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# Exponential stability of numerical solutions for a class of stochastic age-dependent capital system with Poisson jumps

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ABSTRACT

Recently, numerical solutions of stochastic differential equations have received a great deal of attention. Numerical approximation schemes are invaluable tools for exploring their properties. In this paper, we introduce a class of stochastic age-dependent (vintage) capital system with Poisson jumps. We also give the discrete approximate solution with an implicit Euler scheme in time discretization. Using Gronwall's lemma and Barkholder–Davis–Gundy's inequality, some criteria are obtained for the exponential stability of numerical solutions to the stochastic age-dependent capital system with Poisson jumps. It is proved that the numerical approximation solutions converge to the analytic solutions of the equations under the given conditions, where information on the order of approximation is provided. These error bounds imply strong convergence as the timestep tends to zero. A numerical example is used to illustrate the theoretical results.



**Contents**



## 1. Introduction

Deterministic age-dependent (vintage) capital dynamics can often be described by the ordinary partial differential equation:

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$$\begin{cases} \dfrac{\partial K(a,t)}{\partial t} + \dfrac{\partial K(a,t)}{\partial a} = -\mu(a,t)K(a,t) + f(t, K(a,t)) & \text{in } Q, \\ K(0,t) = \phi(t) = \gamma(t)A(t)F\left(L(t), \displaystyle\int_0^A K(a,t)\mathrm{d}a\right), & \text{in } t \in [0,T], \\ K(a,0) = K_0(a), & \text{in } a \in (0,A), \\ N(t) = \displaystyle\int_0^A K(a,t)\mathrm{d}a, & \text{in } t \in [0.T], \end{cases} \tag{1}$$

where $Q = [0; A] \times [0; T]$; the stock of capital goods of age $a$ at time $t$ is denoted by $K(a, t)$, $N(t)$ is the total sum of the capital, $a$ is the age of the capital, the investment $\phi(t)$ in the new capital, and the investment $f(t, K(a, t))$, in the capital of age $a$ are the endogenous (unknown) variables. The maximum physical lifetime of capital $A$, the planning interval of calendar time $[0, T]$, the depreciation rate $\mu(a, t)$ of capital, and the capital density $k_0(a)$ (the initial distribution of capital over age) are given. $\gamma(t)$ denotes the accumulative rate at the moment of $t$; $0 < \gamma(t) < 1$, and $A(t)$ is the technical progress at the moment of $t$. Eq. (1) is a generalization of the deterministic capital equation. Eq. (1) describes the evolution of the composition of the productive capital as a function of purchasing/selling new or used capital. According to Eq. (1), machines of any age between 0 and $A$ can be bought or sold.

Recently, deterministic age-dependent (vintage) capital systems (without stochastically perturbed i.e. $g \equiv 0$) have been studied by many authors. For example, Feichtinger [1] established deterministic (vintage) capital dynamics and the necessary optimality conditions. Feichtinger [2] developed the vintage capital stock model with technological progress, while the model is solved. Goetz [3] has studied the capital replacement decision of a firm as a distributed investment and disinvestment optimal control problem.

In Eq. (1), $K(a, t)$ denotes the riskless capital. However, some important sources of uncertainty may be discontinuous, recurrent, and fluctuating. Such significant events include innovations in techniques, introduction of new products, natural disasters, and changes in laws or government policies. The relationship among these events and the profitability of risky assets can be very complicated. Furthermore, there can be numerous events and economic variables that are potentially related to the profitability of risky assets. Since capital markets are incomplete, asset returns can have discontinuities of an unpredictable size. This is true regardless of the number of securities available for trading. For simplicity, we consider only jump uncertainty in the market. Jump-diffusion uncertainty would then only add to the incompleteness. On the other hand, technological uncertainty is modeled as a Poisson arrival process that reduces the cost of investment, while revenue uncertainty is modeled as a diffusion process [4], since the financial market has one riskless and one risky asset. In order to describe this capital process situation, we suppose that the parameter $f(t, K)$ is stochastically perturbed with

$$f(t,K) + g(t,K)\frac{\mathrm{d}W_t}{\mathrm{d}t} + h(t,K)\frac{\mathrm{d}N_t}{\mathrm{d}t}$$

where $W_t$ is white noise, then this environmentally perturbed system may be described by the following equation

$$\begin{cases} \dfrac{\partial K(a,t)}{\partial t} + \dfrac{\partial K(a,t)}{\partial a} = -\mu(a,t)K(a,t) + f(t, K(a,t)) & \\ \qquad + g(t,K(a,t))\frac{\mathrm{d}W_t}{\mathrm{d}t} + h(t,K(a,t))\frac{\mathrm{d}N_t}{\mathrm{d}t} & \text{in } Q, \\ K(0,t) = \phi(t) = \gamma(t)A(t)F\left(L(t), \displaystyle\int_0^A K(a,t)\mathrm{d}a\right), & \text{in } t \in [0,T], \\ K(a,0) = K_0(a), & \text{in } a \in (0,A), \\ N(t) = \displaystyle\int_0^A K(a,t)\mathrm{d}a, & \text{in } t \in [0.T], \end{cases} \tag{2}$$

