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A New Approach to Mathematical Economics: On Its Structure as a Homomorphism of Gibbs–Falkian Thermodynamics

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DEDICATED TO P. A. SAMUELSON ON THE OCCASION OF HIS 80TH BIRTHDAY

A homomorphism between the theories of thermodynamics and economics is introduced. By means of Gibbs-Falkian thermodynamics it is shown that a systems theory can be set up including economic equations of state, as well as a concise notion of equilibirum. Indicating strict separations with respect to dynamics in economics, commonly used at present, and adopting the physico-dynamical concepts of energy, motion, and momentum, connections between the economic phase space and the related space of events are established.

True theory never substitutes practice—it is practice.

T. S. W. Salomon

INTRODUCTION

For a long time *economic* theory has known attempts to apply methods and definitions from the natural sciences, above all from physics, for the description of economic relations and processes [3]. It seems to be legitimate to combine the different scientific concepts of physics and economics in the face of decreasing natural resources. The authors propose to combine them by way of a mathematically stringent analytical *economic* theory.

Prominent American economists have been fostering such ideas to this very day. Pre-eminent among these are Paul A. Samuelson (M.I.T.) and

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Nicholas Georgescu-Roegen (Vanderbilt University). In Europe two Nobel prize winners, Jan Tinbergen and Tjalling Koopmans, "show the GIBBSian influence." [18, p. 256].

In 1970 Samuelson dedicated a substantial part of his programmatic Nobel prize speech to the adequacy of thermodynamic maximum principles pertaining to economic problems [17]. It should be noted that he once called himself a "grandson of Josiah Willard GIBBS," the celebrated American thermodynamicist at Yale [18]. The main points of Samuelson's approach are self-evident *morphisms between thermodynamics and economics*. Therein a critical point, however, is the entropy notion and, in this context, the Second Law of thermodynamics. This problem is concisely expressed by Samuelson [16]:

> In theoretical economics there is no "irreversibility" concept, which is one reason GEORGESCU-ROEGEN [in 1971] is critical of conventional economics.

This critique and also the reasons for it are expounded by Georgescu-Roegen in his world-renowned book, "The Entropy Law and The Economic Process" [8]. Therein the necessity is emphasized to postulate some sort of Second Law also for economics. But Georgescu-Roegen casts off his belief in to simple analogies or in direct references to physical entropy laws. His respective statement is definite: "The true output of the economic process is not a physical outflow of waste, but the enjoyment of life." And he further points out that without introducing this concept "into our analytical armamentarium we are *not* in the economic world." [8, p. 282]. This idea, specified by Samuelson's approach, was the starting point of the investigation presented here.

Hence, the object of this paper will deal with what is briefly termed "Gibbs-Falkian dynamics," reflected by a *homomorphism* joining the domains of physics and economics. Clearly, there are no simple analogies. The crucial point is, at a first glance, that only the *mathematical structure* of phenomenological thermodynamics has morphisms with theoretical economics. Our views coincide with Samuelson's respective statement [18, p. 263].

The first part of this paper is concerned with the far-reaching consequences of a typical mathematical property for theoretical economics. This property is denoted as *extensitivity* [6, p. 263f] and is well known in physics. Using Gibbs' thermostatics, these consequences may easily be exemplified with regard to some fundamental relations between the variables of any homogeneous domain in Gibbs' meaning [9, p. 63]. The corresponding economic relations are presented for the special case of a production system which is characterized by a modified production function of the CES-type.

In the second part it is intended to explain a thermodynamically motivated concept of equilibrium states in economics. In addition, some suggestions will be made concerning the dynamical behavior of economic systems in an appropriate space of events. The Gibbs-Falkian thermodynamics [21] can be seen as a theory of nonlinear dynamical systems, for which the problem of time and space parameters is conclusively arranged by Noether's theorem.

By the way, the idea to demonstrate *structural correspondences* between the relations among variables of two theories, different in form and subject matter, was first taken up in the authors' publication "Analytische Produktionstheorie" back in 1992 [10].

1.1. *Homomorphisms and Homogeneous Functions*

Let T_1 be the theory of thermostatics, and let T_2, e.g., be the supply-oriented *economic* production theory. A homomorphic mapping between these two theories is proposed, using the common terms of mathematical mapping techniques [12]. Symbolically,

$$\tau_{\text{hom}} : T_1 \rightarrow T_2, \qquad \tau_{\text{hom}} : a \otimes b \mapsto \tau_{\text{hom}}(a) * \tau_{\text{hom}}(b); \tag{1}$$

a, b as elements of T_1; $\otimes$ and $*$ as operations in T_1 and T_2, respectively.

It goes without saying that Samuelson's isomorphisms between parts of the theories are included, if the *correspondence operation* τ_{hom} is one-to-one.

In all parts of physics, as well as in those of economics, the respective domains may be characterized by a set of well-defined "*standard variables.*" Let us dwell for a moment on T_1, that is, on physics! In that science the standard variables are connected among themselves by some characteristic group-theoretical transformation properties, relevant for the set of variables and constitutive for the physical system. In addition, it must be emphasized that all these standard variables are *extensive.*

One of the most important properties of any extensive quantity is its *additivity;* in other words, assume that a reference unit may be divided into subunits. If, for some given quantity, the sum of the values for all parts equals the value for the whole, then the respective quantity is called extensive. Another one of the characteristics of each extensive variables is its *conservation* quality. Unlike other ones, this property is not true for some quantities, e.g., for the heat rates as the most relevant *interacting quantity* in physics. Such quantities, just as all the extensive variables, contain the same constant factor λ to be defined by the ratio of the standard variables X_i and its related modification x_i,

$$X_i = \lambda \cdot x_i, \qquad \lambda > 0; i = 1, 2, \ldots, n.$$

For these variables a *homogeneous function* of degree m may be set up. The quantitative version of this function may be written in the well-known form

$$f(X_1, X_2, \ldots, X_n) \equiv f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^m \cdot f(x_1, x_2, \ldots, x_n) \quad (2)$$

which is generally true.

