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Journal of Algebra 319 (2008) 1154–1248

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Some thoughts on the Jacobian Conjecture, Part II

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Received 6 October 2006

Available online 28 March 2007

Communicated by Ronald Solomon

Dedicated to the Fond Memory of Walter Feit

Abstract

In Abhyankar's Purdue Lectures of 1971, the bivariate Jacobian Conjecture was settled for the case of two plus epsilon characteristic pairs. In the published version, the epsilon part got left out. Now we take care of the omission by preparing for a sharper result with full proof in Part III. The Jacobian Method is applied to giving a new simple proof of Jung's Automorphism Theorem. A detailed description of the Degreewise Newton Polygon is given. Some thoughts on the multivariate Jacobian Conjecture are included.

Keywords: Conjecture; Jacobian

1. Introduction

Although this is the second part of our paper [Ab5], for the convenience of the reader we shall try to make it readable by itself. Especially in this introduction, let us provide lots of motivation. At any rate, the introduction will be an informal discourse, followed by a formal presentation starting in Section 2. Actually, even Sections 2 to 6 will be written in a "readable" manner, which will continue the project of carrying out the implied request made by the referee of the first part (which was seconded by the editor) who suggested an "explicit reference to the author's Engineering Book [Ab3] to soften the otherwise intimidating austere style of the first part (necessitated by the foundational requirements of precision and generality)." Many thanks to Nick Inglis, Ben Kravitz, and Avinash Sathaye for valuable help in the composition of this second part.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.02.056

For a summary of this second part see (1.11). Here let us point out that in Sections 7 to 9 we give an exhaustive treatment of the Degreewise Newton Polygon or the Full Newton Polygon which was initially introduced in our Purdue Lectures of 1971 and discussed in the notes [Ab1] and [Ab4] of those lectures. This is the incarnation of the original Newton Polygon more appropriate as we descend from power series to polynomials. It could be thought of as the convex hull of a finite set of points in the plane (see pictures in Section 9). We propose to give an algorithmic rendering of it in an Operations Research set-up including a complexity analysis at a later opportunity; here "we" means a father and son collaboration.

In Section 4 we give a transparent proof of item (1.2) of the Introduction of Part I which says that every member of a Jacobian pair has at most two points at infinity. In Section 5 we use this to give a very short new proof of Jung's Automorphism Theorem. In Section 6 we settle some cases of the Jacobian Conjecture, and give a simple proof of the equivalences asserted in item (1.3) of the Introduction of Part I. Section 8 ends with property (XII) to be used in Part III for completing the proof of the sharper version of the two plus epsilon characteristic pairs case of the bivariate Jacobian Problem spoken of in Part I. Section 10 has a discussion of the multivariate Jacobian Problem; also see (1.1) and (1.5) below.

(1.1) Bivariate Jacobian Problem. Let us begin with the 2-variable Jacobian Problem. So let $f(X,Y)$ and $g(X,Y)$ be two polynomials in two variables X and Y with coefficients in a field k. We define the Jacobian of f and g with respect to X and Y to be

$$J(f,g)=J_{(X,Y)}(f,g)=\det\begin{pmatrix} f_X & f_Y \\ g_X & g_Y\end{pmatrix}$$

where f_X is the partial derivative of f with respect to X, and so on. Let

$$\theta\!\!\!-$$

denote a generic nonzero constant, i.e., an unspecified element of $k^\times$ where

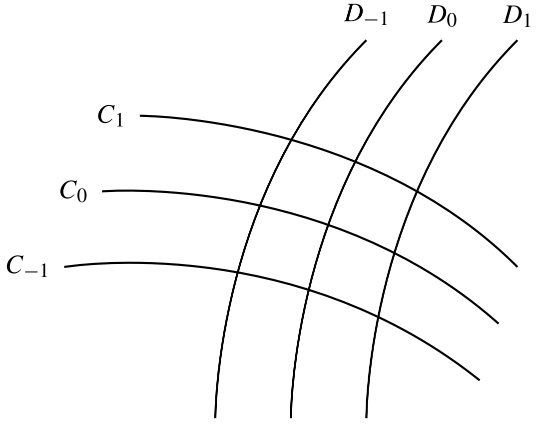
$$k^\times = \text{the set of all nonzero elements of } k.$$

If $J(f,g)=\theta\!\!\!-$ then we call (f,g) a Jacobian pair. Let $\operatorname{ch}(k)$ denote the characteristic of k. The Jacobian Conjecture says that if $\operatorname{ch}(k)=0$ and (f,g) is a Jacobian pair then (f,g) is an automorphic pair, i.e., X and Y are polynomials in f and g.

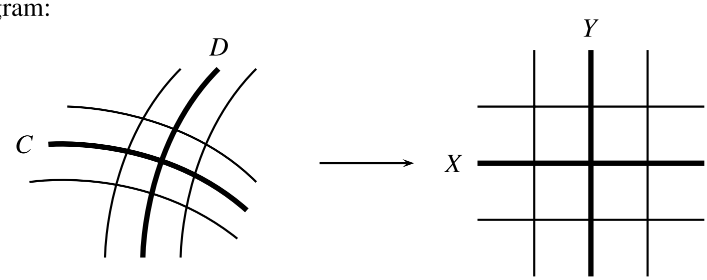
Let us now make some

(1.2) Geometric comments. We can try to think of some properties which all automorphic pairs have and then prove that Jacobian pairs also have the same properties. So we come to the natural question as to what properties automorphic pairs have. Geometrically, instead of thinking about polynomials f and g, we can think of a pair of plane curves C and D defined by the solutions to $f(X,Y)=0$ and $g(X,Y)=0$, respectively.

