The Stability of Spiral Flow Between Coaxial Cylinders

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Abstract—We investigate the hydrodynamic stability for the motion of a viscous incompressible fluid between two coaxial cylinders which move spirally along their common axis assuming that the gap between the cylinders is small and that the Reynolds number is small compared to the Taylor number. We use the variational method to derive a relation between the critical Taylor number and the Reynolds number. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Hydrodynamic stability has been recognized as one of the central problems of fluid mechanics for a long time, and it has many applications in engineering, in meteorology and oceanography, and in astrophysics and geophysics (see, e.g., [1-5]). The goal of this paper is to investigate the hydrodynamic stability for the motion of a viscous incompressible fluid between two coaxial cylinders which move spirally along their common axis.

In general, a fluid flow can be classified as either laminar or turbulent. In a laminar flow the adjacent layers of the fluid move in an orderly way parallel to each other and in the direction of flow. In a turbulent flow, the fluid does not follow a simple pattern but moves in an extremely irregular manner. From a practical point of view, turbulence can be either desirable or unwelcome depending on whether it produces or destroys a desired property of the flow, and the designer of fluid systems may have to make special efforts either to prevent it or to stimulate it. When the average velocity of the fluid in a given system is sufficiently small, a laminar flow dominates. However, as the velocity increases, a condition is eventually reached at which a laminar flow is no longer stable and a transition to a turbulent flow occurs. Hydrodynamic stability is concerned with when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence. A numerical description of such a transition can be achieved by using dimensionless numbers such as the Reynolds number and the Taylor number.

Let $R_1$ (respectively, $R_2$) be the radius of the inner (respectively, outer) cylinder which rotates with constant angular velocity $\Omega_1$ (respectively, $\Omega_2$) and translates in the axial direction with...
constant velocity $W_1$ (respectively, $W_2$). We denote by $R$ the Reynolds number computed using the difference $W_1 - W_2$ of the axial velocities of the cylinders and the width $R_2 - R_1$ of the gap between the two cylinders. Thus, we have

$$R = \frac{(W_1 - W_2)(R_2 - R_1)}{\nu},$$

where $\nu$ is the kinematic viscosity of the fluid. In many cases, the problem of the stability of the given system can be solved by analyzing the Reynolds number. However, when the system involves a rotation of the fluid, there is another dimensionless number which plays a significant role in the theory of stability, called the Taylor number, which uses appropriate angular velocities instead of linear velocities. In our particular flow system it can be given by

$$T = \frac{2(R_2 - R_1)^4(\Omega_1 + \Omega_2) (\Omega_2 R_2^2 - \Omega_1 R_1^2)}{\nu^2(R_2^2 - R_1^2)}.$$

As in the case of the Reynolds number, the motion of the fluid becomes turbulent if the Taylor number is sufficiently high. Thus, there should be a critical value of the Taylor number $T$ such that the flow becomes turbulent whenever $T$ is higher than that number.

The problem of flow between coaxial cylinders has been the subject of many studies (see, e.g., [6]). The stability of a spiral flow between coaxial cylinders has been investigated by Chandrasekhar (cf. [7–9]), Kruger and DiPrima [10] when the axial flow is caused by a pressure gradient. In this paper, we consider the case where the cylinders not only rotate but also move in the axial direction, which induces a spiral motion of the fluid, assuming that the gap between the cylinders is small. Since we are more interested in the effect of the rotation rather than the axial motion of the fluid on the stability of the flow, it is assumed that the Reynolds number is small compared to the Taylor number. We use the variational method to analyze the values of the Taylor number $T$ and derive a relation between the critical Taylor number and the Reynolds number with the aid of the computer package Mathematica.

## 2. BASIC FLOW SOLUTIONS

In this section, we obtain basic flow solutions for the flow of a viscous fluid between two coaxial cylinders of infinite length by solving the Navier-Stokes equations in cylindrical coordinates. We assume that the inner (respectively, outer) cylinder with radius $R_1$ (respectively, $R_2$) rotates with angular velocity $\Omega_1$ (respectively, $\Omega_2$) and moves in the axial direction with speed $W_1$ (respectively, $W_2$) and that the gap between the cylinders is small compared to the radii of the cylinders (see Figure 1).
Using the cylindrical coordinate system \((r, \theta, z)\), the Navier-Stokes equations for the velocity \(\mathbf{v} = (v_r, v_\theta, v_z)\) and the pressure \(p\) can be written in the form

\[
\frac{\partial v_r}{\partial t} + (\mathbf{v} \cdot \nabla) v_r - \frac{v_r^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\Delta v_r}{r^2} - \frac{2 \partial v_\theta}{\partial \theta} - \frac{v_r}{r^2} \right),
\]