For the important special case of f being homogeneous of *degree one* (i.e., $m \equiv 1$), Euler's theorem allows the following differential relation:

$$\sum_{i=1}^{n} X_i \cdot \frac{\partial}{\partial X_i} f(X_1, X_2, \ldots, X_i, \ldots, X_n) = f(X_1, X_2, \ldots, X_i, \ldots, X_n). \quad (3)$$

Put $\xi_i := \partial f / \partial X_i$ for $i = 1, 2, \ldots, n$

$$\Rightarrow \sum_{i=1}^{n} \xi_i \cdot X_i = f(X_1, X_2, \ldots, X_i, \ldots, X_n). \quad (4)$$

Equation (4) opens the way for understanding the statement of Callen, "Extensive parameters play a key role throughout thermodynamic theory." [2, p. 9].

In physics λ is usually identified with *total mass* taking part in the variations of all the relevant variables. All ξ_i are partial derivatives of f with respect to each extensive variable X_i, and as such they do have a separate name. To take a well-known example, the entropy-conjugate variable ξ_i is called the "absolute temperature." Usually all ξ_i are termed "*intensive*" [2, p. 31].

As to economics, the second theory T_2, Euler's theorem clearly offers advantages for homogeneous functions as a means of describing the reality of economic activities. And there cannot be any doubt that the advocates of neoclassical *economic* theory have been well aware of the fact that extensive variables may duly represent relevant items of economics. To a large extent the benefit of these variables is due to their being measured along a *ratio scale*. Here is not the place to detail the conditions of whether any given extensive variable X_i may be classified as a standard variable. Some interesting work on standard variables in physics is reviewed and extended in a recent paper by one of us [22].

Clearly it is necessary to answer the question which one of the *economic* quantities might play the role of the constant λ. In order to apply *linear-homogeneous* functions to economics according to Eq. (3), it is surely adequate to pick out for λ some quantity which yields for all X_i reasonable related variables $x_i = X_i/\lambda$, $i = 1, 2, \ldots, n$.

More than two hundred years of *economic* experience allow the conclusion that in production theory λ may be equated with the so-called *labor force*, that is, the number N of employed (index e), employable, and unemployed (index u) persons in a nation's economy. For the benefit of the observations and statements presented here, it is in no way required to use the labor force as an equivalent for λ.

One might just as well have taken the entire population for that purpose, and N^u would have to include, e.g., the number of children as well, with the obvious advantage of an easier utilization of national accounting data in the sectors of consumption. Other conventions, however, might be considered, e.g., the averaged working hours per annum.

By way of definition the labor force is practically constant within the domain of the other relevant standard variables. Consequently, the definition

$$\lambda \equiv N = N^e + N^u \tag{5}$$

is assumed to be suitable for the modeling of *economic* theories.

Hence, related variables x_i arise from the common standard variables of *production* theory, viz. *capital* (C), *output* (Q), and the *total number of the actual hours of work* (L):

$$x_C = \frac{C}{N}; \qquad x_Q = \frac{Q}{N}; \qquad x_L = \frac{L}{N}. \tag{6}$$

Evidently, the set of variables introduced here will generally not suffice to characterize adequately the highly complex structure of *economic* activities. One will, therefore, have to take into account additional variables, among these even such that are neither extensive variables nor interacting quantities, but rather quantities which are called *intensive*.

By means of example the general *rate of interest* p might be such an intensive variable. Assuming that p is connected in some way with a certain sort of capital, say with $C^{[p]}$, then an *economic* problem could be described and explained by the set of variables $C^{[p]}$, Q, L, and p itself. They refer in each moment to an *economic* state occurring in a domain, wherein the labor force N may be considered sufficiently constant. Furthermore, it should always be remembered with reference to Gibbs [9, p. 67] that N is a significant parameter represented by λ. Under these circumstances simple mathematical operations offer some obvious conclusions which have hardly been drawn systematically right up to this day by traditional economists.

1.2. *Gibbs-Falkian Thermodynamics and Production Theory*

Let there be a so-called Gibbs function between capital $C^{[p]}$ and the other variables according to

$$C^{[p]} = C^{[p]}(Q, L, p \mid \lambda = N). \tag{7}$$

By way of a *Legendre contact transformation* [2, p. 90f] with regard to the intensive variable p, an extensive variable V is generated, being worth attracting our attention to its conjugate economic meaning.

This special case yields

$$C = C^{[p]} - \left.\frac{\partial C^{[p]}}{\partial p}\right|_{Q,L} \cdot p \tag{8}$$

$$\begin{aligned}\Rightarrow C &= C\left(Q, L, \left.\frac{\partial C^{[p]}}{\partial p}\right|_{Q,L} \mid \lambda = N\right)\\ &= C(Q, L, V \mid \lambda = N).\end{aligned} \tag{9}$$

The *formal side* of the subject is obvious: The Legendre transformation replaces the functional equation (7) by (9) which exclusively contains extensive variables. N remains a variable even if it is assumed to be constant under the proposed variations of state.

As to the *economic* theory T_2, however, in the authors' view something really remarkable happens. Apart from the new extensive standard variable

$$V = \left.\frac{\partial C^{[p]}}{\partial p}\right|_{Q,L}$$

which is directly related to p, a second capital quantity arises: the *stock of productive capital* C. The rate of interest p may be seen quite traditionally in connection with capital. The variable p is, a fortiori, quantitatively joined to the set of relevant standard variables for which a closed boundary surface is additionally adapted to new external and economically proper conditions. Within this higher-order area real production takes place per definition. This fact forms one part of *economic* fundamentals, but does not inspire dogmatic *economics* up to now. It is remarkable, nonetheless, that p emerges as a conjugate variable of V from the theory presented here. In physics [6, p. 105f] V is one of the most important variables, as it allows to define *spatially homogeneous* regions known as *phases*. Thus, it seems adequate to term V as the *domain of economic activity*.

