Matrix Power Control Algorithm for Multi-input Multi-output Random Vibration Test

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Abstract:

Both auto-power spectrum and cross-power spectrum need to be controlled in multi-input multi-output (MIMO) random vibration test. During the control process with the difference control algorithm (DCA), a lower triangular matrix is derived from Cholesky decomposition of a reference spectrum matrix. The diagonal elements of the lower triangular matrix (DELM) may become negative. These negative values have no meaning in physical significance and can cause divergence of auto-power spectrum control. A proportional root mean square control algorithm (PRMSCA) provides another method to avoid the divergence caused by negative values of DELM, but PRMSCA cannot control the cross-power spectrum. A new control algorithm named matrix power control algorithm (MPCA) is proposed in the paper. MPCA can guarantee that DELM is always positive in the auto-power spectrum control. MPCA can also control the cross-power spectrum. After these three control algorithms are analyzed, three-input three-output random vibration control tests are implemented on a three-axis vibration shaker. The results show the validity of the proposed MPCA.

Keywords: multi-input multi-output; environmental testing; vibration control; random vibration; auto-power spectrum; cross-power spectrum

1. Introduction

The vibration of structures in real world is generally in multiple directions, but a vast majority of vibration tests are performed in single direction by single shaker. It is found that single shaker cannot duplicate many failures caused by vibration in vehicles or spacecraft. The real world vibration environment should be simulated by multiple shakers in one or more axes. For large or heavy structures, multiple shakers can provide sufficient driving force. For slender structures, multiple shakers can provide proper distribution of excitation energy. It is recognized that multiple shakers will better excite the modes of structures simultaneously than single shaker. More and more attentions have been paid to multiple shaker vibration tests. The “method 527 multi-exciter testing” has been added to MIL-STD-810G. Studies on multi-input multi-output (MIMO) random vibration test control have been made to implement the multiple shaker vibration tests. Many technical papers about MIMO random vibration test control have been presented. In MIMO vibration test control, the cross-power spectrum control plays an important role along with the auto-power spectrum control. This paper proposes a new MIMO random vibration test control algorithm, with which both auto-power spectrum and cross-power spectrum can be controlled with good performance.

Since decades ago, Smallwood has published many papers about multiple shaker random vibration con-
control [1-8]. These papers establish the MIMO random vibration test control theory. In the recent years, Underwood has also published papers boosting the development of MIMO vibration test control techniques [9-11]. Refs. [12]-[16] also discussed MIMO random vibration test control, and showed test or simulation results. The origin of these theories in the mentioned papers is the one proposed by Smallwood [5] in 1999. In this algorithm, the error matrix between control spectrum matrix and reference spectrum matrix is expressed in difference form, so the algorithm is named difference control algorithm (DCA). During the control process using DCA, a lower triangular matrix is derived from Cholesky decomposition of a reference spectrum matrix, and the diagonal elements of the lower triangular matrix (DELTM) may become negative. The negative value has no meaning in physical significance [17] and can cause divergence in the response spectrum control. An algorithm named proportional root square control algorithm (PRMSCA) proposed by He [13] can avoid the divergence of DCA. It was validated by tests that the auto-power spectrum converged rapidly and was steady to reference spectrum by using PRMSCA, but the cross-power spectrum was not controlled. It is necessary to find an algorithm with good control performance of PRMSCA and the ability to control cross-power spectrum meanwhile.

In a random vibration test, the motion of a structure under test will be unique if both the auto-power spectrum and the cross-power spectrum are controlled at the same time point. In order to duplicate the unique motion of the structure under test, a new algorithm named matrix power control algorithm (MPCA) is discussed in this paper. The error matrix between control spectrum matrix and reference spectrum matrix is expressed in multiplication form, and a convergence coefficient is added to MPCA. The coefficient guarantees the convergence of control and improves the control accuracy [18]. MPCA can be regarded as an extension of PRMSCA, because it has the same ability as PRMSCA to control auto-power spectrum. Furthermore MPCA can control the cross-power spectrum. At the end of this paper, the control results of three-axis vibration test using MPCA are shown.

