Homological finiteness properties of monoids, their ideals and maximal subgroups☆

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ARTICLE INFO

ABSTRACT

We consider the general question of how the homological finiteness property left-FP_n (resp. right-FP_n) holding in a monoid influences, and conversely depends on, the property holding in the substructures of that monoid. This is done by giving methods for constructing free resolutions of substructures from free resolutions of their containing monoids, and vice versa. In particular, we show that left-FP_n is inherited by the maximal subgroups in a completely simple minimal ideal, in the case that the minimal ideal has finitely many left ideals. For completely simple semigroups, we prove the converse, and as a corollary, show that a completely simple semigroup is of type left- and right-FP_n if and only if it has finitely many left and right ideals and all of its maximal subgroups are of type FP_n. Also, given an ideal of a monoid, we show that if the ideal has a two-sided identity element then the containing monoid is of type left-FP_n if and only if the ideal is of type left-FP_n. Applying this result, we obtain necessary and sufficient conditions for a Clifford monoid (and more generally a strong semilattice of monoids) to be of type left-FP_n. Examples are provided showing that for each of the results all of the hypotheses are necessary.

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1. Introduction

Let S be a monoid and $\mathbb{Z}S$ be the monoid ring over the integers $\mathbb{Z}$. For $n \geq 0$, the monoid S is of *type left*-FP_n if there is a resolution

$$A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to \mathbb{Z} \to 0$$

of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$ such that $A_0, A_1, \ldots, A_n$ are finitely generated free left $\mathbb{Z}S$-modules. Monoids of type *right*-FP_n are defined dually, working with right $\mathbb{Z}S$-modules.

The property FP_n was introduced for groups by Bieri in [7] and since then has received a great deal of attention in the literature; see [6,8,10,13,39]. One natural line of investigation has been the study of the closure properties of FP_n. Examples include results about the behaviour of FP_n under taking: finite index subgroups or extensions, direct (and semidirect) products, wreath products, HNN extensions, amalgamated free products, and quasi-isometry invariance; see [2,5,7,51].

In monoid and semigroup theory the property FP_n arises naturally in the study of string rewriting systems (i.e. semigroup presentations). The history of rewriting systems in monoids and groups is long and distinguished, and has its roots in the fundamental work of Dehn and Thue. The main focus of this research has been on the so-called complete rewriting systems

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(also called convergent rewriting systems) and in algorithms for computing normal forms. A finite complete rewriting system is a finite presentation for a monoid of a particular form (both confluent and Noetherian) which in particular gives a solution of the word problem for the monoid; see [9] for more details. Therefore it is of considerable interest to develop an understanding of which monoids are presentable by such rewriting systems. Many important classes of groups are known to be presentable by finite complete rewriting systems, including Coxeter groups, surface groups, and many closed three-manifold groups. Rewriting systems continue to receive a lot of attention in the literature; see [15–17,23,25,30,45]. The connection between complete rewriting systems and homological finiteness properties is given by a result of Anick [4] (see also [12]) which shows that a monoid that admits such a presentation must be of type left- and right-FP_∞ (meaning type FP_n for all n). More background on the importance the property FP_n (and other related finiteness conditions) in semigroup theory, and the connections with the theory of string rewriting systems may be found in the survey articles [20,42].

For groups the properties left- and right-FP_n are equivalent, so we simply speak of groups of type FP_n. However, for monoids in general the two notions are independent. Indeed, in [19] Cohen gives an example of a monoid, related to the Thompson group, that is right-FP_∞ but not even left-FP_1. Several other related homological finiteness properties for monoids have been defined, which are all equivalent to FP_n when applied to groups, but are different for monoids in general. A central theme of recent research in this area has been to investigate how these various properties relate to one another; see for example [19,21,35–37,43,44].

On the other hand, in contrast to the situation in group theory, far less attention has been paid to the closure properties of homological finiteness conditions in semigroup and monoid theory, with only a handful of results of this kind having appeared in the literature. For monoid constructions that are direct generalisations of group constructions, perhaps unsurprisingly, some results generalise in a straightforward way from groups to monoids. For example, as observed in [29], the direct product $M \times N$ of two monoids is of type left-FP_n if and only if each of M and N is (this may be proved just as for groups using Künneth theory). Also generalising from groups, it was shown in [43] that left-FP_n is inherited under taking retracts of monoids (which is known to be true for groups more generally for quasi-retracts [2]). However, the study of constructions specific to, and important in, semigroup theory has not yet received serious attention in the literature.

In recent work [35,36] Kobayashi considered the behaviour of left- and right-FP_n for some basic fundamental semigroup-theoretic constructions including left and right zero semigroups (and more generally left and right groups), semilattices, and the process of adjoining a zero element to a monoid. He then used these observations to give examples of monoids that clarify the relationship between the properties left-FP_1, right-FP_1, bi-FP_1, and finite generation (see Section 6 for more on this). The results we obtain here will shed more light on (and in some cases significantly extend) some of the results obtained by Kobayashi in [35,36]. The importance of understanding the closure properties of FP_n is highlighted further still by the work [44] where a monoid is constructed from two groups, it is shown how the homological finiteness properties of the monoid relate to those of the groups, and then combined with [6] this is used to give a counterexample to an open question about homological finiteness properties of string rewriting systems.

In this paper we shall consider the general question of how the property left-FP_n holding in a monoid influences, and conversely depends on, the property holding in the substructures of that monoid. We are particularly interested in relating the property holding in the monoid to the property holding in the subgroups of the monoid, since such results act as a bridge between the homology theory of groups and that of semigroups and monoids. The results we present here complement analogous results regarding cohomology obtained in [1,41].

The paper is structured as follows. After giving some preliminaries in Section 2, we begin our investigation in Section 3 by considering ideals, showing that if T is an ideal of a monoid S, and if T has a two-sided identity element, then S is of type left-FP_n if and only if T is (see Theorem 3). This result is reasonably straightforward to prove. In one direction it generalises the recent observation of Kobayashi [36] that a monoid with a two-sided zero element is of type left- and right-FP_∞. The result does not extend to ideals in general; see Example 1 in Section 6. Applying this result we show (in Theorem 9) that a Clifford monoid S, in the sense of [18], is of type left-FP_n if and only if it has a minimal idempotent e and the maximal subgroup that contains e is of type FP_n (for any undefined concepts we refer the reader to Section 2).

Results relating properties of monoids with those of their subgroups are known to hold for several other important finiteness conditions including being: finitely generated, finitely presented, having finite derivation type, and being residually finite (see [24,26,50]). In particular for each of these properties it is known that if S is a (von Neumann) regular monoid with finitely many idempotents then S has the property if and only if all of its maximal subgroups have the property. In [50, Remark and Open Problem 4.5] the author posed as an open problem the question of whether the corresponding result holds for the property left-FP_n. The results mentioned in the previous paragraph answer this question with a resounding no, since, for instance, whether a Clifford monoid is of type left-FP_n depends only on one of its maximal subgroups (namely the minimal one), and the other maximal subgroups can have any properties that one desires. This leaves the general question of to what extent the property left-FP_n holding in a monoid relates to the property holding in the maximal subgroups of that monoid. As we shall see, it is straightforward to show (see Theorem 2) that the property left-FP_n is inherited by maximal subgroups contained in a completely simple minimal ideal, in the case that the ideal has finitely many left ideals. Without the finiteness assumption on left ideals the result no longer holds (see Example 2 in Section 7). In Sections 4 and 5 we consider the more difficult converse problem, and present results that show how the property left-FP_n holding in a completely simple semigroup relates to the property holding in its maximal subgroups (see Theorem 4, Corollary 1 and Theorem 5). In particular we show that a completely simple semigroup is of type left- and right-FP_n if and only if it has finitely many left and right ideals and all of its maximal subgroups are of type FP_n.

In Section 6 using recent results of Kobayashi [35] we go on to analyse left-FP$_1$ in more detail and, extending [35, Corollary 2.7], we give necessary and sufficient conditions for a completely simple semigroup to be of type left-FP$_1$ (see Theorem 7). In Section 7 we give examples showing the hypotheses of our main results are necessary, and present some further applications. Finally, in Section 8 we discuss some other related homological finiteness properties, including having finite cohomological dimension, and explain how one may construct counterexamples to several other open problems posed in [49,50] regarding these properties.

2. Preliminaries

Free resolutions and the finiteness property FP$_n$

Let us begin by recalling some basic definitions from homology theory that we need; for more details we refer the reader to [31].

Let S be a monoid and let $\mathbb{Z}S$ be the integral monoid ring of S. We have the standard augmentation

$$\epsilon_S : \mathbb{Z}S \to \mathbb{Z}, \quad s \mapsto 1 \, (s \in S)$$

and therefore we can regard $\mathbb{Z}$ as a trivial left $\mathbb{Z}S$-module with the $\mathbb{Z}S$-action via ϵ_S:

$$\lambda \cdot z = \epsilon_S(\lambda)z \quad (\lambda \in \mathbb{Z}S, z \in \mathbb{Z}).$$

A *free resolution* of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$ is a sequence $A_0, A_1, A_2, \ldots$ of free left $\mathbb{Z}S$-modules and homomorphisms $\partial_0 : A_0 \to \mathbb{Z}$ and $\partial_i : A_i \to A_{i-1}$, for $n \geq 1$, such that the sequence

$$\cdots \to A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

is *exact* (i.e. im$\partial_{n+1} = \ker\partial_n$ for $n \geq 0$, and $\partial_0(A_0) = \mathbb{Z}$). We shall often refer to such a resolution simply as a *free left resolution of S*. A monoid S is said to be of type left-FP$_n$ if there is a partial free resolution of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$:

$$A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to \mathbb{Z} \to 0$$

where $A_0, A_1, \ldots, A_n$ are all finitely generated. Dually we can regard $\mathbb{Z}$ as a right $\mathbb{Z}S$-module via ϵ_S, and analogously define free right resolutions, and monoids of type right-FP$_n$. For a semigroup S without a two-sided identity the ring $\mathbb{Z}S$ does not have an identity and the above definition is no longer valid. Thus to extend the notion of FP$_n$ to arbitrary semigroups we utilise the standard device of adjoining an identity element. That is, given a semigroup S we use S^1 to denote the semigroup $S = S \cup \{1\}$ with an identity element 1 adjoined with $1 \notin S$ and where we define $s1 = 1s = s$ for all $s \in S$. Then for a semigroup S without a two-sided identity element we say that S is of type left-FP$_n$ if the monoid S^1 is of type left-FP$_n$.

There are several important differences when working with $\mathbb{Z}S$-modules, where S is a monoid, compared to working with $\mathbb{Z}G$-modules, with G a group. Two of the most important differences are the following.

• For a group G, any left $\mathbb{Z}G$-module can be regarded as a right $\mathbb{Z}G$-module, and conversely, by defining $ug = g^{-1}u$ for any $g \in G$ and u in the module. Moreover applying this operation to a free left resolution of $\mathbb{Z}$ yields a free right resolution of $\mathbb{Z}$ and hence (as mentioned in the introduction) for groups left-FP$_n$ and right-FP$_n$ are equivalent, and we simply speak of property FP$_n$ when working with groups. More generally left- and right-FP$_n$ coincide for inverse semigroups (see [38] for more on inverse semigroups). However, as observed in the introduction, the same is far from being true for arbitrary monoids; see [19].

• If G is a group and H is a subgroup of G then $\mathbb{Z}G$, regarded as a left module over $\mathbb{Z}H$, is free with rank equal to the index of H in G. This fact is fundamental for the proof that FP$_n$ is preserved under taking finite index subgroups or extensions (see [11, Proposition 5.1]). In contrast, given a monoid S and subsemigroup T, where T has with a two-sided identity element, then $\mathbb{Z}S$ regarded as a left module over $\mathbb{Z}T$ will *not* in general be free.
These two fundamental differences are the main reason for the fairly intricate arguments needed to establish some of the results below.

We shall make repeated use of the following consequence of the generalised Schanuel Lemma; see [11, p193].

Lemma 1. *Let S be a monoid. For $n \geq 0$ if*

$$A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

is a partial free resolution of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$, with $A_0, \ldots, A_{n-1}$ finitely generated free left $\mathbb{Z}S$-modules, then S is of type left-FP$_n$ if and only if $\ker \partial_{n-1}$ is finitely generated.

