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Symmetry of some entire solutions to the Allen–Cahn equation in two dimensions

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ABSTRACT

In this paper, we prove even symmetry and monotonicity of certain solutions of Allen–Cahn equation in a half plane. We also show that entire solutions with *finite Morse index* and *four ends* must be evenly symmetric with respect to two orthogonal axes. A classification scheme of general entire solutions with *finite Morse index* is also presented using energy quantization.

1. Introduction

We shall consider entire solutions of the following Allen–Cahn equation

$$u_{xx} + u_{yy} - F'(u) = 0, \qquad |u| \leqslant 1, \qquad (x, y) \in \mathbb{R}^2, \tag{1.1}$$

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where F is a balanced double-well potential, i.e., $F \in C^{2,\beta}([-1,1])$ satisfies $F(1) = F(-1) = 0$ and

$$\begin{cases} F'(-1) = F'(1) = 0, \quad F''(-1) > 0, \quad F''(1) > 0; \\ F'(t) > 0, \quad t \in (-1, t_0); \quad F'(t) < 0, \quad t \in (t_0, 1) \end{cases} \tag{1.2}$$

for some $t_0 \in (0,1)$. Without loss of generality, we may assume that $t_0 = 0$. A typical example of balanced double-well potential is $F(u) = \frac{1}{4}(1-u^2)^2$, $u \in \mathbb{R}$.

It is well known that there exists a unique transition layer solution $g(y)$ (up to translation) to the one dimensional Allen–Cahn equation

$$\begin{cases} g''(s) - F'\big(g(s)\big) = 0, \quad s \in \mathbb{R}, \\ \lim\limits_{s\to\infty} g(s) = 1, \quad \lim\limits_{s\to-\infty} g(s) = -1. \end{cases} \tag{1.3}$$

We may assume that $g(0) = 0$. Indeed, g is a minimizer of the following energy functional

$$\mathbf{E}(v) := \int_{-\infty}^{\infty} \left[\frac{1}{2}\left|v'\right|^2 + F(v)\right] dx$$

in $\mathcal{H} := \{v \in H^1_{loc}(\mathbb{R}){:}\ -1 \leqslant v \leqslant 1,\ \lim_{s\pm\infty} v(s) = \pm 1\}$ and

$$\mathbf{e} := \mathbf{E}(g) = \int_{-1}^{1} \sqrt{2F(u)}\, du < \infty.$$

The solution g is non-degenerate in the sense that the linearized operator has a kernel spanned only by g'.

If u is an evenly symmetric solution in x, we may regard u as a solution in the half plane $\mathbb{R}^2_+ := \{(x,y) \mid x \geqslant 0,\ y \in \mathbb{R}\}$,

$$\begin{cases} u_{xx} + u_{yy} - F'(u) = 0, \quad |u| < 1, \quad (x,y) \in \mathbb{R}^2_+, \\ u_x(0,y) = 0, \quad y \in \mathbb{R}. \end{cases} \tag{1.4}$$

We may also assume that u satisfies the monotone condition

$$u_x(x,y) > 0, \quad x > 0,\ y \in \mathbb{R}. \tag{1.5}$$

Our main theorem states that u must be evenly symmetric with respect to y and monotone for $y > 0$.

Theorem 1.1. *Assume that $u(x,y)$ is even in x and satisfies* (1.4) *and* (1.5)*. Then u is even in y, i.e.,*

$$u(x,y) = u(x,-y), \quad (x,y) \in \mathbb{R}^2_+ \tag{1.6}$$

after a proper translation in y. Moreover, $u_y(x,y) < 0$ for $x > 0$, $y > 0$, and the nodal set of u for $x > x_0$ can be expressed as the graph of two $C^{3,\beta}$ functions $y = \pm k(x)$ which is asymptotically linear, i.e., $k(x) = \kappa x + C + o(1)$ for some constants $\kappa > 0$, C, as x goes to infinity.

In particular, we have

$$\lim_{x\to\infty} u(x,y) = 1, \quad \forall y \in \mathbb{R}. \tag{1.7}$$

This symmetry result may be regarded as the counterpart of De Giorgi conjecture for a half plane. The original De Giorgi conjecture concerns the symmetry of monotone solutions in the entire space, and has been studied intensively in last two decades. The conjecture has been essentially solved (see, e.g., [4,16,3,17,29] and [12]). Fully nonlinear version of similar symmetry results in two dimensions can be found in [14].

We shall prove Theorem 1.1 in three main steps. First, we carry out a preliminary analysis of the nodal set Γ of u and show that Γ can be regarded as graphs of two $C^{3,\beta}$ functions $y = k_i(x)$, $i = 1, 2$ for $x > x_0$ large enough; Second, we show that $k(x)$ must be asymptotically linear. Finally we use the moving plane method to conclude.

We shall also discuss the even symmetry of entire solutions whose asymptotically behavior at infinity are roughly prescribed. For example, we can show that an entire solution with *finite Morse index* and *four ends* must be evenly symmetric in both x and y, after a proper translation and rotation. For a finite integer $m \geqslant 0$, we say that a solution u defined in $\Omega \subset \mathbb{R}^n$ has *finite Morse index m* if m is the maximal dimension of any linear subspace of Sobolev space $H^1(\Omega)$ contained in

$$\mathcal{N} := \left\{ \phi \in H^1(\Omega): \int_\Omega |\nabla \phi|^2 + F''(u)\phi^2 \, dV < 0 \right\} \cap \{0\}. \tag{1.8}$$

If $m = 0$, u is also called a *stable* solution in Ω. If an entire solution u has *finite Morse index*, then it is well known that u must be *stable* outside a large enough ball B_{R_0} (see, e.g., [15,8] and [9]).

An entire solution u is called a solution with $2k$ *ends* for some positive integer k if the nodal set Γ of u outside a large disc $B_R(0)$ consists of $2k$ embedded C^1 curves $\Gamma_i := \{(r_i(t), \theta_i(t)): \forall t \geqslant 0\}$, $1 \leqslant i \leqslant 2k$, in polar coordinates, $r_i(t) \to \infty$ as $t \to \infty$, and

$$\Gamma_i \subset \left\{ (r, \theta): r \geqslant R, \ \theta_i^- < \theta < \theta_i^+, \ 1 \leqslant i \leqslant 2k \right\}$$

where $0 \leqslant \theta_i^- < \theta_i^+ < \theta_{i+1}^- < \theta_{i+1}^+ < 2\pi$, $1 \leqslant i \leqslant 2k - 1$.

We note that there is a similar but slightly different definition of $2k$ *ends* solutions in [11,24] and [23], where the asymptotically behavior of the level sets of $2k$ *ends* solutions are prescribed as straight lines. In their definition, it follows that such solutions must have finite Morse index (see [24]). On the other hand, with the definition of $2k$ *ends* solution in this paper, it can be shown that a $2k$ *ends* solution with finite Morse index must have level sets being asymptotically straight lines if the spreading angle $\theta_i^+ - \theta_i^-$ are less than π (see [19]). The latter condition is believed to be a technical condition, which may be removable. In this sense, the two definitions are almost equivalent among the finite Morse index solution class, with the definition in this paper being slightly more general.

We have the following symmetry result for entire solutions with *four ends*.

Theorem 1.2. *Suppose that u is an entire solution to* (1.1) *with* finite Morse index *and* four ends. *Assume also*

$$0 < \theta_i^+ - \theta_i^- < \pi, \quad 1 \leqslant i \leqslant 4. \tag{1.9}$$

Then, after a proper translation and rotation, u satisfies

$$u(x, y) = u(x, -y) = u(-x, y), \quad \forall (x, y) \in \mathbb{R}^2 \tag{1.10}$$

and

$$u_x(x, y) > 0, \qquad u_y(x, y) < 0, \quad \forall x > 0, \ y > 0 \tag{1.11}$$

and (1.7) *holds. Moreover, there exists an angle* $\Theta = 2\theta \in (0, \pi)$ *such that the nodal set of* u *in the first quadrant is a graph of a* $C^{3,\beta}$ *function* $y = k(x)$ *for* $x > X_0$ *large enough, and*

$$k(x) = x \tan\theta + o(1), \quad \text{as } x \to \infty.$$

There is a simpler version of the above theorem. If we assume that the level sets of a *four end* solution are asymptotically straight lines, then the solution must satisfy (1.10) and (1.11), after proper translation and rotation. We do not need to assume that u is of finite *Morse index*. See Theorem 4.2.