In accordance with Gibbs-Falkian thermodynamics the linear-homogeneous relation (9) is called "Gibbs–Euler function" (GEF). This is a special case of many Gibbs functions [5, p. 21] for the adequate description of the *production* problem in such a way that extensive variables are chosen exclusively.

In any case, if a GEF [21, p. 108] contains all the relevant standard variables pertaining to the problem in question; it also provides all the information about the physical or *economic* system under investigation. Although such an axiom is well within the scope of any theory using deductive reasoning, it can only be confirmed by experience afterwards. Hence, any real system will have to be approximated by a *mathematical model* with a set of variables which is complete by definition. Consequently, the total differential of the dependent variable C can be formed with the help of GEF (9).

Let us repeat. each GEF consists of a complete set of exclusive extensive variables and consequently reflects the system under investigation [5, p. 74]: GEF ⇔ System. Within the framework of the production theory (T_2) the constitutive relations

$$C = \frac{\partial C}{\partial Q} \cdot Q + \frac{\partial C}{\partial L} \cdot L + \frac{\partial C}{\partial V} \cdot V + \frac{\partial C}{\partial N} \cdot N \tag{10}$$

and

$$dC = \frac{\partial C}{\partial Q} \cdot dQ + \frac{\partial C}{\partial L} \cdot dL + \frac{\partial C}{\partial V} \cdot dV \tag{11}$$

hold. They represent the simplest version, as only one "particle number" N is considered, for which the constraint $dN \equiv 0$ is presupposed. Extended versions may easily be realized by the introduction of some additional "particle numbers" corresponding to (5), for instance.

The partial derivatives of productive capital with respect to every other variable, ceteris paribus, can now be named according to their *economic* meanings; for some of them quite new expressions had to be formulated—as there are no correspondences to contemporary economic theory:

$\frac{\partial C}{\partial Q} := \xi_Q,$ Marginal capital output ratio

$-\frac{\partial C}{\partial L} := \xi_L,$ Marginal intensity of capital

$-\frac{\partial C}{\partial V} := p,$ Rate of interest of the economic system [new]

$\frac{\partial C}{\partial N} := \mu,$ Technological potential of a production system [new].

1.3. *Equations of State*

Using the new symbols, Eqs. (10) and (11) can simply be represented by the forms

$$C = \xi_Q \cdot Q - \xi_L \cdot L - p \cdot V + \mu \cdot N \Leftrightarrow \text{Equation for capital} \tag{12}$$

$$dC = \xi_Q \cdot dQ - \xi_L \cdot dL - p \cdot dV; \quad \Leftrightarrow \text{Equation of capital forms} \tag{13}$$

to be subjected to the conservation rule $dN \equiv 0$.

Note that the negative signs with ξ_L and p are mere conventions; these intensive variables are changing their values in directions opposite to those of their respective extensive variables.

Equation (12) may also be called "Euler–Reech function" of economic systems, whilst the Pfaffian (13) is denoted by "Gibbs fundamental equation," in correspondence to Gibbs [9, pp. 86, 357] and conventions in physics [21]. Both are only compatible iff the second part of the total differential dC vanishes identically. Then, the compatibility requirement of (12) and (13) asks for the relation

$$Nd\mu + Qd\xi_Q - Ld\xi_L - Vdp \equiv 0 \tag{14}$$

which is an inevitable consequence of the set of variables introduced above.

In physics this corresponding identity is known as the "Gibbs–Duhem relation"; it is valid both there and in its *homomorphic* counterpart for an *economic* system. From this basic relation one can a fortiori deduce a functional connection

$$\mu = \mu(\xi_Q, \xi_L, p), \tag{15}$$

between only the intensive system variables corresponding to, and independent of, the special mode of the economic behavior observed.

Obviously, the existence of (15) is formally conclusive, if it is admitted that the economic system may be described by the GEF of Eq. (9) given in the form of (12), the equation for productive capital. Equations of this kind, like the one for the *technological potential* μ, should, in the authors' opinion, belong to the basic tools of any macro-economic theory. Nonetheless, one asks oneself why such *marginal system equations* are not known in economics. *Marginal* quantities [8, p. 190] themselves, like ξ_Q or ξ_L, are of course well known in the textbooks; their statistical treatment in many cases, however, is deplorably dubious.

Partial differentiation of the technological potential μ with respect to ξ_Q, ξ_L, and p shows that these derivatives are joined to the extensive standard variables of the production system, for the Gibbs–Duhem relation (14) allows us to compute a set of straightforward differential expressions:

$$\left.\frac{\partial\mu}{\partial\xi_Q}\right|_{\xi_L,p} = -\frac{Q}{N}; \qquad \left.\frac{\partial\mu}{\partial\xi_L}\right|_{\xi_Q,p} = \frac{L}{N}; \qquad \left.\frac{\partial\mu}{\partial p}\right|_{\xi_Q,\xi_L} = \frac{V}{N}. \tag{16}$$

The number of intensive variables, capable of independent variation, is called the number of *degrees of freedom* for any given system [2, p. 50].

The performance of an economic system is traditionally described by a *production function* [15, p. 158f]. If such a mathematical expression can be equated to a GEF of type (9), then both the marginal system equation and the so-called *equations of state* may explicitly be derived from it. The latter are obtained from (16) by differentiating the function μ with respect to each of the intensive quantities and considering (6). Symbolically we have

$$x_Q = x_Q(\xi_Q, \xi_L, p)$$

or

$$x_L = x_L(\xi_Q, \xi_L, p)$$

or

$$x_V = x_V(\xi_Q, \xi_L, p).$$

The set of equations of state is compiled, expressing each related standard variable as a function of the intensive parameters only. These ideas will be expounded in the second part of the paper presented.