One consequence of Lemma 1 is that if a monoid S is of type left-FP$_n$ then there is a partial free resolution:

$$A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$ where $A_0 = \mathbb{Z}S$ and $\partial_0 = \epsilon_S$ is the standard augmentation.

Throughout, given a subset X of a left $\mathbb{Z}S$-module A, where S is a monoid, we use $\langle X \rangle_{\mathbb{Z}S}$ to denote the left $\mathbb{Z}S$-module generated by the set X. Unless otherwise stated, we work with left modules and property left-FP$_n$ throughout.

Green's relations and completely simple semigroups

We now outline some of the basic concepts from semigroup theory that we shall need; for more details we refer the reader to [33,47]. Green's relations were first introduced in [28] and have ever since played a fundamental role in the structure theory of semigroups. For elements x and y of a semigroup S we write $x\mathscr{R}y$ if x and y generate the same principal right ideal, $x\mathscr{L}y$ if they generate the same principal left ideal, and let $\mathscr{H}$ denote the intersection of $\mathscr{R}$ and $\mathscr{L}$. In other words, for $x, y \in S$

$$x\mathscr{R}y \Leftrightarrow xS^1 = yS^1, \qquad x\mathscr{L}y \Leftrightarrow S^1x = S^1y, \qquad x\mathscr{H}y \Leftrightarrow x\mathscr{L}y \wedge x\mathscr{R}y.$$

Each of these relations is an equivalence relation on S which we call the $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-classes of the semigroup, respectively. A semigroup S is said to be (von Neumann) *regular* if for every $x \in S$ there exists $y \in S$ such that $xyx = x$. A semigroup is regular if and only if every $\mathscr{R}$-class (equivalently every $\mathscr{L}$-class) contains at least one idempotent.

We use $E(S)$ to denote the set of idempotents of a semigroup S. Let e be an idempotent in a semigroup S. Then eSe is the largest submonoid of S (with respect to inclusion) whose identity element is e. The group of units G_e of eSe (i.e. the members of eSe that have two-sided inverses in eSe) is the largest subgroup of S (with respect to inclusion) with identity e, and is called the *maximal subgroup of S containing e*. If an $\mathscr{H}$-class H contains an idempotent then H is a maximal subgroup of S, and conversely every maximal subgroup of S arises in this way. From the definitions it is easily seen that the $\mathscr{R}$-classes (resp. $\mathscr{L}$-classes) are in one–one correspondence with principal right (resp. left) ideals of the semigroup.

Of particular importance are those semigroups that have no proper two-sided ideals. A semigroup is called *simple* if it has no proper two-sided ideals, and is called *completely simple* if it is simple and has minimal left and right ideals. It is not hard to see that if a semigroup S has a minimal ideal K then K is a simple semigroup, which is sometimes referred to as the *kernel* of the semigroup (see [33, Proposition 3.1.4]). A *right zero semigroup* is a semigroup U such that $xy = y$ for all $x, y \in U$ (dually one defines left zero semigroup), and a *right* (resp. *left*) *group* is a direct product of a group and a right (resp. left) zero semigroup. A right (resp. left) group is precisely a completely simple semigroup with a single $\mathscr{R}$-class (resp. $\mathscr{L}$-class). A *rectangular band* is a direct product of a left zero semigroup and a right zero semigroup. Free resolutions for completely simple semigroups will be considered in detail in Sections 4 and 5.

3. Resolutions for subsemigroups

In this section, we make some general observations about the relationship between the property left-FP$_n$ holding in a monoid and the same property holding in certain subsemigroups of the monoid. We shall see that if M is an ideal of a monoid S, and M has a two-sided identity element, then passing from free resolutions of S to free resolutions of M is straightforward. In particular, we shall prove the following result.

Theorem 1. *Let S be a monoid, let R be a right ideal of S and suppose that $R \cong M \times B$ where M is a monoid and B is a right zero semigroup. If S is of type left-FP$_n$ and B is finite then M is of type left-FP$_n$.*

Once established, Theorem 1 can then be applied both to maximal subgroups in minimal ideals (Theorem 2) and to ideals with identity (Theorem 3). Before proving Theorem 1 we first need some basic lemmas.

Let S be a monoid and let M be a subsemigroup of S such that M has a two-sided identity element $e \in M$. If A is a left $\mathbb{Z}S$-module then $e \cdot A = eA$ is a left $\mathbb{Z}M$-module with

$$ea_1 + ea_2 = e(a_1 + a_2)\ (a_1, a_2 \in A), \qquad \lambda(ea) = e(\lambda a) \in eA\ (a \in A, \lambda \in \mathbb{Z}M).$$

There is then an obvious functor Φ from the category of left $\mathbb{Z}S$-modules to the category of left $\mathbb{Z}M$-modules defined as follows. For a left $\mathbb{Z}S$-module A we define $\Phi(A) = eA$, and for a left $\mathbb{Z}S$-module homomorphism $\theta : A_2 \to A_1$ we let

$$\Phi(\theta) : eA_2 \to eA_1$$

be the restriction of θ to eA_2. This is well defined since for all $a_2 \in A_2$ we have $\theta(ea_2) = e\theta(a_2) \in eA_1$. It is not hard to see that the functor Φ is exact, so we omit the proof.

Lemma 2. *The functor Φ is exact i.e. if*

$$A_2 \xrightarrow{\theta_2} A_1 \xrightarrow{\theta_1} A_0$$

is an exact sequence of left $\mathbb{Z}S$-modules then

$$\Phi(A_2) \xrightarrow{\Phi(\theta_2)} \Phi(A_1) \xrightarrow{\Phi(\theta_1)} \Phi(A_0)$$

is an exact sequence of left $\mathbb{Z}M$-modules.

For a general subsemigroup M, with a two-sided identity element, of a semigroup S the functor Φ will not map free left $\mathbb{Z}S$-modules to free left $\mathbb{Z}M$-modules. We now show that when S and M satisfy the conditions given in the statement of Theorem 1, then freeness is preserved.

Lemma 3. *Let $R = N \times B$ where N is a monoid and B is a right zero semigroup. Fix $y \in B$ and let $M = \{(n, y) : n \in N\}$. Then, viewed as a left $\mathbb{Z}M$-module, $\mathbb{Z}R$ is free with basis $F = \{(1, b) : b \in B\}$.*

Proof. Clearly each $r \in R$ can be written uniquely in the form $r = mf$ where $m \in M$ and $f \in F$. It follows that each $\alpha \in \mathbb{Z}R$ can be written uniquely in the form $\alpha = \sum_{f \in F} \lambda_f f$ where $\lambda_f \in \mathbb{Z}M$ for $f \in F$. This proves the lemma. $\square$

Lemma 4. *Let S be a monoid with a right ideal R such that $R = N \times B$ where N is a monoid and B is a right zero semigroup. Fix $y \in B$ and let $M = \{(n, y) : n \in N\}$. Then M is a subsemigroup of S with a two-sided identity $e = (1, y)$, and if A is a free left $\mathbb{Z}S$-module of rank r then $\Phi(A) = eA$ is a free left $\mathbb{Z}M$-module of rank $r|B|$.*

Proof. Let $A = \bigoplus_{x \in X} \mathbb{Z}Sx$ be a free left $\mathbb{Z}S$-module with basis X, where X is a non-empty set with $|X| = r$. It follows from the hypotheses that $eS = R$ and therefore $eA = \bigoplus_{x \in X} \mathbb{Z}Rx$ which is a free left $\mathbb{Z}M$-module with basis

$$F \cdot X = \{f \cdot x : f \in F, x \in X\}$$

by Lemma 3, where $F = \{(1, b) : b \in B\}$. Therefore eA is a free left $\mathbb{Z}M$-module of rank $|X||B| = r|B|$. $\square$

Proof of Theorem 1. Suppose S is of type left-FP$_n$ and that B is finite. Let

$$\mathcal{A} : A_n \xrightarrow{\theta_n} A_{n-1} \xrightarrow{\theta_{n-1}} \cdots \xrightarrow{\theta_2} A_1 \xrightarrow{\theta_1} A_0 \xrightarrow{\theta_0} \mathbb{Z} \to 0$$

be a partial free left resolution for S where A_i is a finitely generated free left $\mathbb{Z}S$-module for $i = 0, \ldots, n$. It then follows from Lemmas 2 and 4 that

$$\mathcal{B} : B_n \xrightarrow{\psi_n} B_{n-1} \xrightarrow{\psi_{n-1}} \cdots \xrightarrow{\psi_2} B_1 \xrightarrow{\psi_1} B_0 \xrightarrow{\psi_0} \mathbb{Z} \to 0$$

is a partial free left resolution for M where $B_i = eA_i$ is a finitely generated left $\mathbb{Z}M$-module and ψ_i is the restriction of A_i to B_i, for $i = 0, \ldots, n$. Therefore M is of type left-FP$_n$. $\square$

We can apply Theorem 1 to obtain the following.

Theorem 2. *Let S be a monoid and let H be a maximal subgroup of S contained in a completely simple minimal ideal U of S. If S is of type left-FP$_n$ and U has finitely many left ideals then H is of type FP$_n$.*

Proof. Let $e \in H$ be the identity of H, let $R = eS$ and set $F = E(S) \cap R$. Since U is a minimal ideal and is completely simple it follows that R is an $\mathcal{R}$-class of U which by the Rees theorem [33, Section 3.2] implies that $R \cong H \times F$. But H is a monoid and F, which is a set of $\mathcal{R}$-related idempotents, is a right zero semigroup. Now the result follows by applying Theorem 1. $\square$

The converse of Theorem 2 does not hold. Indeed, if L is an infinite left zero semigroup, then $S = L^1$ has finitely many left ideals and all of its maximal subgroups are trivial (and so are of type left- and right-FP$_\infty$) but S itself is not of type left-FP$_1$ by Theorem 7 below. The same example shows that the converse of Theorem 1 is also not true in general.

Theorem 2 may, in particular, be applied to completely simple semigroups with finitely many left ideals. Necessary and sufficient conditions for such a semigroup to be of type left-FP$_n$ will be given in Theorem 4 below.

The assumption that B is finite is necessary for Theorem 1 to hold (correspondingly the assumption that U has finitely many left ideals is necessary for Theorem 2); see Example 2 in Section 7 below. When B is a singleton, M is an ideal with a two-sided identity, and in this case the converse of Theorem 1 does hold, as we now demonstrate.

Theorem 3. *Let S be a monoid, let T be an ideal of S and suppose that T has a two-sided identity element. Then S is of type left-FP$_n$ if and only if T is of type left-FP$_n$.*

Proof. Applying Theorem 1 in the case $|B| = 1$ proves that if S is of type left-FP$_n$ then so is T. (Alternatively, this direction follows from [43, Theorem 3], since T is a retract of S.)

For the converse, suppose that T is of type left-FP$_n$ and let

$$\mathcal{A} : A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

be a partial free resolution of the trivial left $\mathbb{Z}T$-module $\mathbb{Z}$. Let e be the two-sided identity element of T. Since e is an idempotent in the ring $\mathbb{Z}S$ it follows that $\mathbb{Z}Se = \mathbb{Z}T$ is a projective left $\mathbb{Z}S$-module (see [11, Chapter 1, Section 8]). Since direct sums of projective modules are projective we deduce that $\mathcal{A}$ is a partial projective resolution of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$ where each A_i is a finitely generated left $\mathbb{Z}S$-module. Indeed, that the sequence is exact is immediate, and each A_i is finitely generated as a left $\mathbb{Z}S$-module since each A_i is finitely generated as a left $\mathbb{Z}T$-module and $T \subseteq S$. Finally, each ∂_i (with $i \in \{0, 1, \ldots, n\}$) is a homomorphism of left $\mathbb{Z}S$-modules since for all $a \in A_i$ and $\lambda \in \mathbb{Z}S$, since e is an identity element in T and ∂_i is a left $\mathbb{Z}T$-module homomorphism, we have

$$\partial_i(\lambda a) = \partial_i(\lambda(ea)) = \partial_i((\lambda e)a) = \lambda e \partial_i(a) = \lambda \partial_i(a).$$

Therefore it follows (see [11, Chapter 8, Theorem 4.3]) that S is of type left-FP$_n$. $\square$

Kobayashi [36] recently observed that a monoid with a two-sided zero element is of type left- and right-FP$_\infty$. Kobayashi's result is a special case of Theorem 3 where $T = \{0\}$.

