][j_1,\dots,j_r]$, where $s > i+\lambda$ and $s + r = (i+\lambda) + (\lambda + i + 1 + t)$.*

Proof. By induction on i. If $i = 0$ this follows from Lemma 6.1. Let us suppose it true for $i - 1$. By Lemma 6.1, we have

$$\sum_{\sigma\in S_{\lambda+i+1}} \epsilon_\sigma [a_1,\dots,a_i, x_{\sigma(1)},\dots,x_{\sigma(\lambda)}]$$

$$\times [x_{\sigma(\lambda+1)},\dots,x_{\sigma(\lambda+i+1)}, v_1,\dots,v_t]$$

$$= \sum_{\sigma\in S_{\lambda+i+1}} \epsilon_\sigma \left(\sum_{j=1}^{i} [a_1,\dots,a_{j-1}, x_{\sigma(\lambda+1)}, a_{j+1},\dots,a_i, x_{\sigma(1)},\dots,x_{\sigma(\lambda)}]\right.$$

$$\times [a_j, x_{\sigma(\lambda+2)},\dots,x_{\sigma(\lambda+i+1)}, v_1,\dots,v_t]$$

$$+ \sum_{\tau=1}^{\lambda} [a_1,\dots,a_i, x_{\sigma(1)},\dots,x_{\sigma(\tau-1)}, x_{\sigma(\lambda+1)}, x_{\sigma(\tau+1)},\dots,x_{\sigma(\lambda)}]$$

$$\left.\times [x_{\sigma(\tau)}, x_{\sigma(\lambda+2)},\dots,x_{\sigma(\lambda+i+1)}, v_1,\dots,v_t]\right) + R'$$

$$= R'' - \lambda\left(\sum_{\sigma\in S_{\lambda+i+2}} \epsilon_\sigma [a_1,\dots,a_i, x_{\sigma(1)},\dots,x_{\sigma(\lambda)}]\right.$$

$$\left.\times [x_{\sigma(\lambda+1)},\dots,x_{\sigma(\lambda+i+1)}, v_1,\dots,v_t]\right) + R'.$$

(R' is given by induction.)

Hence $(1 + \lambda)(\sum_{\sigma \in S_{\lambda+i+1}} \epsilon_\sigma[a_1, \ldots, a_i, x_{\sigma(1)}, \ldots, x_{\sigma(\lambda)}][x_{\sigma(\lambda+1)}, \ldots, x_{\sigma(\lambda+i+1)}, v_1, \ldots, v_t] = R' + R''$ and the lemma follows. Let us now consider the ring $S_A = A[\langle x_i, x_j \rangle]$, A a commutative ring. To a given standard tableau

$$T = \begin{pmatrix} i_1 & i_2 & \cdots & i_{h_1} \\ j_1 & j_2 & \cdots & j_{h_2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

with an even number of elements on each row and $\leqslant 2n$, we can associate an element still denoted by T of S_A, $T = [i_1, \ldots, i_{h_1}][j_1, \ldots, j_{h_2}] \ldots$; then, as in [2], one can use Lemma 6.2 to prove

LEMMA 6.3. *The ring S_Q is spanned over Q by the standard tableaux.*

We come now to the linear independence.

LEMMA 6.4. *The standard tableaux in S_A (A any commutative ring) are linearly independent.*

Proof. One proceeds as in paragraph 1, using the linearity of a tableau in each row and reduces by induction the size of the tableaux canceling the Pfaffian $[1, 2, \ldots, 2n]$, which is generically invertible.

We can now prove our main theorem.

THEOREM 6.5. *For any commutative ring A the standard tableaux are a basis of S_A.*

Proof. By Lemma 6.4 it is sufficient to show that the double standard tableaux span S_A. Clearly one is reduced to the case $A = Z$. Now, given $q \in S_Z$ we know, by Lemma 6.3, that

$$q = \sum_i c_i T_i, \qquad c_i \in Q.$$

Now let r be the least integer such that $rc_i \in Z$, each i. We have $rq = \sum_i rc_i T_i$ and, if p/r is a prime, computing our expression mod p(as functions) we have, by the linear independence of the T_i's, that p/rc, $\forall i$, hence a contradiction to minimality of r unless $r = 1$.