Concluding the first part, it should be emphasized that the existence of (15) has a paradigmatical significance for any mathematical economics whatever; provided that economic theory does work with the extensive quantities commonly used; the reverse is also true: unless the marginal system equation relies on empirical data, then these variables are not extensive and cannot mathematically be applied in the traditional manner! But what other alternative will remain in this case?

2

2.1. *Production Function and Marginal System Equation*

To obtain an idea of the precise mathematical structure of an *economic* equation of state, one may revert to a recognized tool of neo-classical economics: that of the so-called *CES production function* (with CES for constant elasticity of substitution). A CES function is homogeneous of degree one and is established originally in the three variables C, Q, and L.

According to the GEF (9) we choose capital as the dependent variable. In the first part of the paper it is argued that there are scientifically eminent reasons to expand the "classical" set of variables by at least two further quantities: the domain of economic activity (V) and the labor force (N). Even if one may treat these quantities as constants, they will yet, despite this, appear as parameters in the respective production function. For that reason, we get an *"extended" CES function* (ECES) written in a structurally *conventional* version

$$C = \left(\frac{1}{k} \cdot Q^{-\sigma} - \frac{l}{k} \cdot L^{-\sigma} - \frac{v}{k} \cdot V^{-\sigma} - \frac{n}{k} \cdot N^{-\sigma}\right)^{1/\sigma}, \tag{17}$$

which is a GEF of the system.

Any ECES function is homogeneous of degree one and homothetic. k, l, v, n, and σ are parameters relevant to the observed production system. Obviously, their units imply the correct dimension for each term of (17).

In connection with the homogeneous Euler–Reech equation the ECES (17) permits the straightforward calculation of the *technological potential of the production system:*

$$\mu := \left.\frac{\partial C}{\partial N}\right|_{Q,L,V} = \frac{n}{k} \cdot \left[\frac{1}{k} \cdot \left(\frac{Q}{N}\right)^{-\sigma} - \frac{l}{k} \cdot \left(\frac{L}{N}\right)^{-\sigma} - \frac{v}{k} \cdot \left(\frac{V}{N}\right)^{-\sigma} - \frac{n}{k}\right]^{-(\sigma+1)/\sigma}. \tag{18}$$

Using the *economic* equations of state, the three related variables Q/N, L/N, and V/N in (18) can easily be replaced by intensive quantities only. Hence, the *marginal system equation* under ECES conditions,

$$\mu = (-n)^{1/\sigma} \cdot (k^{1/(\sigma+1)} - \xi_Q^{\sigma/(\sigma+1)} + l^{1/(\sigma+1)} \cdot \xi_L^{\sigma/(\sigma+1)} + v^{1/(\sigma+1)} \cdot p^{\sigma/(\sigma+1)})^{(\sigma+1)/\sigma}, \tag{19}$$

Results as the immediate inference from the set of standard variables chosen.

Although there are quite a few possibilities to make use of *linear-homogeneous* functions for the description of real economic systems, the ECES function has one undeniable advantage. It leads to a simple algebraic connection between the marginal coefficients like ξ_Q, ξ_L, or μ and their respective *statistical characteristics* L/Q, C/L, or C/N, well known from national accounting:

$$\xi_Q = \frac{1}{k} \cdot \left(\frac{C}{Q}\right)^{\sigma+1}; \qquad \mu = -\frac{n}{k} \cdot \left(\frac{C}{N}\right)^{\sigma+1};$$
$$\xi_L = \frac{l}{k} \cdot \left(\frac{C}{L}\right)^{\sigma+1}; \qquad p = -\frac{v}{k} \cdot \left(\frac{C}{V}\right)^{\sigma+1}.$$

In case our (*economic*) world should work along ECES lines, this attractive property would enable economists to exploit the advantages of the new theory without letting traditional statistical data bases become obsolete.

The reader might ask himself, however, why the analysis presented here puts such a great weight upon the intensive variables. There are some answers as to the *axiomatic* foundation and substantiation of this analysis. But there also exists another reason which grants privileges to the intensive parameters. With their aid it is possible to establish a theoretically consistent definition of equilibrium, in accordance with Samuelson's demand on the mathematical structure of the homomorphic mapping formalism, as proposed in (1). This concept refers methodologically to Gibbsian thermostatics, thereby overcoming for our purposes the affinity of contemporary economics with classical mechanics, deplored by Georgescu-Roegen or Samuelson.

2.2. *Equilibrium States in Economics*

Let Σ_1 and Σ_2 be two economic systems in close contact, each defined by their respective GEFs, and forming a *total system* Σ to be isolated from its environment. Consequently, free interaction is exclusively realized by the reciprocal exchange between the respective variables of Σ_1 and Σ_2 by definition. The basic idea of introducing standard variables that such quantities be *alike* in meaning and definition for every single one of them holds for all sections of the respective science, and this is true for economics, too. Therefore, the reader is justified to regard simultaneously a *production system* Σ_1 and the interacting *consumption system* Σ_2 as parts of an entity Σ. In that case, V_1 (resp. V_2) and L_1 (resp. L_2) are related to their corresponding domains and labor inputs of both the supply and the demand sectors.

Assuming the sum $N_1 + N_2$ to be constant for Σ, the variables N_1 and N_2 refer to the specified labor force of the production system Σ_1 and the number of consumers of the interacting system Σ_2, respectively. In this context it should be emphasized that additional constraints are required, because N_1 contains partial "particle numbers," e.g., the number N_1^e of the employed persons in the production system Σ_1, which will normally also appear as variables in the consumption system Σ_2. This is not the place to discuss the corresponding mathematical complications connected with these constraints.

Now, there is a fundamental assumption, according to which this process Π is initiated and terminated by an inherent trend towards equilibrium in Σ. Equilibrium, however, is a phenomenon which may be manipulated by the observer's length and time standards concerning the real processes running down in the system. They have to be compared with the *variation of time and length scales* occurring *spontaneously* in Π, due either to purely internal changes or to interactions between the natural course of events and the environment.