2. Theory of MPCA

The control target of a random vibration test control is to keep the response power spectrum \( S_{yy} \) identical with the reference spectrum \( R \):

\[
S_{yy} = R
\]  

(1)

So, the driving spectrum matrix is designed as [5,13]

\[
D = ALP
\]  

(2)

where \( A \) is the inverse matrix of frequency response matrix \( G \), named the impedance matrix of the system

\[
A = G^{-1}
\]  

(3)

\( L \) is a lower triangular matrix, which is derived from the Cholesky decomposition of reference matrix \( R \):

\[
R = LL^H
\]  

(4)

From Eq. (4), it is known that the diagonal elements of \( L \) are positive real, and the off-diagonal elements are complex. The superscript “\( H \)” denotes conjugate transpose. \( P \) is a diagonal matrix of dimension \( N \times N \), and is named random phase matrix, whose diagonal element is \( e^{i \theta_i} \) (\( i = 1, 2, \cdots, N \)), where \( \theta_i \) denotes phase angle, which obeys uniform distribution in the interval \([\pi, \pi] \). The response of the system can be expressed as

\[
Y = GD = GALP
\]  

(5)

Then the power spectrum matrix is obtained:

\[
S_{yy} = YY^H = GALPP^HLL^HAA^HG^H = GAL^HL^HAA^HG^H
\]  

(6)

where \( S_{yy} \) is a mathematical expectation in application, and indicates energy power in physics significance, so it is positive definite. Then the Cholesky decomposition of \( S_{yy} \) can be obtained:

\[
S_{yy} = L_SL_S^H
\]  

(7)

The diagonal elements of \( L_S \) in Eq. (7) are positive real, and the off-diagonal elements are complex. Due to the measurement error and the leakage caused by time domain randomization process [19-20] and so on, the impedance matrix in Eq. (2) is not the ideal impedance matrix \( A \). The nonideal impedance matrix is denoted by \( \tilde{A} \), and the error between \( \tilde{A} \) and \( A \) is expressed as

\[
\tilde{A} = A\tilde{E}
\]  

(8)

where \( \tilde{E} \) is the error matrix. By substituting Eqs. (7)-(8) into Eq. (6), one can have

\[
S_{yy} = L_SL_S^H = \tilde{E}LL^H\tilde{E}^H
\]  

(9)

Consequently, the response power spectrum matrix \( S_{yy} \) is not identical with the reference spectrum matrix \( R \) when the error exists. Compare Eq. (4) with Eq. (9), a correction matrix \( E \) is defined, which satisfies

\[
ES_{yy}E^H = R
\]  

(10)

Substituting Eq. (4), Eq. (7) and Eq. (9) into Eq. (10) yields

\[
EL_S^H\tilde{E} = E\tilde{E}LL^HE^H = L^H
\]  

(11)

According to the uniqueness of Cholesky decomposition and comparing both sides of Eq. (11), we can find

\[
E = LL_l^{-1}
\]  

(12)
Because the diagonal elements of $L$ and $L_\delta$ are positive real and the off-diagonal elements are complex, it can be concluded from Eq. (12) that $E$ is a lower triangular matrix, its diagonal elements are positive real, and its off-diagonal elements are complex. Then the correction matrix $E$ can be used to correct $L$. In order to guarantee the convergence of control and improve the control accuracy, a convergence coefficient $\varepsilon$ is introduced at the superscript position of $E$ and thus we have

$$L_{\text{new}} = E^\varepsilon L_{\text{old}}$$ (13)

When $\varepsilon$ is small, it needs more time to make $S_{\text{yy}}$ converge into the allowable boundary of $R$, but the control accuracy is better than that of a bigger $\varepsilon$. The value of $\varepsilon$ can be adjusted according to the need of engineering application, and it can be a constant or a variable number during the control process.[18]

Eq. (7), Eq. (12) and Eq. (13) form the new algorithm—MPCA as shown in Fig. 1.