We now ask when do these curves give curvilinear coordinates. As we shall see, such coordinates play an important role in various aspects of algebraic geometry such as the connection between linear systems and rational transformations. They also play an important role in the Lagrangian and Hamiltonian aspects of classical mechanics which are intimately related to the multivariable Jacobian Problem. So we consider a curvilinear coordinate system which, for illustrative purposes, can be drawn as follows:



We think of these coordinates as two families of curves given by $C_\lambda : f(X,Y) = \lambda$ and $D_\mu : g(X,Y) = \mu$ with λ and μ varying over k. An automorphic pair gives rise to the bunch of pairs which are parts of a curvilinear coordinate system. Simply pick one from the horizontal system and one from the vertical system, and call these C and D as given by the polynomials f and g, i.e., by taking $C = C_0$ and $D = D_0$. Thus an automorphic pair gives a map of a system of coordinates to another system of coordinates, say a map from the curvilinear coordinate system to the Cartesian coordinate system. In other words, via the automorphism of the polynomial ring $k[X,Y]$, we send the curves C and D to the X and Y axes. We illustrate this concept in the following diagram:



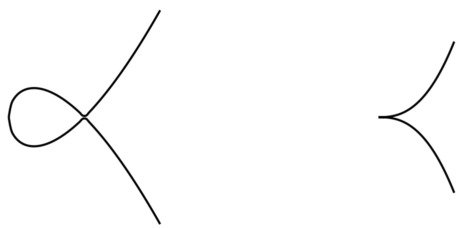
So we conclude that an automorphism of the polynomial ring $k[X,Y]$ corresponds to a biregular map of the affine plane $\mathbb{A}^2$ over k. Now, at finite distance, C and D meet in exactly one point because the X and Y axes meet in one point, and intersection multiplicity is invariant under biregular transformations.

Let N and M be the degrees of the curves C and D, i.e., of the polynomials f and g respectively. Now let us make the hypothesis that $M + N > 2$, i.e., one of the curves is not a line. (Otherwise we are in the uninteresting case of a linear transformation.) Then by Bézout's theorem, C and D meet in MN points. So let us discuss points at ∞, since every plane curve has points at ∞. Since $MN > 1$, but the curves meet at exactly one point at finite distance, they must meet at ∞ in at least one point. Therefore, we conclude that any automorphic pair (with $M+N > 2$) must have a common point at ∞. By making a slight jump in reasoning, we conclude that all of their points at ∞ must be the same, because they must have $MN - 1$ coincidences at ∞. Moreover, C and D should have the same points at ∞ even "counting multiplicities." Let us now look at the degree forms of f and g, i.e., the highest degree terms in the expansions of

f and g. Call them F and G. The factors (roots) of the degree forms give the points at infinity. So F and G must have the same roots because f and g have the same points at ∞. Moreover, because of the counting multiplicities, the degree forms must be powers of each other. Actually, F and G must be the correct powers of each other, i.e., $F^M = \theta G^N$.

We note that C should be irreducible because in some coordinate system it becomes the X-axis. Translating this from a geometric statement into an algebraic one, this means f should be irreducible. But we can actually do better, i.e., f should have exactly one place at ∞ because the X-axis has only one point at ∞, and it is analytically irreducible at that point (which is the definition of having one place at ∞).

To calculate the number of points at infinity of any plane curve $C : f(X, Y) = 0$, we proceed as follows. We first homogenize, transforming $f(X, Y)$ into the homogeneous polynomial $\phi(X, Y, Z)$ obtained by putting $\phi(X, Y, Z) = Z^N f(X/Z, Y/Z)$. Then putting $Z = 0$ will give us the number of points at ∞. (Or another way to do this is simply look at the number of factors of the degree form.) Each point at ∞ has a certain number of places. Let us look at two examples:



The alpha curve (on the left), given by $Y^2 - X^2 - X^3 = 0$, has 2 places (= 2 branches) at the origin, i.e., it factors as a power series into two (nonunit) factors. The cusp (on the right), given by $Y^2 - X^3 = 0$, has 1 place (= 1 branch) at the origin, i.e., it does not factor as a power series.

We note that the number of places at ∞ is far more useful information than the number of points at ∞. This is because the number of places at ∞ is invariant under automorphisms, but the number of points is not. Indeed, by an automorphism, we can always transform a curve C into a curve $\overline{C}$ which has only one point at ∞. This phenomenon is very hard to understand geometrically but very easy algebraically. The automorphism which will do this (at least when f does depend on Y) is $X \mapsto X$ and $Y \mapsto Y + X^r$ for $r \gg 0$. By this we can convert the polynomial $f(X, Y)$ into a polynomial with exactly one point at ∞.

Thus going back to our original purpose, in Section 5 we shall prove that the (Bivariate) Jacobian Conjecture is equivalent to showing that if (f, g) is a Jacobian pair then f has only one point at ∞, and also equivalent to showing that if (f, g) is a Jacobian pair then f has only one place at ∞. As a first step towards this, in Section 4 we shall show that if (f, g) is a Jacobian pair then f has at most two points at ∞; for pictures see (9.2). In Section 5 the two point method will be applied to give a simple new proof of Jung's Automorphism Theorem.

At any rate, since in some coordinate system C becomes the X-axis, we conclude that, assuming (f, g) to be an automorphic pair, C must be an irreducible rational curve which is nonsingular at finite distance. We have already discussed irreducibility. Rationality simply means that the genus of the curve is 0. Consequently, to establish the (Bivariate) Jacobian Conjecture, we should try to show that if (f, g) is a Jacobian pair then $C : f(X, Y) = 0$ is an irreducible rational curve which is nonsingular at finite distance and which meets $D : g(X, Y) = 0$ exactly at one point at finite distance, and there the two curves meet each other transversally.