Finally the theorem on invariants:

THEOREM 6.6. *S_A is the ring of invariants of k $2n$-vectors, $x_1, \ldots, x_k$, under the symplectic group.*

Proof. It is similar to the case of the general linear group. One can assume $k \geqslant 2n$, and consider the open set where $[1, 2, \ldots, 2n]$ is invertible; here the variety with its group action is a product and one uses the standard monomial theory to make the final cancellation.

As in Section 5 we have

THEOREM 6.7. *The ideal of relations among the $\langle x_i, x_j \rangle$ is generated by the Pfaffians of the principal minors of size $2n+2$ of the skew symmetric matrix $|\langle x_i, x_j \rangle|$.*

Proof. Trivial by Theorem 6.5.

7. THE BRAUER–WEYL ALGEBRA

As in Section 4 for the case of the general linear group, one can deduce the structure of the endomorphism ring in $V^{\otimes n}$ with respect to the other classical groups. We shall limit ourselves to the case of the orthogonal group since the symplectic case is similar. So, let $G = O(m, k)$, k being as in paragraph 4. The ring $\mathrm{End}_G(V^{\otimes n})$ is identified to $(V^{\otimes n} \otimes V^{\otimes n})^{*G} \simeq (V^{\otimes 2n})^{*G}$ which is spanned by the contractions $v_1 \otimes \cdots \otimes v_n \otimes u_1 \otimes \cdots \otimes u_n \to \prod (v_i, v_j)(v_h, u_k)(u_s, u_t)$. Now one easily verifies that, if each v_i is matched with a u_j, one has a permutation $\in S_n \subseteq \mathrm{End}_G(V^{\otimes n})$. The other generators for this algebra can be obtained as follows. Let i, j be two indices in $1, 2, \ldots, n$, say 1, 2 for simplicity. Let $\tau_{12} : V^{\otimes n} \to V^{\otimes n}$ be the map

$$\tau_{12} : v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_n \to (v_1, v_2)\, I \otimes v_3 \otimes \cdots \otimes v_n,$$

$I \in V \otimes V$ being the element corresponding to the identity mapping $1_V \in \mathrm{End}(V) \simeq V \otimes V$. Then τ_{12} corresponds to the invariant $(v_1, v_2)(u_1, u_2)(v_3, u_3) \cdots (v_n, u_n)$, and it is a simple matter to show that the σ's and the τ_{ij} generate $\mathrm{End}_G(V^{\otimes n})$ as an algebra over the base ring. In fact as in Section 4, if $\gamma \in \mathrm{End}_G(V^{\otimes n})$ corresponds to a product of scalar products and if μ is a permutation, $\gamma\mu^{-1}$ and $\mu\gamma$ correspond to the same product where the u_i's (resp. the v_j's) are permuted according to μ. Thus $\mathrm{End}_G(V^{\otimes n})$ is spanned by the elements $\sigma_1 \tau_{12} \tau_{34} \cdots \tau_{2k-1,2k} \sigma_2$ one can also write down a multiplication table for such elements that shows that

$$\begin{aligned}
\tau_{ij} &= \tau_{ji}, \\
\sigma\tau_{ij}\sigma^{-1} &= \tau_{\sigma(i)\sigma(j)}, \\
\tau_{ij}\tau_{hk} &= \tau_{hk}\tau_{ij} && \text{if } (i, j) \cap (h, k) = \varnothing; \\
\tau_{ij}\tau_{jk} &= \tau_{ij}(ik) && i \neq k; \\
\tau_{ij}^2 &= m\tau_{ij} && (m = \dim V).
\end{aligned}$$

In characteristic 0 one can base on this algebra the decomposition theory of $V^{\otimes n}$ under $O(V)$.

Note added in proof. We prove here Theorem 5.6 with A any commutative ring. First of all, we need to have Lemma 5.5 when A is a field of char 2.

We go back to the notations of Lemma 5.5. Let V be the variety $U_1 \times U_2$; U_1 being the space of nondegenerate $n \times n$ symmetric matrices, U_2 the space of m–n tuples of vectors $(u_{n+1}', u_{n+2}', \ldots, u_m')$.