In Fig. 1, the frequency response function of the test system is firstly measured, and the initial value of $L$ is calculated from $R$. Then the drive spectrum $D$ can be obtained in frequency domain and transformed to drive signal $x$ in time domain by the time domain randomization process. $y$ is the response of the system, and is used to calculate power spectrum matrix $S_{\text{yy}}$. The control algorithm (here is MPCA) corrects $L$ to get a new drive spectrum $D$.

3. Analysis of Control Algorithms

3.1. Difference control algorithm

3.1.1. Benefit from DCA

By expressing the error between control spectrum and reference spectrum in difference form, it is obtained in Ref. [7] that

$$L_{\text{old}} + \Delta L_{\text{old}}^H = R - S_{\text{yy}} = \hat{E}$$ (14)

where $\hat{E}$ is the error matrix between $R$ and $S_{\text{yy}}$, and $\Delta$ is the correction matrix of $L$:

$$L_{\text{new}} = L_{\text{old}} + \Delta$$ (15)

Solving Eq. (14), one can obtain $\Delta$, whose elements are shown in Eq. (16).

$$\begin{align*}
\delta_{ij} &= \frac{\hat{e}_{ij}}{2l_{ij}} \\
\delta_{jj} &= \frac{\hat{e}_{jj} - \delta_{ij} l_{ij}}{l_{ij}} \\
\hat{e}_{jk} &= \frac{-\sum_{l=1}^{N} (\delta_{jl} l_{lj} + \delta_{lk} l_{kj})}{2l_{jj}} \\
\delta_{jk} &= \frac{-\delta_{jk} l_{jk} - \sum_{l=1}^{N} (\delta_{jl} l_{lj} + \delta_{lk} l_{kj})}{l_{jj}}
\end{align*}$$ (16)

where $\hat{e}_{jk}$, $l_{jk}$ and $\delta_{jk}$ are elements of matrix $\hat{E}$, $L$ and $\Delta$ at the $j$th row and $k$th column, and superscript $**$ denotes conjuguate.

$S_{\text{yy}}$ is a Hermite matrix, whose diagonal elements denote auto-power spectra and off-diagonal elements denote cross-power spectra. The definition “independent variables” will be used to express the independent “absolute value” and “phase” of a complex number, so a real number has only one independent variable, and a complex number has two independent variables. If the dimension of $S_{\text{yy}}$ is $N \times N$, there are $N$ real numbers that denote auto-power spectra, and due to the conjugate symmetry of $S_{\text{yy}}$ there are only $N(N-1)/2$ independent complex numbers that denote cross-power spectra, so there are $N(N-1)$ independent variables that denote cross-power spectra. In totality, there are $N^2$ independent variables that need to be controlled in $S_{\text{yy}}$.

The diagonal elements $\delta_{jj}$ are real, and the off-diagonal elements are complex, so there are $N^2$ independent variables in correction matrix $\Delta$, which means DCA can control all the independent variables of $S_{\text{yy}}$. Namely, DCA can control both auto-power spectra and cross-power spectra.

3.1.2. Limitation of DCA

Although “scale” operation is added, there is still a problem. If $\delta_{jj}$ is a negative number and its absolute value is bigger than that of $l_{jj}$, i.e.,

$$|\delta_{jj}| > |l_{jj}|$$ (17)

However, from Eq. (16), it can be seen that if $l_{jj}(j=1, 2, \cdots, N)$ grows large, $\delta_{kk}(j, k=1, 2, \cdots, N)$ will become smaller because of the division by $l_{jj}$, and this could cause $l_{jj}$ to be “stuck at an unrealistically high level”. If $l_{jj}$ becomes very small, $\delta_{kk}$ may become very large, and the correction will “blow up”. Ref. [7] suggested adding “scale” operation to prevent $\delta_{kk}$ becoming too large or too small.