We proceed to prove that, assuming (f, g) to be a Jacobian pair with $M + N > 2$, we have $F^M = \theta G^N$. First we note that, denoting determinants by $| \, |$, we have

$$\begin{vmatrix} f_X & f_Y \\ g_X & g_Y \end{vmatrix} = \begin{vmatrix} F_X & F_Y \\ G_X & G_Y \end{vmatrix} + \text{terms of degree} < M + N - 2$$

and hence

$$\begin{vmatrix} F_X & F_Y \\ G_X & G_Y \end{vmatrix} = 0.$$

By an age-old criterion for dependence of functions, which will be discussed in (1.9) and (1.10), the above equation implies that F and G are algebraically dependent. This criterion, in fact, is the reason that Jacobians were first introduced. The desired relation $F^M = \theta G^N$ can be deduced from the algebraic dependence of F and G. However, we prefer to give a more direct proof of the said relation.

Continuing with our direct proof, we take the matrix

$$\begin{pmatrix} F_X & F_Y \\ G_X & G_Y \end{pmatrix}$$

whose determinant is 0, and multiply the first column by X and the second column by Y. Then adding the columns of the new matrix we get the sums $XF_X + YF_Y$ and $XG_X + YG_Y$ which, by Euler's theorem on homogeneous functions, are equal to NF and MG respectively. Thus we have a new determinant

$$\begin{vmatrix} NF & F_Y \\ MG & G_Y \end{vmatrix}$$

which equals zero. By dehomogenizing and equating the resulting determinant to zero we get the one variable equation

$$N\overline{F}\,\overline{G}_Y - M\overline{G}\,\overline{F}_Y = 0$$

where $\overline{F}(Y) = F(1, Y)$ and $\overline{G}(Y) = G(1, Y)$. So we have removed a variable. Multiplying the above equation by $\frac{-\overline{F}^{M-1}}{\overline{G}^{N+1}}$ and invoking the quotient rule in the sense of ordinary calculus we get

$$\left(\frac{\overline{F}^M}{\overline{G}^N}\right)_Y = 0$$

and hence

$$\overline{F}^M = \theta \overline{G}^N.$$

Now homogenizing, i.e., substituting Y/X for Y and then multiplying both sides by X^{MN}, we get the desired relation $F^M = \theta G^N$. Thus we have proved the:

(1.3) Jacobian Lemma. *If f, g is a Jacobian pair in $k[X, Y]$ with $\mathrm{ch}(k) = 0$, and if $M + N > 2$ where (N, M) are the degrees of (f, g), then for the homogeneous polynomials (F, G) which are the degree forms of (f, g) we have $F^M = \not\theta G^N$.*

In the above proof we did not use the fact that F, G are the degree forms of the Jacobian pair f, g, but only the facts that they are homogeneous polynomials of degrees N, M and their Jacobian is zero. In other words, we have proved the:

(1.4) Eulerian Lemma. *If $F = F(X, Y)$ and $G = G(X, Y)$ are nonzero homogeneous polynomials of degrees N and M over a field k of characteristic zero such that $J_{(X,Y)}(F, G) = 0$, then $F^M = \not\theta G^N$.*

As we shall see in Section 6, this Eulerian Method can be used to settle the Jacobian Conjecture when M and N are coprime, and to show that the Jacobian Conjecture is equivalent to saying that the degree pair (N, M) of any Jacobian pair (f, g) is principal, i.e., either M divides N, or N divides M. The proof of the coprime case will also include a proof when $\mathrm{GCD}(N, M) =$ a prime number or 4.

(1.5) Trivariate Jacobian problem. Now suppose we have a Jacobian triple, i.e. polynomials, f, g, h in X, Y, Z with coefficients in k satisfying the Jacobian Condition $J_{(X,Y,Z)}(f, g, h) = \not\theta$. Then, again assuming $\mathrm{ch}(k) = 0$ and extending the Jacobian Problem to 3 variables, we would like to show that (f, g, h) is an automorphic triple, i.e., $k[X, Y, Z] = k[f, g, h]$. In particular we want show that the surface given by $f(X, Y, Z) = 0$ is irreducible, nonsingular at finite distance and, in a generalized sense of genus, has genus 0. There is no clear method of generalizing the genus to surfaces, but we would like the genus to be 0 in every sense that we can think of generalizing it. In this case, we run into many problems. For example, we do not know much about how to tell what the singularities of a surface look like, how many there are, or how to calculate the genus in terms of these singularities. As can be imagined, there are even more problems in higher dimensions.

(1.6) Multivariate Jacobian Problem. Generalizing to more variables, let $f_1(X_1, \ldots, X_n)$, $\ldots, f_n(X_1, \ldots, X_n)$ be rational functions in $X_1, \ldots, X_n$ with coefficients in a field k, where n is any positive integer. Their Jacobian

$$\frac{J(f_1, \ldots, f_n)}{J(X_1, \ldots, X_n)} = \det\left(\frac{\partial f_i}{\partial X_j}\right)$$

is the determinant of the $n \times n$ Jacobian matrix

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(X_1, \ldots, X_n)} = \left(\frac{\partial f_i}{\partial X_j}\right)$$

whose (i, j)th entry is the (formal) partial derivative

$$(f_i)_{X_j} = \frac{\partial f_i}{\partial X_j} \in k(X_1, \ldots, X_n).$$

As alternative notations for the Jacobian of $f_1, \dots, f_n$ relative to $X_1, \dots, X_n$, we may write

$$J(f_1, \dots, f_n) = J_{(X_1,\dots,X_n)}(f_1, \dots, f_n) = \frac{J(f_1, \dots, f_n)}{J(X_1, \dots, X_n)}.$$

By a Jacobian n-tuple (over k) we mean $f_1, \dots, f_n$ in $k[X_1, \dots, X_n]$ such that $J(f_1, \dots, f_n) = \not{0}$. By an automorphic n-tuple (over k) we mean $f_1, \dots, f_n$ in $k[X_1, \dots, X_n]$ such that $k[f_1, \dots, f_n] = k[X_1, \dots, X_n]$.