where $f(t, K(a, t)) + g(t, K(a, t))\frac{\mathrm{d}W_t}{\mathrm{d}t} + h(t, K(a, t))\frac{\mathrm{d}N_t}{\mathrm{d}t}$ denotes effects of external environment for capital system, such as innovations in techniques, introduction of new products, natural disasters, and changes in laws and government policies, and so on. The effects of the external environment has the deterministic and random parts which depend on $t$ and $K(a, t)$. $h(t, K(a, t))$ is a jump coefficient and $N_t$ is a scalar Poisson process with intensity $\lambda_t$.

Eq. (1) is a generalization of the deterministic age-dependent capital system. A new stochastic age-dependent (vintage) capital system is given by model (2). It is an extension of Eq. (1). The effects of the stochastic environmental noise considerations lead to a stochastic age-dependent capital system (2), which is more realistic.

However, to the best of our knowledge, there are no numerical methods available for stochastic partial differential equations with Poisson jumps. Thus, numerical approximation schemes are invaluable tools for exploring their properties. In this paper, we use the recent mathematical technique on the stochastic capital system to estimate its numerical solutions. Some mathematical results may be found in [5–7]. We shall extend the idea from the papers [8,9] to the numerical solutions for a stochastic age-dependent capital system with Poisson jumps. The main purpose of this paper is to investigate the convergence of numerical approximation of a stochastic age-dependent capital system with Poisson jumps under the given conditions. Our work differs from that in Refs. [1–3] in that (a) numerical analysis is considered, and (b) Poisson jumps are involved.

In Section 2, we shall collect some basic preliminary results which are essential for the Euler approximation analysis, and Euler approximation is introduced. In Section 3, we give the main result that the Euler method is exponentially stable in the mean square sense under some conditions, and the proof of this main result is completed. In Section 4, a numerical example is provided to illustrate the theoretical results. A conclusion is given in Section 5.

## 2. Preliminaries and approximation

Throughout this paper, let

$$V = H^1([0, A]) \equiv \left\{ \varphi | \varphi \in L^2([0, A]), \frac{\partial \varphi}{\partial a} \in L^2([0, A]), \text{where } \frac{\partial \varphi}{\partial a} \text{ are generalized partial derivatives} \right\}.$$

$V$ is a Sobolev space. $H = L^2([0, A])$ such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

Then $V' = H^{-1}([0, A])$ the dual space of V. We denote by $|\cdot|$ and $\|\cdot\|$ the norms in $V$ and $V'$ respectively; by $\langle \cdot, \cdot \rangle$ the duality product between $V, V'$, and by $(\cdot, \cdot)$ the scalar product in $H$. $K$ is a real separable Hilbert space. For an operator $B \in \mathscr{L}(K, H)$ is the space of all bounded linear operators from $K$ into $H$, we denote by $\|B\|_2$ the Hilbert–Schmidt norm, i.e.

$$\|B\|_2^2 = \operatorname{tr}(BWB^T).$$

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all $P$-null sets). We assume that Poisson process $N_t$ is independent of the Brownian motion $W(t)$.

Let $C = C([0, T]; H)$ be the space of all continuous functions from $[0, T]$ into $H$ with sup-norm $\|\psi\|_C = \sup_{0 \leq s \leq T} |\psi(s)|$, $L_V^p = L^p([0, T]; V)$ and $L_H^p = L^p([0, T]; H)$.

**Definition 2.1.** Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ be the stochastic basis and $W_t$ a Wiener process. Suppose that $K_0$ is a random variable such that $E|K_0|^2 < \infty$. A stochastic process $K_t \equiv K(a, t)$ is said to be a solution on $\Omega$ to the stochastic age-structured capital system for $t \in [0, T]$ if the following conditions are satisfied:

(1) $K_t$ is a $\mathscr{F}_t$-measurable random variable;

(2) $K_t \in I^p(0, T; V) \bigcap L^2(\Omega; C(0, T; V))$, $p > 1, T > 0$, where $I^p(0, T; V)$ denotes the space of all $V$-valued processes $(K_t)_{t \in [0,T]}$ (we will write $K_t$ for short) measurable (from $[0, T] \times \Omega$ into $V$), and satisfying

$$E \int_0^T \|K_t\|^p \mathrm{d}t < \infty.$$

Here $C(0, T; V)$ denotes the space of all continuous functions from $[0, T]$ to $V$.

