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Two-Point Boundary Value Problems on In-homogeneous Time-Scale Linear Dynamic Processes

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1. INTRODUCTION

It is a well-known fact that the theory of differential equations provides a broad mathematical basis for an understanding of continuous time dynamic processes. But the theory of difference equations is of very little help because of lack of qualitative features. There are many results on continuous time dynamical systems which are needed in a discrete time context. In most of the situations those results, not the proofs, are merely translated into the discrete time language and then they are applied without any care. This is very dangerous and in some prominent cases things may go wrong. One should be careful in dealing with such problems because continuous time orbits and discrete time orbits are topologicallly different. For a general treatment and for the notation, we refer to B. Aulbach and S. Hilger [2]. In this paper, by a time-scale we mean any closed subset of R, where R is the real line. We also mean that any closed subset of R is a time-scale and call our time-scales in-homogeneous. As pointed out by B. Aulbach and S. Hilger, it provides a new direction of research in dynamical processes with in-homogeneous time-scale.

This paper is organised as follows. In Section 2, we briefly describe some salient features of time-scales, i.e., closed subsets of R, functions defined on time-scales, operations with those functions, and the qualitative theory of linear dynamic processes. Section 3 presents the variation of parameters formula and a few notions on boundary value problems. Existence and unicity of solutions to two-point boundary value problems are discussed in Section 4. The properties of the Green's matrix are also discussed in this section. Section 5 generalises the results obtained in Section 4 to multi-point boundary value problems.

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2. Preliminaries

We denote the time-scales by the symbol T. As subsets of R time-scales carry an ordered structure and topological structure in a canonical way. A time-scale T may be either bounded below or above. All order theoretical notions such as bounds, supremum and infimum, and intervals are available in T as in the case of R. If A is a bounded subset of T then A has both infimum and supremum belonging to T not necessarily to A. By an interval we mean the intersection of the real interval with a given time-scale. If A is any subset of T which is open in R it is also open in T. The converse is not in general true. The jump operators introduced on a time-scale T may be connected or disconnected. To overcome this topological difficulty the concept of jump operators is introduced in the following way. The operators σ and ρ from T to T, defined by [1]

$$\sigma: \begin{cases} T \to T \\ t \to \operatorname{Inf}\{s \in T: s > t\} \end{cases}$$

$$\rho: \begin{cases} T \to T \\ t - \operatorname{Sup}\{s \in T: s < t\} \end{cases}$$

are called jump operators. If σ is bounded above and ρ is bounded below then we define

$$\sigma(\max T) = \max T$$

and

$$\rho(\min T) = \min T.$$

These operators allow us to classify the point of time-scale T. A point $t \in T$ is said to be right-dense if $\sigma(t) = t$ and left-dense if $\rho(t) = t$, right-scattered if $\sigma(t) > t$ and left-scattered if $\rho(t) < t$.

Definition 2.1. Let T be a time-scale, X a real Banach space, and $f: T \to X$. We say that f is differentiable at a point $t_0 \in T$, if there exists an $a \in X$ such that for any $\varepsilon > 0$ there exists a neighbourhood U of t_0 such that,

$$|f(\sigma(t_0)) - f(t) - (\sigma(t_0) - t)a| \leqslant \varepsilon\, |\sigma(t_0) - t) \qquad \forall t \in U$$

or more specifically, f is differentiable at t_0 if

$$\lim_{t \to \sigma(t_0)} \frac{f(t) - f(\sigma(t_0))}{t - \sigma(t_0)} = \frac{df}{dt}(t_0) = f^{\Delta}(t_0)$$

provided the limit exists.

If f is differentiable for every $t \in T$ we say that $f: T \to X$ is differentiable on T.

RESULT 2.1. *Let $f: T \to X$ and $t_0 \in T$. Then if f is differentiable at t_0 then f is continuous at t_0.*

Proof. By hypothesis,

$$f^{\Delta}(t_0) = \lim_{t \to \sigma(t_0)} \frac{f(t) - f(\sigma(t_0))}{t - \sigma(t_0)};$$

consider,

$$f(t) - f(\sigma(t_0)) = \frac{f(t) - f(\sigma(t_0))}{t - \sigma(t_0)} (t - \sigma(t_0))$$

$$\lim_{t \to \sigma(t_0)} [f(t) - f(\sigma(t_0))] = f^{\Delta}(t_0) 0$$

$$= 0;$$

$$\lim_{t \to \sigma(t_0)} f(t) = f(\sigma(t_0)).$$

If T has a left-scattered maximum T^* then at T^* any function $f: T \to X$ is differentiable and the definition of differentiability can be satisfied with any $a \in X$.