Although the “scale” process is added, there is still a problem. If $\delta_{jj}$ is a negative number and its absolute value is bigger than that of $l_{jj}$, i.e.,
3.2. Proportional root mean square control algorithm

3.2.1. Benefit from PRMSCA

Ref. [13] shows a different correction method of \( L \): defining a diagonal correction matrix \( \Delta_{\text{diag}} \) with its diagonal elements being

\[
\Delta_{\text{diag},jj} = \sqrt{\frac{s_{yy,ij}}{r_{jj}}} \quad (j = 1, 2, \ldots, N) \tag{19}
\]

where \( s_{yy,ij} \) and \( r_{jj} \) are the diagonal elements of \( S_{yy} \) and \( R \) respectively. From Eq. (19), \( \Delta_{\text{diag},jj} > 0 \). Without controlling cross-power spectrum, \( R \) is a diagonal matrix, and so is \( L \). The correct method can be expressed as

\[
L_{\text{new}} = \Delta_{\text{diag}} L_{\text{old}} \tag{20}
\]

The diagonal elements of \( L_{\text{new}} \) are the products of the diagonal elements of \( \Delta_{\text{diag}} \) and \( L_{\text{old}} \) (Eq. (20)), so the diagonal elements of \( L \) will always be positive, and will not have the “negative value problem” as DCA. Test results show that PRMSCA has good performance on controlling auto-power spectrum.

3.2.2. Limitation of PRMSCA

From Eq. (19), it can be seen that \( \Delta_{\text{diag}} \) is a real diagonal matrix, and has only \( N \) independent variables, so PRMSCA can only control \( N \) independent variables of \( S_{yy} \). That means, PRMSCA can only control the diagonal elements of \( S_{yy} \) which denote the auto-power spectrum.

3.3. Matrix power control algorithm

3.3.1. Improvements of MPCA

DCA can control both auto-power spectrum and cross-power spectrum, but needs “scale” operation and still has the “negative value problem” in the control process. PRMSCA can avoid “negative value problem” by keeping \( l_{ij} \) always positive, but it is unable to control cross-power spectra for the sake of correction matrix \( \Delta_{\text{diag}} \) which has no off-diagonal elements.

We have tried to add off-diagonal elements to the correction matrix of PRMSCA, but found it difficult to get explicit equation of correction matrix. The reasons for this difficulty are the two assumptions in difference forms. One is the product of \( G \) and \( A \), the other is the error matrix between \( S_{yy} \) and \( R \):

\[
GA = I + \hat{E} \tag{21}
\]

\[
\hat{E} = R - S_{yy} \tag{22}
\]

where \( I \) is the unit matrix. These two assumptions are changed into the following forms in MPCA:

\[
GA = \hat{I} \hat{E} = \hat{E} \tag{23}
\]

\[
E S_{yy} E^H = R \tag{24}
\]

Furthermore an unique calculation of correction matrix \( E \) is obtained from \( L \) and \( L_{S} \) (Eq. (12)), but not from \( S_{yy} \) and \( R \) as in DCA (Eq. (16)) and PRMSCA (Eq. (19)).

All of the above improvements help MPCA absorb the advantages of both DCA and PRMSCA.

3.3.2. Benefit from MPCA

To simplify the expression, let \( e = 1 \), and the elements of \( L_{\text{new}} \) can be obtained from Eq. (13):

\[
l_{jk,\text{new}} = \sum_{i=k}^{N} e_{ij} l_{ik,\text{old}} \quad (1 \leq j \leq N, 1 \leq k \leq j) \tag{25}
\]

When \( k=j \), Eq. (25) can be written as

\[
l_{jj,\text{new}} = e_{jj} l_{jj,\text{old}} \tag{26}
\]

Because the diagonal elements of \( E \) are positive real, \( l_{ij} \) is always positive real (Eq. (26)), which is the same with that of PRMSCA (Eq. (20)). MPCA has the same feature as PRMSCA in auto-power spectrum control. This feature guarantees the steady convergence of auto-power spectrum control.

It has been proved that \( E \) is a lower triangular matrix with its diagonal elements being positive real and off-diagonal elements being complex, so there are \( N^2 \) independent variables in correction matrix \( E \), which is the same as the correction matrix \( \Delta \) of DCA. In a word, MPCA can avoid “negative value problem” in auto-power spectrum control and can control cross-power spectrum.

Furthermore, MPCA does not need “scale” operation and will never be “stuck” or “blown up”. This will make MPCA more easily to be used in engineering application.