(3) It satisfies the equation:

$$\langle K_t, v \rangle + \int_0^t \left\langle \frac{\partial K_s}{\partial a}, v \right\rangle \mathrm{d}s = (K_0, v) - \int_0^t \langle \mu(a, s) K_s, v \rangle \mathrm{d}s + \int_0^t \langle f(s, K_s), v \rangle \mathrm{d}s + \int_0^t \langle g(s, K_s), v \rangle \mathrm{d}W_s + \int_0^t \langle h(s, K_s) \mathrm{d}N_s, v \rangle \tag{3}$$

for all $v \in V$, $t \in [0, T]$, a.e. $\omega \in \Omega$, where the stochastic integral is understood in the Itô sense.

$A$ is the maximal age of the capital, so

$$K(a, t) \equiv K_t = 0, \quad \forall r \geq A.$$

Let $\Delta t = \frac{T}{N}$, for system (2) the discrete approximate solution on $t = 0, \Delta t, 2\Delta t, \ldots, N\Delta t$ is defined by the iterative scheme

$$Q_t^{n+1} - Q_t^n - \frac{\partial Q_t^{n+1}}{\partial a} \Delta t = -\mu(a, t) Q_t^n \Delta t + f(t, Q_t^n) \Delta t + g(t, Q_t^n) \Delta W_n + h(t, Q_t^n) \Delta N_n. \tag{4}$$

Here, $Q_t^n$ is the approximation of $K(a, t_n)$, for $t_n = n\Delta t$, the time increment is $\Delta t = \frac{T}{N} \ll 1$, with $\Delta W_n = W(t_{n+1}) - W(t_n)$ and $\Delta N_n = N(t_{n+1}) - N(t_n)$ denoting the increments of the Brownian motion and the Poisson processes, respectively.

For convenience, we shall extend the discrete numerical solution to continuous time. We first define the step function

$$Z_t \equiv Z(a, t) = \sum_{k=0}^{N-1} Q_t^k \mathbf{1}_{[k\Delta t, (k+1)\Delta t)}(t), \tag{5}$$

where $\mathbf{1}_G$ is the indicator function for the set $G$. Then we define

$$Q_t - K_0 + \int_0^t \frac{\partial Q_s}{\partial a} \mathrm{d}s = -\int_0^t \mu(a, s) Z_s \mathrm{d}s + \int_0^t f(s, Z_s) \mathrm{d}s + \int_0^t g(s, Z_s) \mathrm{d}W_s + \int_0^t h(s, Z_s) \mathrm{d}N_s, \tag{6}$$

with $Q_0 = K(a, 0)$, $Q_t = Q(a, t)$. It is straightforward to check that $Z(a, t_k) = Q_t^k = Q(a, t_k)$. First, we state the assumptions about the stochastic age-dependent capital system with Poisson jumps that will be considered:

(i) $\mu(a, t)$ is non-negative measurable in $Q$, $\gamma(t)$ and $A(t)$ are non-negative continuous in $[0; T]$ such that

$$\begin{cases} 0 \le \mu_0 \le \mu(a, t) \le \bar{\mu} < \infty, & \text{in } Q, \\ \text{Let } \gamma(t)A(t) \le \eta;\ \eta \text{ is a non-negative constant;} & \text{in } [0, T], \end{cases}$$

where $\int_0^A \mu(a, t)\mathrm{d}a = +\infty$.

(ii) $f(t, 0) = 0,\ g(t, 0) = 0$, (and) $h(t, 0) = 0$, $t \in [0, T]$;

(iii) (Lipschitz condition) there exists a positive constant $K$ such that $x, y \in H$

$$|f(t, y) - f(t, x)| \vee \|g(t, y) - g(t, x)\|_2 \vee |h(t, y) - h(t, x)| \le K|y - x|;$$

(iv)

$$\begin{cases} F(L, N) \ge 0 (F(L, 0) = 0), & \dfrac{\partial F}{\partial L} > 0, \\ 0 < \dfrac{\partial F}{\partial N} < F_1, & \text{where } F_1 \text{ is a positive constant.} \end{cases}$$

In an analogous way to the corresponding proof presented in [10], we may establish the following existence and uniqueness conclusion: under the conditions (i)–(iv), Eq. (3) has a unique continuous solution $K(a, t)$ on $(a, t) \in Q$.

**Definition 2.2.** Suppose that $K_0$ is a random variable such that $E|K_0|^2 < \infty$. For a given step size $\Delta > 0$, a numerical method is said to be exponentially stable in mean square on Eq. (2) if there is a pair of positive constants $\gamma$ and $\bar{N}$, such that with initial data $K_0$,

$$E|Q_t^n|^2 \le \bar{N} E|K_0|^2 \mathrm{e}^{-\gamma n \Delta}, \quad \forall n = 0, 1, 2, \ldots. \tag{7}$$

**Lemma 2.3** (*Burkholder–Davis–Gundy's Inequality [11]*). *There exist universal constants $c_p$ and $C_p$ for any $0 < p < \infty$ such that for every continuous local martingale $M$ vanishing at zero and any stopping time $\tau$.*

$$c_p E\left(\langle M, M\rangle_\tau^{p/2}\right) \le E\left(\sup_{0 \le s \le \tau} |M_s|^p\right) \le C_p E\left(\langle M, M\rangle_\tau^{p/2}\right).$$

## 3. The main results

In this section, we provide some lemmas which are necessary for the proof of our result. Because $Q_t$ is the discrete numerical solution of Eq. (2), we first study properties of $Q_t$.