THEOREM 2.1. *If f is continuous at t_0 and t_0 is right-scattered then f is differentiable at t_0 with derivative*

$$f^{\Delta}(t_0) = \frac{f(\sigma(t_0)) - f(t_0)}{\mu^*(t_0)},$$

where $\mu^*: T^k \to R_0^+$ defined by

$$\mu^*(t) = \mu(\sigma(t), t)$$

is called grainyness [2].

Proof. By hypothesis, we have

$$\lim_{t \to \sigma(t_0)} f(t) = f(\sigma(t_0))$$

and t_0 is right-scattered implies $\sigma(t_0) - t_0 > 0$. Now consider,

$$\lim_{\sigma(t_0) \to t_0} \frac{f(\sigma(t_0)) - f(t_0)}{\sigma(t_0) - t_0} = f^{\Delta}(\sigma(t_0)).$$

If f and g are two differentiable functions at t_0 then fg is differentiable at t_0 and $(fg)^{\Delta}(t_0) = f(\sigma(t_0))\, g^{\Delta}(t_0) + f^{\Delta}(t_0)\, g(t_0)$.

A function $g: T^k \to X$ is rd-continuous if it is continuous in every right-dense point $t \in T^k$ and if $\lim_{s \to t^-} g(s)$ exists for each left-dense $t \in T^k$.

We say that a mapping $A: T^k \to B(X)$ is regressive if for each $t \in T^k$ the mapping $A(t)\, \mu^*(t) + \mathrm{id}: X \to X$ is invertible, where $B(X)$ is a real Banach space.

3. Variation of Parameters Formula

Definition 3.1 Let $A: T^k \to B(X)$ be regressive and rd-continuous and let $b: T^k \to X$ be rd-continuous. Then a mapping $\phi: T \to X$ is said to be a solution of the linear dynamic equation

$$x^{\Delta} = A(t)x + b(t) \tag{3.1}$$

if

$$\phi^{\Delta}(t) = A(t)\, \phi(t) + b(t) \qquad \text{for all} \quad t \in T^k.$$

The associated homogeneous equation for (3.1) is

$$x^{\Delta} = A(t)x. \tag{3.2}$$

If a solution of $x(t)$ of (3.1) satisfies the condition $x(\tau) = x_0$ for a pair of $(\tau, x_0) \in T \times X$, it is called a solution of the initial value problem:

$$\begin{aligned} x^{\Delta} &= A(t)x + b(t) \\ x(\tau) &= x_0. \end{aligned} \tag{3.3}$$

Theorem 3.1. *If $A: T^k \to B(X)$ is regressive and rd-continuous and $b: T^k \to X$ is rd-continuous, then a particular solution $\bar{x}(t)$ of*

$$x^{\Delta} = A(t)x + b(t)$$

is of the form

$$\int_{\tau}^{t} \phi_A(t, \sigma(s))\, b(s)\, \Delta s,$$

where $\phi_A(\cdot, \tau)$ denotes the solution of (3.2).

Theorem 3.2. *If $\bar{x}(t)$ is any particular solution of the in-homogeneous problem and $\phi_A(\cdot, \tau)$ denotes the solution of*

$$x^{\Delta} = A(t)x$$

satisfying $x(\tau) = I$, *then any solution* $x(t)$ *of* (3.3) *has the form*

$$x(t) = \phi_A(t, \tau)x_0 + \int_\tau^t \phi_A(t, \sigma(s))\, b(s)\, \Delta s.$$

Proof. We seek a particular solution of

$$x^\Delta = A(t)x + b(t)$$

in the form,

$$\bar{x}(t) = \phi_A(t)\, K(t).$$

Now substituting the value of $\bar{x}(t)$ in (3.1) we get

$$[\phi_A(t)\, K(t)]^\Delta = A(t)\, \phi_A(t)\, K(t) + b(t)$$
$$\phi_A(\sigma(t))\, K^\Delta(t) + \phi_A^\Delta(t)\, K(t) = A(t)\, \phi_A(t)\, K(t) + b(t)$$
$$\phi_A(\sigma(t))\, K^\Delta(t) + A(t)\, \phi_A(t)\, K(t) = A(t)\, \phi_A(t)\, K(t) + b(t)$$
$$\phi_A(\sigma(t))\, K^\Delta(t) = b(t)$$
$$K^\Delta(t) = \phi_A^{-1}(\sigma(t))\, b(t)$$
$$K(t) = \int_\tau^t \phi_A^{-1}\,(\sigma(s))\, b(s)\, \Delta s.$$