An entire solution u with *four ends* may be called a *saddle* solution. The above theorem may be regarded as a form of De Giorgi conjecture for saddle solutions. The angle Θ may be called the *contact angle* of u (see [19] for more discussion).

The condition (1.9) is a technical condition and is believed to be unnecessary. However, we need it for the proof of an energy bound in Lemma 5.1 for a functional

$$\mathcal{E}_R(u) := \int\limits_{B_R} \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) dx\,dy. \tag{1.12}$$

If we assume the energy bound (5.2) in Lemma 5.1 directly instead of (1.9), the conclusion of Theorem 1.2 still holds. Indeed, we have the following general energy quantization result. Note that a different energy quantization phenomenon has been shown for Ginzburg–Landau equation (see [5]).

Theorem 1.3. *Assume that* u *is an entire solution of* (1.1) *with* finite Morse index. *Then there holds either*

$$\lim_{R\to\infty} \mathcal{E}_R(u)/R = \infty, \tag{1.13}$$

or

$$\lim_{R\to\infty} \mathcal{E}_R(u)/R = 2k\mathbf{e} \tag{1.14}$$

for some positive integer k.

In the latter case, u *must be an entire solution with* $2k$ ends, *and the nodal set of* u *must be asymptotically straight lines. Moreover, if we denote the directions of these lines by* $\nu_i = \langle \cos\theta_i, \sin\theta_i \rangle$, $1 \leqslant i \leqslant 2k$, *then*

$$\sum_{i=1}^{2k} \nu_i = \langle 0, 0 \rangle. \tag{1.15}$$

It is suspected that the first case in Theorem 1.3 may not happen at all. It would be interesting to show that only (1.14) holds and for a given configuration ν_i, $1 \leqslant i \leqslant k$ there exist only two corresponding solutions with opposite signs after a proper translation. All entire solutions with *finite Morse index* could then be classified accordingly.

We note that the existence of entire solutions with *finite Morse index* and $2k$ *ends* has been shown in [10] for $k = 2$ and in [2] for general k, where the nodal sets are straight lines (see also [31,18,22] for more discussion on these solutions). More general solutions with nodal sets being almost parallel lines are found in [13]. It was also pointed out in [13] that there may not be any symmetry for entire solutions with six or more ends. Note also that (1.15) implies (1.9) for $k = 2$. Similar saddle solutions for vector valued Allen–Cahn equation are also constructed in [1].

It is noted that in [11], the moduli space of all $2k$ *end* solutions is studied. Since the submission and the posting of the original version of this paper, there have been new developments in the study of the moduli space of all *four end* solutions in [24] and [23]. Because *four end* solutions are evenly symmetric, their asymptotic behavior is determined by the *contact angle* Θ of the asymptotic straight half lines. It is shown in [24] that for any connected component in the moduli space of *four end* solutions the *contact angle* Θ can range from 0 to π. In addition, in [23] it is proven that there is only one connected component in the moduli space, which connects the saddle solution with crossing nodal lines in [10] and the solutions with almost parallel nodal lines in [13]. Thus the study of all *four end* solutions is parallel to results in the theory of minimal surfaces developed in [28]. (See [24] for a detailed discussion on this similarity.)

For a given $\Theta \in (0, \pi)$, the uniqueness of *four ends* entire solutions with *contact angle* Θ is still unknown. It is stated in [11] and [13] that the formal dimension of the moduli space of entire solutions with $2k$ *ends* is $2k$. For $k = 2$, it means that there is a local uniqueness of *saddle* solutions with a fixed *contact angle*, up to a translation and rotation. However, the global uniqueness is a very different and more difficult question.

The paper is organized as follows. In Section 2, some preliminary results for entire solutions of Allen–Cahn equation in all dimensions shall be stated. In Section 3, we will prove Theorem 1.1. In Section 4, a simpler version of Theorem 1.2 shall be proven. Theorem 1.3 and the energy quantization property will be proven in Section 5.

2. Some basic properties

In this section we shall state some useful properties of entire solutions to the Allen–Cahn equation. We first state a gradient estimate (1.1) for all dimensions which was proven in [26].

Proposition 2.1. *Assume that* $F(s) \geqslant 0$, $\forall s \in [-1, 1]$. *Suppose that* u *is a solution to* (1.1)*. Then*

$$|\nabla u|^2(x, y) \leqslant 2F\big(u(x, y)\big), \quad (x, y) \in \mathbb{R}^n. \tag{2.1}$$

It is also well known that u has the following exponential decay with respect to distance from the level set.

Proposition 2.2. *Assume that* u *is a solution to* (1.1)*. Then there exists constants* C *and* $\nu > 0$ *such that*

$$\big|u^2 - 1\big| + |\nabla u| + |\nabla^2 u| \leqslant Ce^{-\nu d(x, y)} \tag{2.2}$$

where $d(x, y)$ *is the distance to the nodal set* Γ *of* u.

This property can be proven by comparing u with a solution $u_R > 0$ of the Allen–Cahn equation in a ball B_R centered at (x, y) with zero boundary condition, where $R = d(x, y)$. (See, e.g., [16].)

The following monotonicity property of energy is shown in [27].

Proposition 2.3. *Assume that* u *is a solution to* (1.1)*. Then* $\mathcal{E}_R(u)/R$ *is increasing in* R.

3. Even symmetry of solutions on a half plane

We now consider an entire solution u which is even in x. Note that u may be regarded as a solution of (1.4) on a half plane.

We first study the limit of $u(x, y)$ as x goes to infinity.

Define

$$u^\tau(x, y) := u(\tau + x, y), \quad x \geqslant -\tau, \ \forall y \in \mathbb{R}.$$

It is easy to see that $u^\tau(x, y)$ converges to some function $u^+(y) > -1$ in $C^3_{loc}(\mathbb{R}^2)$ as τ goes to infinity, and $u^+(y)$ satisfies one dimensional Allen–Cahn equation

$$u_{yy} - F'(u) = 0, \quad y \in \mathbb{R}. \tag{3.1}$$

Let

$$\sigma^\tau(x,y) = \frac{u_x^\tau(x,y)}{u_x^\tau(0,0)} > 0, \quad \forall x \geqslant -\tau,\ y \in \mathbb{R}.$$

By the Harnack inequality and the gradient estimate for elliptic equations, we know that $\sigma^\tau(x,y)$ converges to $\sigma^*(x,y) > 0$ in $C^2_{loc}(\mathbb{R}^2)$ as τ goes to infinity, and $\sigma^*(x,y)$ satisfies the linearized equation of Allen–Cahn equation

$$\sigma_{xx} + \sigma_{yy} - F''\big(u^+(y)\big)\sigma = 0, \quad (x,y) \in \mathbb{R}^2. \tag{3.2}$$

Hence u^+ is a stable solution of (3.1) and $u^+ \not\equiv 0$. Then, by solving (3.1) explicitly, we know that there are three possibilities for u^+:

(i) $u^+ \equiv 1$;
(ii) $u^+(y) = g(y-K)$ for some constant K;
(iii) $u^+(y) = g(K-y)$ for some constant K.

The next goal is to show that only (i) holds. To do so, we shall prove several basic properties for u. The first property is an energy estimate of u on a line.