3.4. Summary

Three control algorithms have been discussed on two issues:

1. The ability to avoid “negative value problem”. PRMSCA and MPCA are able to keep \( l_{ij} \) always positive real, which will bring steady convergence of auto-power spectrum control. However, DCA does not have this ability.

2. The ability to control cross-power spectra. DCA and MPCA can control the cross-power spectra, but
PRMSCA cannot.

The features of the three control algorithms are summarized in Table 1.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Able to avoid &quot;negative value problem&quot;</td>
<td>×</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Able to control cross-power spectra</td>
<td>√</td>
<td>×</td>
<td>√</td>
</tr>
</tbody>
</table>

MPCA absorbs the advantages of both DCA and PRMSCA. MPCA can keep $P_j$ always positive real which will bring steady convergence of auto-power spectrum control, and correction matrix of MPCA has $N^2$ independent variables, which will allow MPCA to control both auto-power spectra and cross-power spectra simultaneously.

4. Test

4.1. Parameter setting

Three-input three-output random vibration tests were implemented on a three-axis vibration shaker. Control Points 1-3 were defined in x, y and z directions respectively upon the center of shaker table. Fig. 2 shows the test equipment: Shinken G-6080-3HT-020 three-axis vibration shaker, Agilent VXI numerical signal transmitting and sampling system, PCB acceleration sensors, etc.

![Test equipment](image)

Fig. 2 Test equipment.

Settings of parameters of reference spectrum are as follows. Frequency band was 2 000 Hz, spectrum line 400. The auto-power spectrum of control Point 1 was the same as that of control Point 3 (see Fig. 3(a)); that of control Point 2 is shown in Fig. 3(b). In Fig. 3(a), the slope of the reference spectrum is 3 dB/Oct at 20-100 Hz, the power spectrum density (PSD) is $2 \times 10^{-7} g^2 \cdot Hz^{-1}$ at 100-1 000 Hz, and the slop is $-3$ dB/Oct at 1 000-2 000 Hz. In Fig. 3(b), the slop is 3 dB/Oct at 20-200 Hz, and the PSD $1 \times 10^{-7} g^2 \cdot Hz^{-1}$ at 200-2 000 Hz. The error boundary of control was generally set with $\pm 3$ dB as alarm boundary and $\pm 6$ dB as abort boundary.

![Reference spectrum of auto-power spectrum](image)

Fig. 3 Reference spectrum of auto-power spectrum.

Cross-power spectrum control means to control coherence and phase of cross-power spectrum. The reference spectrum of coherence and phase was set as constants listed in Table 2. There is no general setting of error boundary for coherence and phase, so boundaries were not set in this paper.

<table>
<thead>
<tr>
<th>Control point</th>
<th>Coherence</th>
<th>Phase/°</th>
</tr>
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<tbody>
<tr>
<td>1 and 2</td>
<td>0.1</td>
<td>-45</td>
</tr>
<tr>
<td>1 and 3</td>
<td>0.6</td>
<td>90</td>
</tr>
<tr>
<td>2 and 3</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

4.2. Test results

4.2.1. DCA control results

DCA was used firstly. Part of the control results are shown in Fig. 4, where the outside dash dot lines denote $\pm 6$ dB abort boundary of reference spectrum, the dash lines next to dash dot lines denote $\pm 3$ dB alarm boundary, and the dot line in the middle denotes reference spectrum, and the solid line denotes auto-power spectrum of response. During the control process, the auto-power spectrum of control Point 2 at 1 285 Hz grew higher and higher. At the 8th correction of $L$, the auto-power spectrum (see Fig. 4) is $3.726 \times 10^{-7} g^2 \cdot Hz^{-1}$, 5.7 dB above the reference value.

![DCA control result of auto-power spectrum of control Point 2](image)

Fig. 4 DCA control result of auto-power spectrum of control Point 2.
By recording the correction of $L$, it is found that $l_{22}$ becomes negative at the 5th correction (see Fig. 5). At the 4th correction, $l_{22,\text{old}}=3.112 \times 10^{-5}$, $\delta_{22}=-9.763 \times 10^{-5}$, $l_{22,\text{new}}=-6.651 \times 10^{-5}$. When the $l_{22,\text{new}}$ is applied to Eq. (6), the negative value is used with its absolute value. The control spectrum will go higher, and $l_{22}$ will be corrected to be smaller, as shown in Fig. 5.