The particular solution is

$$\bar{x}(t) = \phi_A(t)\, K(t)$$
$$= \int_\tau^t \phi_A(t)\, \phi_A^{-1}\,(\sigma(s))\, b(s)\, \Delta s$$
$$= \int_\tau^t \phi_A(t, \sigma(s))\, b(s)\, \Delta s.$$

Thus any solution of (3.3) is

$$x(t) = \phi_A(t)C + \int_\tau^t \phi_A(t, \sigma(s))\, b(s)\, \Delta s,$$

where C is a constant n-vector and will be determined from the initial conditions. Thus,

$$x(\tau) = \phi_A(\tau)C$$
$$C = \phi_A^{-1}(\tau)\, x(\tau) = \phi_A^{-1}(\tau)x_0$$
$$x(t) = \phi_A(t)\, \phi_A^{-1}(\tau)x_0 + \int_\tau^t \phi_A(t, \sigma(s))\, b(s)\, \Delta s$$
$$= \phi_A(t, \tau)x_0 + \int_\tau^t \phi_A(t, \sigma(s))\, b(s)\, \Delta s.$$

DEFINITION 3.2. If $\phi_A(t)$ is a fundamental matrix of (3.2) then the matrix D defined by

$$D = M\phi_A(a) + N\phi_A(b)$$

is called the characteristic matrix for the boundary value problem (3.1) satisfying the boundary conditions $Mx(a) + Nx(b) = 0$.

DEFINITION 3.3 The number of linearly independent solutions of the boundary value problem is called the index of compatibility of the boundary value problem. If the index of compatibility of the boundary value problem is zero then we say that the boundary value problem is incompatible.

THEOREM 3.3. *If the boundary value problem*

$$x^{\Delta} = A(t)x + b(t)$$
$$Mx(a) + Nx(b) = 0$$

has a characteristic matrix of rank r *then its index of compatibility is* $(n - r)$.

Proof. Let $\phi_A(t)$ be a fundamental matrix of

$$x^{\Delta} = A(t)x.$$

Then any solution is given by $\phi_A(t)C$, where C is a constant n-vector and will be determined uniquely from the boundary conditions. Substituting the general solution of

$$x^{\Delta} = A(t)x$$

in the boundary condition matrix we get

$$[M\phi_A(a) + N\phi_A(b)]\, C = 0$$
$$DC = 0$$

since the rank of D is r, and DC has $(n - r)$ linearly independent solutions. Hence, the index of compatibility of the boundary value problem is $(n - r)$.

4. MAIN RESULT

We establish our main result, namely, the existence and unicity of solutions to two-point boundary value problems on an in-homogeneous time-scale of a system of the linear dynamic process by utilising the variation of parameters formula established in Section 2. The solution is established as an integral representation of the Green's matrix and the properties of the Green's matrix are also studied. Our main result is stated in the following theorem.

THEOREM 4.1. *Let $A: T^k \to B(X)$ be rd-continuous and regressive and let $b: T^k \to X$ be continuous. Then there exists a unique solution of the linear dynamic equation,*

$$x^{\Delta} = A(t)x + b(t)$$

satisfying the general boundary conditions,

$$Mx(a) + Nx(b) = \alpha,$$

where M and N are constant square matrices of order n, and $x(t)$ is given by

$$x(t) = \phi_A(t)\, D^{-1}\alpha + \int_a^b G(t, \sigma(s))\, b(s)\, \Delta s,$$

where $G(t, \sigma(s))$ is the Green's matrix for the homogeneous linear dynamic boundary value problem.