An analogous result to Theorem 3 regarding cohomology was proved in [1].

Example 1 in Section 6 below shows that Theorem 3 does not hold if we remove the assumption that the ideal T has a two-sided identity element. Theorem 3 will be applied below in Section 7 to prove Theorem 9 which characterises the property left-FP$_n$ for Clifford monoids (and more generally strong semilattices of monoids).

4. Resolutions for completely simple semigroups I

Let U be a completely simple semigroup, let H be a maximal subgroup of U, and let $S = U^1$. As we saw above in Section 3, given a free left resolution for S we may construct a free left resolution for H. In particular, if the free left resolution for S is finitely generated up to dimension n, and U has only finitely many left ideals, then the free left resolution for H will be finitely generated up to dimension n also i.e. the property left-FP_n will be inherited by H from S.

In this section and the one that follows it we shall consider the converse problem. Given a partial free resolution for the group H we show how to construct a partial free resolution for $S = U^1$, and then use this to prove the following.

Theorem 4. *Let U be a completely simple semigroup with finitely many left ideals and let H be a maximal subgroup of U. Then U is of type left-FP_n if and only if U has finitely many right ideals and the group H is of type FP_n.*

Examples will be provided in Sections 6 and 7 showing that this theorem fails if any of the hypotheses are lifted. Theorem 4 has the following immediate consequence.

Corollary 1. *Let U be a finitely generated completely simple semigroup. Then the following are equivalent:*

 (i) *U is of type left-FP_n;*
 (ii) *U is of type right-FP_n;*
(iii) *All maximal subgroups of U are of type FP_n.*

We also have the following.

Theorem 5. *Let U be a completely simple semigroup and let H be a maximal subgroup of U. Then U is of type left-FP_n and right-FP_n if and only if U has finitely many left and right ideals and the group H is of type FP_n.*

Proof. Suppose that U is of type left- and right-FP_n. Then in particular U is of type left- and right-FP_1 which, by Theorem 7 (or alternatively [35, Theorem 2.6]), implies that U has finitely many left and right ideals. Then by Theorem 4 it follows that H is of type FP_n.

The converse is a direct corollary of Theorem 4. □

The rest of this section, and the one that follows it, will be dedicated to the proof of Theorem 4. Results analogous to those above for other finiteness properties, including automaticity and finite derivation type, have appeared in the literature; see [14,22,40]. Other important recent work on completely simple semigroups includes [34].

Let us outline our strategy for proving Theorem 4. We shall adopt standard notation for completely simple semigroups. Let U be a completely simple semigroup. We assume that the $\mathscr{R}$- and $\mathscr{L}$-classes of U are indexed by sets I and Ω respectively so that

$$U = \bigcup_{i \in I} R_i = \bigcup_{\omega \in \Omega} L_\omega.$$

The $\mathscr{H}$-classes of U are the sets $H_{i\omega} = R_i \cap L_\omega$ for $i \in I$ and $\omega \in \Omega$. Every $\mathscr{H}$-class of U contains an idempotent, we use $e_{i\omega}$ to denote the idempotent of $H_{i\omega}$ which is exactly the identity of the group $H_{i\omega}$. All the group $\mathscr{H}$-classes $H_{i\omega}$ ($i \in I$, $\omega \in \Omega$) are isomorphic to a fixed group G, called the *Schützenberger group* of U. The best way to visualise a completely simple semigroup U is as a rectangular grid tiled with $|I| \times |\Omega|$ squares, representing the $\mathscr{H}$-classes, with each row of squares representing an $\mathscr{R}$-class, and each column of squares representing an $\mathscr{L}$-class (this is sometimes referred to as an *egg box diagram*).

The following proposition lists some basic properties of completely simple semigroups.

Proposition 1. *Let U be a completely simple semigroup with set of $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-classes $\{R_i : i \in I\}$, $\{L_\omega : \omega \in \Omega\}$ and $\{H_{i\omega} : i \in I, \omega \in \Omega\}$, respectively.*

 (i) *If $x \in H_{i\omega}$ and $y \in H_{j\mu}$ then $xy \in H_{i\mu}$.*
 (ii) *Each idempotent is a left identity in its $\mathscr{R}$-class and dually a right identity in its $\mathscr{L}$-class. In other words $e_{i\omega}s = s$ for all $s \in R_i$, and $se_{i\omega} = s$ for all $s \in L_\omega$.*
(iii) *For all $i, j \in I$ and $\omega, \mu \in \Omega$ we have $H_{i\omega} \cong H_{j\mu}$.*

The Rees theorem (see [33, Section 3.2] or originally [46]) characterises completely simple semigroups as Rees matrix semigroups over groups. We make use of Rees's result in Section 6.

Let us fix some notation that will remain in force throughout this section. Let U be a completely simple semigroup with set of $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-classes $\{R_i : i \in I\}$, $\{L_\omega : \omega \in \Omega\}$ and $\{H_{i\omega} : i \in I, \omega \in \Omega\}$, respectively. We assume throughout this section that U has only finitely many left ideals, which is equivalent to saying that the set Ω is finite. We suppose that the index sets I and Ω each contain the distinguished symbol 1 and let $R = R_1$, $L = L_1$ and $H = H_{1,1} = R_1 \cap L_1 = R \cap L$.

Recall (see Section 2) that the maximal subgroups of U are precisely the $\mathscr{H}$-classes of U. So $H = H_{1,1}$ is both a fixed maximal subgroup of U and also an $\mathscr{H}$-class of U. Also, by Proposition 1(iii) every other maximal subgroup of U is isomorphic to H.

Define $S = U^1$ (the completely simple semigroup U with an identity adjoined) and set $T = L \cup \{1\} \subseteq S$ which is a submonoid of S. Note that L is a completely simple semigroup with underlying group H. In fact, L is a left group. We have the inclusions $H \leq T \leq S$. This allows us to break down the problem of relating S with H into two stages: first we relate S and T (in Proposition 3), and then we relate T and H (in Proposition 4). Theorem 4 then follows by combining Propositions 3 and 4.

It is important to observe that with the above definitions $\mathbb{Z}T$ is *not* free when viewed as a left $\mathbb{Z}H$-module, and also $\mathbb{Z}S$ is *not* free when viewed as a left $\mathbb{Z}T$-module.

The rest of this section will be devoted to the problem of relating free left resolutions for S with those for T, with the main result being Proposition 2, from which Proposition 3 is a consequence.

Let $e \in H$ be the idempotent in the $\mathscr{H}$-class H, and set let F denote the set of all other idempotents in R i.e. $F = \{E(U) \setminus \{e\}\} \cap R$. Note that F is finite since the index set Ω is assumed to be finite. For $y \in U$ we use L_y to denote the $\mathscr{L}$-class of U containing y. Similarly we define R_y and H_y.

The following self-evident fact will be used repeatedly throughout the section.

Lemma 5. *Every element $\lambda \in \mathbb{Z}S$ may be written uniquely in the form*

$$\lambda = \lambda^{(1)} + \sum_{f \in F} \lambda^{(f)}$$

where $\lambda^{(1)} \in \mathbb{Z}T$ and $\lambda^{(f)} \in \mathbb{Z}L_f$ for all $f \in F$.

The general observation given in the next lemma will be important to us.

Lemma 6. *Let A be a left $\mathbb{Z}S$-module. Then A is finitely generated as a left $\mathbb{Z}S$-module if and only if A is finitely generated as a left $\mathbb{Z}T$-module.*

Proof. For the non-trivial direction of the proof let X be a finite generating set for A as a left $\mathbb{Z}S$-module. For every $\lambda \in \mathbb{Z}S$, decomposing as in Lemma 5, we obtain

$$\lambda = \lambda^{(1)} + \sum_{f \in F} \lambda^{(f)} = \lambda^{(1)} + \sum_{f \in F} \lambda^{(f)} ef$$

where $\lambda^{(1)} \in \mathbb{Z}T$, $\lambda^{(f)} \in \mathbb{Z}L_f$ $(f \in F)$ and by Proposition 1 for all $f \in F$, $\lambda^{(f)} = \lambda^{(f)}f = \lambda^{(f)}ef$ and $\lambda^{(f)}e \in \mathbb{Z}L \subseteq \mathbb{Z}T$. It follows that for all $x \in X$ and $\lambda \in \mathbb{Z}S$:

$$\lambda x = \lambda^{(1)}x + \sum_{f \in F} (\lambda^{(f)}e)(fx) \in \langle X \cup FX \rangle_{\mathbb{Z}T}.$$

Therefore

$$X \cup FX = X \cup \{fx : f \in F, x \in X\} \subseteq A$$

is a generating set for A regarded as a left $\mathbb{Z}T$-module, where $X \cup FX$ is finite since X and F are both finite. □

For every partial free left resolution of S we shall associate a partial free left resolution of T and mappings θ and ϕ relating the two resolutions. Let

$$\mathcal{A} : A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

be a partial free resolution of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$, where $A_0 = \mathbb{Z}S$, $\partial_0 = \epsilon_S$ is the standard augmentation, $\ker \epsilon_S = \langle X_0 \rangle_{\mathbb{Z}T}$, and for $j = 1, \ldots, n$

$$A_j = \bigoplus_{x \in X_{j-1}} \mathbb{Z}S[x]$$

where $\ker(\partial_{j-1}) = \langle X_{j-1} \rangle_{\mathbb{Z}T}$ and $\partial_j : A_j \to A_{j-1}$ is the left $\mathbb{Z}S$-module homomorphism extending $[x] \mapsto x$ $(x \in X_{j-1})$. Note here that each X_{j-1} has been chosen so that $\langle X_{j-1} \rangle_{\mathbb{Z}T} = \ker(\partial_{j-1})$ and not just $\langle X_{j-1} \rangle_{\mathbb{Z}S} = \ker(\partial_{j-1})$. By Lemmas 1 and 6 it follows that if S is of type left-FP$_n$ then such a partial resolution $\mathcal{A}$ exists with $|X_i| < \infty$ for all $0 \leq i \leq n - 1$.

Using $\mathcal{A}$, our aim is to construct a resolution $\mathcal{B}$ for T. Define

$$B_0 = \mathbb{Z}T[e] \oplus \bigoplus_{f \in F} \mathbb{Z}T[f],$$

and, using the natural decomposition given in Lemma 5, define a mapping $\theta : A_0 \to B_0$ by

$$\theta(\lambda) = \lambda^{(1)}[e] + \sum_{f \in F} \lambda^{(f)}e[f] \quad (\lambda \in A_0 = \mathbb{Z}S).$$

Then for $1 \leq m \leq n$ define

$$B_m = \left(\bigoplus_{x \in X_{m-1}} \mathbb{Z}T[x] \right) \oplus \left(\bigoplus_{\substack{i=m-1,\ldots,0 \\ x \in X_i, f \in F}} \mathbb{Z}T[f, x] \right) \oplus \left(\bigoplus_{f \in F} \mathbb{Z}T[f] \right),$$

and a mapping $\theta : A_m \to B_m$ given by

$$\theta\left(\sum_{x \in X_{m-1}} \lambda_x[x]\right) = \sum_{x \in X_{m-1}} \lambda_x^{(1)}[x] + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_x^{(f)}e[f,x]$$

where $\lambda_x \in \mathbb{Z}S$ for $x \in X_{m-1}$. The fact that $\theta(A_m)$ is a subset of B_m follows from the definition of θ along with Proposition 1. We view θ as a mapping with domain $\cup_{0 \le i \le n} A_i$ such that for every $1 \le m \le n$ the restriction of θ to A_m maps A_m to B_m.

We may now state the main result of this section which relates free resolutions of S with free resolutions of T.