Accordingly the characteristic times of these inherent scales, which prove to be necessary for the adjustment of the respective *equilibrium state*, are strongly influenced by the details of the natural exchange behavior between Σ_1 and Σ_2. Obviously there is a narrow relationship between these so-called *relaxation times* and the well-known time measures introduced by A. Marshall, in order to relate the impact of different demand conditions to the corresponding new equilibrium level of the prices. In his textbook Samuelson has concisely illustrated some typical adjustment processes by means of a few qualitatively varying "Marshall time intervals" [15, p. 24].

For reasons of simplicity it is assumed that, during the interaction between Σ_1 and Σ_2, the standard variables L_1 and V_1, as well as L_2 and V_2, remain constant. Without additional exterior influence both partial systems will tend towards a definite mutual state of equilibrium, whereas the following constraints are valid for the aggregate system Σ:

$$\begin{aligned} &\Sigma = \Sigma_1 \cup \Sigma_2 \\ &dC = dC_1 + dC_2 \equiv 0, \qquad dN = dN_1 + dN_2 \equiv 0 \\ &dV = dV_1 + dV_2 \equiv 0, \qquad dL = dL_1 + dL_2 \equiv 0. \end{aligned} \tag{20}$$

Assuming a constant total capital stock for both the supply and demand subsystems, Gibbs' fundamental equations for Σ_1 and Σ_2 are resolved with respect to the output differentials dQ_1 and dQ_2. Then the sum of both differentials may be written as

$$\begin{aligned} dQ &= dQ_1 + dQ_2 \\ &= \left(\frac{1}{\xi_{Q_1}} - \frac{1}{\xi_{Q_2}}\right) \cdot dC_1 - \left(\frac{\mu_1}{\xi_{Q_1}} - \frac{\mu_2}{\xi_{Q_2}}\right) \cdot dN_1 . \end{aligned} \tag{21}$$

In accordance with the principle of equilibrium in thermodynamics it is evident that we define *economic equilibrium* by the maximum of Q with regard to the exchange quantities C_1 and N_1:

$$\left.\frac{\partial Q}{\partial C_1}\right|_{N_1} \overset{!}{=} 0, \qquad \left.\frac{\partial Q}{\partial N_1}\right|_{C_1} \overset{!}{=} 0. \tag{22}$$

These conditions of equilibrium have two consequences:

1. Normally, the aggregate system Σ does not fulfill the constraint $dQ \equiv 0$, in contrast to the differentials dC and dN.

2. As can easily be recognized from (21), the equilibrium state is distinguished by the corresponding values of the respective intensive variables:

$$\xi_{Q_1} = \xi_{Q_2} =: \xi_Q \tag{23}$$
$$\mu_1 = \mu_2 =: \mu.$$

But the following implication also holds:

$$dL_1 = dL_2 = dV_1 = dV_2 \equiv 0 \Rightarrow \begin{cases} \xi_{L_1} \neq \xi_{L_2} \\ p_1 \neq p_2. \end{cases} \tag{24}$$

With respect to the chosen constraints, the results in (24) indicate an inequality in the interest rates (and also in the marginal coefficient ξ_L). This case corresponds to the so-called *osmotic* pressure equilibrium in physics which dominates the flora.

For the *economic* case Samuelson has worked out the causes of the difference between the interest rates of the supply and demand sectors. It is significant that this difference will only vanish if the domains V_1 (resp. V_2) may freely be exchanged between Σ_1 (resp. Σ_2). Obviously, this case can only lead to equal values of p_1 and p_2; they are supposed to attain equilibrium much faster than the variables in (23).

For an exercise the reader may use the equation of state (19) derived from the ECES production function for the two separate systems, each one furnished with a different set of parameters. The connection between these two equations yields the qualifying relation for equilibrium, joining ξ_{L_1} and ξ_{L_2} at a constant value of ξ_Q, and vice versa.

2.3. *From the Domain of Economic Activity to the Space of Events*

Special attention should be turned to the variable V. It fixes the domain of the production resp. consumption system under investigation, especially its open or closed boundaries with the domains of surrounding economic systems.

Obviously V must not be interpreted in a geometric way. This variable defines the *volume* of a multi-dimensional abstract space, by means of which the notions of *closed* and *open systems* may be determined for the explanation of interaction between systems. Consequently, it is often useful to substitute the *geometrical* measure of extension $\mathcal{V}$ by a term to be denoted as the specified *domain* V of the scientific discipline regarded.

For thermodynamics Falk has proved that such a "physical," i.e., non-geometrical, domain can be introduced. This measure refers to an extension which is not fixed with wall-like boundaries. But, in principle its unit may be chosen arbitrarily; i.e., it does not correspond to the common unit of volume expressed by a length scale convention [6, p. 368f], together with an integer dimension: Fractals, therefore, are admissible in principle. For this reason, Gibbs–Falkian thermodynamics is independent of some grave theoretical problems associated with a mechanically founded, but yet questionable, priority of perceptions in geometrical terms over concepts to be strictly based on ideas of matter. Clearly, this a fortiori holds for theoretical economics, because economic processes recorded by means of real space coordinates used in physics are irrelevant, in general. To understand this comparatively awkward theoretical concept better, one should remember that the definitions of a system and the corresponding GEF are equivalent.

By way of construction the GEF neither contains a *time variable* nor a *position vector*. It consists instead exclusively of extensive variables which span the so-called *phase space*. Therein the relevant economic processes can be described completely by sequences of state; in this context a state χ_0 is defined to be an actual real-valued realization (C_0, Q_0, L_0, V_0, ..., N_0) of a given set of standard variables.