Proof. Any solution of the linear dynamic equation

$$x^{\Delta} = A(t)x + b(t)$$

is given by

$$x(t) = \phi_A(t)C + \int_a^t \phi_A(t, \sigma(s))\, b(s)\, \Delta s.$$

Substituting the general form of $x(t)$ in the boundary condition matrix we get,

$$[M\phi_A(a) + N\phi_A(b)]\,C + N\int_a^b \phi_A(t, \sigma(s))\, b(s)\, \Delta s = \alpha$$

$$DC = \alpha - N\int_a^b \phi_A(t, \sigma(s))\, b(s)\, \Delta s$$

$$C = D^{-1}\alpha - D^{-1}N\int_a^b \phi_A(t, \sigma(s))\, b(s)\, \Delta s.$$

Hence

$$
\begin{aligned}
x(t) &= \phi_A(t)\, D^{-1}\alpha - \phi_A(t)\, D^{-1}N \int_a^b \phi_A(t, \sigma(s))\, b(s)\, \Delta s \\
&\quad + \int_a^t \phi_A(t, \sigma(s))\, b(s)\, \Delta s \\
&= \phi_A(t)\, D^{-1}\alpha - \phi_A(t)\, D^{-1}N\phi_A(b) \int_a^b \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&\quad + \phi_A(t) \int_a^t \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&= \phi_A(t)\, D^{-1}\alpha - \phi_A(t)\, D^{-1}N\phi_A(b) \left[\int_a^t \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \right. \\
&\quad \left. + \int_t^b \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \right] + \phi_A(t) \int_a^t \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&= \phi_A(t)\, D^{-1}\alpha + \phi_A(t)[I - D^{-1}N\phi_A(b)] \int_a^t \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&\quad - \phi_A(t)\, D^{-1}N\phi_A(b) \int_t^b \phi_A^{-1}(\sigma(s))\, \Delta s \\
&= \phi_A(t)\, D^{-1}\alpha + \phi_A(t)[D^{-1}D - D^{-1}N\phi_A(b)] \int_a^t \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&\quad - \phi_A(t)\, D^{-1}N\phi_A(b) \int_t^b \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&= \phi_A(t)\, D^{-1}\alpha + \phi_A(t)\, D^{-1}[D - N\phi_A(b)] \int_a^t \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&\quad - \phi_A(t)\, D^{-1}N\phi_A(b) \int_t^b \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&= \phi_A(t)\, D^{-1}\alpha + \phi_A(t)\, D^{-1}M\phi_A(a) \int_a^t \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&\quad - \phi_A(t)\, D^{-1}N\phi_A(b) \int_t^b \phi_A^{-1}(\sigma(s))\, b(s)\, \Delta s \\
&= \phi_A(t)\, D^{-1}\alpha + \int_a^b G(t, \sigma(s))\, b(s)\, \Delta s,
\end{aligned}
$$

where $G(t, \sigma(s))$ is the Green's matrix given by

$$
G(t, (s)) = \begin{cases} \phi_A(t)\, D^{-1}M\phi_A(a)\, \phi_A^{-1}(\sigma(s)), & a \leqslant s < t \leqslant b \\ -\phi_A(t)\, D^{-1}N\phi_A(b)\, \phi_A^{-1}((s)), & a \leqslant t < s \leqslant b \end{cases}
$$

which can be written conveniently as

$$G(t, \sigma(s)) = \begin{cases} \phi_A(t)\, D^{-1} M \phi_A(a, \sigma(s)), & a \leqslant s < t \leqslant b \\ -\phi_A(t)\, D^{-1} N \phi_A(b, \sigma(s)), & a \leqslant t < s \leqslant b. \end{cases} \tag{4.1}$$

THEOREM 4.2. *The green's matrix G has the following properties*:

(i) *The components of* $G(t, \sigma(s))$ *when regarded as a function of t with fixed s have continuous first derivatives on* $[a, s)$ *and* $(s, b]$. *At the point* $t = s$, *G has an upward jump discontinuity*, i.e.,

$$G(s^+, \sigma(s)) - G(s^-, \sigma(s)) = \phi_A(s, \sigma(s)) = \phi_A(s)\, \phi_A^{-1}(\sigma(s)).$$

If $\phi_A^{-1}(\sigma(s))$ *is rd-continuous then G has a finite jump discontinuity of unit magnitude* [4].

(ii) *G is a formal solution of the homogeneous boundary value problem and it fails to be a true solution because of the discontinuity at* $t = s$.

(iii) *G is unique with properties* (i) *and* (ii).