Consider the map $\pi\colon W \to V$ given by $\pi\colon (u_1, \ldots, u_m) \to (gg^t; gu_{n+1}, gu_{n+2}, \ldots, gu_m)$; where g is the $n \times n$ matrix with rows $u_1, \ldots, u_n$. The image of π is the set $\bar{V} = \bar{U}_1 \times U_2$, $\bar{U}_1$ being the set of diagonalizable nondegenerate bilinear symmetric forms. If char $A \neq 2$ we have $U_1 = \bar{U}_1$, in characteristic 2 we will show that the complement of $\bar{U}_1 \times U_2$ in V is closed of codimension $\geqslant 2$. This is clearly sufficient for the proof of Lemma 5.5.

We work by induction on n. The group $Gl(n, -)$ acts on U_1 by $CAC^t (C \in Gl(n, K)$, $A \in U_1(K)$, K a field).

$\bar{U}_1$ is the orbit of 1. Clearly $\bar{U}_1$ contains an open set and so, being an orbit, it is open. Let us write $A = (a_{ij})$ for a symmetric matrix and consider the open sets U^i, $i = 1, \ldots, n$ defined by $a_{ii} \neq 0$. It will be enough to show that the complement of U_1 has codimension $\geqslant 2$ in each U^i. Let $i = 1$ for implicity.

The first step for putting the form (Ax, y) in diagonal form is the change of basis

$$e_1' = e_1, \quad e_i' = e_1 - (a_{i1}/a_{11})e_1, \qquad i > 1.$$

($e_1, \ldots, e_n$ the canonical basis).

By this base change the matrix of the form (Ax, y) becomes

$$\begin{pmatrix} a_{ii} & 0 \\ 0 & \bar{A} \end{pmatrix}.$$

The map $j\colon A \to \bar{A}$ is a morphism from U_1' to the nondegenerate symmetric $n - 1 \times n - 1$ matrices. The fibers of j have constant dimension $n - 1$ and clearly the form associated to A can be put in diagonal form if this is true for $\bar{A}$.

Thus the inductive hypothesis concludes the proof that the complement of $\bar{U}_1 \times U_2$ in V is closed of codimension $\geqslant 2$.

The argument given in Theorem 5.6 works therefore also for A, a field of characteristic 2. We complete now the proof of Theorem 5.6 for A an arbitrary commutative ring.

If μ is the map classifying the given action, the formal invariants are the elements a with $\mu(a) = a \otimes 1$ (cf. Section 2).

Thus, if Theorem 5.6 is proved for a base ring B it is also proved for any base ring A flat over B, in particular, if A is an algebra over a field. Now let A be arbitrary and f a formal invariant. Consider $\bar{A} = A \otimes_Z Q$, $f \otimes 1 \in \bar{A}$ is a formal invariant for which Theorem 5.6 holds, thus $f \otimes 1 = \sum c_i T_i$, $c_i \in \bar{A}$, T_i's d-tableaux.

Let $A' = A \otimes 1 \subset \bar{A}$ and m the minimum integer with $mc_i \in A \otimes 1 = A'$ for all c_i.

We claim $m = 1$, otherwise assume $m = pq$ and p a prime. We reduce mod p and in $A'/pA'[u_{ij}]$ we have $\sum \overline{mc_i} T_i = 0$. Since the T_i's are linearly independent over A'/pA', we must have $mc_i \in pA'$ for all i, thus $mc_i = pa_i$ and $p(qc_i - a_i) = 0$ in A'. Since A' is torsion free $qc_i = a_i \in A'$ and we have a contradiction to the assumption that $m = pq$.

We can conclude thus, that there is a linear combination g of tableaux with coefficients in A such that $f' = f - g$ has coefficients in the kernel I of the map $A \to A \otimes_Z Q$.

Replacing f with f' we may assume that f is a torsion element. By decomposing the torsion of A into its primary parts we may assume furthermore that f has p torsion for some prime p, i.e., $p^k f = 0$. We work by induction on k, if $k = 1$ we are in an algebra over $Z/(p)$ and Theorem 5.6 holds, otherwise let $J = \{a \in A \mid p^{k-1} a = 0\}$.

In A/J the image of f is killed by p, hence it is a linear combination of double tableaux, thus we can again modify f to make it killed by p^{k-1} and then finish by induction.

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