**Lemma 3.1.** *Under assumptions* (i)–(iv), *for any $T > 0$,*

$$\sup_{0 \le t \le T} E|Q_t|^2 \le C_{1T}, \tag{8}$$

*where $C_{1T}$ is a positive constant independent of $\Delta t$, but depends on $Q_0$ and $T$.*

**Proof.** From Eq. (6), applying Itô's formula to $|Q_t|^2$ yields

$$\begin{aligned} |Q_t|^2 = {} & |Q_0|^2 + 2\int_0^t \left\langle -\frac{\partial Q_s}{\partial a}, Q_s \right\rangle \mathrm{d}s - 2\int_0^t (\mu(a, s)Z_s, Q_s)\mathrm{d}s + 2\int_0^t (f(s, Z_s), Q_s)\mathrm{d}s \\ & + 2\int_0^t (Q_s, g(s, Z_s))\mathrm{d}W_s + 2\int_0^t (Q_s, h(s, Z_s))\mathrm{d}N_s + \int_0^t \|g(s, Z_s)\|_2^2 \mathrm{d}s + \lambda \int_0^t |h(s, Z_s)|^2 \mathrm{d}s \\ \le {} & |Q_0|^2 + 2\int_0^t \left\langle -\frac{\partial Q_s}{\partial a}, Q_s \right\rangle \mathrm{d}s - 2\int_0^t (\mu(a, s)Z_s, Q_s)\mathrm{d}s + 2\int_0^t (f(s, Z_s), Q_s)\mathrm{d}s \\ & + 2\int_0^t (Q_s, g(s, Z_s))\mathrm{d}W_s + 2\int_0^t (Q_s, h(s, Z_s))\mathrm{d}\bar{N}_s + 2\lambda \int_0^t (Q_s, h(s, Z_s))\mathrm{d}s \\ & + \int_0^t \|g(s, Z_s)\|_2^2 \mathrm{d}s + \lambda \int_0^t |h(s, Z_s)|^2 \mathrm{d}s \end{aligned}$$

where $N_t = \bar{N}_t - \lambda t$ is a compensated Poisson process.

Since

$$-\left\langle \frac{\partial Q_s}{\partial a}, Q_s \right\rangle = -\int_0^A Q_s d_a(Q_s) = \frac{1}{2}\gamma^2(s)A^2(s)\left[F\left(L(s), \int_0^A Q_s \mathrm{d}a\right) - F(L(s), 0)\right]^2$$
$$\leq \frac{1}{2}\eta^2\left(\frac{\partial F(L, N)}{\partial N}\bigg|_y\right)^2\left(\int_0^A Q_s \mathrm{d}a\right) \leq \frac{1}{2}AF_1^2\eta^2|Q_s|^2$$

where $y \in (0, \int_0^A Q_s \mathrm{d}a)$.

Therefore, by conditions (i) and (iii), we get that

$$|Q_t|^2 \leq |Q_0|^2 + AF_1^2\eta^2\int_0^t |Q_s|^2\mathrm{d}s + \int_0^t |f(s, Z_s)|^2\mathrm{d}s + 2\bar{\mu}\int_0^t |Q_s|\,|Z_s|\mathrm{d}s + \int_0^t |Q_s|^2\mathrm{d}s$$
$$+ \int_0^t \|g(s, Z_s)\|_2^2\mathrm{d}s + 2\int_0^t (Q_s, g(s, Z_s))\mathrm{d}W_s + 2\int_0^t (Q_s, h(s, Z_s))\mathrm{d}\bar{N}_s + 2\lambda\int_0^t (Q_s, h(s, Z_s))\mathrm{d}s$$
$$+ \lambda\int_0^t |h(s, Z_s)|^2\mathrm{d}s.$$

Now, it follows that for any $t \in [0, T]$

$$E\sup_{0\leq s\leq t}|Q_s|^2 \leq E|Q_0|^2 + (AF_1^2\eta^2 + \bar{\mu} + \lambda + 1)\int_0^t E\sup_{0\leq s\leq t}|Q_s|^2\mathrm{d}s + \bar{\mu}\int_0^t E|Z_s|^2\mathrm{d}s$$
$$+ \int_0^t E|f(s, Z_s)|^2\mathrm{d}s + \int_0^t E\|g(s, Z_s)\|_2^2\mathrm{d}s + 2\lambda\int_0^t E|h(s, Z_s)|^2\mathrm{d}s$$
$$+ 2E\sup_{0\leq s\leq t}\int_0^s (Q_\tau, g(\tau, Z_\tau))\mathrm{d}W_\tau + 2E\sup_{0\leq s\leq t}\int_0^s (Q_\tau, h(\tau, Z_\tau))\mathrm{d}\bar{N}_\tau.$$