(2.1)

\[
\frac{\partial v_\theta}{\partial t} + (\mathbf{v} \cdot \nabla) v_\theta + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left( \frac{\Delta v_\theta}{r^2} + \frac{2 \partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \right),
\]

(2.2)

\[
\frac{\partial v_z}{\partial t} + (\mathbf{v} \cdot \nabla) v_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \Delta v_z,
\]

(2.3)

where \(\rho\) is the density, \(\nu\) denotes the kinematic viscosity, and

\[
(\mathbf{v} \cdot \nabla) f = v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r} \frac{\partial f}{\partial \theta} + v_z \frac{\partial f}{\partial z},
\]

\[
\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.
\]

The continuity equation in cylindrical coordinates is given by

\[
\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0.
\]

(2.4)

We assume that the flow is steady and axisymmetric and that it has constant properties along the axial direction. Then equations (2.1)–(2.4) have solutions of the form

\[
v_r = 0, \quad v_\theta = V(r), \quad v_z = W(r), \quad p = P(r).
\]

(2.5)

Thus, the continuity equation is automatically satisfied, and the Navier-Stokes equations reduce to

\[
\frac{dP}{dr} = \frac{\rho V^2}{r},
\]

(2.6)

\[
\frac{d^2V}{dr^2} + \frac{1}{r} \frac{dV}{dr} - \frac{V}{r^2} = 0,
\]

(2.7)

\[
\frac{d^2W}{dr^2} + \frac{1}{r} \frac{dW}{dr} = 0.
\]

(2.8)

From (2.7) and the boundary conditions \(V(R_1) = \Omega_1 R_1\) and \(V(R_2) = \Omega_2 R_2\), we obtain

\[
V(r) = Ar + \frac{B}{r},
\]

(2.9)

where

\[
A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad B = \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}.
\]

(2.10)

Solving (2.8) with \(W(R_1) = W_1\) and \(W(R_2) = W_2\), we obtain the axial component of the velocity

\[
W(r) = \frac{(W_2 - W_1) \ln(r/R_1)}{\ln(R_2/R_1)} + W_1
\]

(2.11)

for the basic flow.
3. PERTURBATION EQUATIONS

In order to consider the stability of our flow system we consider an infinitesimal perturbation of the basic flow given by (2.5) by assuming that the perturbed flow is given by

\[ v_r = u', \quad v_0 = V(r) + v', \quad v_z = W(r) + w', \quad p = P(r) + p'. \]  

Substituting these in (2.1)-(2.4), we obtain the linearized equations of motion of the form

\[ \frac{\partial u'}{\partial t} + \frac{V}{r} \frac{\partial u'}{\partial \theta} + \frac{W}{r} \frac{\partial u'}{\partial z} - \frac{2Vv'}{r} = -\frac{1}{\rho} \frac{\partial p'}{\partial r} + \nu \left( \frac{\Delta u'}{\rho^2} - \frac{2}{r^2} \frac{\partial u'}{\partial r} - \frac{u'}{r^2} \right), \]

\[ \frac{\partial v'}{\partial t} + \frac{u'}{r} \frac{\partial v'}{\partial \theta} + \frac{V}{r} \frac{\partial v'}{\partial z} + \frac{1}{\rho} \frac{\partial p'}{\partial r} + \nu \left( \frac{\Delta v'}{\rho^2} + \frac{2}{r^2} \frac{\partial u'}{\partial r} - \frac{v'}{r^2} \right), \]

\[ \frac{\partial w'}{\partial t} + \frac{u'}{r} \frac{\partial w'}{\partial \theta} + \frac{W}{r} \frac{\partial w'}{\partial z} = -\frac{1}{\rho} \frac{\partial p'}{\partial z} + \nu \Delta w', \]

\[ \frac{1}{r} \frac{\partial}{\partial r} (ru') + \frac{1}{\rho} \frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z} = 0. \]  