![Fig. 5](image1.png)

Fig. 5 Correction of $l_{22}$ at 1285 Hz with DCA, PRMSCA and MPCA.

Theoretically, DCA is able to control the cross-power spectrum, but due to the negative value of $l_{22}$ at 1285 Hz, the coherence between control Point 1 and Point 3 droppes down to 0.22 at 1285 Hz (see Fig. 6), 0.38 below the reference value 0.6.

![Fig. 6](image2.png)

Fig. 6 DCA control result of cross-power spectrum between control Point 1 and Point 3.

4.2.2. PRMSCA control results

PRMSCA is designed to avoid the negative value of $l_{jj}$, and it succeeds in controlling auto-power spectrum. The auto-power spectrum control result at control Point 2 is shown in Fig. 7. The spectrum line at 1285 Hz keeps well within $\pm 3$ dB alarm boundary of reference spectrum.

![Fig. 7](image3.png)

Fig. 7 PRMSCA control result of auto-power spectrum of control Point 2.

The correction of $L$ is also plotted in Fig. 5. $l_{22}$ is always positive. PRMSCA does not control cross-power spectra, so $l_{22}$ is not close to zero when $L$ is corrected.

As analyzed in Section 3.2.2, PRMSCA cannot control cross-power spectra. The coherence (see Fig. 8) is 0.32 at 30 Hz, and is a little lower than the reference at high frequencies. The coherence is almost what it is at the beginning of the test.

![Fig. 8](image4.png)

Fig. 8 PRMSCA control result of cross-power spectrum between control Point 1 and Point 3.

4.2.3. MPCA control results

MPCA was used finally. The correction of $L$ is also plotted in Fig. 5. $l_{22}$ is always positive. Let $\epsilon=0.5$, and the control result of auto-power spectrum is shown in Fig. 9. As Fig. 9(b) shows, the divergence of auto-power spectrum of control Point 2 at 1285 Hz does not exist.

![Fig. 9](image5.png)

Fig. 9 MPCA control result of auto-power spectrum.
Seen from Fig. 9, all auto-power spectra are well controlled within ±3 dB alarm boundary.

The control result of cross-power spectrum is shown in Fig. 10.

![Coherence and Phase Plots](image)

**Fig. 10** MPCA control result of cross-power spectrum.

Compared with DCA (see Fig. 6) and PRMSCA (see Fig. 8), the control result of MPCA (see Fig. 10(b)) is better. The coherence curve is more close to the reference value than that of DCA and PRMSCA.

The phase of control results are not compared, because they are all well controlled by DCA, PRMSCA and MPCA. The phase in Fig. 10(c) looks “bad” controlled, which is because the corresponding coherence is set to zero. Compared with DCA and PRMSCA, MPCA can control auto-power spectrum and cross-power spectrum better. For the sake of the length of this paper, other control results of DCA and PRMSCA are not shown. The rest control results of MPCA are shown in Fig. 9(a) and Fig. 9(c), and Fig. 10(a) and Fig. 10(c). They are all well controlled by MPCA.

### 5. Conclusions

A new control algorithm for MIMO random vibration tests named MPCA is discussed in this paper. The theory of this algorithm is presented, and a comparing analysis with other two algorithms is made. Finally, three-input three-output random vibration tests are implemented. The “negative value problem” is discussed based on the test results of DCA, PRMSCA and MPCA.

From the theoretical analysis and comparing tests, it is known that MPCA is steady, convergent, able to control both auto-power spectrum and cross-power spectrum and does not need “scale” process, so MPCA is a better choice for MIMO random vibration test control.

The idea of assumptions used in MPCA can also be used in other MIMO vibration test controls.

The auto-power spectrum and cross-power spectrum are regarded as equally important in MPCA in this paper. When they are not equally important, how to add the weights of importance into MPCA? This will be studied in the future.

### References


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