Using condition (iii) yields

$$E\sup_{0\leq s\leq t}|Q_s|^2 \leq E|Q_0|^2 + (AF_1^2\eta^2 + \bar{\mu} + \lambda + 1)\int_0^t E\sup_{0\leq s\leq t}|Q_s|^2\mathrm{d}s + (\bar{\mu} + 2K^2 + 2\lambda K^2)\int_0^t |Z_s|^2$$
$$+ 2E\sup_{0\leq s\leq t}\int_0^s (Q_\tau, g(\tau, Z_\tau))\mathrm{d}W_\tau + 2E\sup_{0\leq s\leq t}\int_0^s (Q_\tau, h(\tau, Z_\tau))\mathrm{d}\bar{N}_\tau. \tag{9}$$

By Burkholder–Davis–Gundy's inequality, we have

$$E\left[\sup_{0\leq s\leq t}\int_0^s (Q_\tau, g(\tau, Z_\tau))\mathrm{d}W_\tau\right] \leq \frac{1}{8}E[\sup_{0\leq s\leq t}|Q_s|^2] + K'\cdot K^2\int_0^t E|Z_s|^2\mathrm{d}s, \tag{10}$$

$$E\left[\sup_{0\leq s\leq t}\int_0^s (Q_\tau, h(\tau, Z_\tau))\mathrm{d}\bar{N}_\tau\right] \leq \frac{1}{8}E[\sup_{0\leq s\leq t}|Q_s|^2] + K_1'\cdot K^2\int_0^t E|Z_s|^2\mathrm{d}s, \tag{11}$$

where $K'$ and $K_1'$ are positive constants. Thus, it follows from (9), (10) and (11)

$$E\sup_{0\leq s\leq t}|Q_s|^2 \leq 2(AF_1^2\eta^2 + 2\bar{\mu} + \lambda + 1 + 2K^2 + 2K_1K^2 + 4\lambda K^2)\int_0^t E\sup_{0\leq r\leq s}|Q_r|^2\mathrm{d}s + 2E|Q_0|^2, \quad \forall t \in [0, T],$$

where $K_1 = \max\{K', K_1'\}$. Now, Gronwall's lemma obviously implies the required result. The proof is complete. □

**Lemma 3.2.** *Under the assumptions* (i)–(iv), *for any* $T > 0$,

$$E\sup_{0\leq t\leq T}|Q_t - Z_t|^2 \leq C_2\Delta t\sup_{t\in[0,T]}E|Q_t|^2. \tag{12}$$

**Proof.** For $\forall t \in [0, T]$, there exists an integer $k$ such that $t \in [k\Delta t, (k+1)\Delta t)$. We have

$$Q_t - Z_t = Q_t - Q_t^k = -\int_{k\Delta t}^t \frac{\partial Q_s}{\partial a}\mathrm{d}s - \int_{k\Delta t}^t \mu(a, s)Z_s\mathrm{d}s + \int_{k\Delta t}^t f(s, Z_s)\mathrm{d}s + \int_{k\Delta t}^t g(s, Z_s)\mathrm{d}W_s$$
$$+ \int_{k\Delta t}^t h(s, Z_s)\mathrm{d}N_s.$$

Thus,

$$|Q_t - Z_t|^2 \le 5\left|\int_{k\Delta t}^t \frac{\partial Q_s}{\partial a}\mathrm{d}s\right|^2 + 5\left|\int_{k\Delta t}^t \mu(a,s)Z_s\mathrm{d}s\right|^2 + 5\left|\int_{k\Delta t}^t f(s,Z_s)\mathrm{d}s\right|^2 + 5\left|\int_{k\Delta t}^t g(s,Z_s)\mathrm{d}W_s\right|^2 + 5\left|\int_{k\Delta t}^t h(s,Z_s)\mathrm{d}N_s\right|^2.$$

Now, Cauchy–Schwarz's inequality and the assumptions (i)–(iii) give

$$\begin{aligned}|Q_t - Z_t|^2 &\le 5\Delta t\int_{k\Delta t}^t \left|\frac{\partial Q_s}{\partial a}\right|^2 \mathrm{d}s + 5\bar{\mu}^2\Delta t\int_{k\Delta t}^t |Z_s|^2\mathrm{d}s + 5\Delta t\int_{k\Delta t}^t |f(s,Z_s)|^2\mathrm{d}s \\ &\quad + 5\left|\int_{k\Delta t}^t g(s,Z_s)\mathrm{d}W_s\right|^2 + 10\left|\int_{k\Delta t}^t h(s,Z_s)\mathrm{d}\bar{N}_s\right|^2 + 10\left|\lambda\int_{k\Delta t}^t h(s,Z_s)\mathrm{d}s\right|^2 \\ &\le 5\Delta t\int_{k\Delta t}^t \left|\frac{\partial Q_s}{\partial a}\right|^2 \mathrm{d}s + 5\bar{\mu}^2\Delta t\int_{k\Delta t}^t |Z_s|^2\mathrm{d}s + 5(1+2\lambda^2)K^2\Delta t\int_{k\Delta t}^t |Z_s|^2\mathrm{d}s \\ &\quad + 5\left|\int_{k\Delta t}^t g(s,Z_s)\mathrm{d}W_s\right|^2 + 10\left|\int_{k\Delta t}^t h(s,Z_s)\mathrm{d}\bar{N}_s\right|^2,\end{aligned}$$