Proposition 2. *Let U be a completely simple semigroup with finitely many left ideals, let L be an $\mathscr{L}$-class of U, and set $S = U^1$ and $T = L^1$. Let*

$$\mathcal{A} : A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0 = \epsilon_S} \mathbb{Z} \to 0$$

be a partial free resolution of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$ where $A_0 = \mathbb{Z}S$, $\partial_0 = \epsilon_S$ is the standard augmentation, $\ker \epsilon_S = \langle X_0 \rangle_{\mathbb{Z}T}$ and for $1 \le j \le n$

$$A_j = \bigoplus_{x \in X_{j-1}} \mathbb{Z}S[x]$$

where $\ker(\partial_{j-1}) = \langle X_{j-1} \rangle_{\mathbb{Z}T}$ and $\partial_j : A_j \to A_{j-1}$ is the left $\mathbb{Z}S$-module homomorphism extending $[x] \mapsto x$ $(x \in X_{j-1})$. Then with the above notation:

$$\mathcal{B} : B_n \xrightarrow{\partial'_n} B_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} B_1 \xrightarrow{\partial'_1} B_0 \xrightarrow{\partial'_0} \mathbb{Z} \to 0$$

is a partial free resolution of the trivial left $\mathbb{Z}T$-module $\mathbb{Z}$ where $\partial'_0 : B_0 \to \mathbb{Z}$ is the left $\mathbb{Z}S$-module homomorphism extending:

$$\partial'_0\left(\lambda_e[e] + \sum_{f \in F} \lambda_f[f]\right) = \epsilon_T(\lambda_e) + \sum_{f \in F} \epsilon_T(\lambda_f),$$

where ϵ_T is the standard augmentation, and for $1 \le m \le n$, $\partial'_m : B_m \to B_{m-1}$ is the left $\mathbb{Z}T$-module homomorphism extending: $\partial'_m([x]) = \theta(x)$ $(x \in X_{m-1})$, and for $f \in F$ and $x \in X_i$,

$$\partial'_m([f,x]) = \begin{cases} \theta(fx) & \text{if } i = m-1 \\ (1-e)[f,x] & \text{if } i = m-2, m-4, \dots \\ e[f,x] & \text{if } i = m-3, m-5, \dots \end{cases}$$

and

$$\partial'_m([f]) = \begin{cases} e[f] & \text{if } m \text{ is even} \\ (1-e)[f] & \text{if } m \text{ is odd}. \end{cases}$$

Furthermore $\ker \partial_0$ is finitely generated if and only if $\ker \partial'_0$ is finitely generated; and if $A_0, A_1, \dots, A_n$ are all finitely generated then $B_0, B_1, \dots, B_n$ are finitely generated as well, in which case $\ker \partial_n$ is finitely generated if and only if $\ker \partial'_n$ is finitely generated.

Remark 1. Note that the definition of ∂'_m given in Proposition 2 makes sense since for $x \in X_{m-1}$ and $f \in F$ we have $x, fx \in A_{m-1}$ which by definition of θ implies

$$\theta(x), \theta(fx) \in \bigoplus_{x \in X_{m-2}} \mathbb{Z}T[x] \oplus \bigoplus_{\substack{x \in X_{m-2} \\ f \in F}} \mathbb{Z}T[f,x] \subseteq B_{m-1}. \tag{1}$$

Before embarking on the proof of Proposition 2 we shall give one of its important consequences.

Proposition 3. *Let U be a completely simple semigroup and let L be a left ideal of U. If U has finitely many left ideals then U is of type left-FP$_n$ if and only if L is of type left-FP$_n$.*

Proof. This is equivalent (in the above notation) to proving that if F is finite then $S = U^1$ is of type left-FP$_n$ if and only if $T = L^1$ is of type left-FP$_n$. We prove the result by induction on n. The base case $n = 1$ is easy to verify directly. Alternatively, it follows from a more general result that we prove below (see Theorem 7 in Section 6). Now let $n > 1$ and assume inductively that the result holds for values strictly less than n.

If S is of type left-FP$_n$ then by Lemmas 1 and 6 the partial resolution $\mathcal{A}$ given in the statement of Proposition 2 may be chosen so that X_i is finite for all $0 \le i \le n-1$. By Proposition 2, $\mathcal{B}$ is then a partial free resolution of the trivial left $\mathbb{Z}T$-module $\mathbb{Z}$ with each B_i free of finite rank. Therefore T is of type left-FP$_n$.

Conversely suppose that T is of type left-FP$_n$. In particular T is of type left-FP$_{n-1}$ which by induction implies that S is of type left-FP$_{n-1}$. By Lemma 1 and Proposition 2, using the definitions given in Proposition 2, we have that

$$\mathcal{A} : A_{n-1} \xrightarrow{\partial_{n-1}} A_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0 = \epsilon_S} \mathbb{Z} \rightarrow 0,$$

$$\mathcal{B} : B_{n-1} \xrightarrow{\partial'_{n-1}} B_{n-2} \xrightarrow{\partial'_{n-2}} \cdots \xrightarrow{\partial'_1} B_0 \xrightarrow{\partial'_0} \mathbb{Z} \rightarrow 0$$

are both partial free resolutions with A_j and B_j finitely generated for $j = 0, 1, \ldots, n-1$. By Lemma 1 since T is of type left-FP$_n$ it follows that $\ker \partial'_{n-1}$ is finitely generated which by the last clause in the statement of Proposition 2 implies that $\ker \partial_{n-1}$ is finitely generated (both as a left $\mathbb{Z}S$-module and a left $\mathbb{Z}T$-module, by Lemma 6). It now follows from Lemma 1 that S is of type left-FP$_n$. $\square$

We now work through several technical lemmas which will then be utilised at the end of the section where we prove Proposition 2.

First we define a mapping $\phi : B_m \rightarrow A_m$ which taken together with $\theta : A_m \rightarrow B_m$ (defined above) will help clarify the relationship between $\mathcal{A}$ and $\mathcal{B}$.

Define $\phi : B_0 \rightarrow A_0$ by:

$$\phi\left(\lambda_e[e] + \sum_{f \in F} \lambda_f[f]\right) = \lambda_e + \sum_{f \in F} \lambda_f f,$$

and for $1 \leq m \leq n$ define $\phi : B_m \rightarrow A_m$ by:

$$\phi\left(\sum_{x \in X_{m-1}} \lambda_x[x] + \sum_{\substack{i = m-1, \ldots, 0 \\ x \in X_i, f \in F}} \lambda_{f,x}[f, x] + \sum_{f \in F} \lambda_f[f]\right) = \sum_{x \in X_{m-1}} \lambda_x[x] + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_{f,x} f[x].$$

In a similar way as for θ above, we view ϕ as a map with domain $\cup_{0 \leq i \leq n} B_i$ where the restriction of ϕ to B_m maps B_m to A_m. The relationship between the mappings thus far defined is illustrated below:

$$
\begin{array}{ccccccccccc}
A_n & \xrightarrow{\partial_n} & A_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_1} & A_0 & \xrightarrow{\partial_0 = \epsilon_S} & \mathbb{Z} & \rightarrow & 0 \\
\theta \downarrow\uparrow \phi & & \theta \downarrow\uparrow \phi & & & & \theta \downarrow\uparrow \phi & & & & \\
B_n & \xrightarrow{\partial'_n} & B_{n-1} & \xrightarrow{\partial'_{n-1}} & \cdots & \xrightarrow{\partial'_1} & B_0 & \xrightarrow{\partial'_0} & \mathbb{Z} & \rightarrow & 0.
\end{array}
$$

Lemma 7. *For all $0 \leq m \leq n$ the mapping $\theta : A_m \rightarrow B_m$ is a homomorphism of left $\mathbb{Z}T$-modules.*

Proof. For all $\lambda, \mu \in \mathbb{Z}S$ and $f \in F$, decomposing as in Lemma 5, it is easily seen that

$$(\lambda + \mu)^{(1)} = \lambda^{(1)} + \mu^{(1)} \quad \text{and} \quad (\lambda + \mu)^{(f)} = \lambda^{(f)} + \mu^{(f)}.$$

From this and the definition of θ it follows that

$$\theta(\alpha + \beta) = \theta(\alpha) + \theta(\beta)$$

for all $\alpha, \beta \in A_m$. Hence θ is a homomorphism of abelian groups.

By Proposition 1, under its action on S by left multiplication, T stabilises itself and each of the sets L_f ($f \in F$) setwise. It follows that for all $\mu \in \mathbb{Z}S, \lambda \in \mathbb{Z}T$ and $f \in F$:

$$(\lambda \cdot \mu)^{(1)} = \lambda \cdot \mu^{(1)} \quad \text{and} \quad (\lambda \cdot \mu)^{(f)} = \lambda \cdot \mu^{(f)}.$$

From this and the definition of θ we conclude that $\lambda \cdot \theta(\alpha) = \theta(\lambda \cdot \alpha)$ for all $\lambda \in \mathbb{Z}T$ and $\alpha \in A_m$. $\square$

Lemma 8. *For all $0 \leq m \leq n$ the mapping $\phi : B_m \rightarrow A_m$ is a homomorphism of left $\mathbb{Z}T$-modules.*

Proof. This follows easily from the definition of ϕ. $\square$

Lemma 9. *For $0 \leq m \leq n$ the composition $\phi\theta : A_m \rightarrow A_m$ is the identity map on A_m. In particular, θ is injective.*

Proof. Suppose that $m \geq 1$, the case $m = 0$ may be handled similarly. Let $\sum_{x \in X_{m-1}} \lambda_x[x] \in A_m$, where $\lambda_x \in \mathbb{Z}S$ for $x \in X_{m-1}$. Then from the definitions of ϕ and θ we have

$$\phi\theta\left(\sum_{x \in X_{m-1}} \lambda_x[x]\right) = \phi\left(\sum_{x \in X_{m-1}} \lambda_x^{(1)}[x] + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_x^{(f)} e[f, x]\right)$$

$$= \sum_{x \in X_{m-1}} \lambda_x^{(1)}[x] + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_x^{(f)} ef[x] = \sum_{x \in X_{m-1}} \lambda_x[x]$$

since, by Proposition 1, for $f \in F$ and $x \in X_{m-1}$ we have $\lambda_x^{(f)} ef = \lambda_x^{(f)} f = \lambda_x^{(f)}$, because $\lambda_x^{(f)} \in L_f$ and f is a right identity in its $\mathscr{L}$-class. $\square$

Lemma 10. *The mapping ϕ satisfies $\phi(\ker \partial'_m) \subseteq \ker \partial_m$ for all $0 \leq m \leq n$.*

Proof. The fact that $\phi(\ker \partial'_0)$ is a subset of $\ker \partial_0$ is an easy consequence of the definition of $\phi : B_0 \to A_0$. This deals with the case $m = 0$. Now let $l \in \ker \partial'_m$ for some $1 \leq m \leq n$. Say:

$$l = \sum_{x \in X_{m-1}} \lambda_x [x] + \sum_{\substack{i=m-1,\ldots,0, \\ x \in X_i, f \in F}} \lambda_{f,x}[f, x] + \sum_{f \in F} \lambda_f [f],$$

where each of the terms λ_x, $\lambda_{f,x}$ and λ_f belongs to $\mathbb{Z}T$. Suppose that m is odd, the case when m is even is dealt with similarly. Then since $l \in \ker \partial'_m$ applying ∂'_m gives

$$\sum_{x \in X_{m-1}} \lambda_x \theta(x) + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_{f,x} \theta(fx) + \sum_{\substack{i=m-2,m-4,\ldots \\ x \in X_i, f \in F}} \lambda_{f,x}(1-e)[f, x]$$

$$+ \sum_{\substack{i=m-3,m-5,\ldots \\ x \in X_i, f \in F}} \lambda_{f,x} e[f, x] + \sum_{f \in F} \lambda_f (1-e)[f] = \partial'_m(l) = 0. \tag{2}$$

From this along with (1) and the definition of $\theta : A_{m-1} \to B_{m-1}$ we deduce:

$$\begin{aligned}
\lambda_{f,x} e &= 0 && \text{for } f \in F, x \in X_i \ (i = m-3, m-5, \ldots) \\
\lambda_{f,x}(1-e) &= 0 && \text{for } f \in F, x \in X_i \ (i = m-4, m-6, \ldots) \\
\lambda_f (1-e) &= 0 && \text{for } f \in F,
\end{aligned}$$

and hence Eq. (2) becomes:

$$\sum_{x \in X_{m-1}} \lambda_x \theta(x) + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_{f,x} \theta(fx) + \sum_{\substack{x \in X_{m-2} \\ f \in F}} \lambda_{f,x}(1-e)[f, x] = 0. \tag{3}$$