The Jacobian Conjecture predicts that if $\mathrm{ch}(k) = 0$ then every Jacobian n-tuple is an automorphic n-tuple; we call this JC_n.

(1.7) Automorphic tuple. The converse of the Jacobian Conjecture says that every automorphic n-tuple is a Jacobian n-tuple, and this is what motivates the Jacobian Conjecture in the first place. To prove the said converse, we recall the (formal) Jacobian chain rule which generalizes the (formal) derivative chain rule and which says that given any n-variable rational functions $f_i(X_1, \dots, X_n) \in k(X_1, \dots, X_n)$ and $g_i(Y_1, \dots, Y_n) \in k(Y_1, \dots, Y_n)$ for $1 \leqslant i \leqslant n$, we have

$$(1) \qquad J_{(X_1,\dots,X_n)}(g_1, \dots, g_n) = J_{(f_1,\dots,f_n)}(g_1, \dots, g_n) J_{(X_1,\dots,X_n)}(f_1, \dots, f_n)$$

where by definition we have $J_{(X_1,\dots,X_n)}(g_1, \dots, g_n) = J_{(X_1,\dots,X_n)}(h_1, \dots, h_n)$ with $h_i(X_1, \dots, X_n) = g_i(f_1(X_1, \dots, X_n), \dots, f_n(X_1, \dots, X_n))$, and where by definition $J_{(f_1,\dots,f_n)}(g_1, \dots, g_n) = j(f_1(X_1, \dots, X_n), \dots, f_n(X_1, \dots, X_n))$ with $j(Y_1, \dots, Y_n) = J_{(Y_1,\dots,Y_n)}(g_1, \dots, g_n)$. The said converse follows by noting that if $f_1, \dots, f_n$ is an automorphic n-tuple, then we can find $g_i(Y_1, \dots, Y_n) \in k[Y_1, \dots, Y_n]$ for $1 \leqslant i \leqslant n$ such that $h_i(X_1, \dots, X_n) = X_i$ for $1 \leqslant i \leqslant n$, and now the LHS of the above chain rule equals 1, and both the factors of the RHS are polynomials in $X_1, \dots, X_n$ with coefficients in k and therefore, by invoking the fact that the degree of the product of any two polynomials equals the sum of their degrees, we see that each one of the said two factors must be reduced to a nonzero element of k.

(1.8) Chain rule. To put the above converse in proper perspective, let L/k be a separably generated field extension whose transcendence degree is a positive integer n, and let $X_1, \dots, X_n$ be a separating transcendence basis of L/k. In other words, let us assume that L/k has a transcendence basis $X_1, \dots, X_n$ such that $L/k(X_1, \dots, X_n)$ is separable algebraic. Now for $1 \leqslant i \leqslant n$, the derivation $\partial/\partial X_i$ of $k(X_1, \dots, X_n)$ can be uniquely extended to a derivation of L/k, and denoting this extension again by $\partial/\partial X_i$, we get a basis $\partial/\partial X_1, \dots, \partial/\partial X_n$ of $\mathrm{Der}_k(L, L)$ as a vector space over L; see L1§12(N1) and L6§6(E14)–(E20) of [Ab6]. Now the definitions of the Jacobian and Jacobian matrix given in (1.6) carries over, and (1.7)(1) is reincarnated as the (generalized) Jacobian chain rule

$$(2) \qquad J_{(X_1,\dots,X_n)}(g_1, \dots, g_n) = J_{(f_1,\dots,f_n)}(g_1, \dots, g_n) J_{(X_1,\dots,X_n)}(f_1, \dots, f_n)$$

where $f_1, \dots, f_n$ is any other separating transcendence basis of L/k, and $g_1, \dots, g_n$ are any elements of L. This follows from the (generalized) derivative chain rule which says that for $1 \leqslant i \leqslant n$ and any $z \in L$ we have

$$(3) \qquad \frac{\partial z}{\partial X_i} = \sum_{1 \leqslant j \leqslant n} \frac{\partial z}{\partial f_j} \frac{\partial f_j}{\partial X_i}.$$

Now (3) is obvious for $z \in k(f_1, \dots, f_n)$ and it is true for all $z \in L$ because both sides of (3) are k-derivations of L which is separable algebraic over $k(f_1, \dots, f_n)$; see L6§6(E12) of [Ab6].

(1.9) Jacobian Criterion. Having generalized the chain rules, let us put the said converse in proper perspective by proving the Jacobian Criterion which says that in the situation of (1.6), any given elements $y_1, \dots, y_n$ in L form a separating transcendence basis of L/k iff $J_{(X_1,\dots,X_n)}(y_1, \dots, y_n) \neq 0$. Namely, if $y_1, \dots, y_n$ is a separating transcendence basis of L/k then by taking $f_i = y_i$ and $g_i = X_i$ for $1 \leqslant i \leqslant n$ in (2) we see that the LHS of (2) equals 1 and hence both the factors of the RHS of (2) must be nonzero. Before turning to the proof of the "reverse implication," note that clearly there exists a nonzero irreducible $(n+1)$-variable polynomial $\phi(X_1, \dots, X_n, Y)$ over k which is unique up to a nonzero multiplier in k and which is such that $\phi(X_1, \dots, X_n, y_1) = 0$; also we can find $0 \neq \psi(X_1, \dots, X_n) \in k[X_1, \dots, X_n]$ such that $\phi(X_1, \dots, X_n, Y)/\psi(X_1, \dots, X_n)$ is the minimal polynomial of y_1 over $k(X_1, \dots, X_n)$; see the proof of L5§5(Q32)(T142.2) of [Ab6]. Since y_1 is separable algebraic over $k(X_1, \dots, X_n)$, we must have $\phi_Y(X_1, \dots, X_n, y_1) \neq 0$. The equation $\phi(X_1, \dots, X_n, y_1) = 0$ gives the equation $\phi_{X_1}(X_1, \dots, X_n, y_1) + \phi_Y(X_1, \dots, X_n, y_1)(y_1)_{X_1} = 0$, and hence

$$(y_1)_{X_1} = -\phi_{X_1}(X_1, \dots, X_n, y_1)/\phi_Y(X_1, \dots, X_n, y_1).$$