We assume that the disturbances are of the form

\[ u' = u(r)e^{i(st + kz)}, \quad v' = v(r)e^{i(st + kz)}, \quad p' = p(r)e^{i(st + kz)} \]  

for some parameters \( s \) and \( k \). Then by (3.5) we have

\[ w' = \frac{i}{k} \left( \frac{\partial}{\partial r} (ru) \right) e^{i(st + kz)} = \frac{i}{k} \left( \frac{du}{dr} + \frac{u}{r} \right) e^{i(st + kz)}. \]  

Substituting (3.6) and (3.7) in (3.2)-(3.4), we obtain

\[ \frac{1}{\rho v} \frac{dp}{dr} = \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - k^2 - \frac{is}{\nu} - \frac{ik}{\nu} W \right) u - \frac{2V}{r} v, \]

\[ v = \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - k^2 - \frac{is}{\nu} - \frac{ik}{\nu} W \right) v = \frac{1}{\nu} \left( \frac{dV}{dr} + \frac{V}{r} \right) u, \]

\[ \frac{k^2}{\rho v} \frac{dp}{dr} = \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - k^2 - \frac{is}{\nu} - \frac{ik}{\nu} W \right) \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) u \]

\[ - \frac{ik}{\nu} \frac{dW}{dr} \left( \frac{d}{dr} + \frac{1}{r} \right) u + \frac{ik}{\nu} \frac{d}{dr} \left( \frac{dW}{dr} \right). \]  

By eliminating \( \frac{dp}{dr} \) from (3.8) and (3.10), we get

\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - k^2 - \frac{is}{\nu} - \frac{ik}{\nu} W \right) \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - k^2 \right) u \]

\[ + \frac{ik}{\nu} \left( \frac{d^2W}{dr^2} - \frac{1}{r} \frac{dW}{dr} \right) u = \frac{2k^2V}{r^2} v. \]  

4. EIGENVALUE PROBLEMS

In this section, we derive the eigenvalue problem for \( u = u(r), v = v(r) \), and \( p = p(r) \) given in (3.6). We set

\[ \delta = R_2 - R_1, \quad R_0 = \frac{(R_1 + R_2)}{2}, \quad x = \frac{(r - R_0)}{\delta}, \]
where $\delta$ is assumed to be small as was mentioned before. Using (2.9) and (2.10), we obtain

\[
\frac{V}{r} = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} - \frac{(\Omega_2 - \Omega_1)R_2^2 R_1^2}{R_2^2 - R_1^2} \frac{1}{r^2} \\
= \frac{1}{R_2^2 - R_1^2} \left[ \Omega_2 R_2^2 \left( 1 - \frac{R_1^2}{r^2} \right) - \Omega_1 R_1^2 \left( 1 - \frac{R_2^2}{r^2} \right) \right].
\]

However, since $\delta/R_0$ is small, we see that

\[
R_2^2 - R_1^2 = \left( R_0 + \frac{\delta}{2} \right)^2 - \left( R_0 - \frac{\delta}{2} \right)^2 \approx 2\delta R_0,
\]

\[
R_2^2 \left( 1 - \frac{R_1^2}{r^2} \right) = \left( R_0 + \frac{\delta}{2} \right)^2 \left[ 1 - \frac{(R_0 - \delta/2)^2}{(R_0 + \delta/2)^2} \right] \\
\approx R_0^2 \left( 1 + \frac{\delta}{R_0} \right) \left[ 1 - \left( 1 - \frac{\delta}{R_0} - \frac{2\delta}{R_0} \right) \right] \approx R_0 (1 + 2\delta) R_0,
\]

\[
R_1^2 \left( 1 - \frac{R_2^2}{r^2} \right) = \left( R_0 + \frac{\delta}{2} \right)^2 \left[ 1 - \frac{(R_0 + \delta/2)^2}{(R_0 + \delta/2)^2} \right] \\
\approx R_0^2 \left( 1 - \frac{\delta}{R_0} \right) \left[ 1 - \left( 1 + \frac{\delta}{R_0} - \frac{2\delta}{R_0} \right) \right] \approx R_0 (-1 + 2\delta).
\]

Thus, we have

\[
\frac{V}{r} \approx \frac{1}{2\delta R_0} \left[ \frac{\Omega_2 R_m (1 + 2\delta) \delta - \Omega_1 R_0 (-1 + 2\delta) \delta}{\Omega_2 + \Omega_1} \right]
\]

\[
= \left[ \frac{\Omega_2}{\Omega_1} \right] - \frac{(\Omega_2 + \Omega_1) x}{2}.
\]