(When $m = 1$ the third term here is $\sum_{f \in F} \lambda_f(1-e)[f]$ but the rest of the argument follows the same lines as below.) It follows from Eq. (3) and Lemma 7 that

$$\sum_{\substack{x \in X_{m-2} \\ f \in F}} \lambda_{f,x}(1-e)[f, x] \in \theta(A_{m-1}) \subseteq B_{m-1},$$

which from the definition of θ implies that $\lambda_{f,x}(1-e) \in \mathbb{Z}L$, for all $f \in F$ and $x \in X_{m-2}$. Therefore $\lambda_{f,x}(1-e) = \lambda_{f,x}(1-e)e = 0$ since e is a right identity in the $\mathscr{L}$-class L. Now substituting this back into Eq. (3) and applying Lemma 7 gives

$$\begin{aligned}
\theta(\partial_m(\phi(l))) &= \theta\left(\sum_{x \in X_{m-1}} \lambda_x x + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_{f,x} fx \right) \\
&= \sum_{x \in X_{m-1}} \lambda_x \theta(x) + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_{f,x} \theta(fx) = 0.
\end{aligned}$$

But by Lemma 9, θ is injective and so since $\theta(0) = 0 = \theta(\partial_m(\phi(l)))$ we conclude $\partial_m(\phi(l)) = 0$. $\quad\square$

Lemma 11. *The mapping θ satisfies $\theta(\ker \partial_m) \subseteq \ker \partial'_m$ for all $0 \leq m \leq n$.*

Proof. It is an immediate consequence of the definition of $\theta : A_0 \to B_0$ that $\theta(\ker \partial_0) \subseteq \ker \partial'_0$. This shows that the result holds when $m = 0$. Next let $1 \leq m \leq n$ and take an arbitrary element $\alpha = \sum_{x \in X_{m-1}} \lambda_x [x] \in \ker \partial_m$. Then from the definitions and Lemma 7 we have:

$$\begin{aligned}
\partial'_m\left(\theta\left(\sum_{x \in X_{m-1}} \lambda_x[x] \right) \right) &= \partial'_m\left(\sum_{x \in X_{m-1}} \lambda_x^{(1)}[x] + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_x^{(f)} e[f, x] \right) \\
&= \sum_{x \in X_{m-1}} \lambda_x^{(1)} \theta(x) + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_x^{(f)} e\theta(fx) \\
&= \theta\left(\sum_{x \in X_{m-1}} \lambda_x^{(1)} x + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_x^{(f)} efx \right) \\
&= \theta\left(\sum_{x \in X_{m-1}} \lambda_x x \right) = \theta(\partial_m(\alpha)) = \theta(0) = 0,
\end{aligned}$$

where $\lambda_x^{(f)} ef = \lambda_x^{(f)} f = \lambda_x^{(f)}$ by Proposition 1 since $\lambda_x^{(f)} \in \mathbb{Z}L_f$. $\quad\square$

The next result relates generating sets of kernels from the sequence $\mathcal{B}$ with generating sets of kernels from the sequence $\mathcal{A}$.

Lemma 12. *Let Y be a subset of $\ker \partial'_m$ for some $0 \leq m \leq n$. If $\langle Y \rangle_{\mathbb{Z}T} = \ker \partial'_m$ then $\langle \phi(Y) \rangle_{\mathbb{Z}T} = \ker \partial_m$.*

Proof. It follows from Lemma 10 that $\phi(Y) \subseteq \ker \partial_m$. Let $\alpha \in \ker \partial_m$ be arbitrary. Since $\phi\theta$ is the identity on A_m by Lemma 9, we have $\alpha = \phi\theta(\alpha)$ where $\theta(\alpha) \in \ker \partial'_m = \langle Y \rangle_{\mathbb{Z}T}$ by Lemma 11. Write:

$$\theta(\alpha) = \sum_{y \in Y} \gamma_y y, \quad (\gamma_y \in \mathbb{Z}T).$$

Then by Lemma 8:

$$\alpha = \phi\theta(\alpha) = \phi\left(\sum_{y \in Y} \gamma_y y\right) = \sum_{y \in Y} \gamma_y \phi(y) \in \langle \phi(Y) \rangle_{\mathbb{Z}T}.$$

Since α was arbitrary it follows that $\langle \phi(Y) \rangle_{\mathbb{Z}T} = \ker \partial_m$. $\quad\square$

We are now in a position to prove the main result of this section.

Proof of Proposition 2. For $0 \leq m \leq n$, we define the following subset Y_m of B_m:

$$
\begin{aligned}
Y_m = \theta(X_m) \ &\cup \{(1-e)[f,x] : f \in F, \ x \in X_i \ (i = m-1, m-3, \ldots)\} \\
&\cup \{e[f,x] : f \in F, \ x \in X_i \ (i = m-2, m-4, \ldots)\} \\
&\cup Q,
\end{aligned}
$$

where

$$Q = \begin{cases} \{(1-e)[f] : f \in F\} & \text{if } m \text{ is even} \\ \{e[f] : f \in F\} & \text{if } m \text{ is odd.} \end{cases}$$

Claim. *For $0 \leq m \leq n$ we have $Y_m \subseteq \ker \partial'_m$ and $\langle Y_m \rangle_{\mathbb{Z}T} = \ker \partial'_m$.*

Proof of Claim. Suppose that m is odd, the case when m is even is similar. In particular the case $m = 0$ may be dealt with using a similar argument.

Since $X_m \subseteq \ker \partial_m$ it follows from Lemma 11 that $\theta(X_m) \subseteq \ker \partial'_m$. It then rapidly follows from the definition of ∂'_m and Lemma 7 that $Y_m \subseteq \ker \partial'_m$.

To see that $\langle Y_m \rangle_{\mathbb{Z}T} = \ker \partial'_m$ let $l \in \ker \partial'_m$ be arbitrary, say

$$l = \sum_{x \in X_{m-1}} \lambda_x [x] + \sum_{\substack{i = m-1, \ldots, 0, \\ x \in X_i, f \in F}} \lambda_{f,x} [f,x] + \sum_{f \in F} \lambda_f [f],$$

where each of the terms λ_x, $\lambda_{f,x}$ and λ_f belongs to $\mathbb{Z}T$. Exactly as in the proof of Lemma 10 from $\partial'_m(l) = 0$ we deduce:

$$
\begin{aligned}
\lambda_{f,x} e &= 0 \quad \text{for } f \in F \setminus \{e\}, x \in X_i \ (i = m-3, m-5, \ldots) \\
\lambda_{f,x}(1-e) &= 0 \quad \text{for } f \in F \setminus \{e\}, x \in X_i \ (i = m-2, m-4, \ldots) \\
\lambda_f(1-e) &= 0.
\end{aligned}
$$

It then follows from the definition of Y_m, and since m is odd, that

$$
\begin{aligned}
&\sum_{\substack{i = m-2, m-3 \ldots, 0 \\ x \in X_i, f \in F}} \lambda_{f,x} [f,x] + \sum_{f \in F} \lambda_f [f] \\
&= \sum_{\substack{i = m-2, m-4, \ldots \\ x \in X_i, f \in F}} \lambda_{f,x} e[f,x] + \sum_{\substack{i = m-3, m-5, \ldots \\ x \in X_i, f \in F}} \lambda_{f,x}(1-e)[f,x] + \sum_{f \in F} \lambda_f e[f] \\
&\in \langle Y_m \rangle_{\mathbb{Z}T}.
\end{aligned}
$$

In other words $l - l_1 \in \langle Y_m \rangle_{\mathbb{Z}T}$ where

$$l_1 = \sum_{x \in X_{m-1}} \lambda_x [x] + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_{f,x} [f,x] \in \ker \partial'_m.$$

Since $l_1 \in \ker \partial'_m$, by Lemma 10 we have $\phi(l_1) \in \ker \partial_m = \langle X_m \rangle_{\mathbb{Z}T}$, and applying Lemma 7:

$$\theta(\phi(l_1)) = \sum_{x \in X_{m-1}} \lambda_x [x] + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_{f,x} e[f,x] \in \langle \theta(X_m) \rangle_{\mathbb{Z}T} \subseteq \langle Y_m \rangle_{\mathbb{Z}T}.$$

We conclude, again by inspection of Y_m, that

$$l_1 = \theta(\phi(l_1)) + \sum_{\substack{x \in X_{m-1} \\ f \in F}} \lambda_{f,x}(1 - e)[f, x] \in \langle Y_m \rangle_{\mathbb{Z}T},$$

and hence $l = (l - l_1) + l_1 \in \langle Y_m \rangle_{\mathbb{Z}T}$, completing the proof of the claim. $\square$

Returning to the proof of Proposition 2, by definition each of $B_0, B_1, \ldots, B_n$ is a free left $\mathbb{Z}T$-module. Next we must show that $\mathcal{B}$ is exact. It is an immediate consequence of the definitions that $\partial'_0(B_0) = \mathbb{Z}$. Now let $1 \leq j \leq n - 1$ and consider $\text{im}\partial'_{j+1}$. For all $x \in X_j \subseteq \ker(\partial_j)$ and $f \in F$ we have $fx \in \ker(\partial_j)$ and so by Lemma 11 and the claim:

$$\theta(fx) \in \theta(\ker(\partial_j)) \subseteq \ker \partial'_j = \langle Y_j \rangle_{\mathbb{Z}T}.$$

Using this observation, it is then easy to verify that $\text{im}\partial'_{j+1} \subseteq \langle Y_j \rangle_{\mathbb{Z}T}$. From the definition of ∂'_{j+1} we see that $Y_j \subseteq \text{im}\partial'_{j+1}$. Therefore, applying the above claim we conclude that $\mathcal{B}$ is exact.

The last clauses in the statement of Proposition 2 follow from the claim and Lemma 12. $\square$

5. Resolutions for completely simple semigroups II

In this section, we finish off the proof of Theorem 4. In Section 4 above, in the notation of that section, we saw how to pass between free resolutions of S and free resolutions of T. We now go on to consider the relationship between resolutions of T and those of H with the aim of proving the following result.

Proposition 4. *Let L be a left group and let H be a maximal subgroup of L. Then L is of type left-FP_n if and only if L has finitely many idempotents, and H is of type FP_n.*

Recall that a left group is the direct product of a left zero semigroup and a group. We note that a recent result of Kobayashi [35, Corollary 2.7] is exactly Proposition 4 in the special case $n = 1$.

Note that Proposition 4 does not simply follow from the dual of Proposition 3, since all statements are about *left* $\mathbb{Z}S$-modules.

Proposition 4 will follow from Proposition 5 and Theorem 7 which will be proved below. We continue using the same notation that was introduced above in Section 4, with the exception of F which will be used to denote a different set of idempotents from before. So $T = L \cup \{1\} \leq S$ where $L = L_1$ is a completely simple semigroup with a single $\mathcal{L}$-class. Fix an idempotent e in L, let $F = E(L)$ and set $H = H_e$. For $f \in F$ we use H_f to denote the $\mathcal{H}$-class of f. Note that now F is the set of idempotents of an $\mathcal{L}$-class, while in the previous section above F was used to denote a set of $\mathcal{R}$-related idempotents.

One direction of Proposition 4 is straightforward. Suppose that T is of type left-FP_n. Then H is of type left-FP_n by Theorem 2. Also, since T is of type left-FP_1, it follows from Theorem 7 below (see also [35, Theorem 2.6]) that L has finitely many idempotents.

The main result of this section is the following which, when combined with the previous paragraph, has Proposition 4 as a consequence.