Therefore

$$\begin{cases} (y_1)_{X_1} \neq 0 \\ \quad \Rightarrow \phi_{X_1}(X_1, \dots, X_n, y_1) \neq 0 \\ \quad \Rightarrow X_1 \text{ is separable algebraic over } k(y_1, X_2, \dots, X_n) \\ \quad \Rightarrow y_1, X_2, \dots, X_n \text{ is a separating transcendence basis of } L/k. \end{cases}$$

Thus

(•) $$\begin{cases} (y_1)_{X_1} \neq 0 \\ \quad \Rightarrow y_1, X_2, \dots, X_n \text{ is a separating transcendence basis of } L/k. \end{cases}$$

Now by induction on n we shall prove the "reverse implication"

(••) $$\begin{cases} J_{(X_1,\dots,X_n)}(y_1, \dots, y_n) \neq 0 \\ \quad \Rightarrow y_1, \dots, y_n \text{ is a separating transcendence basis of } L/k. \end{cases}$$

For $n = 1$ we are reduced to (•). So let $n > 1$ and assume for $n - 1$. We are supposing $J_{(X_1,\dots,X_n)}(y_1, \dots, y_n) \neq 0$, and hence upon expanding the LHS by its first row we must have $(y_1)_{X_j} \neq 0$ for some $j \in \{1, \dots, n\}$. Relabeling $X_1, \dots, X_n$ suitably we may assume that $(y_1)_{X_1} \neq 0$. Now $y_1, X_2, \dots, X_n$ is a separating transcendence basis of L/k by (•), and hence by (1.6) we get

$$J_{(y_1,X_2,\dots,X_n)}(y_1, \dots, y_n) = J_{(X_1,\dots,X_n)}(y_1, \dots, y_n) J_{(y_1,X_2,\dots,X_n)}(X_1, \dots, X_n).$$

By what we have proved above we see that the second factor of the above RHS is nonzero, and by assumption so is the first factor. Therefore the LHS must be nonzero, i.e.,

$$J_{(y_1,X_2,\dots,X_n)}(y_1, \dots, y_n) \neq 0.$$

Now $X_2, \dots, X_n$ is a separating transcendence basis of $L/k(y_1)$, the first row of the Jacobian matrix $\frac{\partial(y_1,\dots,y_n)}{\partial(y_1,X_2,\dots,X_n)}$ is $(1, 0, \dots, 0)$ and its minor obtained by deleting the first row and the first column coincides with the Jacobian matrix $\frac{\partial(y_2,\dots,y_n)}{\partial(X_2,\dots,X_n)}$. Consequently the above displayed inequality implies the inequality

$$J_{(X_2,\dots,X_n)}(y_2, \dots, y_n) \neq 0.$$

Therefore the induction hypothesis tells that $y_2, \dots, y_n$ is a separating transcendence basis of $L/k(y_1)$. It follows that $y_1, \dots, y_n$ is a separating transcendence basis of L/k.

(1.10) Functional dependence. Now (••) has the following simpler proof in case of $\mathrm{ch}(k) = 0$, when it suffices to show that: (•••) if the elements $y_1, \dots, y_n$ are algebraically dependent over k then $J_{(X_1,\dots,X_n)}(y_1, \dots, y_n) = 0$. Namely, let $\Phi(Y_1, \dots, Y_n)$ be a nonzero polynomial over k such that $\Phi(y_1, \dots, y_n) = 0$. Choose Φ to be of smallest (total) degree and let that degree be e. Upon taking the X_j partial derivative of both sides of the equation $\Phi(y_1, \dots, y_n) = 0$ we obtain the n homogeneous linear equations $\sum_{1\leqslant i\leqslant n}(y_i)_{X_j}\Phi_{Y_i}(y_1, \dots, y_n) = 0$ which have the nontrivial solution $\Phi_{Y_1}(y_1, \dots, y_n), \dots, \Phi_{Y_n}(y_1, \dots, y_n)$, and hence by Cramer's Rule the Jacobian $J_{(X_1,\dots,X_n)}(y_1, \dots, y_n)$, which is clearly the determinant of their coefficients, must equal to zero. To show that the said solution is nontrivial, pick $i \in 1, \dots, n$ so that the exponent of Y_i is positive in a monomial of degree e whose coefficient in Φ is nonzero. Now $\mathrm{ch}(k) = 0$ tells us that Φ_{Y_i} is a nonzero polynomial of degree less than e and hence $\Phi_{Y_i}(y_1, \dots, y_n) \neq 0$.

The implication (•••) continues to be true if the assumption $\mathrm{ch}(k) = 0$ is replaced by the weaker assumption that the field k is perfect. To see this let $\mathrm{ch}(k) = p \neq 0$. If Φ does not belong to $k[Y_1^p, \dots, Y_n^p]$ then the above argument works. Otherwise, assuming k to be perfect, we can write $\Phi = \Psi^p$ where $\Psi(Y_1, \dots, Y_n)$ is a nonzero polynomial over k of degree e/p with $\Phi(y_1, \dots, y_n) = 0$ which contradicts the minimality of e.