On the other hand, by (2.11), we obtain

\[
W(r) = \frac{(W_2 - W_1) \ln(r/R_1)}{\ln(R_2/R_1)} \\
\approx \frac{(W_2 - W_1) \ln(1 + (x - 1/2)(\delta/R_0))}{\ln(1 + \delta/R_0)} \approx (W_1 - W_2) \left( \frac{1}{2} - x \right).
\]

We also have

\[
\frac{d^2W}{dr^2} - \frac{1}{r} \frac{dW}{dr} = -\frac{2(W_2 - W_1)}{\ln(R_2/R_1)} \frac{1}{r^2} \approx -\frac{2(W_2 - W_1)}{\ln(1 + \delta/R_0)} \frac{1}{R_0^2} \approx \frac{2(W_1 - W_2)}{\delta R_0}.
\]

Now we denote by

\[
R = (W_1 - W_2) \frac{\delta}{\nu}
\]

the Reynolds number for the difference $W_1 - W_2$ in axial velocities of the cylinders and set

\[
D = \frac{d}{dx}, \quad \omega = \frac{\Omega_2}{\Omega_1}.
\]

Then, using the fact that $\delta$ is small, we obtain

\[
\frac{d}{dr} + \frac{1}{r} = \frac{1}{\delta} D + \frac{1}{R_0 + x\delta} \approx \frac{1}{\delta} D,
\]

\[
\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} = \frac{1}{\delta^2} D^2 + \frac{1}{(R_0 + x\delta)^2} D - \frac{1}{(R_0 + x\delta)^2} \approx \frac{1}{\delta^2} D^2.
\]
Now if we set
\[ \xi = k\delta, \quad \eta = \frac{s\delta^2}{\nu}, \]
equations (3.9) and (3.11) reduce to
\[ \left( D^2 - \xi^2 - i\eta - i\xi R \left( \frac{1}{2} - x \right) \right) v = \left( \frac{2\xi^2}{\nu} \right) u, \]
\[ \left( D^2 - \xi^2 - i\eta - i\xi R \left( \frac{1}{2} - x \right) \right) \left( D^2 - a^2 \right) u + i\xi R \left( \frac{2\delta}{R_0} \right) u \]
\[ = \frac{\Omega_1 (1 + \omega) \xi^2 \delta^2}{\nu} \left[ 1 - 2 \left( \frac{1 - \omega}{1 + \omega} \right) x \right] v. \]

We rescale \( u \) by introducing
\[ \tilde{u} = \frac{(1 + \omega)\Omega_1 \xi^2 \delta^2}{\nu} u. \]

Since \( \delta \) is small, suppressing the term \( i\xi R(2\delta/R_0)u \) in (4.3) and using the Taylor number
\[ T = \frac{2(1 + \omega)A\Omega_1 \delta^4}{\nu^2}, \]
we obtain the eigenvalue problem consisting of the equations
\[ \left( D^2 - \xi^2 - i\eta - i\xi R \left( \frac{1}{2} - x \right) \right) v = -T\xi^2 \tilde{u}, \]
\[ \left( D^2 - \xi^2 - i\eta - i\xi R \left( \frac{1}{2} - x \right) \right) \left( D^2 - \xi^2 \right) \tilde{u} = \left[ 1 - 2 \left( \frac{1 - \omega}{1 + \omega} \right) x \right] v, \]
with boundary conditions
\[ \tilde{u} = v = D\tilde{u} = 0 \]
for \( x = \pm 1/2. \)

We consider the parameters \( \varepsilon \) and \( \lambda \) given by
\[ \varepsilon = R\xi, \quad \lambda = T\xi^2, \]
and assume that \( \tilde{u}, v, \) and \( \lambda \) can be written in the form
\[ \tilde{u} = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \]
\[ v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots, \]
\[ \lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots. \]

Then from (4.4) and (4.5), we obtain
\[ \left( D^2 - \xi^2 - i\eta - i\varepsilon \left( \frac{1}{2} - x \right) \right) \left( u_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots \right) \]
\[ = -\left( \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots \right) \left( u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \right), \]
\[ \left( D^2 - \xi^2 - i\eta \right) \left( D^2 - \xi^2 \right) \left( D^2 - \xi^2 \right) \left( u_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots \right) \]
\[ = \left( v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots \right). \]