Proposition 5. *Let L be a left group, H be a maximal subgroup of L with identity e, and set $F = E(L)$ and $T = L^1$. Let*

$$\mathcal{A} : A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0 = \epsilon_H} \mathbb{Z} \to 0$$

be a partial free resolution of the trivial left $\mathbb{Z}H$-module $\mathbb{Z}$ where $A_0 = \mathbb{Z}H$, $\partial_0 = \epsilon_H$ is the standard augmentation, $\ker \epsilon_H = \langle X_0 \rangle_{\mathbb{Z}H}$ and for $1 \leq m \leq n$

$$A_m = \bigoplus_{x \in X_{m-1}} \mathbb{Z}H[x]$$

where $\ker(\partial_{m-1}) = \langle X_{m-1} \rangle_{\mathbb{Z}H}$ and $\partial_m : A_m \to A_{m-1}$ is the left $\mathbb{Z}H$-module homomorphism extending $[x] \mapsto x$ $(x \in X_{m-1})$. Then

$$\mathcal{B} : B_n \xrightarrow{\partial'_n} B_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} B_1 \xrightarrow{\partial'_1} B_0 \xrightarrow{\partial'_0 = \epsilon_T} \mathbb{Z} \to 0$$

is a partial free resolution of the trivial left $\mathbb{Z}T$-module $\mathbb{Z}$ where $B_0 = \mathbb{Z}T$, $\partial'_0 = \epsilon_T$ is the standard augmentation, and for $1 \leq m \leq n$:

$$B_m = \bigoplus_{\substack{i = m-1, \ldots, 0 \\ x \in X_i}} \mathbb{Z}T[x] \oplus \bigoplus_{f \in F} \mathbb{Z}T[f],$$

where, viewing $A_m \subseteq B_m$ under the natural inclusion arising from $\mathbb{Z}H \subseteq \mathbb{Z}T$, $\partial'_m : B_m \to B_{m-1}$ is the left $\mathbb{Z}T$-module homomorphism extending:

$$\partial'_m([x]) = \begin{cases} x & \text{if } x \in X_{m-1} \\ (1 - e)[x] & \text{if } x \in X_i, i = m-2, \ m-4, \ldots \\ e[x] & \text{if } x \in X_i, i = m-3, \ m-5, \ldots \end{cases}$$

and

$$\partial'_m([f]) = \begin{cases} f - 1 & \text{if } m = 1 \\ (f - 1)[f] & \text{if } m \text{ odd and } m \neq 1 \\ f[f] & \text{if } m \text{ even.} \end{cases}$$

Moreover if F is finite, and $A_1, \ldots, A_n$ are all finitely generated then $B_1, \ldots, B_n$ are finitely generated. Thus if H is of type left-FP_n then T is of type left-FP_n.

Proof. Let $Y_0 = X_0 \cup \{(1 - f) : f \in F\}$, and then for $1 \leq m \leq n$ define:

$$Y_m = X_m \ \cup \ \{(1 - e)[x] : x \in X_i \ (i = m - 1, m - 3, \ldots)\}$$
$$\cup \ \{e[x] : x \in X_i \ (i = m - 2, m - 4, \ldots)\}$$
$$\cup \ Q,$$

where

$$Q = \begin{cases} \{(1 - f)[f] : f \in F\} & \text{if } m \text{ is even} \\ \{f[f] : f \in F\} & \text{if } m \text{ is odd.} \end{cases}$$

Claim. For $0 \leq m \leq n$ we have $Y_m \subseteq \ker \partial'_m$ and $\langle Y_m \rangle_{\mathbb{Z}T} = \ker \partial'_m$.

Proof of Claim. We prove the result for m odd. The proof in the case m even (and in particular the case $m = 0$) is similar.

First we must verify that Y_m is a subset of $\ker \partial'_m$. From the definitions we see that ∂_m is the restriction of ∂'_m to $A_m \subseteq B_m$, and it follows from this that $X_m \subseteq \ker \partial'_m$ since $X_m \subseteq \ker \partial_m$. For $x \in X_{m-1}$ since e is a left identity for H we have

$$\partial'_m((1 - e)[x]) = (1 - e)x = 0 \quad (x \in X_{m-1}).$$

It is then easily verified from the definition of ∂'_m that the remaining members of Y_m belong to $\ker \partial'_m$. This proves $Y_m \subseteq \ker \partial'_m$.

To see that $\langle Y_m \rangle_{\mathbb{Z}T} = \ker \partial'_m$, let $\alpha \in \ker \partial'_m$ be arbitrary, say:

$$\alpha = \sum_{\substack{i = m-1, \ldots, 0 \\ x \in X_i}} \lambda_x[x] + \sum_{f \in F} \lambda_f[f],$$

where each $\lambda_x, \lambda_f \in \mathbb{Z}T$, and since m is odd $\partial'_m(\alpha) = 0$ implies

$$\sum_{x \in X_{m-1}} \lambda_x x + \sum_{\substack{i = m-2, m-4, \ldots \\ x \in X_i}} \lambda_x(1 - e)[x] + \sum_{\substack{i = m-3, m-5, \ldots \\ x \in X_i}} \lambda_x e[x] + \sum_{f \in F} \lambda_f(f - 1)[f] = 0. \tag{4}$$

Recalling that

$$X_{m-1} \subseteq \ker \partial_{m-1} \subseteq A_{m-1} = \bigoplus_{x \in X_{m-2}} \mathbb{Z}H[x], \tag{5}$$

it is then immediate from Eq. (4) that

$$\begin{aligned} \lambda_x e &= 0 \quad \text{for } x \in X_i \ (i = m - 3, m - 5, \ldots) \\ \lambda_x(1 - e) &= 0 \quad \text{for } x \in X_i \ (i = m - 4, m - 6, \ldots) \\ \lambda_f(f - 1) &= 0 \quad \text{for } f \in F. \end{aligned}$$

(When $m = 1$ things work slightly differently, as we shall explain below.) From these observations, along with the definition of Y_m, we deduce:

$$\sum_{\substack{i = m-3, m-4, \ldots, 0 \\ x \in X_i}} \lambda_x[x] + \sum_{f \in F} \lambda_f[f] = \sum_{\substack{i = m-3, m-5, \ldots \\ x \in X_i}} \lambda_x(1 - e)[x] + \sum_{\substack{i = m-4, m-6, \ldots \\ x \in X_i}} \lambda_x e[x] + \sum_{f \in F} \lambda_f f[f] \in \langle Y_m \rangle_{\mathbb{Z}T}.$$

In other words $\alpha - \alpha_1 \in \langle Y_m \rangle_{\mathbb{Z}T}$ where:

$$\alpha_1 = \sum_{x \in X_{m-1}} \lambda_x[x] + \sum_{x \in X_{m-2}} \lambda_x[x], \tag{6}$$

and

$$\sum_{x \in X_{m-1}} \lambda_x x + \sum_{x \in X_{m-2}} \lambda_x(1 - e)[x] = \partial'_m(\alpha_1) = 0. \tag{7}$$

Now

$$\alpha_1 = \left(\sum_{x \in X_{m-1}} \lambda_x(1 - e)[x] + \sum_{x \in X_{m-2}} \lambda_x e[x] \right) + \alpha_2$$

where

$$\alpha_2 = \sum_{x \in X_{m-1}} \mu_x[x] + \sum_{x \in X_{m-2}} \sigma_x[x],$$

with $\mu_x = \lambda_x e \in \mathbb{Z}L$ $(x \in X_{m-1})$, and $\sigma_x = \lambda_x(1 - e) \in \mathbb{Z}$ $(x \in X_{m-2})$ since $l(1 - e) = 0$ for all $l \in L$. Since

$$(1 - e)[x]\ (x \in X_{m-1}), \quad e[x]\ (x \in X_{m-2}),$$

all belong to Y_m we deduce that $\alpha_1 - \alpha_2 \in \langle Y_m \rangle_{\mathbb{Z}T}$. Applying ∂'_m to α_2 gives:

$$\sum_{x \in X_{m-1}} \mu_x x + \sum_{x \in X_{m-2}} \sigma_x(1 - e)[x] = \partial'_m(\alpha_2) = 0, \tag{8}$$

where $\mu_x \in \mathbb{Z}L$ $(x \in X_{m-1})$ and $\sigma_x \in \mathbb{Z}$ $(x \in X_{m-2})$. For each $x \in X_{m-1}$, since $\mu_x \in \mathbb{Z}L$ it decomposes uniquely in the following way

$$\mu_x = \sum_{f \in F} \mu_x^{(f)}, \quad (\mu_x^{(f)} \in \mathbb{Z}H_f). \tag{9}$$

In the next step of the argument our aim is to deduce $\sigma_x = 0$ for all $x \in X_{m-2}$. First we consider what happens when $m \geq 3$ (recall that m is odd by assumption), and then we shall explain how to modify the argument in the case $m = 1$.

Suppose $m \geq 3$. Then, for each $x \in X_{m-2}$ by considering the coefficient of $[x]$ in Eq. (8), and recalling (5) and the facts $\mu_x \in \mathbb{Z}L$ $(x \in X_{m-1})$ and $\sigma_x \in \mathbb{Z}$ $(x \in X_{m-2})$, we immediately deduce that $\sigma_x = 0$ for all $x \in X_{m-2}$.

Turning our attention to the special case when $m = 1$. In this case the above argument leads to:

$$\alpha_2 = \sum_{x \in X_0} \mu_x[x] + \sum_{f \in F} \sigma_f[f]$$

where $\mu_x \in \mathbb{Z}L$ $(x \in X_0)$ and $\sigma_f \in \mathbb{Z}$ $(f \in F)$, and

$$\sum_{x \in X_0} \mu_x x + \sum_{f \in F} \sigma_f(1 - f) = \partial'_1(\alpha_2) = 0. \tag{10}$$

For every $f \in F$ and $x \in X_0 \subseteq \mathbb{Z}H$ since H stabilises H_f under its action by right multiplication it follows that, in the notation of (9), $\mu_x^{(f)} x \in \mathbb{Z}H_f$. Thus from equation (10) we conclude that for each $f \in f$:

$$\sum_{x \in X_0} \mu_x^{(f)} x - \sigma_f f = 0, \tag{11}$$

where $\mu_x^{(f)} \in \mathbb{Z}H_f$ and $\sigma_f \in \mathbb{Z}$. But $X_0 \subseteq \ker \epsilon_T$ where ϵ_T is the standard augmentation, and so from Eq. (11) it follows that $\sigma_f f \in \ker \epsilon_T$ and so since $\sigma_f \in \mathbb{Z}$ we have $\sigma_f = \epsilon_T(\sigma_f f) = 0$ for all $f \in F$.

Thus both in the case $m = 1$ and $m \geq 3$ we conclude that

$$\alpha_2 = \sum_{x \in X_{m-1}} \mu_x[x], \tag{12}$$

where $\mu_x \in \mathbb{Z}L$ $(x \in X_{m-1})$, and

$$\sum_{x \in X_{m-1}} \mu_x x = \partial'_m(\alpha_2) = 0.$$

For all $l \in L$ and $x \in X_{m-1}$, from (5) we deduce

$$lx \in \bigoplus_{x \in X_{m-2}} \mathbb{Z}H_f[x] \Leftrightarrow l \in H_f,$$

and it follows that

$$\alpha_2 = \sum_{f \in F} \alpha_2^{(f)}$$

where, using the decomposition (9), $\alpha_2^{(f)} = \sum_{x \in X_{m-1}} \mu_x^{(f)}[x] \in \ker \partial'_m$ for every $f \in F$. Let $f \in F$ be arbitrary. Then

$$\alpha_2^{(f)} = \sum_{x \in X_{m-1}} \mu_x^{(f)}[x] \quad (\mu_x^{(f)} \in \mathbb{Z}H_f),$$

and

$$\sum_{x \in X_{m-1}} \mu_x^{(f)} x = \partial'_m(\alpha_2^{(f)}) = 0. \tag{13}$$

To complete the proof of the claim it suffices to show that $\alpha_2^{(f)} \in \langle Y_m \rangle_{\mathbb{Z}T}$. Clearly

$$f \cdot X_m = fX_m \subseteq \langle Y_m \rangle_{\mathbb{Z}T}.$$

From Eq. (13) we deduce that $e\alpha_2^{(f)}$ belongs to $\ker \partial_m = \langle X_m \rangle_{\mathbb{Z}H}$. So we can write:

$$e\alpha_2^{(f)} = \mu_1 z_1 + \cdots + \mu_r z_r, \quad (\mu_i \in \mathbb{Z}H, z_i \in X_m).$$

But then since $fe = f$ and f is a left identity for $\alpha_2^{(f)}$ we obtain

$$\begin{aligned}
\alpha_2^{(f)} = fe\alpha_2^{(f)} &= f(\mu_1 z_1 + \cdots + \mu_r z_r) \\
&= (f\mu_1)(fz_1) + \cdots + (f\mu_r)(fz_r) \in \langle fX_m \rangle_{\mathbb{Z}T} \subseteq \langle Y_m \rangle_{\mathbb{Z}T},
\end{aligned}$$

since $f\mu_i \in \mathbb{Z}H_f$ $(i = 1, \ldots, r)$ and f is a right identity for L_f. Since f was arbitrary this shows that $\alpha_2 \in \langle Y_m \rangle_{\mathbb{Z}T}$, completing the proof of the claim. $\square$

Returning to the proof of Proposition 5, by definition each of $B_0, B_1, \ldots, B_n$ is a free left $\mathbb{Z}T$-module. To complete the proof we must show that $\mathscr{B}$ is exact. Clearly ϵ_T maps B_0 onto $\mathbb{Z}$. Now let $1 \le j \le n - 1$ and consider $\mathrm{im}\partial_{j+1}'$. It follows from the claim that $Y_j \subseteq \ker \partial_j'$ which, along with the definitions of Y_j, B_j and ∂_j', shows that $\mathrm{im}\partial_{j+1}' \subseteq \langle Y_j \rangle_{\mathbb{Z}T} \subseteq \ker \partial_j'$. Since Y_j is a subset of $\mathrm{im}\partial_{j+1}'$ it then follows from the claim that the sequence $\mathscr{B}$ is exact.