whence applying Burkholder–Davis–Gundy's inequality and conditions (ii)–(iii) lead to

$$E\sup_{t\in[0,T]}\left|\int_{k\Delta t}^t g(s,Z_s)\mathrm{d}W_s\right|^2 \le C_3\int_{k\Delta t}^t E\sup_{s\in[0,T]}|Z_s|^2\mathrm{d}s,$$

and

$$E\sup_{t\in[0,T]}\left|\int_{k\Delta t}^t h(s,Z_s)\mathrm{d}\bar{N}_s\right|^2 \le C_3\int_{k\Delta t}^t E\sup_{s\in[0,T]}|Z_s|^2\mathrm{d}s,$$

where $C_3$ is a constant. Because the differential operator $\frac{\partial}{\partial a}$ is a bounded linear operator, we obtain

$$E\sup_{t\in[0,T]}|Q_t - Z_t|^2 \le 5C_4\Delta t\sup_{t\in[0,T]}E|Q_s|^2 + 5[\bar{\mu}^2\Delta t + (1+2\lambda^2)K^2\Delta t + 3C_3]\Delta t\sup_{t\in[0,T]}E|Q_t|^2,$$

where $C_4$ is a constant. The result (12) is obtained.

We are now in a position to prove a strong convergence result. □

**Lemma 3.3.** *Under assumptions* (i)–(iv), *for any* $T > 0$,

$$\sup_{0\le t\le T}E|Q_t - K_t|^2 \le C_T\Delta t\sup_{t\in[0,T]}E|Q_s|^2, \tag{13}$$

*where $C_T$ is independent of $\Delta t$, but depends on $T$.*

**Proof.** Combining (2) with (6) gives

$$\begin{aligned}K_t - Q_t &= -\int_0^t \frac{\partial(K_s - Q_s)}{\partial a}\mathrm{d}s - \int_0^t \mu(a,s)(K_s - Z_s)\mathrm{d}s + \int_0^t (f(s,K_s) - f(s,Z_s))\mathrm{d}s \\ &\quad + \int_0^t (g(s,K_s) - g(s,Z_s))\mathrm{d}W_s + \int_0^t (h(s,K_s) - h(s,Z_s))\mathrm{d}N_s.\end{aligned}$$

Therefore using Itô's formula, along with Cauchy–Schwarz's inequality, (i)–(iv) yields,

$$\begin{aligned}d|K_t - Q_t|^2 &= -2\left\langle K_t - Q_t, \frac{\partial(K_t - Q_t)}{\partial a}\right\rangle\mathrm{d}t - 2\langle K_t - Q_t, \mu(a,t)(K_t - Z_t)\mathrm{d}t\rangle \\ &\quad + 2\langle K_t - Q_t, f(t,K_t) - f(t,Z_t)\rangle\mathrm{d}t + \|g(t,K_t) - g(t,Z_t)\|_2^2\mathrm{d}t + \lambda|h(t,K_t) - h(t,Z_t)|^2\mathrm{d}t \\ &\quad + 2(K_t - Q_t, (g(t,K_t) - g(t,Z_t))\mathrm{d}W_t) + 2(K_t - Q_t, (h(t,K_t) - h(t,Z_t))\mathrm{d}N_t) \\ &\le AF_1^2\eta^2|K_t - Q_t|^2\mathrm{d}t + 2\bar{\mu}|K_t - Q_t|\,|K_t - Z_t|\mathrm{d}t + 2(\lambda + K)|K_t - Q_t|\,|K_t - Z_t|\mathrm{d}t \\ &\quad + (1+\lambda)K^2|K_t - Z_t|^2\mathrm{d}t + 2(K_t - Q_t, (g(t,K_t) - g(t,Z_t))\mathrm{d}W_t) \\ &\quad + 2(K_t - Q_t, (h(t,K_t) - h(t,Z_t))\mathrm{d}\bar{N}_t),\end{aligned}$$

where d$P$ is the differential of $P$ relative to $t$. Hence, for any $t \in [0, T]$,

$$E \sup_{s\in[0,T]} |K_s - Q_s|^2 \le (AF_1^2\eta^2 + \bar{\mu} + \lambda + K) \int_0^T E \sup_{r\in[0,T]} |K_r - Q_r|^2 \mathrm{d}t + [\bar{\mu} + K + (1+\lambda)K^2] \times E \int_0^T |K_t - Z_t|^2 \mathrm{d}t + 2E \sup_{s\in[0,T]} \int_0^s (K_t - Q_t, (g(t, K_t) - g(t, Z_t))\mathrm{d}W_t) + 2E \sup_{s\in[0,T]} \int_0^s (K_t - Q_t, (h(t, K_t) - h(t, Z_t))\mathrm{d}\bar{N}_t). \tag{14}$$