3.1. Energy estimate

We first show a simple but important lemma regarding the energy of u on y-axis.

Lemma 3.1. *Suppose that u is a solution to* (1.4) *and* (1.5)*. Then*

$$\int_{\mathbb{R}} \left[F\big(u(0,y)\big) + \frac{1}{2}u_y^2(0,y)\right] dy < 3\mathbf{e}. \tag{3.3}$$

Proof. Define

$$h(y) = \int_0^\infty u_y u_x \, dx, \quad \forall y \in \mathbb{R}.$$

In view of (2.1) and the positivity of u_x, it is easy to see that $h(y)$ is well defined and

$$\big|h(y)\big| < \int_0^\infty \sqrt{2F\big(u(x,y)\big)} \cdot u_x \, dx \leqslant \mathbf{e} - G\big(u(0,y)\big) < \mathbf{e}, \quad \forall y \in \mathbb{R}$$

where

$$G(t) := \int_{-1}^t \sqrt{2F(s)}\, ds, \quad \forall t \in [-1,1]. \tag{3.4}$$

Differentiating $h(y)$ with respect to y and using (1.4), we obtain

$$h'(y) = \int_0^\infty (u_{yy}u_x + u_y u_{xy})\,dx$$
$$= \int_0^\infty \left[\frac{\partial}{\partial x}\left(F(u) - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2\right)\right]dx$$
$$= \left[F\big(u^+(y)\big) + \frac{1}{2}\big(u_y^+\big)^2(y)\right] - \left[F\big(u(0,y)\big) + \frac{1}{2}u_y^2(0,y)\right]. \tag{3.5}$$

Here we have used the facts $u_x(0,y) = 0$ and $\lim_{x\to\infty} u_x(x,y) = 0$, $\forall y \in \mathbb{R}$. Then, we derive

$$\int_a^b \left[F\big(u(0,y)\big) + \frac{1}{2}u_y^2(0,y)\right]dy$$
$$= \int_a^b \left[F\big(u^+(y)\big) + \frac{1}{2}\big(u_y^+\big)^2(y)\right]dy + \big(h(a) - h(b)\big). \tag{3.6}$$

Define

$$\rho(x) = \int_{\mathbb{R}} \left[F\big(u(x,y)\big) + \frac{1}{2}u_y^2(x,y) - \frac{1}{2}u_x^2(x,y)\right]dy \tag{3.7}$$

and

$$\rho^+ = \int_{\mathbb{R}} \left[F\big(u^+(y)\big) + \frac{1}{2}\big(u_y^+\big)^2(y)\right]dy. \tag{3.8}$$

Then, letting $a \to -\infty$ and $b \to +\infty$ in (3.6), in view of the bound of $h(y)$ we obtain

$$\rho(0) = \rho^+ + \lim_{a\to-\infty} h(a) - \lim_{b\to\infty} h(b) \leqslant 3\mathbf{e}. \tag{3.9}$$

Therefore, (3.3) is proven. □

3.2. A Hamiltonian identity

Next we shall show a Hamiltonian identity for solutions of (1.4).

Lemma 3.2. *Assume that $u(x,y)$ satisfies (1.4) and (1.5). Then*

$$\rho(x) = \rho(0), \quad \forall x \in \mathbb{R}_+. \tag{3.10}$$

Proof. By (3.3) and the boundedness of u in $C^3(\mathbb{R}^n)$, we know that the following limits exist

$$v^+ := \lim_{y\to\infty} u(0,y), \qquad v^- := \lim_{y\to-\infty} u(0,y)$$

and

$$\left|v^+\right| = 1, \qquad \left|v^-\right| = 1.$$

Indeed, by the standard translation argument it can be shown that

$$v^+_\infty(x,y) := \lim_{t\to\infty} u(x,y+t), \qquad v^-_\infty(x,y) := \lim_{t\to-\infty} u(x,y+t)$$

exist and are solutions to (1.4), and hence

$$v^+_\infty(x,y) \equiv v^+, \qquad v^-_\infty(x,y) \equiv v^-, \quad (x,y)\in\mathbb{R}^2.$$

In particular,

$$\begin{aligned}
&\lim_{y\to\infty} u_x(x,y) = \left(v^+_\infty\right)_x(x,y) = 0, \qquad \lim_{y\to-\infty} u_x(x,y) = \left(v^-_\infty\right)_x(x,y) = 0,\\
&\lim_{y\to\infty} u_y(x,y) = \left(v^+_\infty\right)_y(x,y) = 0, \qquad \lim_{y\to-\infty} u_y(x,y) = \left(v^-_\infty\right)_y(x,y) = 0.
\end{aligned} \tag{3.11}$$

Define

$$h_R(y) := \int_0^R u_y u_x \, dx, \quad \forall y\in\mathbb{R}.$$

Then, in view of (3.11), we have

$$\lim_{|y|\to\infty} h_R(y) = 0, \quad \forall R \geqslant 0.$$

As before, differentiating $h_R(y)$ with respect to y and using (1.4), we can obtain

$$\begin{aligned}
h'_R(y) &= \int_0^R (u_{yy}u_x + u_y u_{xy})\,dx\\
&= \int_0^R \left[\frac{\partial}{\partial x}\left(F(u) - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2\right)\right] dx\\
&= \left[F\big(u(R,y)\big) + \frac{1}{2}u_y^2(R,y) - \frac{1}{2}u_x^2(R,y)\right] - \left[F\big(u(0,y)\big) + \frac{1}{2}u_y^2(0,y)\right].
\end{aligned}$$

Then, integrating the above with respect to y in $\mathbb{R}$, we derive

$$\rho(0) - \rho(R) = \lim_{a\to-\infty} h_R(a) - \lim_{b\to\infty} h_R(b) = 0. \tag{3.12}$$

The lemma is proven. □

The Hamiltonian identity (3.10) for Allen–Cahn equation was first formulated and proved in [19] by using an elementary approach. It also follows, at least formally, from a general balancing formula (or conservation law) for variational problems under deformations generated by a smooth vector field. See [25,30] and references therein. The Hamiltonian identities in general can be rigorously proven using the general balance formula once the behavior of solutions at infinity can be taken care of. However, in the proof of (3.10) a more refined argument is needed due to the lack of information on the asymptotical behavior of the solution to begin with. For example, we do not know ahead of time whether the nodal set of u is asymptotically straight lines or not. Indeed, we need the Hamiltonian identities to show that the nodal set is asymptotically straight.

This general idea of balance formula or conservation law is also used to prove Pohazaev identity. A nice presentation of Hamiltonian identities using this idea can be found in [11] and [24]. We note that the Hamiltonian identity is sometimes more fundamental than Pohazaev identity due to it's simple form and the information on lower dimensional spaces (see [19,20] and [21]).

We can indeed show the following limit.

Lemma 3.3.

$$\lim_{|y|\to\infty} u(x,y) = -1, \quad \forall x \in \mathbb{R}. \tag{3.13}$$

Proof. We shall show the lemma by considering different cases.

In Case (i), i.e., $u^+ \equiv 1$, there are four possibilities:

(1) $v^+ = 1$, $v^- = 1$;
(2) $v^+ = -1$, $v^- = 1$;
(3) $v^+ = 1$, $v^- = -1$;
(4) $v^+ = -1$, $v^- = -1$.

From (3.9) and the Hamiltonian identity (3.10) we have

$$\lim_{a\to-\infty} h(a) - \lim_{b\to\infty} h(b) + \rho^+ = \rho(0) = \lim_{x\to\infty} \rho(x). \tag{3.14}$$

In subcase (1), we can estimate

$$\lim_{a\to-\infty} \big|h(a)\big| \leqslant \lim_{a\to-\infty} \big[G(1) - G\big(u(0,a)\big)\big] = 0$$

and

$$\lim_{b\to\infty} \big|h(b)\big| \leqslant \lim_{b\to\infty} \big[G(1) - G\big(u(0,b)\big)\big] = 0.$$

Then (3.9) becomes

$$\rho(0) = \lim_{a\to-\infty} h(a) - \lim_{b\to\infty} h(b) = 0.$$

This is a contradiction, and therefore subcase (1) is excluded.