The above characteristic zero proof of (•••) is inspired by the standard proof of the age-old calculus version of the Jacobian Criterion where algebraic dependence is replaced by "functional dependence" and it is asserted that n functions $y_1, \dots, y_n$ of n variables $X_1, \dots, X_n$ are dependent iff their Jacobian $J_{(X_1,\dots,X_n)}(y_1, \dots, y_n)$ is identically zero. For instance see the books of Edwards [Edw], Gibson [Gib], Goursat [Gou], Kaplan [Kap], or Phillips [Phi]. This indeed was the birth certificate of Jacobians which, as discussed in these books, were immediately used, by Stokes (1819–1903) and others, for changing variables in multiple integrals. At any rate, all this belongs to the nineteenth century (1800–1900), which indeed was the finest century for the development of our beloved science of mathematics.

(1.11) Notation and summary. Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ are the sets of all nonnegative integers, integers, rationals, and reals, respectively, whereas $\mathbb{N}_+ = \mathbb{Z}_+ \subset \mathbb{Q}_+ \subset \mathbb{R}_+$ are the sets of all positive elements in them. Moreover $S^n = S \times \cdots \times S$ (n times) is the set of all n-tuples with entries in a set S.

Section 2 introduces isobaric polynomials. Section 3 generalizes Eulerian Lemma (1.4) and applies it. Sections 4 to 6 are described in (1.2) to (1.4). Section 8 proves the similarity of the Newton Polygons of a Jacobian pair, and Section 6 shows the Jacobian Conjecture to be equivalent to their being triangles.

2. Isobaric polynomials

Let us now generalize the Jacobian Lemma (1.3) from homogeneous to weighted homogeneous polynomials, or as they used to be called, to isobaric polynomials. To explain what these are, consider the polynomials

$$Y^5 + X^2Y^3 + X^4Y$$

and

$$Y^5 + XY^3 + X^2Y.$$

The first is homogeneous because all its terms are of the same (total) degree, namely 5. The second is not because its terms have different degrees, namely 5, 4, 3. But by regarding X to be twice as important, or twice as heavy, as Y, the second becomes ISOBARIC of weight 5, or as we may say, weighted homogeneous of weight 5 relative to the weight system (2, 1). Clearly the isobaricness is unchanged for proportional weight systems. For instance, the second polynomial is isobaric of weight 15 for the weight system (6, 3).

Thus, without loss of generality, we may restrict our attention to weight systems (w_1, w_2) where w_1 and w_2 are coprime integers with $w_1 > 0$. In this case by putting $w = -w_2/w_1$ we see that w_1 and w_2 are uniquely determined as the denominator and the negative numerator of the rational number w when it is written out as a fraction in reduced form. Now, without ambiguity, we may speak of w-homogeneous polynomials and their w-degrees. It is convenient to define these notions also for meromorphic polynomials, i.e., members of the ring $k[X, X^{-1}, Y, Y^{-1}]$ where X, Y are indeterminates over a field k; we may think of these as polynomials in which we allow the exponents to be positive as well as negative integers. Namely, for any

$$\theta = \theta(X, Y) = \sum_{(i,j)\in\mathbb{Z}^2} \theta_{ij} X^i Y^i \in k\left[X, X^{-1}, Y, Y^{-1}\right] \quad \text{with } \theta_{ij} \in k$$

we define the *support* of θ by putting

$$\mathrm{Supp}(\theta) = \left\{(i, j) \in \mathbb{Z}^2\colon\ \theta_{i,j} \neq 0\right\}.$$

Noting that $\mathrm{Supp}(\theta)$ is a finite subset of $\mathbb{Z}^2$ (and: $\mathrm{Supp}(\theta) = \emptyset \Leftrightarrow \theta = 0$), we define the *$w$-degree* of θ by putting

$$\deg_w \theta = \max\left\{iw_1 + jw_2\colon\ (i, j) \in \mathrm{Supp}(\theta)\right\}$$

and we define the *w-degree form* of θ by putting

$$\theta_w^+ = \theta_w^+(X, Y) = \sum_{\{(i,j)\in\mathrm{Supp}(\theta)\colon\ iw_1+jw_2=\deg_w\theta\}} \theta_{ij} X^i Y^j$$

with the understanding that

$$\text{if } \theta = 0 \quad \text{then } \deg_w \theta = -\infty \text{ and } \theta_w^+ = 0.$$

We say that θ is *w-homogeneous* to mean that $\theta_w^+ = \theta$.

We define the *w-version* of θ (i.e., the jacked up version of θ) by putting

$$\theta_w^0 = \theta_w^0(X, Y) = \theta\big(X^{w_1}, Y^{w_2}\big) \in k\big[X, X^{-1}, Y, Y^{-1}\big]$$

and we note that

$$\deg_w \theta = \deg\big(\theta_w^0\big) \quad \text{and} \quad \big(\theta_w^+\big)_w^0 = \text{degree form of } \theta_w^0$$

and

$$\theta \text{ is } w\text{-homogeneous} \quad \Leftrightarrow \quad \theta_w^0 \text{ is homogeneous}$$

where the usual notions of degree, degree form, and homogeneity are extended from $k[X, Y]$ to $k[X, X^{-1}, Y, Y^{-1}]$ in an obvious manner. Pictures in (9.4).

By a *w-automorphic pair* we mean a pair of w-homogeneous elements (x, y) in $k[X, Y]$ such that (x, y) is an automorphic pair, i.e., such that $k[x, y] = k[X, Y]$.