Collecting the zeroth order terms for \( \varepsilon \), we obtain the eigenvalue problem for the unperturbed system in the form
\[ (D^2 - \xi^2 - i\eta) v_0 = -\lambda_0 u_0, \]
\[ (D^2 - \xi^2 - i\eta) (D^2 - \xi^2) u_0 = v_0. \]
subject to the boundary conditions

\[ u_0 = v_0 = Du_0 = 0 \quad (4.12) \]

for \( x = \pm 1/2 \). By considering the first-order terms, we obtain the eigenvalue problem for the first-order perturbations in the form

\[ (D^2 - \xi^2 - i\eta) v_1 = -\lambda_0 u_1 - \lambda_1 u_0 + i \left( \frac{1}{2} - x \right) v_0, \quad (4.13) \]

\[ (D^2 - \xi^2 - i\eta)(D^2 - \xi^2)u_1 = v_1 + i \left( \frac{1}{2} - x \right)(D^2 - \xi^2)u_0 \quad (4.14) \]

with boundary conditions

\[ u_1 = v_1 = Du_1 = 0 \quad (4.15) \]

for \( x = \pm 1/2 \).

### 5. Eigenvalues for the Unperturbed System

In this section, we discuss the eigenvalue problem (4.10)-(4.12) for the unperturbed system using the variational method. Let \( u_0^{(k)}, v_0^{(k)} \) (respectively, \( u_0^{(\ell)}, v_0^{(\ell)} \)) be solutions of that eigenvalue problem for the eigenvalue \( \lambda_0^{(k)} \) (respectively, \( \lambda_0^{(\ell)} \)). Then we have

\[ (D^2 - \xi^2 - i\eta) v_0^{(k)} = -\lambda_0^{(k)} u_0^{(k)}, \quad (5.1) \]

\[ (D^2 - \xi^2 - i\eta)(D^2 - \xi^2)u_0^{(k)} = v_0^{(k)} \quad (5.2) \]

with boundary conditions

\[ u_0^{(k)} = v_0^{(k)} = Du_0^{(k)} = 0 \quad (5.3) \]

for \( x = \pm 1/2 \). We also obtain similar equations and boundary conditions by replacing superscript \( (k) \) by \( (\ell) \) in (5.1)-(5.3). Thus, we have

\[ \int_{-1/2}^{1/2} v_0^{(\ell)} (D^2 - \xi^2 - i\eta) v_0^{(k)} \, dx = -\lambda_0^{(k)} \int_{-1/2}^{1/2} u_0^{(k)} v_0^{( \ell) } \, dx, \quad (5.4) \]

\[ \int_{-1/2}^{1/2} v_0^{(\ell)} (D^2 - \xi^2 - i\eta)(D^2 - \xi^2)u_0^{(k)} \, dx = \int_{-1/2}^{1/2} u_0^{(k)} v_0^{( \ell) } \, dx. \quad (5.5) \]

Taking integration by parts, we see that

\[ \int_{-1/2}^{1/2} v_0^{(\ell)} (D^2 - \xi^2 - i\eta) v_0^{(k)} \, dx = -\int_{-1/2}^{1/2} \left[ \left( Dv_0^{(k)} \right) \left( Dv_0^{(\ell)} \right) + \left( \xi^2 + i\eta \right) v_0^{(k)} v_0^{( \ell) } \right] \, dx, \quad (5.6) \]

\[ \int_{-1/2}^{1/2} u_0^{(k)} (D^2 - \xi^2 - i\eta)(D^2 - \xi^2)u_0^{(\ell)} \, dx \]

\[ = \int_{-1/2}^{1/2} \left[ \left( (D^2 - \xi^2) u_0^{(k)} \right) \left( (D^2 - \xi^2) u_0^{(\ell)} \right) + i\eta \left( \left( Du_0^{(k)} \right) \left( Du_0^{(\ell)} \right) + \xi^2 u_0^{(k)} u_0^{(\ell)} \right) \right] \, dx. \quad (5.7) \]