In subcase (2), we can estimate

$$\lim_{a\to-\infty} \big|h(a)\big| \leqslant \lim_{a\to-\infty} \big[G(1) - G\big(u(0,a)\big)\big] \leqslant \mathbf{e}$$

and

$$\lim_{b\to\infty}\left|h(b)\right| \leqslant \lim_{b\to\infty}\left[G(1)-G\left(u(0,b)\right)\right]=0.$$

Then (3.9) becomes

$$\rho(0)\leqslant \mathbf{e}.$$

On the other hand, by the definition of $\mathbf{e}$, we have $\rho(0)\geqslant \mathbf{e}$. Then we have $u(0,y)=g(\pm y+K_1)$ for some $K_1\in\mathbb{R}$. Then $u(x,y)-u(0,y)$ is nonnegative and satisfies a linearized equation of (1.1). By the Harnack inequality, we can derive $u(x,y)\equiv u(0,y)$. This contradicts with (1.5), and hence subcase (2) is excluded. Subcase (3) is similar to subcase (2). The lemma then follows easily from (3.11).

In Case (ii), i.e., $u^+(y)=g(y-K)$, in view of the monotone condition (1.5) we know only subcases (2) and (4) are possible. If subcase (2) happens, then (3.9) becomes

$$\rho(0)=\mathbf{e}+\lim_{a\to-\infty}h(a)-\lim_{b\to\infty}h(b)=\mathbf{e}.$$

Since $\rho(0)\geqslant\mathbf{e}$, we get a contradiction immediately as in Case (i). Therefore subcase (2) is excluded.

Case (iii) is similar to Case (ii). In all cases, we have proved that only subcase (4) holds. Hence (3.13) is proven. □

In the level set analysis below, we shall focus on Case (i): $u^+\equiv 1$. The other two cases can be discussed similarly with minor modifications, and can be excluded eventually at the end of this section.

In view of (1.5) and (3.13), the nodal set Γ of u can be represented by the graph of a function $x=\gamma(y)$ which is defined for $y\leqslant K_1$, and $y\geqslant K_2$ with $K_1\leqslant K_2$ and is C^3. By Lemma (3.3), we also know

$$\lim_{|y|\to\infty}\gamma(y)=\infty. \tag{3.15}$$

3.3. *The slope of the level set has a limit*

First we show the limits of $\gamma'(y)$ exist as $y\to\pm\infty$.

Lemma 3.4. *There exist $\theta_1\in[0,\pi/2]$ and $\theta_2\in[-\pi/2,0]$ such that*

$$\lim_{y\to\infty}\gamma'(y)=\tan\theta_1,\qquad \lim_{y\to-\infty}\gamma'(y)=\tan\theta_2. \tag{3.16}$$

Here we use the convention that $\tan(\pi/2)=\infty$, $\tan(-\pi/2)=-\infty$.

Proof. We first show that u behaves like a one dimensional solution along the level set curve γ as y goes to infinity. For any sequence $\{y_m\}$ and constant $\theta\in[-\pi/2,\pi/2]$ with $|y_m|\to\infty$ and

$$\lim_{m\to\infty}\gamma'(y_m)=\tan\theta,$$

we define

$$u^m(x,y):=u\left(x+\gamma(y_m),y+y_m\right),\quad x\geqslant-\gamma(y_m),\ y\in\mathbb{R}.$$

Then u^m converges to u^* in $C^3_{loc}(\mathbb{R}^2)$ after taking a subsequence if necessary, where u^* is a solution of (1.1) with $\frac{\partial u^*}{\partial x}(x,y) \geqslant 0$, $(x,y) \in \mathbb{R}^2$. By the Harnack inequality, we know that either $\frac{\partial u^*}{\partial x}(x,y) \equiv 0$, or $\frac{\partial u^*}{\partial x}(x,y) > 0$, $(x,y) \in \mathbb{R}^2$. In the first case, we define

$$\sigma^m(x,y) = \frac{u^m_x(x,y)}{u^m_x(0,0)} > 0, \quad \forall x \geqslant -\gamma(y_m),\ y \in \mathbb{R}.$$

By the Harnack inequality and the gradient estimate for elliptic equations, we know that $\sigma^m(x,y)$ converges along a subsequence to $\sigma^*(x,y) > 0$ in $C^2_{loc}(\mathbb{R}^2)$ as m goes to infinity. Furthermore, $\sigma^*(x,y)$ satisfies the linearized equation of Allen–Cahn equation at u^*

$$\sigma_{xx} + \sigma_{yy} - F''\big(u^*\big)\sigma = 0, \quad (x,y) \in \mathbb{R}^2. \tag{3.17}$$

Hence u^* is stable in both cases. By the De Giorgi conjecture for $n = 2$ [16], we know that u^* depends only on one direction. Since $u^*(0,0) = 0$, we conclude

$$u^*(x,y) = g(x\cos\theta - y\sin\theta), \quad \forall (x,y) \in \mathbb{R}^2. \tag{3.18}$$

Note that straightforward computations can lead to

$$\rho^*(\theta) := \int\limits_{\mathbb{R}} \left[F\big(u^*(x,y)\big) + \frac{1}{2}\big(u^*_y\big)^2(x,y) - \frac{1}{2}\big(u^*_x\big)^2(x,y) \right] dy = \mathbf{e}\sin\theta. \tag{3.19}$$

(See, e.g., [19].) This relation between the angle θ and ρ shall be used to show (3.16), intuitively by using Hamiltonian identity (3.10). However, due to the involvement of two interfaces which may not be separated far enough, we need to argue more carefully as follows.

Let

$$\limsup_{y\to\infty} \gamma'(y) = \tan\theta_1 \tag{3.20}$$

for some $\theta_1 \in [0, \pi/2]$.

If $\liminf_{y\to\infty} \gamma'(y) = \tan\theta_0 < \tan\theta_1$ for some $\theta_0 \in [-\pi/2, \theta_1)$, then, for any fixed $\theta \in (\theta_0, \theta_1)$ there exists a sequence $\{y_m\}$ with $\lim_{m\to\infty} \gamma'(y_m) = \tan\theta$ and $y_m \to \infty$ as $m \to \infty$.

For any fixed $R > 0$, by the monotone condition (1.5) and (3.18) we have

$$\begin{aligned}
\lim_{m\to\infty} h(y_m) &= \lim_{m\to\infty} \int\limits_{\gamma(y_m)-R}^{\gamma(y_m)+R} u_x u_y \, dx \\
&\quad + O(1)\cdot \lim_{m\to\infty}\big[G(1) - G\big(u\big(\gamma(y_m)+R, y_m\big)\big)\big] \\
&\quad + O(1)\cdot \lim_{m\to\infty}\big[G\big(u\big(\gamma(y_m)-R, y_m\big)\big) - G\big(u(0,y_m)\big)\big] \\
&= -\sin\theta\big[G\big(g(R\cos\theta)\big) - G\big(g(-R\cos\theta)\big)\big] + O(1)\big[G\big(g(-R)\big)\big]
\end{aligned} \tag{3.21}$$

where G is defined in (3.4) and $O(1)$ is with respect to $R \to \infty$. Letting R go to infinity, we obtain

$$\lim_{m\to\infty} h(y_m) = -\mathbf{e}\sin\theta.$$

By (3.6), we know that $\lim_{a\to\infty} h(a)$ exists and hence

$$\lim_{y\to\infty} h(y) = -\mathbf{e}\sin\theta.$$

This leads to

$$\lim_{y\to\infty} \gamma'(y) = \tan\theta$$

which contradicts (3.20). Therefore the first limit in (3.16) is proven.