By Burkholder–Davis–Gundy's inequality, we have

$$E \sup_{s\in[0,T]} \int_0^s (K_t - Q_t, (g(t, K_t) - g(t, Z_t))\mathrm{d}W_t) \le \frac{1}{8} E[\sup_{0\le s\le T} |K_t - Q_t|^2] + K' \int_0^T E|K_t - Z_t|^2 \mathrm{d}t, \tag{15}$$

and

$$E \sup_{s\in[0,T]} \int_0^s (K_t - Q_t, (h(t, K_t) - h(t, Z_t))\mathrm{d}\bar{N}_t) \le \frac{1}{8} E[\sup_{0\le s\le T} |K_t - Q_t|^2] + K_1' \int_0^T E|K_t - Z_t|^2 \mathrm{d}t, \tag{16}$$

where $K'$ and $K_1'$ are positive constants. Let $K_1 = \max\{K', K_1'\}$, inserting (15) and (16) into (14) gives

$$E \sup_{s\in[0,T]} |K_s - Q_s|^2 \le (AF_1^2\eta^2 + \bar{\mu} + \lambda + K) \int_0^T E \sup_{r\in[0,s]} |K_r - Q_r|^2 \mathrm{d}s + [\bar{\mu} + K + (1+\lambda)K^2 + 4K_1] \times \int_0^T E|K_t - Z_t|^2 \mathrm{d}t + \frac{1}{2} E \sup_{s\in[0,T]} |K_s - Q_s|^2.$$

Hence,

$$\begin{aligned} E \sup_{s\in[0,T]} |K_s - Q_s|^2 &\le 2(AF_1^2\eta^2 + \bar{\mu} + \lambda + K) \int_0^T E \sup_{r\in[0,s]} |K_r - Q_r|^2 \mathrm{d}s + 2[\bar{\mu} + K + (1+\lambda)K^2 + 4K_1] \\ &\quad \times \int_0^T E|K_t - Z_t|^2 \mathrm{d}t \\ &\le 2(\bar{\beta}^2 A + \bar{\mu} + \lambda + K) \int_0^T E \sup_{r\in[0,s]} |K_r - Q_r|^2 \mathrm{d}s + 4[\bar{\mu} + K + (1+\lambda)K^2 + 4K_1] \\ &\quad \times \int_0^T (|Q_t - Z_t|^2 + |K_t - Q_t|^2) \mathrm{d}t \\ &\le 2[AF_1^2\eta^2 + \lambda + 3\bar{\mu} + 3K + 2(1+\lambda)K^2 + 8K_1] \int_0^T E \sup_{r\in[0,s]} |K_r - Q_r|^2 \mathrm{d}s \\ &\quad + 4[\bar{\mu} + K + (1+\lambda)K^2 + 8K_1] \int_0^T E|Q_t - Z_t|^2 \mathrm{d}t. \end{aligned}$$

Applying Lemma 3.2, we obtain a bound of the form

$$E \sup_{s\in[0,T]} |K_s - Q_s|^2 \le D_1 \Delta t + D_2 \int_0^T E \sup_{r\in[0,s]} |K_r - Q_r|^2 \mathrm{d}s,$$

where $D_1 = 4(AF_1^2\eta^2 + K + (1+\lambda)K^2 + 8K_1)TC_2 \sup_{t\in[0,T]} E|Q_t|^2$, and $D_2 = 2[AF_1^2\eta^2 + 3\bar{\mu} + \lambda + 3K + 2(1+\lambda)K^2 + 8K_1]$. By applying Gronwall's inequality, we have the following inequality

$$E\left(\sup_{s\in[0,T]} |K_s - Q_s|^2\right) \le D_1 \Delta t \exp(D_2 T).$$

By Lemma 3.1, (13) is obtained. The proof is proved. □

**Lemma 3.4.** *Under assumptions* (i)–(iii), *the trivial solution of Eq.* (2) *is exponentially stable in mean square. That is, there is a pair of positive constants* $\gamma$ *and* $M$ *such that, for any* $K_0$

$$E|K_t|^2 \le ME|K_0|^2 \mathrm{e}^{-\gamma t}, \quad \forall t \ge 0. \tag{17}$$

The proof of this lemma is analogous to that of Theorem 3.2 in [12].