Using (5.4)-(5.7), we obtain

\[ \int_{-1/2}^{1/2} \left[ \left( Dv_0^{(k)} \right) \left( Dv_0^{(\ell)} \right) + \left( \xi^2 + i\eta \right) v_0^{(k)} v_0^{( \ell) } \right] \, dx \]

\[ = \lambda_0^{(k)} \int_{-1/2}^{1/2} \left[ \left( (D^2 - \xi^2) u_0^{(k)} \right) \left( (D^2 - \xi^2) u_0^{(\ell)} \right) + i\eta \left( \left( Du_0^{(k)} \right) \left( Du_0^{(\ell)} \right) + \xi^2 u_0^{(k)} u_0^{(\ell)} \right) \right] \, dx. \quad (5.8) \]
Subtracting this equation from the one obtained by interchanging superscripts \((k)\) and \((\ell)\), we have

\[
\left(\lambda_0^{(k)} - \lambda_0^{(\ell)}\right) \int_{-1/2}^{1/2} \left[ \left((D^2 - \xi^2) u_0^{(k)}\right) \left((D^2 - \xi^2) u_0^{(\ell)}\right) + i\eta \left((Du_0^{(k)}) (Du_0^{(\ell)}) + \xi^2 u_0^{(k)} u_0^{(\ell)}\right) \right] dx = 0.
\]

Hence using this, (5.5) and (5.7), we obtain

\[
\left(\lambda_0^{(k)} - \lambda_0^{(\ell)}\right) \int_{-1/2}^{1/2} u_0^{(k)} v_0^{(\ell)} dx = 0.
\]

Thus, we see that

\[
\int_{-1/2}^{1/2} u_0^{(k)} v_0^{(\ell)} dx = \int_{-1/2}^{1/2} v_0^{(k)} u_0^{(\ell)} dx = 0
\]

for \(k \neq \ell\). On the other hand, if

\[
\lambda_0 = \lambda_0^{(k)} = \lambda_0^{(\ell)}, \quad u_0 = u_0^{(k)} = u_0^{(\ell)}, \quad v_0 = v_0^{(k)} = v_0^{(\ell)},
\]

then by (5.8), we have

\[
\lambda_0 = \frac{\int_{-1/2}^{1/2} \left[ (Dv_0)^2 + (\xi^2 + i\eta) v_0^2 \right] dx}{\int_{-1/2}^{1/2} \left[ ((D^2 - \xi^2) u_0)^2 + i\eta ((Du_0)^2 + \xi^2 u_0^2) \right] dx}.
\]

Now we shall compute an approximate value of the lowest eigenvalue \(\lambda_0^*\) for the unperturbed system by choosing an appropriate trial function. As a candidate for the trial function for \(v_0\) which is simple for calculations and satisfies the boundary conditions (4.12), we shall use

\[
v_0 = \cos \pi x.
\]

Then by (4.11), we have

\[
(D^2 - \xi^2 - i\eta) (D^2 - \xi^2) u_0 = \cos \pi x.
\]

Solving this equation together with the boundary conditions

\[
u_0 \left(\frac{1}{2}\right) = Du_0 \left(\frac{1}{2}\right), \quad u_0 \left(-\frac{1}{2}\right) = Du_0 \left(-\frac{1}{2}\right),
\]

we obtain

\[
u_0 = \gamma \left[ \cos \pi x + \pi \Delta \left( \frac{\cosh \xi x}{\cosh (\xi/2)} - \frac{\cosh q x}{\cosh (q/2)} \right) \right],
\]

where

\[
q = \sqrt{\xi^2 + i\eta}, \quad \gamma = \frac{1}{(\pi^2 + \xi^2)(\pi^2 + q^2)}, \quad \Delta = \frac{1}{\xi \tanh (\xi/2) - q \tanh (q/2)}.
\]

Substituting (5.11) and (5.12) in (5.10), we obtain

\[
\lambda_0^* = \frac{(\pi^2 + \xi^2)(\pi^2 + q^2)^2}{1 + 4i\pi^2 \Delta \gamma \eta}
\]

as an approximate value of the lowest eigenvalue for the unperturbed system. Since \(q, \gamma,\) and \(\Delta\) are functions of \(\xi\) and \(\eta\) given by (4.1), we see that \(\lambda_0^*\) is also a function of \(\xi\) and \(\eta\).
6. EIGENVALUES FOR THE FIRST-ORDER PERTURBATION

We now consider the eigenvalue problem (4.13)-(4.15) for the first-order perturbation, again by using the variational method. Eliminating $u_1$ from (4.13) and (4.14), we obtain

$$(D^2 - q^2)^2 (D^2 - \xi^2) v_1 = -\lambda_0 v_1 - i\lambda_0 \left(\frac{1}{2} - x\right) (D^2 - \xi^2) u_0 - \lambda_1 (D^2 - q^2) (D^2 - \xi^2) u_0 + i (D^2 - q^2) (D^2 - \xi^2) \left(\frac{1}{2} - x\right) v_0.$$  