Similarly, we can show the second limit in (3.16). □

Furthermore, by (3.14) we have

$$\mathbf{e}(\sin\theta_1 - \sin\theta_2) = \rho(0) = \lim_{R\to\infty} \rho(R). \tag{3.22}$$

We note that in Case (ii), the above discussion can be modified with $\theta_2 = -\pi/2$ and $y \to -\infty$ being replaced by $y \to K$. Similar modifications can be done for Case (iii) with $\theta_1 = \pi/2$.

3.4. The limits of slopes differ by a sign

We shall show that the limits of the slopes of the level set differ only by a sign, i.e., $\theta_1 = -\theta_2 \in (0, \pi/2)$.

Lemma 3.5. *There holds*

$$\theta_1 = -\theta_2. \tag{3.23}$$

Proof. Recall that u is an *even* solution in $\mathbb{R}^2$ with respect to x.

Let us choose an angle $\theta \in (0, \pi/2)$, $\theta \neq \theta_1, -\theta_2$ and a Cartesian coordinate system (z_1, z_2) such that z_1-axis and y-axis form an angle θ. In other words, we have $x = z_1\sin\theta + z_2\cos\theta$, $y = z_1\cos\theta - z_2\sin\theta$. By (2.2), we know that

$$\left|u^2(z_1, z_2) - 1\right| + \left|\nabla u(z_1, z_2)\right| + \left|\nabla^2 u(z_{1,2})\right| \leqslant Ce^{-\nu_1|z_1|}, \quad \forall z_1 \in \mathbb{R} \tag{3.24}$$

for some positive constants $\nu_1 > 0$ and C.

Therefore, there holds a Hamiltonian identity like (3.10) with respect to z. Namely,

$$\bar\rho(\theta, z_2) := \int_{\mathbb{R}} \left[F\big(u(z_1, z_2)\big) + \frac{1}{2}u_{z_1}^2(z_1, z_2) - \frac{1}{2}u_{z_2}^2(z_1, z_2)\right] dz_1 = \bar\rho(\theta, 0) < \infty. \tag{3.25}$$

(The proof is similar to (3.10); See also Theorem 1.1 in [19].) When $\theta > \theta_1$, $\theta > -\theta_2$, a straightforward computation can lead to

$$\begin{cases} \lim\limits_{z_2\to\infty} \bar\rho(\theta, z_2) = \mathbf{e}\big(\sin(\theta - \theta_1) + \sin(\theta + \theta_1)\big); \\ \lim\limits_{z_2\to-\infty} \bar\rho(\theta, z_2) = \mathbf{e}\big(\sin(\theta - \theta_2) + \sin(\theta + \theta_2)\big). \end{cases} \tag{3.26}$$

Then we have

$$\sin(\theta-\theta_1)+\sin(\theta+\theta_1)=\sin(\theta-\theta_2)+\sin(\theta+\theta_2)$$

and hence $\theta_1=-\theta_2$.

The same conclusion can be reached if θ is in other range compared to $\theta_1, -\theta_2$, with only slight difference in the expression in (3.26). The details is left to the reader. See also (2.12) in [19]. □

Since $\rho(0)>0$, an easy consequence of Lemma 3.5 and (3.22) is $\theta_1=-\theta_2>0$. Next we shall show $\theta_1<\pi/2$.

If $\theta_1=\pi/2$, we choose $\theta\in(0,\pi/2)$ and carry out the same computation as (3.26) to obtain

$$\bar\rho(\theta,0)=\lim_{z_2\to\infty}\bar\rho(\theta,z_2)=2\mathbf{e}\sin(\pi/2-\theta). \tag{3.27}$$

Letting $\theta\to\pi/2$, we obtain

$$\lim_{\theta\to\pi/2}\bar\rho(\theta,0)=0.$$

On the other hand, by (2.1) we have

$$\lim_{\theta\to\pi/2}\bar\rho(\theta,0)\geqslant\int_{\mathbb{R}}\frac{1}{2}u_x^2(x,0)\,dx>0.$$

This is a contradiction, and hence proves $\theta_1<\pi/2$.

3.5. The level set is asymptotically a straight line

Below we quote a lemma from [19] on the asymptotical behavior of the level set.

Lemma 3.6. *Suppose that $u(y_1,y_2)$ is a solution of (1.4) in a cone $\mathcal{C}:=\{y\in\mathbb{R}^2\colon\ |y_1|\leqslant y_2\tan\alpha_0,\ y_2\geqslant M>0\}$ for some $0<\alpha_0<\pi/2$. The nodal set of u in $\mathcal{C}$ is given by the graph of a function $y_1=k(y_2)$. Assume*

$$\lim_{y_2\to\infty}k'(y_2)=0. \tag{3.28}$$

Then there is a finite number A such that

$$\lim_{y_2\to\infty}k(y_2)=A_1. \tag{3.29}$$

The lemma can be shown in three steps. First, we show that an energy of u on a line segment $[-y_2\tan\alpha, y_2\tan\alpha_0]$, $\alpha\in(0,\alpha_0)$ is exponentially close to $\mathbf{e}$ as y_2 tends to ∞. Second, we construct an optimal approximation of $u(\cdot,y_2)$ by a shift of the one dimensional solution $g(y_1-l(y_2))$, and show that the error is exponentially small in L^2 norm as y_2 goes to infinity. Finally, we deduce that the shift $l(y_2)$ has a finite limit, and then conclude that $k(y_2)$ has a finite limit. For the details of the proof, the reader is referred to [19].

Now we choose the coordinate system (y_1,y_2) so that y_2-axis form an angle θ_1 with y-axis, and $\alpha_0<\min\{\pi/2-\theta_1,\theta_1\}$. Using Lemma 3.4 and Lemma 3.6, we conclude that

$$\gamma(y)=(\tan\theta_1)y+A_2+o(1),\quad \text{as } y\to\infty. \tag{3.30}$$

Similarly, we can show

$$\gamma(y) = -(\tan\theta_1)y + A_3 + o(1), \quad \text{as } y \to -\infty. \tag{3.31}$$

Then, for Y_0 large enough, the inverse functions of $\gamma(y)$ for $y > Y_0$ and $y < Y_0$ exist, and may be written as $y = k_1(x)$, $y = k_2(x)$ respectively. Moreover,

$$k_1(x) = \kappa x + B_1 + o(1), \qquad k_2(x) = -\kappa x + B_2 + o(1) \tag{3.32}$$

as $x \to \infty$, where $\kappa = \cot\theta_1$ is a positive (finite) constant, and B_1, B_2 are constants.

3.6. *The moving plane method*

Next we shall use the moving plane method to show the even symmetry of u with respect to y. Due to the fact that the asymptotical behavior of u is not homogeneous near infinity, in particular, there is a transition layer along the nodal set, the classic moving plane method has to be carefully modified. Indeed, we have to use the exact asymptotical formulas of the nodal sets $y = k_i(x), i = 1, 2$ near infinity as well the asymptotical behavior of u along these curves.

For this purpose, we define $u_\lambda(x, y) := u(x, 2\lambda - y)$ and $w_\lambda := u_\lambda - u$ in $D_\lambda := \{(x, y)\colon\ x \geqslant 0, y \geqslant \lambda\}$.