Nevertheless, the natural sciences, briefly what we call T_1, are making use of a second device for the tracing of a process. The extensive variables are regarded as functions in a "space of events," spanned by time and the *three geometrical* space coordinates. Mathematically, the connection of the phase space and the space of events is a relation; on the other hand, the inverse relation is a many-to-one mapping. This means that in each point of the *space–time continuum* all extensive variables possess unique values. But the reverse is not true: not each one of the possible states in the phase space can be mapped into a unique point of space–time. In strict analogy to T_1, an *economic space of events* may be generated, for which the following relations hold:

$$
\begin{aligned}
&\text{Relation } \Gamma\colon (C, Q, L, V, N) \rightarrow (t_{\text{ec}}, \eta_1, \eta_2, \ldots, \eta_s) \\
&\quad \text{with } \Gamma\colon (C_0, Q_0, L_0, V_0, N_0) \mapsto (t_{\text{ec}\,o}, \eta_{10}, \eta_{20}, \ldots, \eta_{so}). \\
&\text{Function } \Gamma^{-1}\colon (t_{\text{ec}}, \eta_1, \eta_2, \ldots, \eta_s) \underset{\text{"into"}}{\rightarrow} (C, Q, L, V, N) \\
&\quad \text{with } \Gamma^{-1}\colon (t_{\text{ec}\,o}, \eta_{10}, \eta_{20}, \ldots, \eta_{so}) \underset{\text{"into"}}{\mapsto} (C_0, Q_0, L_0, V_0, N_0)]
\end{aligned}
\tag{25}
$$

This calls for some remarks: the *economic time parameter* t_{ec} may, but does not have to, coincide with the common linear Newton time t of physics.

With respect to spatial coordinates, the more abstract definition of the non-geometrical measure of extension mentioned above offers great advantages for mathematical economics. Assuming that each *coordinate* of the economic space of events η_j ($j = 1, \ldots, s$) has to be independent of the other ones in reality, or per definition, and is, moreover, subject to certain *transformation rules* and scaling laws, then the set of these coordinates may be selected freely.

Remembering the direct connection between the domain of economic activity V and the rate of interest p, and specific evaluation of economic processes depends on the researcher's free choice of the coordinates η_j. In other words, the construction of the space of events implies the preference of some parameters to many others. Therefore, the alleged property of p to be *value-oriented*, results in a natural way from the theory presented here. Additionally, it is in accordance with the main results of von Böhm-Bawerk's famous fundamental analysis of phenomena concerning the rate of interest [1, p. 444f].

Adopting the notion of "volume" in physics, there are the following connections between the volume element and the spatial coordinates:

$$dV := \prod_{j=1}^{s} d\eta_j. \tag{26}$$

In addition to (26) the coordinates η_j will form the set of components for the position vector:

$$\mathbf{r}_{\text{ec}} = \mathbf{r}(\eta_1, \eta_2, \ldots, \eta_j, \ldots, \eta_s). \tag{27}$$

Now, this introduction of $\mathbf{r}_{\text{ec}}$ allows to realize the *kinematic idea of motion* in economics; the velocity may then be defined as variation of position along with that of time, taking place in the space of events:

$$\mathbf{v}_{\text{kin}} := \frac{d\mathbf{r}_{\text{ec}}}{dt_{\text{ec}}}. \tag{28}$$

Despite this, the reader may remember a crucial argument voiced by Reichenbach in 1960, "It was not inherent in the nature of reality that space should be described by coordinates; it was a subjective assignment whose empirical implications had to be examined." [20, p. 5]

Consequently, motion had immediately to be introduced into the original phase space representation; it is included, however, without using space and time coordinates. For this reason, Falk suggested substituting the kinematic concept by a dynamic one, realized for physical items by Gibbs-Falkian thermodynamics.

Assigning a vector-valued *velocity* $\mathbf{v}_{ec}$ to the dynamical processes in economics, the conjugate extensive standard variable $\mathbf{P}_{ec}$ is generated by a Legendre transformation. Thus, the equation of capital forms (13) is extended by the additional term $\mathbf{v}_{ec} \cdot d\mathbf{P}_{ec}$, expressed as a scalar product. Consequently, the velocity $\mathbf{v}_{ec}$ results from the Gibbs fundamental equation as

$$\mathbf{v}_{ec} = \frac{\partial C}{\partial \mathbf{P}_{ec}}\bigg|_{Q,L,V,N} \tag{29}$$

Remarkably, the partial derivative in (29) considers the influence of all the variables involved in the respective process within the phase space. In analogy to T_l the variable $\mathbf{P}_{ec}$ is denoted as economic momentum.

If the two ideas of velocity in the *economic* phase space and in the *economic* space of events are to be compatible, one has to postulate the identity of $\mathbf{v}_{ec}$ and $\mathbf{v}_{kin}$. This leads to the condition

$$\mathbf{v}_{kin} \overset{!}{\equiv} \mathbf{v}_{ec} \Rightarrow \frac{d\mathbf{r}_{ec}}{dt_{ec}} \equiv \frac{\partial C}{\partial \mathbf{P}_{ec}}\bigg|_{Q,L,V,N}, \tag{30}$$

a form in which it will not be necessary to regard the vector $\mathbf{r}_{ec}$ as a *geometrical* measure of distance.

2.4. *On Traditional Approaches to Dynamics*

Things have reached a point, where the question of *dynamics in economics* arises to be discussed in principle. Despite the great effort to develop a genuine mathematical theory of such dynamics, the dogmatic education as well as daily practice are dominated by an understanding of this notion, founded on the concept of equilibrium states and joined by non-equilibrium processes. It is noteworthy that such an idea is obviously motivated by the paradigmata of classical physics. Therefore, the so-called *comparative statics* [19, p. 58f], characterized by appropriate variation methods [19], is much appreciated by mathematical economists. For this reason A. Marshall's popular time concept mentioned above is confusing, rather than offering a way to explain different adjustments to the equilibrium states. But also theories, sometimes denoted as *quantitative Verlaufsanalyse,* whose early precursor is von Thünen, miss the core of the question "How then can dynamics be quantitatively described in mathematical economics, provided that it can be mapped at all by non-equilibrium processes?"