The last clause in the statement of the proposition follows since if $X_0, \ldots, X_{m-1}$ (for some $1 \le m \le n$) are all finite, and F is finite, then from its definition B_m will clearly be finitely generated. $\square$

Proof of Theorem 4. Let L be the $\mathscr{L}$-class of U that contains H. Then:

$$\begin{aligned}
&U \text{ is of type left-FP}_n \\
\Longleftrightarrow \quad &L \text{ is of type left-FP}_n &&\text{(by Proposition 3)} \\
\Longleftrightarrow \quad &|E(L)| < \infty \ \& \ H \text{ is of type FP}_n &&\text{(by Proposition 4)} \\
\Longleftrightarrow \quad &U \text{ has finitely many right ideals} \\
&\& \ H \text{ is of type FP}_n. \quad \square
\end{aligned}$$

6. Kobayashi's criterion and the property FP₁

In this section, we turn our attention to the particular case $n = 1$ and examine the behaviour of the property FP$_1$ in more detail. As mentioned in the introduction, a group is of type FP$_1$ if and only if it is finitely generated. For monoids this is no longer the case. In a recent paper [35] Kobayashi characterised the property left-FP$_1$ for monoids in the following way.

Let S be a semigroup and let A be a subset of S. A subsemigroup T of S is called *right unitary* if $st \in T$ implies $s \in T$ for any $t \in T$ and $s \in S$. The intersection of two right unitary subsemigroups is clearly right unitary, so we may speak of the right unitary subsemigroup of S generated by A, which we denote by $\langle A \rangle_{r.u.}$. We say that S is *right unitarily finitely generated* if there is a finite subset A of S such that $\langle A \rangle_{r.u.} = S$. For a subset A of a semigroup S we use $\langle A \rangle$ to denote the subsemigroup of S generated by A.

The *right Cayley graph* $\Gamma_r(S, A)$ of S with respect to a subset A of S is the directed labelled graph with vertices the elements of S, and a directed edge from x to y labelled by $a \in A$ if and only if $xa = y$ in S. (Note here that we do not insist that A is a generating set for S.) We write this as $x \xrightarrow{a} y$. We say that $\Gamma_r(S, A)$ is *connected* if between any two vertices there is an undirected path. Kobayashi's characterisation of the property left-FP$_1$ for monoids may be stated as follows.

Theorem 6. *[35, Proposition 2.4 & Theorem 2.6] A monoid S is of type left-*FP$_1$ *if and only if there is a finite subset A of S such that one of the following equivalent conditions is satisfied:*

(i) *S is right unitarily generated by A i.e. $\langle A \rangle_{r.u.} = S$;*
(ii) *the right Cayley graph $\Gamma_r(S, A)$ is connected.*

Kobayashi's characterisation allows for a more detailed analysis of FP$_n$ for monoids in the special case that $n = 1$. In this section, among other things we use his criterion as a tool to characterise completely simple semigroups of type left-FP$_1$ in terms of the number of right ideals and the subsemigroup generated by the idempotents. The following concept will play an important role.

Definition 1 (*Relative Rank*)**.** Let S be a semigroup and let T be a subsemigroup of S. Then we define:

$$\mathrm{rank}(S : T) = \inf_{A \subseteq S}\{|A| : \langle T \cup A \rangle = S\}$$

which we call the *relative rank of T in S*.

It is not so surprising that the idea of relative rank arises here, since it is a notion that is central for understanding generating sets of completely simple semigroups; see [27,48].

Theorem 7. *Let U be a completely simple semigroup, let H be a maximal subgroup of U and let $\langle E(U)\rangle$ be the subsemigroup generated by the idempotents of U. Then U is of type left-FP_1 if and only if the following two conditions are satisfied:*

(i) *U has finitely many right ideals, and*
(ii) *the subgroup K of H generated by $\langle E(S)\rangle \cap H$ has finite relative rank in H.*

Proof. Let $S = U^1$. By the Rees theorem (see [33, Chapter 3], or originally [46]) we may identify U with a Rees matrix semigroup $M[G; I, \Omega; P]$ over a group G where $G \cong H$ and $P = (p_{\omega i})_{\omega \in \Omega, i \in I}$ is an $\Omega \times I$ matrix with entries from G. This semigroup has elements $U = I \times G \times \Omega$ and multiplication given by:

$$(i, g, \omega)(j, h, \mu) = (i, gp_{\omega j}h, \mu).$$

Moreover, we may assume that the matrix P is in normal form i.e. $p_{1\omega} = p_{i1} = 1$ for all $i \in I$ and $\omega \in \Omega$ (see [33, Chapter 3, Section 4]) and we may also suppose without loss of generality that $H = H_{11}$. It is well known and easy to prove (see [32] for example) that the subgroup K of H generated by $\langle E(S)\rangle \cap H$ consists of all triples $(1, g, 1)$ where g belongs to the subgroup of G generated by the entries in the matrix P.

Suppose that I is finite (i.e. that U has finitely many right ideals) and K has finite relative rank in H. Let X be a finite subset of H such that $\langle X \cup K\rangle = H$. Let F be the set of all idempotents in some fixed $\mathscr{L}$-class L of U. We claim that $F \cup X$ is a right unitary generating set for U. First observe that $E(U) \subseteq \langle F \cup X\rangle_{r.u.}$. Indeed, given any $e \in E(U)$ there exists $f \in F$ such that $e \mathscr{R} f$ and so $ef = f$ and thus $e \in \langle F \cup X\rangle_{r.u.}$. Therefore

$$\begin{aligned}
\langle F \cup X\rangle_{r.u.} &= \langle F \cup X \cup E(U)\rangle_{r.u.} = \langle X \cup E(U)\rangle_{r.u.} \\
&\supseteq \langle X \cup E(U)\rangle \supseteq \langle (X \cup (\langle E(U)\rangle \cap H)) \cup E(U)\rangle \\
&= \langle H \cup E(U)\rangle = U.
\end{aligned}$$

Hence $\langle F \cup X\rangle_{r.u.} = S$ and since $F \cup X$ is finite it follows from Theorem 6 that S is of type left-FP_1.

For the converse, suppose that S is of type left-FP_1. Let A be a finite subset of S such that $\Gamma_r(S, A)$ is connected. Also, we may assume that A is chosen so that $(i, g, \omega) \in A$ if and only if $(i, g^{-1}, \omega) \in A$ for all $i \in I$, $\omega \in \Omega$ and $g \in G$. Since for any collection of $\mathscr{R}$-classes $R_1, \ldots, R_k$ of U the union $R_1 \cup \cdots \cup R_k \cup \{1\}$ is a right unitary submonoid of S it follows that A must intersect every $\mathscr{R}$-class of S, and thus I is finite. To complete the proof we have to show that K has finite relative rank in H.

We claim that $\langle A \cup E(S)\rangle = S$. To see this, suppose for the sake of a contradiction that $\langle A \cup E(S)\rangle \subsetneq S$. Since $\Gamma_r(S, A)$ is connected (as an undirected graph) it follows that there exist $u, v \in S$ such that $u \in \langle A \cup E(S)\rangle$, $v \notin \langle A \cup E(S)\rangle$ and u adjacent to v in $\Gamma_r(S, A)$. So there exists $a \in A$ such that either $ua = v$ or $va = u$. Since $v \notin \langle A \cup E(S)\rangle$ we cannot have $ua = v$ and therefore must have $va = u$. Clearly none of a, v or u is equal to 1. Let

$$v = (i, g, \omega), \quad a = (j, \alpha, \mu), \quad u = (k, \beta, \nu),$$

where $i, j, k \in I$, $\omega, \mu, \nu \in \Omega$ and $g, \alpha, \beta \in G$. Then

$$(i, gp_{\omega j}\alpha, \mu) = (i, g, \omega)(j, \alpha, \mu) = va = u = (k, \beta, \nu),$$

and so

$$i = k, \quad \mu = \nu, \quad \& \quad g = \beta\alpha^{-1}p_{\omega j}^{-1}.$$

Therefore since the matrix P is in normal form:

$$\begin{aligned}
v = (i, g, \omega) &= (i, \beta\alpha^{-1}p_{\omega j}^{-1}, \omega) = (i, \beta p_{\mu 1}1p_{1j}\alpha^{-1}p_{\mu 1}1p_{1j}p_{\omega j}^{-1}, \omega) \\
&= (i, \beta, \mu)(1, 1, 1)(j, \alpha^{-1}, \mu)(1, 1, 1)(j, p_{\omega j}^{-1}, \omega) \in \langle A \cup E(S)\rangle,
\end{aligned}$$

since $(i, \beta, \mu) = (k, \beta, \nu) = u \in \langle A \cup E(S)\rangle$, $(j, \alpha^{-1}, \mu) \in A$ and $(j, p_{\omega j}^{-1}, \omega) \in E(S)$. This is a contradiction. We conclude that $\langle A \cup E(S)\rangle = S$. In particular $H \subseteq \langle A \cup E(S)\rangle$ so given $g \in G$ we can write

$$(1, g, 1) = (i_1, g_1, \omega_1)(i_2, g_2, \omega_2) \cdots (i_k, g_k, \omega_k)$$

where $i_1 = 1$, $\omega_k = 1$ and each $(i_r, g_r, \omega_r) \in A \cup E(S)$. It follows that

$$g = g_1 p_{\omega_1 i_2} g_2 \cdots p_{\omega_{k-1} i_k} g_k$$

where each g_i is either the inverse in G of an entry from P, or is the middle entry of some triple from A. It follows that if B is the set of all middle entries of elements of A then $N \cup B$ generates G where N is the subgroup of G generated by the entries in the matrix P. Since A is finite it follows that B is finite, therefore K has finite relative rank in H and this completes the proof of the theorem. □

We leave as an open question the problem of extending Theorem 7 to values of n greater than 1. It seems likely that the formulation of such a result will need the introduction of the notion of a subgroup K being of *relative type-FP_n* in a group G.

The following result shows how left-FP_1 holding in an ideal of a semigroup influences the same property holding in the semigroup.

Proposition 6. *Let S be a monoid and let J be a left ideal of S. If J is of type left-FP_1 then so is S.*

Proof. Let $T = J \cup \{1\} \leq S$. Since J is left-FP_1 there is a non-empty finite subset A of T such that $\Gamma_r(T, A)$ is connected. Let $a \in A \cap J$. Then for all $s \in S$ we have $sa \in J$, since J is a left ideal, and it follows that $\Gamma_r(S, A)$ is connected and hence S is of type left-FP_1. $\square$

The following example shows that the converse of this result fails, even for two-sided ideals and as a consequence also shows that Theorem 3 above does not hold if we remove the assumption that the ideal has a two-sided identity element. Moreover, it shows that the finiteness assumption on J in Theorem 8 below really is necessary.