Given any f in $k[X, Y]$, we say that f has *one or two points at infinity* in the w-weighted sense if there is a w-automorphic pair (x, y) such that $f_w^+ = \not\theta x^i$ or $\not\theta x^i y^j$ with i, j in $\mathbb{N}_+$ respectively. If one of the above holds then we say that f has *at most two points at infinity* in the w-weighted sense. Pictures in (9.2).

Given any f, g in $k[X, Y]$ with $f \neq 0$, we say that f is *w-similar* to g to mean that $g \neq 0$ and

$$\big(f_w^+\big)^{\deg_w g} = \not\theta \big(g_w^+\big)^{\deg_w f}.$$

Given any f, g in $k[X, Y]$ with $f \neq 0 \neq g$, we define the *w-lag* of (f, g) by putting

$$\operatorname{lag}_w(f, g) = \begin{cases} \deg_w(fg) - \deg_w(XY) - \deg_w J(f, g) & \text{if } J(f, g) \neq 0, \\ \infty & \text{if } J(f, g) = 0. \end{cases}$$

In the above definitions, the reference to w may be dropped when $w = -1$. For motivation of these definitions see the various lemmas proved in Sections 3 and 4.

As an example of isobaricness for more than two variables, let us look at the Sylvester Resultant of the two univariate polynomials $a_0Y^n + a_1Y^{n-1} + \cdots + a_n$ and $b_0Y^m + b_1Y^{m-1} + \cdots + b_m$. Then the said resultant, as a polynomial in $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$, by assigning weight i to a_i for $0 \leqslant i \leqslant n$, and j to b_j for $0 \leqslant j \leqslant m$, becomes isobaric of weight mn. This indeed is the genesis of Bézout's theorem. Namely, in case of plane curves, we let a_i and b_j be polynomials in X of degrees at most i and j respectively, and more generally, in case of hypersurfaces in the space of dimension r, we let a_i and b_j be polynomials in $X_1, \ldots, X_{r-1}$ of degrees at most i and j respectively. See Observations (O3) to (O5) of Section §1 of Lecture 4 of [Ab6].

In greater detail, the hyperspatial Bézout says that, considering the PRIMALS (= hypersurfaces) C and D given by the equations $a_0Y^n + a_1Y^{n-1} + \cdots + a_n = 0$ and $b_0Y^m + b_1Y^{m-1} + \cdots + b_m = 0$, and assuming them to be devoid of common components, i.e., assuming that the polynomials do not have nonconstant common factors, the primals C and D meet in a finite number of irreducible SECUNDUMS (= varieties of dimension $r - 2$) $P_1, \ldots, P_h$ of degrees $d_1, \ldots, d_h$, and letting e_i be the intersection multiplicity of C and D at P_i, we have $d_1e_1 + \cdots + d_he_h = mn$. For $r = 3$, the space Bézout says that, assuming them to be devoid of common components, the surfaces C and D meet in a finite number of irreducible space curves $P_1, \ldots, P_h$ of degrees $d_1, \ldots, d_h$, and letting e_i be the intersection multiplicity of C and D at P_i, we have

$d_1e_1 + \cdots + d_he_h = mn$. For $r = 2$, the plane Bézout says that, assuming them to be devoid of common components, the curves C and D meet in a finite number of points $P_1, \ldots, P_h$ of degrees $d_1, \ldots, d_h$, and letting e_i be the intersection multiplicity of C and D at P_i, we have $d_1e_1 + \cdots + d_he_h = mn$; note that in this case of plane curves (i.e., for $r = 2$) if the ground field k is algebraically closed then $d_1 = \cdots = d_h = 1$; otherwise, a point in the plane is (or corresponds to) a maximal ideal P in $k[X, Y]$ and its degree is the field degree $[k[X, Y]/P : k]$. See [Ab0]. Also see Semple and Roth [SeR]; note that their primals are the same as our primals, but their secunda are our secundums of degree one, and we may call these linear secundums. At any rate, this charming book of Semple and Roth is highly recommended as a "readable" book on algebraic geometry.

As an easy result about isobaricness for several variables, let us prove the

Generalized Euler Theorem (2.1). *Let F be a nonzero meromorphic polynomial in a finite number of indeterminates $X_1, \ldots, X_r$ over the field k, i.e., let F be a nonzero member of $K[X_1, X_1^{-1}, \ldots, X_r, X_r^{-1}]$. Assume that F is isobaric of weight N when we assign weight $W_i \in \mathbb{Z}$ to X_i for $1 \leqslant i \leqslant r$. In other words assume that $F = \sum_{(j_1,\ldots,j_r)\in\Omega} F_{j_1\ldots j_r} X_1^{j_1} \ldots X_r^{j_r}$ with $0 \neq F_{j_1\ldots j_r} \in K$ where Ω is a nonempty finite subset of $\mathbb{Z}^r$ such that for all $(j_1, \ldots, j_r) \in \Omega$ we have $j_1W_1 + \cdots + j_rW_r = N$. Then we have $\sum_{1\leqslant i\leqslant r} W_iX_iF_{X_i} = NF$.*

Proof. By linearity it suffices to prove this when F is a monomial $X_1^{j_1} \ldots X_r^{j_r}$. But then $\sum_{1\leqslant i\leqslant r} W_iX_iF_{X_i} = \sum_{1\leqslant i\leqslant r} W_ij_iF = NF$. $\square$

3. Isobaric Jacobians

Let

$$w = -w_2/w_1 \quad \text{where } w_1 \text{ and } w_2 \text{ are coprime integers with } w_1 > 0$$

[for coprime cf. Remark–Definition (4.8) of Section 4] and let

$$\begin{cases} F, G \text{ be nonzero } w\text{-homogeneous members of } k[X, X^{-1}, Y, Y^{-1}] \\ \quad \text{where } k \text{ is a field of characteristic } 0. \end{cases}$$

We shall now prove some Lemmas about $J(F, G)$, except that Lemma (3.2) will be more general. We start off by proving the following generalization of (1.4).