This point demonstrates the whole problematic nature of any purely mathematical approach to dynamics. Even in modern thermodynamics there is no general agreement on the connotation of the term *non-equilibrium.* As a rule, moving systems are described by separating artificially the mode of *motion* from the pertaining fluid element, whose physical properties are determined with reference to Gibbs' *thermostatics* and its special equilibrium concept, denoted as the *principle of local thermodynamic equilibrium* [24, pp. 68, 145]. In other words: By this "principle" any serious discussion about the crucial meaning of motion related to non-equilibrium flows in space and time is prevented from the very beginning.

This argument may be amplified by resorting to the equilibrium problem presented above for the case of economics. It is ill-posed, as the reader will not find any indication of a way by which the dominating trend towards equilibrium states might be realized. "While this problem is unsolved today, some steps have been made towards solving it" [23, p. 452]. Maxwell suggested that with increasing time the internal distribution function of the local gas properties should approach a corresponding one which is appropriate to kinetic equilibrium, characterized by the celebrated Maxwell–Boltzmann velocity distribution function. For highly sophisticated molecular models he proved that kinetic equilibrium in a gas with a spatially homogeneous initial distribution function is coupled with the exponential disappearance of the dissipative influences due to friction and diffusion. The rate, however, whereby the equilibrium state approaches its limit, remains theoretically unknown [23, p. 459]. For practical applications the theory of relaxation processes is extremely complicated [24, p. 262f]; therefore the corresponding relaxation times will have to be found experimentally.

There is only one important exception: Near any equilibrium state the assumption of definite variations around this state is allowed without prescribing such space- and time-dependent constraints as mentioned.

This special case demonstrates that the results of appropriate variation methods are independent of any chosen process realization. Furthermore, they can be labeled by eventually very different *relaxation times* classifying any transition behavior *near* equilibrium. At present many applications in economics are investigated by means of such variation methods using the Lagrangian as the basic information; see, e.g., [19].

In pure mathematical economics these methods are combined with concepts concerning some dynamical models and their modifications. The study of approaches to this subject has experienced a period of rapid growth in recent years. This has resulted in the publication of numerous books and papers.

The basic model of conventional economic dynamics deals with the trajectories of motion of an economy in the phase space, sometimes also called the *product space*. It is clear that, generally, there are many technologically feasible paths of motion starting from a given state of the economy, propelled by certain forces. Therefore, the preceding states do not uniquely determine the possible future states. For this reason, the transition from one state χ_ζ to the next state $\chi_{\xi+1}$ is given by a point-set mapping, being subject to some mathematically and economically specified requirements. For instance, the substantial convexity requirement follows from the well-known "law of diminishing returns" which has been analyzed by Samuelson [14] and a legion of others. In order to determine a unique set of admissible trajectories starting from a given point, the theory needs appropriate optimality criteria.

The solutions of the respective problems constitute an optimal state, or, in the dynamic case, an *optimal trajectory*. Characterization theorems set up the necessary conditions, or even the necessary and sufficient conditions, for extrema.

The stationarity of a trajectory is defined by the relation $\chi_{\zeta+1} = \alpha\chi_\zeta$, where α depends on some positive functional for all time parameters ζ. Clearly, this convention, as a purely mathematical formalism, has nothing to do with the *thermodynamical* notion of a steady state, which is the most relevant item of non-equilibrium states in physics.

One realizes that the mathematical models of economic equilibrium are less known than those of economic dynamics. Predominantly, the analysis is specialized for equilibrium states and equilibrium trajectories that are associated with an economy consisting of different sectors and the interactions between them. In this context a special economic situation deserves to attract attention: consider two subsystems playing the double roles of consumers and producers; the former strive to maximize their "profits" or "incomes" and the latter intend to maximize their utility function.

The concept of optimal trajectories evolved gradually. However, most of all this theoretical work on economic dynamics is mainly effected by means of two different methodological tools:

(1) The mathematical apparatus of superlinear and sublinear functionals, used by Makarov, for instance [11].

(2) The mathematical apparatus of convex–concave positive homogeneous functions developed by Rockafellar [13].

Concrete applications of the theory of economic dynamics sketched here are practically never available. The reader will find many proved theorems and lemmata, mainly on extremal trajectories, almost exclusively designed by use of von Neumann's well-known economic model, introduced in 1937, and improved substantially by GALE in 1956 [7]. This model is chiefly constructed with regard to the mapping rule requested and mentioned above. By the way, the renowned Leontief model is a special case of the von Neumann version.

2.5. *On an Alternative Way to Dynamics in Economics*

No doubt, there is a striking difference between the actual mathematical theories of economic dynamics and the new approach presented here. It manifests the dogmatic embodiment of traditional economic views in classical mechanics. This means that kinematic considerations and strategies of optimization are dominant, compared with concepts, for which motion and structural elements of the economic system regarded are the original properties.

There is still another point to be mentioned: The traditional mathematical theories of economic dynamics hold in principle for the case where the "time parameters" ζ takes on values in an *arbitrarily* ordered set. The use of ζ arises from the fact that the phase space has to be normally submitted to parametrization in order to control the sequence of economic states by facilities needed for data acquisition and storage in national accounting.

Such a property of ζ coincides only by accident with the notion of time used commonly in science and technology. The differences are significant: Usually, the *physical time coordinate t* relates to its values in a continuously ordered set; t unambiguously differs from the notion of space coordinates, and, especially, it is intimately attached to some physical properties in a definite manner. It is assumed that these three items are also true in economics. Consequently, the space of events is spanned by the coordinates t_{ec} and $\mathbf{r}_{ec}$ corresponding to (25) and (27).