Proof. The Green's matrix G defined in the above theorem may be conveniently written as

$$G(t, \sigma(s)) = \begin{cases} \phi_A(t) H_+, & s < t \\ \phi_A(t) H_-, & s > t, \end{cases} \tag{4.2}$$

where H_+ and H_- are free from t and are given by

$$H_+ = D^{-1} M \phi_A(a, \sigma(s))$$

and

$$H_- = -D^{-1} N \phi_A(b, \sigma(s)).$$

Consider,

$$\begin{aligned} H_+ - H_- &= D^{-1} M \phi_A(a, \sigma(s)) + D^{-1} N \phi_A(b, \sigma(s)) \\ &= D^{-1} M \phi_A(a)\, \phi_A^{-}(\sigma(s)) \\ &\quad + D^{-1} N \phi_A(b)\, \phi_A^{-1}(\sigma(s)) \\ &= D^{-1} [M \phi_A(a) + N \phi_A(b)]\, \phi_A^{-1}(\sigma(s)) \\ &= D^{-1} D \phi_A^{-1}(\sigma(s)) \\ &= \phi_A^{-1}(\sigma(s)) \end{aligned}$$

$$\begin{aligned} \therefore G(s^+, \sigma(s)) - G(s^-, \sigma(s)) &= \phi_A(s)\, H_+ - \phi_A(s) H_- \\ &= \phi_A(s)[H_+ - H_-] \\ &= \phi_A(s)\, \phi_A^{-1}(\sigma(s)). \end{aligned}$$

If $\phi_A^{-1}(\sigma(s))$ is rd-continuous then

$$\phi_A^{-1}(\sigma(s)) = \phi_A^{-1}(s)$$

in which case

$$\begin{aligned} G(s^+, \sigma(s)) - G(s^-, \sigma(s)) &= \phi_A(s)\, \phi_A^{-1}(s) \\ &= I. \end{aligned}$$

(ii) The representation of $G(t, \sigma(s))$ by (4.2) shows that $G(t, \sigma(s))$ is a formal solution of (3.2); it fails to be a true solution because of discontinuity at $t = s$. $G(t, \sigma(s))$ satisfies homogeneous boundary conditions for,

$$\begin{aligned} &MG(a, \sigma(s)) + NG(b, \sigma(s)) \\ &\quad = -M\phi_A(a)\, D^{-1}N\phi_A(b, \sigma(s)) + N\phi_A(b)\, D^{-1}M\phi_A(a, \sigma(s)) \\ &\quad = -M\phi_A(a)\, D^{-1}N\phi_A(b)\, \phi_A^{-1}(\sigma(s)) \\ &\qquad + N\phi_A(b)\, D^{-1}M\phi_A(a)\, \phi_A^{-1}(\sigma(s)) \\ &\quad = -M\phi_A(a)\, D^{-1}[D - M\phi_A(a)]\, \phi_A^{-1}(\sigma(s)) \\ &\qquad + N\phi_A(b)\, D^{-1}M\phi_A(a)\, \phi_A^{-1}(\sigma(s)) \\ &\quad = -M\phi_A(a)\, \phi_A^{-1}(\sigma(s)) + M\phi_A(a)\, D^{-1}M\phi_A(a)\, \phi_A^{-1}(\sigma(s)) \\ &\qquad + [D - M\phi_A(a)]\, D^{-1}M\phi_A(a)\, \phi_A^{-1}(\sigma(s)) \\ &\quad = -M\phi_A(a, \sigma(s)) + M\phi_A(a)\, D^{-1}M\phi_A(a, \sigma(s)) \\ &\qquad + M\phi_A(a, \sigma(s)) - M\phi_A(a)\, D^{-1}M\phi_A(a, \sigma(s)) \\ &\quad = 0. \end{aligned}$$

Thus G is a formal solution of the boundary value problem.

(iii) Now to prove G is unique with properties (i) and (ii). Let $G_1(t, \sigma(s))$ be another Green's matrix with properties (i) and (ii).

Let $X(t, \sigma(s)) = G(t, \sigma(s)) - G_1(t, \sigma(s))$. At $t = s$ we have,

$$\begin{aligned} &X(s^+, \sigma(s)) - X(s^-, \sigma(s)) \\ &\quad = G(s^+, \sigma(s)) - G_1(s^+, \sigma(s)) - G(s^-, \sigma(s)) - G_1(s^-, \sigma(s)) \\ &\quad = G(s^+, \sigma(s)) - G(s^-, \sigma(s)) - G_1(s^+, \sigma(s)) - G_1(s^-, \sigma(s)) \\ &\quad = \phi(s, \sigma(s)) - \phi(s, \sigma(s)) \\ &\quad = 0. \end{aligned}$$

Thus $X(t, \sigma(s))$ has a removable discontinuity at $t = s$.