Lemma 3.7. *When λ is sufficiently large, there holds $w_\lambda > 0$ in D_λ.*

Proof. We first fix X_0 sufficiently large so that $k_1(x), k_2(x)$ are well defined. By the property of double-well potential (1.2), there exists a sufficiently small constant $\delta > 0$ such that $F''(t) > 0$, $t \in [-1, -1 + \delta] \cup [1 - \delta, 1]$. There is also a sufficiently large constant $R_1 > 0$ such that $-1 < g(s) \leqslant -1 + \delta/2$, $\forall s < -R_1$ and $1 - \delta/2 \leqslant g(s) < 1$, $\forall s > R_1$, where g is the one dimensional solution in (1.3). By (3.32) and (3.18), there exist X_1, R_2 sufficiently large such that for $x > X_1$,

$$\begin{cases} u(x, y) < -1 + \delta, & \text{if } y > k_1(x) + R_2, \text{ or } y < -k_2(x) - R_2, \\ u(x, y) > 1 - \delta, & \text{if } 0 < y < k_1(x) - R_2, \text{ or } -k_2(x) + R_2 < y < 0, \\ \left|u(x, y) + g(y\sin\theta_1 - x\cos\theta_1 - B_1\sin\theta_1)\right| \leqslant \delta/2, & \text{if } k_1(x) - R_2 < y < k_1(x) + R_2, \\ \left|u(x, y) - g(y\sin\theta_1 + x\cos\theta_1 - B_2\sin\theta_1)\right| \leqslant \delta/2, & \text{if } k_2(x) - R_2 < y < k_2(x) + R_2. \end{cases} \tag{3.33}$$

When $\lambda > \lambda_1$ is sufficiently large, by (3.32) we have

$$k_2^\lambda(x) := 2\lambda - k_2(x) \geqslant k_1(y) + R_2, \quad \forall x \geqslant X_1.$$

By Lemma 3.3, we can also choose λ_1 so that

$$u(x, y) < -1 + \delta, \quad 0 < x < X_1,\ y > \lambda_1.$$

We claim that $w_\lambda \geqslant 0$ in D_λ for $\lambda > \lambda_1$, and shall show this claim in the following three subsets of D_λ respectively:

$$D_\lambda^+ := \left\{(x, y)\colon\ 0 < x < X_1,\ y > \lambda,\ \text{or } x > X_1,\ y > k_2^\lambda(x)\right\},$$
$$D_\lambda^- := \left\{(x, y)\colon\ x > X_1,\ y < k_1(x)\right\},$$
$$D_\lambda^0 := \left\{(x, y)\colon\ x > X_1,\ k_1(x) < y < k_2^\lambda(x)\right\}.$$

If the claim is not true in D_λ^+, then there exists a sequence of points $\{(x_m, y_m)\}_{m=1}^\infty \in D_\lambda^+$ such that

$$\lim_{m\to\infty} w_\lambda(x_m, y_m) = \lim_{m\to\infty}\big(u_\lambda(x_m, y_m) - u(x_m, y_m)\big) = \inf_{D_\lambda^+} w_\lambda(x, y) < 0.$$

It can be seen from (3.33) that $u_\lambda(x_m, y_m) < u(x_m, y_m) < -1+\delta$ when m is large enough. Then we can use the standard translating arguments to obtain a contradiction as follows. Define $w_\lambda^m(x, y) := w_\lambda(x+x_m, y+y_m)$ in $D_\lambda^+ - (x_m, y_m)$. Then w_λ^m converges to $w_\lambda^\infty(x, y)$ in $C^3_{loc}(D^\infty)$ for some piecewise Lipschitz domain D^∞ in $\mathbb{R}^2$ which contains a small ball centered at the origin. Furthermore, w_λ^∞ attains its negative minimum at the origin and satisfies a linearized equation

$$w_{xx} + w_{yy} - F''\big(\xi(x, y)\big)w = 0, \quad (x, y) \in D^\infty \tag{3.34}$$

where $\xi(x, y) = su(x, y) + (1-s)u_\lambda(x, y)$ for some $s(x, y) \in (0, 1)$ and $F''(\xi(0, 0)) > 0$. This is a contradiction, which leads to the claim in D_λ^+.

Similarly, the claim can be shown in D_λ^- by the strong maximum principle, due to the fact that $u_\lambda > 1-\delta$ in D_λ^- as in (3.33). The claim is also true in D_λ^0 when λ is large enough, due to the last two estimates in (3.33).

Then, using the strong maximum principle (or the Harnack inequality) to an elliptic equation satisfied by w_λ which is similar to (3.34), the lemma is proven. □

Now we define

$$\Lambda = \inf\big\{\lambda:\ u_\lambda(x, y) > u(x, y),\ (x, y) \in D_\lambda\big\}.$$

Lemma 3.8. *There holds*

$$\Lambda = (B_1 + B_2)/2$$

where B_1, B_2 are as in (3.32).

Proof. We shall prove this lemma by contradiction. Suppose the lemma does not hold. By (3.32), we can easily see that $\Lambda > (B_1 + B_2)/2$ and $w_\Lambda > 0$, $\forall (x, y) \in D_\Lambda$. Then there exists a sequence of numbers $\{\lambda_m\}$ such that $\lambda_m < \Lambda$, and $\lim_{m\to\infty} \lambda_m = \Lambda$ and the infimum of w_{λ_m} in D_{λ_m} is negative. Using (3.18) and the translating arguments as above, we can show that the infimum of w_{λ_m} in D_{λ_m} is achieved at a point (x_m, y_m), i.e.,

$$w_{\lambda_m}(x_m, y_m) = \inf_{D_{\lambda_m}} w_{\lambda_m} < 0. \tag{3.35}$$

Since w_{λ_m} satisfies an elliptic equation similar to (3.34) with $\xi(x_m, y_m) = su(x_m, y_m) + (1-s)u_{\lambda_m}(x_m, y_m)$ for some $s \in (0, 1)$, by the strong maximum principle we know that $u(x_m, y_m) > -1+\delta$ and hence $y_m - k_1(x_m) < R_2$ if $x_m > X_1$. By (3.18) and the assumption $\Lambda > (B_1 + B_2)/2$, we know $x_m < X_2$ for some constant X_2 independent of m. Therefore there exists a subsequence of $\{m\}$ (still denoted by itself) such that (x_m, y_m) converges to $(x_0, y_0) \in \bar{D}_\Lambda$ and w_{λ_m} converges to w_Λ in $C^3_{loc}(D_\Lambda)$ as well as in $C^3(B_1(x_0, y_0) \cap \bar{D}_\Lambda)$. It is easy to see that $\nabla w_\Lambda(x_0, y_0) = 0$. Furthermore, w_Λ is an even function in x and satisfies an elliptic equation similar to (3.34) in D_Λ, by the Harnack inequality we can see that (x_0, y_0) is not on the y-axis. Hence (x_0, y_0) must be on the portion of boundary $\{(x, y):\ y = \Lambda\}$ of D_Λ. Then by the Hopf Lemma, we have $\frac{\partial}{\partial y} w_\Lambda(x_0, y_0) > 0$. This is a contradiction, which proves the lemma. □

We note that $u_\Lambda \geqslant u$ in D_Λ and $u_y(x,\lambda) = -\frac{1}{2}\frac{\partial}{\partial y} w_\lambda(x,\lambda) < 0$, $\forall x \in \mathbb{R}$ when $\lambda > \Lambda$. Similarly, we can use the moving plane method from below, i.e., repeating the above procedure for $w_\lambda := u_\lambda - u$ in $D_\lambda^c := \{(x,y)\colon\ x > 0,\ \ y < \lambda\}$, and conclude $u_\Lambda \geqslant u$ in D_Λ^c and hence $u_\Lambda \leqslant u$ in D_Λ. Therefore, Theorem 1.1 is proven.

4. Even symmetry of entire solutions with four ends

We shall show that certain entire solutions of (1.1) with *four ends* must be evenly symmetric with respect to both x-axis and y-axis after a proper translation and rotation. First we consider the case that the *four ends* are asymptotically straight lines, i.e., on each 0-level set Γ_i there holds

$$y = \tan(\theta_i)x + A_i + o(1) \quad \text{as } x \to \infty,\ 1 \leqslant i \leqslant 4 \tag{4.1}$$

where $0 < \theta_i < \theta_{i+1} < 2\pi$, and $\theta_i \neq \pi/2$, $\theta_i \neq \pi/2$, $i = 1,2,3,4$. Without loss of generality, after a proper rotation we may also assume that $0 < \theta_1 = 2\pi - \theta_4 < \pi/2$ and $\theta_2 \neq \pi$, $\theta_3 \neq \pi$.