Using (4.10) and (4.11), we have

$$(D^2 - q^2)^2 (D^2 - \xi^2) v_1 = -\lambda_0 v_1 - \lambda_1 v_0 + i\mathcal{L} v_0,$$  

(6.1)

where

$$\mathcal{L} = \left(\frac{1}{2} - x\right) (D^2 - \xi^2) (D^2 - q^2) + (D^2 - q^2) (D^2 - \xi^2) \left(\frac{1}{2} - x\right).$$

Let $\{v_0^{(j)}, v_0^{(j)}\}$ be the set of eigenfunctions corresponding to the set $\{\lambda_0^{(j)}\}$ of eigenvalues for the unperturbed system (4.10) and (4.11) so that we have

$$(D^2 - q^2) v_0^{(j)} = -\lambda_0^{(j)} v_0^{(j)},$$

$$(D^2 - q^2) (D^2 - \xi^2) v_0^{(j)} = v_0^{(j)}.$$  

From these relations we see that

$$(D^2 - q^2)^2 (D^2 - \xi^2) v_0^{(j)} = -\lambda_0^{(j)} v_0^{(j)}. $$  

(6.2)

Since $v_1$ and the $v_0^{(j)}$ satisfy the same boundary conditions, $v_1$ has a series expansion of the form

$$v_1 = \sum_{j=0}^{\infty} c_j v_0^{(j)}.$$  

(6.3)

Substituting (6.3) into (6.1) and using (6.2), we obtain

$$-\sum_{j=0}^{\infty} \lambda_0^{(j)} c_j v_0^{(j)} = -\lambda_0^* \sum_{j=0}^{\infty} c_j v_0^{(j)} - \lambda_1^* v_0 + i\mathcal{L} v_0,$$  

where $\lambda_1^* = \lambda_1^{(0)}$ is the lowest eigenvalue for the first-order perturbation. Applying the operator $\int_{-1/2}^{1/2} \cdot d x$ and using the orthogonality relations given by (5.9), we obtain

$$-\lambda_0^* c_m \int_{-1/2}^{1/2} u_0^{(m)} v_0^{(m)} d x = -\lambda_0^* c_m \int_{-1/2}^{1/2} u_0^{(m)} v_0^{(m)} d x - \lambda_1^* \int_{-1/2}^{1/2} u_0^{(m)} v_0 d x + i \int_{-1/2}^{1/2} u_0^{(m)} \mathcal{L} v_0 d x.$$  

If we set $m = 0$ and use the fact that $\lambda_0^* = \lambda_0^{(0)}$ and $\lambda_1^* = \lambda_1^{(0)}$, we obtain the formula

$$\lambda_1^* = \frac{i \int_{-1/2}^{1/2} u_0 \mathcal{L} v_0 d x}{\int_{-1/2}^{1/2} u_0 v_0 d x},$$  

(6.4)

for the lowest eigenvalue $\lambda_1^*$ for the first-order perturbation.
Now we shall calculate an approximate value of $\lambda_1^*$ using $u_0$ and $v_0$ given by (5.12) and (5.11), respectively. Thus, we have

$$\int_{-1/2}^{1/2} u_0 v_0 \, dx = \frac{\gamma}{2} \left( 1 + 4i\pi^2 \Delta \gamma \eta \right),$$

$$\int_{-1/2}^{1/2} u_0 \mathcal{L} v_0 \, dx = \frac{\lambda_0 \gamma^2}{2} \left[ \frac{\pi^2 + \xi^2}{2} + 2\pi^2 \Delta^2 \xi \tanh \left( \frac{\xi}{2} \right) - \frac{\pi^2 \Delta^2 (q^2 + \xi^2)}{q} \tanh \left( \frac{q}{2} \right) \right] + \frac{i\pi^2 \Delta^2 \eta}{2 \cosh^2 \left( \frac{q}{2} \right)} + \gamma \left[ \frac{(\pi^2 + \xi^2)(\pi^2 + q^2)}{4} + i\pi^2 \Delta \eta \right].$$

Hence, we obtain

$$\lambda_1^* = i \left( 1 + 4i\pi^2 \Delta \gamma \eta \right)^{-1} \left\{ \lambda_0 \gamma \left[ \frac{\pi^2 + \xi^2}{2} + 2\pi^2 \Delta^2 \xi \tanh \left( \frac{\xi}{2} \right) \right] - \frac{\pi^2 \Delta^2 (q^2 + \xi^2)}{q} \tanh \left( \frac{q}{2} \right) + \frac{i\pi^2 \Delta^2 \eta}{2 \cosh^2 \left( \frac{q}{2} \right)} + 2\gamma \left[ \frac{(\pi^2 + \xi^2)(\pi^2 + q^2)}{4} + i\pi^2 \Delta \eta \right] \right\}$$

as an approximate value of the lowest eigenvalue for the first-order perturbation. As in the case of $\lambda_0^*$, we see that $\lambda_1^*$ is also a function of $\xi$ and $\eta$.