By regarding X appropriately we can make $X(t, \sigma(s))$ rd-continuous for all $t \in [a, b]$.

Now the boundary conditions are linear and X is a linear combination of G and G_1,

$$\begin{aligned} MX(a, \sigma(s)) + NX(b, \sigma(s)) & \\ &= M[G(a, \sigma(s)) - G_1(a, \sigma(s))] + N[G(b, \sigma(s)) - G_1(b, \sigma(s))] \\ &= [MG(a, \sigma(s)) + NG(b, \sigma(s))] \\ &\quad - [MG_1(a, \sigma(s)) + NG_1(b, \sigma(s))] \\ &= \phi_A(s, \sigma(s)) - \phi_A(s, \sigma(s)) \\ &= 0. \end{aligned}$$

For each fixed s, $X(t, \sigma(s))$ is a solution of the homogeneous boundary conditions.

Note that if $X(t, \sigma(s))$ is rd-continuous and posesses continuous first order derivatives at all points except at $t = s$, then $X(t, \sigma(s))$ is also a solution of the homogeneous boundary value problem.

Note that continuity implies rd-continuity but the converse need not be true.

Generalizations to Multi-Point Boundary Value Problem

In this section we generalise the results obtained in the previous section to multi-point boundary value problems. The proofs of the theorems are simple consequences of the previous theorems and hence will not be given. We consider the following multi-point boundary value problem,

$$x^{\Delta} = A(t)x + b(t) \tag{3.1}$$

$$\sum_{i=1}^{n} M_i x(t_i) = \alpha, \tag{5.1}$$

where $a = t_1 < t_2 < \cdots < t_n = b$ and where the differential equation satisfies all the conditions assumed in the previous sections and the M_i's $(i = 1, 2, ..., n)$ are constant square matrices of order n.

Definition 5.1. If $\phi_A(t)$ is a fundamental matrix of (3.2) then the matrix D_1 defined by

$$D_1 = \sum_{i=1}^{n} M_i \phi_A(t_i)$$

is called the characteristic matrix for the boundary value problem (3.1), (5.1).

THEOREM 5.1. *Let $A: T^k \to B(X)$ be rd-continuous and regressive and let $b: T^k \to X$ be continuous. Then there exists a unique solution of the linear dynamic equation* (3.1) *satisfying the general n-point boundary conditions* (5.1) *and is given by,*

$$x(t) = \phi_A(t)\, D_1^{-1} \alpha + \int_a^b G(t, \sigma(s))\, b(s)\, \Delta s,$$

where $G(t, \sigma(s))$ is the Green's matrix for the homogeneous linear dynamic equation (3.1) *satisfying the general n-point boundary conditions* (5.1) *and is given by,*

$$G(t, \sigma(s)) = \begin{cases} \phi_A(t)\, D_1^{-1} \sum_{j=l}^{l} M_j \phi_A(\xi_j, \sigma(s)), & t \leqslant s \\ -\phi_A(t)\, D_1^{-1} \sum_{j=l+1}^{n} M_j \phi_A(\xi_j, \sigma(s)), & t \geqslant s \end{cases}$$

for $t \in [\xi_l, \xi_{l+1}]$, $1 \leqslant l \leqslant n-1$.

THEOREM 5.2. *The Green's matrix G has the following properties.*

(i) *The components of $G(t, \sigma(s))$ when regarded as a function of t with fixed s have continuous first derivatives on $[a, s)$ and $(s, b]$. At the point $t = s$, G has an upward jump discontinuity,* i.e.,

$$\begin{aligned} G(s^+, \sigma(s)) - G(s^-, \sigma(s)) &= \phi_A(s, \sigma(s)) \\ &= \phi_A(s)\, \phi_A^{-1}(\sigma(s)). \end{aligned}$$

If $\phi_A^{-1}(\sigma(s))$ is rd-continuous then G has a finite jump discontinuity of unit magnitude.

(ii) *G is a formal solution of the homogeneous multi-point boundary value problem and it fails to be a true solution because of the discontinuity at $t = s$.*

(iii) *G is unique with properties* (i) and (ii).

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