By Proposition 2.2, we know that Hamiltonian identity (3.10) holds. Moreover, in view of (3.18), on a fixed cone $\{(r,\theta) = (x,y)\colon\ \theta_{i-1} + \delta < \theta < \theta_{i+1} - \delta\}$ with a sufficiently small $\delta > 0$ there holds

$$\bigl|u(x,y) - g(x\sin\theta_i - y\cos\theta_i + A_i\cos\theta_i)\bigr| \to 0, \quad \text{uniformly as } r \to \infty. \tag{4.2}$$

As in (3.19), by Hamiltonian identity (3.10) we can easily obtain that

$$\rho(x) = \mathbf{e}(\cos\theta_1 + \cos\theta_4) = \mathbf{e}(-\cos\theta_2 - \cos\theta_3).$$

Similarly, when x-axis is replaced by y-axis in Hamiltonian identity (3.10), we obtain

$$\mathbf{e}(\sin\theta_1 + \sin\theta_2) = \mathbf{e}(-\sin\theta_3 - \sin\theta_4).$$

We can easily derive that

$$\pi - \theta_2 = \theta_1 = \theta_3 - \pi.$$

Now we follow the moving plane procedure as in the proof of Theorem 1.1. It can be shown that Lemma 3.4 still holds with D_λ being modified as $\{(x,y)\colon\ y \geqslant \lambda\}$. Furthermore, Lemma 3.5 also holds with

$$\Lambda = \max\bigl\{(A_1 + A_4)/2, (A_2 + A_3)/2\bigr\}.$$

Without loss of generality, after proper translation in y we may assume that $\Lambda = A_1 + A_4 = 0 \geqslant A_2 + A_3$.

Next we shall show

$$A_2 + A_3 = 0. \tag{4.3}$$

For this purpose, let us now state another Hamiltonian identity for u, which was used in [6] and [21] for solutions of nonlinear Schrodinger equation before. A similar identity for certain parabolic equations is also used in [7] and may be regarded as conservation of moment.

Define

$$E(x) = \int_{\mathbb{R}} y\left[F\bigl(u(x,y)\bigr) + \frac{1}{2}u_y^2(x,y) - \frac{1}{2}u_x^2(x,y)\right]dy. \tag{4.4}$$

Then, by (2.2), $E(x)$ is well defined. We have

Proposition 4.1.

$$E(x) \equiv C, \quad x \in \mathbb{R}. \tag{4.5}$$

The proof of this Hamiltonian identity is based on (2.2) and is similar to those in [6] and [21]. The details is left to the reader.

Now, using (4.2), straightforward computations can lead to

$$\lim_{x \to \infty} E(x) = (A_1 + A_4)\mathbf{e} \cos\theta_1 = 0$$

and

$$\lim_{x \to -\infty} E(x) = (A_2 + A_3)\mathbf{e} \cos\theta_1.$$

Therefore, (4.3) is proven.

The moving plane method then leads to the even symmetry and monotonicity of u in y. Repeating the above arguments with x and y switched, we can show the even symmetry and monotonicity of u in x. Therefore, we have shown

Theorem 4.2. *Assume that u is an entire solution with* four ends *satisfying* (4.1)*. Then, after a proper translation and rotation, u satisfies* (1.10) *and* (1.11)*.*

5. Energy quantization of entire solutions

In this section we shall show that (4.1) holds under very mild conditions on u. Indeed, we shall consider entire solutions with $2k$ *ends* in general and show some energy quantization properties for entire solutions with *finite Morse index*.

Lemma 5.1. *Suppose u is an entire solution of* (1.1) *with* $2k$ ends*. Assume*

$$\theta_i^+ - \theta_i^- < \pi, \quad 1 \leqslant i \leqslant 2k. \tag{5.1}$$

Then

$$\mathcal{E}_R(u) \leqslant CR, \quad \forall R \tag{5.2}$$

for some positive constant C.

Proof. We only need to focus on conic region $\mathcal{C}_1$ and show

$$\int_{B_R \cap \mathcal{C}_1} \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) dx\,dy \leqslant CR, \quad \forall R.$$

Without loss of generality, we may assume

$$0 < \theta_1^- < \pi/2, \qquad \pi/2 < \theta_1^+ < \pi. \tag{5.3}$$

Choose $0 < \alpha^- < \theta_1^-$, $\theta_1^+ < \alpha^+ < \pi$ and let $\mathcal{C}_1^+ := \{(r,\theta):\ \alpha^- < \theta < \alpha^+\}$. Define

$$\rho_1(y) := \int_{y\cot\alpha^+}^{y\cot\alpha^-} \left[F(u) + \frac{1}{2}\left(u_x^2 - u_y^2\right)\right] dx.$$

Then, in view of (2.2), it is easy to see that

$$\begin{aligned} \left|\rho_1'(y)\right| &= \left|\left[F(u) + \frac{1}{2}\left(u_x^2 - u_y^2\right) + u_x u_y\right]\Big|_{x=y\cot\alpha^+}^{x=y\cot\alpha^-}\right| \\ &\leqslant Ce^{-\mu_1 y}, \quad \forall y \geqslant R_0 \end{aligned}$$

for some positive constants C, μ_1. Hence we have

$$\left|\rho_1(R_1) - \rho_1(R_2)\right| \leqslant Ce^{-\mu_1 R_1}, \quad \forall R_1 \leqslant R_2 \tag{5.4}$$

for some constant $C > 0$. In particular, we have

$$\left|\rho_1(y)\right| \leqslant C, \quad \forall y \geqslant R_0.$$

By (2.1), we have

$$F(u) + \frac{1}{2}\left(u_x^2 - u_y^2\right) \geqslant \frac{1}{2}u_x^2.$$

Hence

$$\int_{B_R \cap \mathcal{C}_1^+} u_x^2\, dx\, dy \leqslant CR < \infty \tag{5.5}$$

for some constant $C > 0$.

Now we choose another Cartesian coordinates (x', y') so that the x'-axis is a small rotation of x-axis and (5.1) and (5.3) still hold. Then we can obtain

$$\int_{B_R \cap \mathcal{C}_1^+} u_{x'}^2\, dx\, dy = \int_{B_R \cap \mathcal{C}_1^+} u_{x'}^2\, dx'\, dy' \leqslant C < \infty.$$

Therefore we obtain

$$\begin{aligned} &\int_{B_R \cap \mathcal{C}_1^+} \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) dx\, dy \\ &\leqslant \int_{B_R \cap \mathcal{C}_1^+} \left(F(u) + \frac{1}{2}\left(u_x^2 - u_y^2\right)\right) dx\, dy + C \int_{B_R \cap \mathcal{C}_1^+} \left(u_x^2 + u_{x'}^2\right) dx\, dy \\ &\leqslant CR, \quad \forall R > 0. \end{aligned}$$

Similarly, we can show that this estimate holds for all $i \in [1, 2k]$.

In view of (2.2), it is easy to see that

$$\int\limits_{\mathbb{R}^2\setminus\cup_{i=1}2k\mathcal{C}_i^+}\left(\frac{1}{2}|\nabla u|^2+F(u)\right)dx\,dy\leqslant\int\limits_0^\infty Cre^{-\mu r}\,dr<C$$

for some constant $C>0$. Hence (5.2) is proven. □

In [27], Modica showed Proposition 2.3 which says that $\mathcal{E}_R(u)/R$ is increasing in R. It follows immediately that $\lim_{R\to\infty}\mathcal{E}_R(u)/R$ exists. Indeed, we can show the following energy quantization property for entire solutions with *finite Morse index*.