### 7. NUMERICAL RESULTS

As was noted before, approximate values of the eigenvalues for first- and second-order perturbations given by (5.13) and (6.5), respectively, are functions of $\xi = k\delta$ and $\eta = s\delta^2/\nu$, where $s$ and $t$ are the parameters in (3.6) and $\delta = R_2 - R_1$ is the width of the gap between the cylinders. Thus by (4.7), an approximate value of the lowest eigenvalue can be written in the form

$$\lambda^* = \lambda_0^* + \varepsilon \lambda^* = \lambda_0^* + R \xi \lambda^*.$$

Since $\lambda^*$ is a real number, we obtain

$$R \xi = -\frac{\text{Im} \lambda_0^*}{\text{Im} \lambda_1^*},$$

where $\text{Im}$ denotes the imaginary part; hence, we obtain an expression of the Reynolds number $R$ in the form

$$R = -\frac{\text{Im} \lambda_0^*}{\xi \text{Im} \lambda_1^*}. \quad (7.1)$$

As for the Taylor number, by (4.6) we have

$$T = \frac{\lambda^*}{\xi^2} = \frac{1}{\xi^2} \left[ \text{Re} \lambda_0^* - \frac{\text{Im} \lambda_0^*}{\text{Im} \lambda_1^*} \cdot \text{Re} \lambda_1^* \right], \quad (7.2)$$

where $\text{Re}$ denotes the real part. Thus, we see that both $R$ and $T$ are functions of $\xi$ and $\eta$.

One way of analyzing the effect of the rotational motion on the stability of our system would be to find the critical Taylor number by minimizing $T$ with respect to $\xi$ when $\eta$ is fixed. However, in this paper we consider instead the values of $T$ and $R$ as functions of $\eta$ for a specific value of $\xi$. In fact, we use $\xi = 3.1$, which is known to be the value of $\xi$ minimizing $T$ when $\eta = 0$ (see [7, p. 376]). More precisely, the critical Taylor number $T$ is obtained when $\xi = 3.1$ and $\eta = 0$. It can be shown that $R$ is an odd function and $T$ is an even function of $\eta$. 
We have used the computer package Mathematica to analyze the dependence of $R$ on $\eta$. It can be seen that $R$ is positive when $\eta$ is negative. The graph in Figure 2 shows the values of $R$ for various values of $(-\eta)$ when $|\eta|$ is small.

From Figure 2, we obtain an approximate relation

$$\eta \approx -2R$$

(7.3)

for $|R| \leq 12$. The dependence of $T$ on $(-\eta)$ is given in Figure 3, which has also been obtained with the help of Mathematica computer package.

From Figures 2 and 3, we see that the critical Taylor number and the Reynolds number are related approximately by the relation

$$T \approx 1715 + 4.5(R^2 - 0.037|R|^3)$$

(7.4)

for $|R| \leq 12$.

8. CONCLUDING REMARKS

In our system the axial motion of the fluid is caused by the axial motions of the cylinders. The stability of a similar system has been investigated by Chandrasekhar (cf. [7–9]), Kruger
and Di Prima [10] when the cylinders only rotate and the axial flow of the fluid is induced by a pressure gradient. For example, in [8] Chandrasekhar obtained

\[ \eta = -3.72R, \quad T = 1715 + 27.9R^2. \]  \hspace{1cm} (8.1)

Note that the Reynolds number in (8.1) was defined using the average velocity for the axial flow of the fluid.

We can compare the relations in (8.1) of Chandrasekhar with our relations (7.3) and (7.4) in the previous section. As is expected, the general behavior of \( T \) and \( R \) in both cases agree, and the critical Taylor number for both systems is given by \( T = 1715 \), when \( \eta = 0 \). However, we see that the formulas are somewhat different even if we take into account the fact that Chandrasekhar used the average velocity for his definition of the Reynolds number.

**REFERENCES**