Example 1. Let S be a finitely generated monoid with a minimal ideal R where R is isomorphic to a rectangular band $A \times B$ where A and B are both infinite sets. Since S is finitely generated it is, in particular, of type left-FP_1. However, by Theorem 6, R is neither of type left-FP_1 nor right-FP_1.

Such an example S may be constructed in the following way. First let U be the submonoid of the full transformation monoid $\mathcal{T}(\mathbb{N})$ (the semigroup of all maps from $\mathbb{N}$ to $\mathbb{N}$ under composition) generated by α which maps $i \mapsto i + 1$ ($i \in \mathbb{N}$) and the constant map γ with image 1. The monoid U is finitely generated, and has a minimal ideal isomorphic to an infinite right zero semigroup. Let V be the dual of U, which is finitely generated and has a minimal ideal that is an infinite left zero semigroup. Then define $S = U \times V$ which has a minimal ideal that is a rectangular band with infinitely many left and right ideals. Also S is finitely generated since it is a direct product of two finitely generated monoids.

Combining Theorem 7 and Proposition 6 we obtain the following.

Theorem 8. *Let S be a monoid with a minimal ideal J that is completely simple and has finitely many right ideals. Let H be a maximal subgroup of S in J and K be the subgroup of H generated by $\langle E(J) \rangle \cap H$. Then S is of type left-FP_1 if and only if K has finite relative rank in H.*

Proof. One direction is a direct consequence of Theorem 7 and Proposition 6.

For the converse, suppose that S is of type left-FP_1. By Theorem 7 to complete the proof it suffices to show that $T = J \cup \{1\} \leq S$ is of type left-FP_1. Let A be a finite subset of S such that $\Gamma_r(S, A)$ is connected as an undirected graph. Fix an $\mathcal{L}$-class L of J and let $F = E(S) \cap L$ which is finite by assumption. Define $B = F \cup AF$ which is a finite subset of J. We shall now prove that B is a right unitary generating set for $T = J \cup \{1\}$.

Let $e \in F$ be arbitrary and let $R = R_e$ be its $\mathcal{R}$-class. We claim that for every $x \in R$ there is a path in $\Gamma_r(T, B)$ from x to e. Let $x \in R$. Since $\Gamma_r(S, A)$ is connected it follows that there is a sequence

$$1 = y_0, y_1, \ldots, y_r = x$$

of elements of S such that for all i, y_i and y_{i+1} are connected by an arc (in some direction) in $\Gamma_r(S, A)$. Now consider the sequence:

$$z_0 = e = e1 = ey_0, \qquad z_1 = ey_1, \ldots, \qquad z_r = ey_r = ex = x,$$

recalling that e is a left identity in its $\mathcal{R}$-class. For all i since $e, ey_i \in J$ and J is a completely simple semigroup, it follows from Proposition 1(i) that $e(ey_i) \mathcal{R} e$. Therefore $ey_i = eey_i \in R$ for all i. Now for any $a \in A$, $y_i a = y_{i+1}$ implies $(ey_i)a = (ey_{i+1})$, while $y_{i+1}a = y_i$ implies $(ey_{i+1})a = (ey_i)$. So the sequence $(z_i)_{0 \leq i \leq r}$ is a path in $\Gamma(S, A)$ contained in R, beginning at e and terminating at x. Consider a typical arc in this path: $ua = v$. This implies $u(ae) = v(e)$ and hence, since $ae, e \in B$ by definition, u and v are joined by a path of length at most 2 in $\Gamma_r(T, B)$. We conclude that in $\Gamma_r(T, B)$ there is a path from every vertex $x \in R$ to e. Since R was an arbitrary $\mathcal{R}$-class, the same is true for every $\mathcal{R}$-class of J. Also every pair of idempotents e_i and e_j of F are connected by a path in $\Gamma_r(T, B)$ of length 2 via $1 \in T$ since $e_i, e_j \in B$. We conclude that $\Gamma_r(T, B)$ is connected and so T is of type left-FP_1. $\square$

In particular, in Theorem 8, if $E(J)$ is finite then K has finite relative rank in H if and only if H is finitely generated which gives the following result.

Corollary 2. *Let S be a semigroup with finitely many left and right ideals, let J be the unique minimal ideal of S, and let H be a maximal subgroup of S in J. Then the following are equivalent:*

(i) *S is of type left-FP_1;*
(ii) *S is of type right-FP_1;*
(iii) *the group H is finitely generated (equivalently, H is of type FP_1).*

Currently we do not know whether Corollary 2 holds for left-FP_n for values of n greater than one. We do however know that it holds in one direction, passing from S to H, by virtue of Theorem 2.

7. Further applications and examples

In this section, we give some examples showing that the finiteness conditions imposed in the main results of the paper really are necessary. We also give some further applications of our results.

The following example shows that without the finiteness assumption on the number of left ideals, Theorem 2 (and therefore also Theorem 4) no longer holds.

Example 2. Let G be the free group over X where $X = \{x_i : i \in \mathbb{N}\}$. Let $S = M[G; I, \Omega; P]$ be the Rees matrix semigroup over G with structure matrix

$$P = \begin{pmatrix} 1 & 1 \\ 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \end{pmatrix}.$$

Then by Theorem 7, S is of type left-FP_1. But G is not of type FP_1 since G is an infinitely generated group.

Simple semigroups

In Theorem 5 we proved that if a completely simple semigroup is of type left- and right-FP_n then all of its maximal subgroups are of type FP_n. We shall now see that this result does not extend to simple semigroups in general.

We begin by quoting a well-known result regarding FP_n for amalgamated free products of groups; see [7] for a proof.

Proposition 7. *Let G be the amalgamated free product $A *_C B$ of groups A, B and C, and let $n \in \mathbb{N}$. If G is FP_n and A and B are FP_{n-1} then C is FP_{n-1}.*

Example 3. Let S be the monoid defined by the following finite presentation:

$$\langle a_1, a_2, a_3, a_4, a_1', a_2', a_3', a_4', b, c \mid a_j a_j' = a_j' a_j = 1, a_1 a_2 = a_3 a_4, bc = 1,$$
$$ba_j = a_j^2 b, a_j c = c a_j^2 \ (j = 1, 2, 3, 4) \rangle.$$

Then from [50, Proposition 3.3] the group of G units of S is defined by the following group presentation:

$$\langle a_1, a_2, a_3, a_4 \mid a_1^{2^i} a_2^{2^i} = a_3^{2^i} a_4^{2^i} \ (i = 0, 1, 2, 3, \ldots) \rangle.$$

As observed in [50] S is a Bruck–Reilly extension (see [33, Chapter 5] for a definition of Bruck–Reilly extension) of the group G and consequently S is simple and every maximal subgroup of S is isomorphic to G. Also, G is the free product with amalgamation of A_1, A_2 (both free groups of rank 2) where the amalgamated subgroup is not finitely generated. Therefore it follows by Proposition 7 that G is not of type FP_2. Recalling (see [42, Proposition 5.6] for instance) that every finitely presented monoid is of type left- and right-FP_2 we obtain the following.

Proposition 8. *There exists a simple monoid that is finitely presented, and hence of type left- and right-FP_2, none of whose maximal subgroups are of type FP_2.*

This leads naturally to the following question: is it true that for every $n \geq 1$ there is a simple monoid of type left- and right-FP_n none of whose maximal subgroups are of type FP_n?

Strong semilattice of monoids

We may apply our results to another fundamental construction from semigroup theory, the so-called *strong semilattices of monoids*. Let $Y = (Y, \leq)$ be a semilattice, and let A_α ($\alpha \in Y$), be a family of disjoint monoids indexed by Y. Denote by 1_α the identity of A_α. Suppose that for any two elements $\alpha, \beta \in Y$, $\beta \leq \alpha$, there exists a homomorphism $\phi_{\alpha,\beta} : A_\alpha \to A_\beta$ such that

(1) $\phi_{\alpha,\alpha}$ is the identity homomorphism on A_α
(2) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$, for all $\alpha, \beta, \gamma \in Y$ with $\gamma \leq \beta \leq \alpha$.

The set $S = \cup_{\alpha \in Y} A_\alpha$ can then be made into a semigroup by defining

$$ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}), \quad a \in A_\alpha, b \in A_\beta.$$

When all A_α are groups then we obtain exactly the Clifford monoids (originally introduced in [18]) from this construction (a *Clifford monoid* is a regular semigroup whose idempotents are central). More details on this construction may be found in [33, Chapter 4].

The following result characterises the property left-FP_n for strong semilattices of monoids, and so in particular for Clifford monoids.

Theorem 9. *Let $S = \mathcal{S}[Y; A_\alpha, \phi_{\alpha,\beta}]$ be a strong semilattice of monoids. Then S is of type left-FP_n if and only if Y has a minimal element e, and the monoid A_e is of type left-FP_n.*

Proof. Suppose S is of type left-FP_n. Then in particular S is of type left-FP_1 and hence the semilattice Y, which is a retract of S, is also of type left-FP_1 by [43, Theorem 3]. (In fact, left-FP_1 is even preserved by arbitrary homomorphic images, which is easily seen from Theorem 6.) By Theorem 6 this means that Y is right unitarily finitely generated. Let A be a finite right unitary generating set for Y. Let X be the subsemilattice generated by A, which is finite since A is finite, and let z be the unique minimal element of X. Define $Z = \{y \in Y : zy = yz = z\}$. Clearly Z is a subsemigroup of S, and is right unitary since for $x \in Z$ and $y \in Y$, $yx \in Z$ implies $yxz = z$ and so $yz = z$ which gives $y \in Z$. Since A is a right unitary generating set for Y it follows that $Z = Y$ and so z is the unique minimal element of Y. The result now follows by applying Theorem 3. □

For inverse semigroups it is known that left-FP_n and right-FP_n are equivalent. In particular this is true for Clifford monoids. The above theorem applies to Clifford monoids.

Corollary 3. *A Clifford monoid is of type FP_n if and only if it has a minimal idempotent e and the maximal subgroup G_e containing e is of type FP_n.*

Another related application of the results of Section 3 is the following.

Corollary 4. *Let S be an inverse semigroup with a minimal idempotent e, and let G be the maximal subgroup of S containing e. Then S is of type FP_n if and only if G is of type FP_n.*

8. Other homological finiteness properties

We conclude the paper with some remarks about some other homological finiteness properties of monoids.

The property bi-FP_n

In [3] Alonso and Hermiller introduced a property which they called *bi-FP_n* (the same property is called *weak bi-FP_n* in [43]).

A monoid M is said to be of type *bi-FP_n* if there is a finite rank length n resolution of $\mathbb{Z}$ by $(\mathbb{Z}M, \mathbb{Z}M)$-bimodules.

Pride [43] showed that a monoid is of type bi-FP_n (in the sense of Alonso and Hermiller) if and only if it is of type left- and right-FP_n. Therefore an alternative way or expressing Theorem 5 above is as follows.

Theorem 10. *A completely simple semigroup S is of type bi-FP_n if and only if it has finitely many left and right ideals and all of its maximal subgroups are of type FP_n.*

Cohomological dimension

Several problems regarding closure properties of homological finiteness conditions of monoids were posed in [50, Remark and Open Problem 4.5] and in [49, Open Problem 11.1(i)]. Specifically, in [50, Remark and Open Problem 4.5] it was asked whether for a regular semigroup S with finitely many left and right ideals whether S has property left-FP_n (resp. finite cohomological dimension) if and only if all maximal subgroups of S have property left-FP_n (resp. finite cohomological dimension). We have already observed above that the first of these questions, concerning property left-FP_n, has a negative answer. We may similarly answer negatively the question about cohomological dimension using [29, Theorem 1] which states that a monoid has left and right cohomological dimension zero if and only if it has a two-sided zero element. Therefore, by taking a group G with infinite cohomological dimension and adjoining a zero element we obtain a regular monoid with finitely many left and right ideals, and with finite cohomological dimension, but with a maximal subgroup that has infinite cohomological dimension.

In exactly the same way we see that neither the property left-FP_n, nor that of having finite cohomological dimension, is inherited by subsemigroups with finite Rees index (the Rees index of a subsemigroup T of a semigroup S is defined as the cardinality of the complement $S \setminus T$). This answers negatively two further open problems that were posed in [49, Open Problem 11.1(i)].

Acknowledgements

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