Clearly, any *non-equilibrium process*, as the common manifestation of dynamics, presupposes the comprehension of *motion*. In view of a phase space representation, the *capital form of motion* $\mathbf{v}_{ec} \cdot d\mathbf{P}_{ec}$ is defined to be the relevant term in the respective Gibbs fundamental equation, by which *motion* is immediately expressed. According to Eq. (29) the velocity of the non-equilibrium process is, above all, dependent on all the relevant variables of the *economic* phase space. As opposed to this, *motion* as a kinematic phenomenon in the space of events needs appropriate time and space coordinates as process parameters.

The study of *economic* conservation laws is still in its infancy relative to its counterparts in physics and engineering. Yet this is an area where there is great interest, and rapid progress is being made. In economics, the *conservative law* has its roots in the most celebrated article of Frank Ramsey (1928). But it was Paul A. Samuelson (1970) who first explicitly introduced the concept of conservation law to theoretical economics. [20, p. 135]

In the last two decades the application of the Noether theorem has been instrumental for the discovery of some "hidden" invariances in *economic* quantities. An excellent review of the Noether theorem for continuous and discrete models is presented by Sato [20, pp. 33f, 76f, 136f]. He and other authors also offer some informative applications regarding fundamental scientific problems in economics [20, pp. 41f, 86f].

By way of Noether's theorem one is able to establish direct connections between the parameters t_{ec} (resp. the "position" vector $\mathbf{r}_{ec}$) and some possible properties of conservation for the corresponding extensive standard variables, if certain conditions prevail. One might think of the relations between capital C and economic time t_{ec} as well as of the new quantity $\mathbf{P}_{ec}$ and the "position" vector $\mathbf{r}_{ec}$.

Now, according to Eq. (30), both modes of description can be combined by extending the Pfaffian (13) to a complete partial differential equation in space–time coordinates, such that the *capital rate equation*

$$DC = \zeta_Q DQ - \zeta_L DL - pDV + \mu DN + \mathbf{v}_{ec} \cdot D\mathbf{P}_{ec} \tag{31}$$

results, which requires the convention $DN \equiv 0$ for the labor force N according to Eq. (5).

For simplicity the abbreviation D is introduced for the *substantial* or *material derivative*, as defined in continuum mechanics; the convention

$$D := \frac{\partial}{\partial t_{ec}} + \mathbf{v}_{ec} \cdot \frac{\partial}{\partial \mathbf{r}_{ec}} \tag{32}$$

combines in one operator both the infinitesimal time variations and the local displacements caused by motion.

Equation (31) forms the starting point of a set of differential equations for the description of an *economic continuum* in the coordinates $\mathbf{r}_{ec}$ and t_{ec}, along with the equations of state pertaining to the problem in question.

This set contains balance equations of the general local form

$$\rho D x_i + \frac{\partial}{\partial \mathbf{r}_{ec}} \cdot \mathbf{j}_{x_i} = b_{x_i}, \qquad i = 1, 2, \ldots, n, \tag{33}$$

where ρ denotes a "density" and $\mathbf{j}_{x_i}$ and b_{x_i} are the flux tensor and the source property, respectively. Both quantities refer to the related variable x_i, according to (6), and including the additional related variables $x_{\mathbf{P}} := \mathbf{P}_{ec}/N$ and $\rho := (V/N)^{-1}$.

It should be noted that the *tensorial flux* $\mathbf{j}_{x_i}$ is only vanishing for *closed* systems. The same is true for the *source term* b_{x_i} in the special case that $X_i := N \cdot x_i$ is conserved. One might think of quantities like capital C or *economic* momentum $\mathbf{P}_{ec}$ and their relation to the *economic* time t_{ec} or the position vector $\mathbf{r}_{ec}$, if certain conditions prevail.

Assuming complete knowledge of the pertaining *constitutive equations* for the fluxes and source terms, the set of balance equations can be solved for the given initial and boundary conditions. There are additional constraints for $\mathbf{j}_{x_i}$ and b_{x_i}, resulting from the balance equations substituted in (31) and subject to certain "zero sum" principles.

By this tool a possibility will be opened to describe, e.g., connections between the rate of interest p and the "nature" of productive capital or other variables, not to forget the dynamical behavior of economic systems, which are open towards their environment.

Finally, it should be emphasized that some economists consider discrete models to be more realistic and more suitable for empirical applications than continuous ones. In this case, the latter serve as instructive approximations, allowing rational discussion on non-stationary and stationary economic processes and interactions with their environments. Depending on the given boundary conditions, future scenarios as to real and virtual potentialities of many *economic* systems may be systematically investigated in view of their behavior in space and time.

CONCLUSION

Some theory of rational economics is constituted by key terms that can be mathematically represented by extensive variables according to the Euler theorem for homogeneous functions. Then a linear-homogeneous Gibbs-Euler function will emerge after all the non-extensive variables have been transformed into extensive ones by Legendre transformations. From the GEF characterizing an economic system, the existence of functional relationships between the marginal intensive variables—so-called equations of state—follows automatically. To the authors' knowledge, there is no precedent within the realm of traditional theory.

Taking, e.g., the CES production function that is well known to economists, it is easy to derive the related algebraic expression for the marginal quantities. For such results an extension of this type of production function is theoretically inevitable. The so-called ECES function additionally contains two new extensive variables: the labor force N and the domain of economic activity V, both taking into account that any economic theory should deal with "people" in an "economic society" with open boundaries. All the conjugate marginal quantities, chiefly resulting from a mere mathematical formalism, are exemplified by the equilibrium conditions for the interaction between producers' and consumers' subsystems.

Finally, the results of the homomorphic mapping τ_{hom} are extended towards economic processes being run dynamically. Such a way can be opened by transitions from the phase space to the economic space of events. Its coordinates define the domain V, according to the researcher's requirements. Incorporating the idea of "motion" in economics, the coordinates allow us to substantiate quantitatively the non-equilibrium behavior in space and time.

Concluding, a capital rate equation for the description of economic dynamics is proposed as a starting point for further research.

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