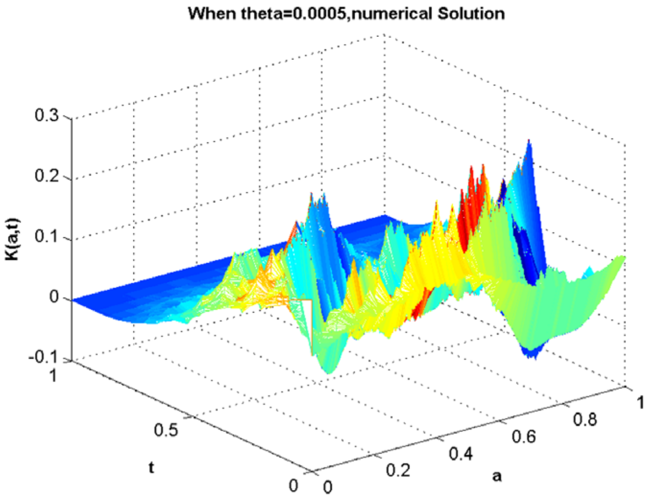


**Fig. 1.** Numerical simulations of stochastic age-dependent capital system.

Now we are in a position to give the main result.

**Theorem 3.5.** *Under assumptions* (i)–(iv)*, the Euler method applied to Eq.* (2) *is exponentially stable in mean square.*

The proof of this theorem is analogous to that of Theorem 2.2 in [13].

## 4. An example

Consider the following stochastic age-dependent capital system with Poisson jumps

$$\begin{cases} \dfrac{\partial K}{\partial t}+\dfrac{\partial K}{\partial a}=-\dfrac{1}{(1-a)^2}K+2Kt-tK\mathrm{d}W_t+K\mathrm{d}N_t, & \text{in } (0,A)\times(0,T),\\ K(0,t)=\dfrac{t^2}{(1-t)^2}\displaystyle\int_0^1 K(a,t)\mathrm{d}a, & \text{in } (0,T),\\ K(a,0)=\exp\left(-\dfrac{1}{1-a}\right), & \text{in } (0,A),\\ N(t)=\displaystyle\int_0^1 K(a,t)\mathrm{d}a, & \text{in } (0,T). \end{cases} \tag{18}$$

Here $W_t$ is a real standard Brownian motion, $N_t$ is a scalar Poisson process with intensity 1. Take $T=1, A=1$ in Eq. (18). We can set this problem in our formulation by taking $H=L^2([0,1]\times[0,1]), V=W_0^1([0,1])$ (a Sobolev space with elements satisfying the boundary conditions above), $\mu(a,t)=\frac{1}{(1-a)^2}, \gamma(t)A(t)=t^2, F(L(t),N(t))=\frac{1}{(1-t)^2}\int_0^1 K(a,t)\mathrm{d}a, L(t)=\frac{1}{(1-t)^2}, f(t,K)=2Kt, h(t,K)=K$ and $g(t,K)=-tK, K(a,0)=\exp\left(-\frac{1}{1-a}\right)$.

Clearly, the operators $f, g$ and $h$ satisfy conditions (ii) and (iii), $F(L,N)$ and $\mu(a,t)$ satisfy conditions (i) and (iv). Consequently, the approximate solution will converge to the true solution of (18) for any $(a,t)\in(0,1)\times(0,T)$ in the sense of Theorem 3.5.

Obviously, $K(a,t)$ in (18) cannot be solved explicitly. It is necessary to know the numerical approximation $Q(a,t)$ of $K(a,t)$. Take $\Delta t=0.005, \Delta a=0.05$. Fig. 1 are numerical simulations of the stochastic age-dependent capital system with Poisson jumps with 1000 experiments, where $K(a,t)=EQ(a,t)=\frac{1}{1000}\sum_{k=1}^{1000}Q_k(a,t)$. This clearly reveals the age-dependent capital system tendency.

## 5. Conclusion

Some important sources of uncertainty may be discontinuous, recurrent, and fluctuating. Such significant events include innovations in techniques, introduction of new products, natural disasters, and changes in laws or government policies. The relationship among these events and the profitability of risky assets can be very complicated. Furthermore, there can be numerous events and economic variables that are potentially related to the profitability of risky assets. In order to describe this situation, this paper introduces a class of stochastic age-dependent capital dynamic system. To the best of our knowledge, there are no numerical methods available for stochastic partial differential equations with Poisson jumps. Thus, numerical approximation schemes are invaluable tools for exploring its properties. In this paper, we extend the idea from the papers [8,9] to the numerical solutions for a stochastic age-dependent capital system with Poisson jumps. The main purpose of this paper is to investigate the convergence of numerical approximation of a stochastic age-dependent capital system with Poisson jumps under the given conditions. Using the recent mathematical technique for the stochastic differential equations, we obtain the condition which can ensure the approximate solution that converges to the true solution for a stochastic age-dependent capital system. At the same time, we propose the numerical solution for stochastic age-dependent capital system with Poisson jumps. The approach is based on constructing a discrete-time approximation to the exact solution by considering the jump time. An example has demonstrated our theory.

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