Lemma 5.2. *Assume that u is an entire solution of* (1.1) *with* finite Morse index *and* $2k$ *ends. Assume also the technical condition* (5.1)*. Then the nodal sets Γ of u are asymptotically straight lines, i.e., there exist $\theta_i\in[\theta_i^-,\theta_i^+]$, $1\leqslant i\leqslant 2k$ such that on Γ_i,*

$$y=\tan(\theta_i)x+A_i+o(1)\quad \textit{as } x\to\infty,\ 1\leqslant i\leqslant 2k \tag{5.6}$$

where $\theta_i\neq\pi/2$, $\theta_i\neq 3\pi/2$, $\forall i\in[1,2k]$ after a proper rotation. Moreover, (1.14) *holds.*

Proof. It is easy to see that $u_\epsilon(x):=u(x/\epsilon)$ is a critical point of functional

$$\mathcal{E}_{\epsilon,R}(u)=\int\limits_{B_R\setminus B_{1/(2R)}}\left(\frac{\epsilon}{2}|\nabla u|^2+\frac{1}{\epsilon}F(u)\right)dx\,dy. \tag{5.7}$$

Fix $R=1$, u_ϵ is a *stable* critical point of (5.7) with $\mathcal{E}_{\epsilon,1}(u_\epsilon)<C<\infty$. By a Γ-convergence result of Tonegawa (Theorem 5 in [32]), there exists a sequence ϵ_n and a union L of N non-intersecting lines of $B_1\setminus B_{1/2}$ such that

$$\epsilon_n\cdot\big(\Gamma\cap(B_{R/\epsilon_n}\setminus B_{1/(2\epsilon_n R)})\big)\to L\quad\text{in Hausdorff distance as } n\to\infty. \tag{5.8}$$

Now fix $R=2,3,\ldots$ and repeat the argument above for a subsequence of $\{\epsilon_n\}$ in the previous step, by the diagonal procedure we can find a subsequence, still denoted by ϵ_n, such that (5.8) holds for all $R=1,2,\ldots.$ Therefore L must be the union of N different rays starting from the origin, and

$$\lim_{R\to\infty}\mathcal{E}_R(u)/R=N\mathbf{e}. \tag{5.9}$$

Fix a ray in L. Without loss of generality, we may assume it to be the positive x-axis which belongs to $\mathcal{C}_1$ after some rotation. Then, for any fixed small angles $\alpha_2>\alpha_1>0$, there exists a sequence of conic regions $\mathcal{C}_{R_n,M_n,\alpha_i}:=\{(x,y):\ R_n\leqslant x\leqslant M_n,\ |y|\leqslant\tan\alpha_i\}$, $i=1,2$ such that $R_n\to\infty$, $M_n/R_n\to\infty$ and

$$\mathcal{C}_{R_n,M_n,\alpha_2}\cap\Gamma\subset\mathcal{C}_{R_n,M_n,\alpha_1}.$$

On the other hand, thanks to the stability of u in $\mathbb{R}^2\subset B_{R_0}$ when R_0 is large enough, by similar arguments to the proof of (3.18) we can show that

$$\mathcal{C}_{R_n,M_n,\alpha_2}\cap\Gamma=\big\{(x,y):\ y=k(x),\ R_n\leqslant x\leqslant M_n\big\}$$

for some C^2 function $k(x)$ and

$$\max_{x\in[R_n,M_n]}\left|k'(x)\right| < \tan\alpha_1, \qquad \max_{x\in[R_n,M_n]}\left|k''(x)\right| \to 0, \quad \text{as } n\to\infty. \tag{5.10}$$

Moreover,

$$\left\|u(x,y)-g\big(y-k(x)\big)\right\|_{C^2(\mathcal{C}_{R_n,M_n,\alpha_2})} \to 0, \quad \text{as } n\to\infty.$$

We may also assume that $k'(R_n)\to 0$. We claim that when n is large enough, M_n can be chosen as any number $R > R_n$ and (5.10) still holds. If this is not true, we can choose M_n such that (5.10) holds but $k'(M_n) = \tan\alpha_1$. We claim that $(\mathcal{C}_{R_n,M_n,\alpha_2} \setminus \mathcal{C}_{R_n,2M_n,(\alpha_1+\alpha_2)/2}) \cap B_{M_n}(M_n, k(M_n))$ is empty. If we assume otherwise, without loss of generality, we may assume that M_n is the first such sequence related to a ray in L. Now we use $\epsilon_n = 1/M_n$ as in (5.8), and obtain the limit as L' which is the union of at least $N+1$ rays. This is a contradiction to (5.9). Hence the claim is true. Then, using the modified Hamiltonian identity in $\mathcal{C}_{R_n,M_n,\alpha_2}$ as in (5.4) with the y-axis being replaced by the tangential direction of $k(x)$ at $(M_n, k(M_n))$, we obtain

$$\mathbf{e} \leqslant \mathbf{e}\cos\alpha_1 + o(1), \quad \text{as } n\to\infty.$$

This is a contradiction, and hence proves that M_n can be chosen as any $R > R_n$ when n large enough. Therefore

$$\mathcal{C}_{R_n,\infty,\alpha_2} \cap \Gamma = \big\{(x,y):\ y = k(x),\ x > R_n\big\}$$

and

$$\left|k'(x)\right| < \tan\alpha_1, \quad x > R_n.$$

Since $\alpha_1 > 0$ is arbitrary, we obtain that

$$\lim_{x\to\infty} k'(x) = 0.$$

Now use Lemma 3.6, we conclude that $\Gamma \cap \mathcal{C}_1$ is asymptotically straight line. The lemma then follows. □

Remark 5.3. Given that u satisfies the condition in Theorem 1.2. If we assume further that, after a proper rotation, the level set in $\mathcal{C}_i$ outside a large ball B_R is a graph of a C^2 function $k(x)$, i.e.,

$$\Gamma \cap \mathcal{C}_i \cap B_R^c = \big\{(x,y):\ y = k(x),\ x > R\big\}, \quad 1 \leqslant i \leqslant 2k, \tag{5.11}$$

then the conclusion of Lemma 5.2 can be shown directly without using the result in [32]. We just start the proof from (5.10) with $M_n = \infty$ and exploits the modified Hamiltonian identity. The details is omitted.

Theorem 1.2 follows from Lemma 5.2 and Theorem 4.2 directly. If we replace (1.9) in Theorem 1.2 by (5.2), the conclusion of Theorem 1.2 still holds.

Proof of Theorem 1.3. If (1.13) does not hold, by the monotonicity formula of Modica we know that (5.2) must be true. Using the Γ-convergence result of Tonegawa as in the proof of Lemma 5.2, we know that there exists a sequence $\{R_n\}$ such that $R_n \to \infty$ and

$$\frac{1}{R_n} \cdot (\Gamma \cap B_{MR_n}) \to L \quad \text{in Hausdorff distance as } n \to \infty \tag{5.12}$$

for any $M > 0$, where L is the union of N rays from the origin. Moreover, (5.9) holds. It follows that Γ must be asymptotically straight lines at infinity, as in the proof of Lemma (5.2). Note that Γ is a union of C^2 curves except at singular points where u and ∇u both vanish, and u u behaves like harmonic function near these singular points. Therefore N must be an even positive integer $2k$. We denote the directions of these lines by $\nu_i = (\cos\theta_i, \sin\theta_i)$, $1 \leqslant i \leqslant 2k$ with $0 < \theta_i < \theta_{i+1} < 2\pi$, $1 \leqslant i \leqslant 2k-1$, after a proper rotation. Using Hamiltonian identity similar to (3.26) but with more terms (see also [19]), we obtain

$$\sum_{i=1}^{2k} \mathbf{e} \sin(\theta_i + \theta) = 0 \tag{5.13}$$

for almost all θ. Hence (1.15) holds. The proof of Theorem 1.3 is complete. $\square$

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