Generalized Eulerian Lemma (3.1). *We have*

$$J(F, G) = 0 \quad \Leftrightarrow \quad F \text{ is } w\text{-similar to } G.$$

Proof. Consider the Jacobian matrix

$$\begin{pmatrix} F_X & F_Y \\ G_X & G_Y \end{pmatrix}$$

and multiply the first column by w_1X and the second column by w_2Y. Then adding the columns of the new matrix we get the sums $w_1XF_X + w_2YF_Y$ and $w_1XG_X + w_2YG_Y$ which, by (2.1), are equal to NF and MG where N and M are the w-degrees of F and G respectively. This gives

$$w_1XJ(F,G) = \begin{vmatrix} NF & F_Y \\ MG & G_Y \end{vmatrix} = NFG_Y - MGF_Y.$$

Multiplying all sides by $\frac{-F^{M-1}}{G^{N+1}}$, by the quotient rule for derivatives, we get

$$-w_1XJ(F,G)\left(\frac{F^{M-1}}{G^{N+1}}\right) = \left(\frac{F^M}{G^N}\right)_Y$$

and hence (because $w_1 \neq 0$):

$$J(F,G) = 0 \quad \Leftrightarrow \quad \left(\frac{F^M}{G^N}\right)_Y = 0.$$

Consequently it suffices to show that

$$\left(\frac{F^M}{G^N}\right)_Y = 0 \quad \Leftrightarrow \quad F^M = \theta G^N.$$

Now

$$\left(\frac{F^M}{G^N}\right)_Y = 0 \quad \Rightarrow \quad F^M = C(X)G^N \quad \text{with } C(X) \in k(X)^\times.$$

Writing $C(X)$ in reduced form we have $C(X) = B(X)/A(X)$ where $A(X)$, $B(X)$ in $k[X]$ are such that either $A(0) \neq 0$ or $B(0) \neq 0$. Thus

$$\left(\frac{F^M}{G^N}\right)_Y = 0 \quad \Rightarrow \quad A(X)F^M = B(X)G^N \quad \text{with either } A(0) \neq 0 \text{ or } B(0) \neq 0.$$

"Comparing terms" of weight MN [cf. Remark (3.6) below] we see that

$$\left(\frac{F^M}{G^N}\right)_Y = 0 \quad \Rightarrow \quad A(0)F^M = B(0)G^N \quad \text{with either } A(0) \neq 0 \text{ or } B(0) \neq 0.$$

Since $F^M \neq 0 \neq G^N$, we must have $A(0) \neq 0 \neq B(0)$ and hence, upon taking $\theta = B(0)/A(0)$, by the above implication we get $(\frac{F^M}{G^N})_Y = 0 \Rightarrow F^M = \theta G^N$. The reverse implication $(\frac{F^M}{G^N})_Y = 0 \Leftarrow F^M = \theta G^N$ is obvious. □

Lemma (3.2). *Given any A, B in $k(X,Y)$ we have the following.*

(3.2.1) *If* $A = A_1 \dots A_n$ *and* $B = B_1 \dots B_m$ *where* n, m *are positive integers and* $A_1, \dots, A_n$, $B_1, \dots, B_m$ *elements in* $k(X, Y)$ *then*

$$J(A, B) = \sum_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m} (A/A_i)(B/B_j) J(A_i, B_j)$$

where by temporary convention

$$A/A_i = A_1 \dots A_{i-1} A_{i+1} \dots A_n \quad \text{and} \quad B/B_j = B_1 \dots B_{j-1} B_{j+1} \dots B_m.$$

(3.2.2) *For any* $\Gamma = \Gamma(X, Y)$ *and* $\Delta = \Delta(X, Y)$ *in* $k[X, Y]$ *we have*

$$J\big(\Gamma(A, B), \Delta(A, B)\big) = J(A, B) J_{(A,B)}\big(\Gamma(A, B), \Delta(A, B)\big)$$

where the unsubscripted (= the first two) Jacobians are relative to (X, Y) *while the last Jacobian is obtained by substituting* (A, B) *for* (X, Y) *in* $J(\Gamma, \Delta) \in k[X, Y]$.

(3.2.3) *For any* $\Theta(X, Y) \in k[X, Y]$ *we have* $J(A, \Theta(A, B)) = \Theta_Y(A, B) J(A, B)$ *where* $\Theta_Y(A, B)$ *is obtained by substituting* (A, B) *for* (X, Y) *in the* Y*-partial of* $\Theta(X, Y)$. *For any* α, β *in* k *and* p, q, r *in* $\mathbb{Z}$ *we have* $J(A, \alpha A^r + \beta A^p B^q) = q\beta A^p B^{q-1} J(A, B)$ *and* $J(\alpha B^r + \beta A^p B^q, B) = p\beta A^{p-1} B^q J(A, B)$.

(3.2.4) *For any* $\gamma(X), \delta(X)$ *in* $k[X]$ *we have* $J(\gamma(A), \delta(A)) = 0$.

(3.2.5) *Assume that* $A = C^p \widehat{A}$ *and* $B = C^q \widehat{B}$ *where* p, q *in* $\mathbb{N}$ *with* $p + q \geqslant 2$ *and* $\widehat{A}, \widehat{B}, C$ *in* $k[X, Y]$. *Then* $J(A, B) = C^{p+q-1} \widehat{C}$ *for some* $\widehat{C} \in k[X, Y]$.

(3.2.6) *If* A *and* B *are nonzero* w*-homogeneous members of* $k[X, X^{-1}, Y, Y^{-1}]$ *with* $J(A, B) \neq 0$ *then* $J(A, B)$ *is a* w*-homogeneous member of* $k[X, X^{-1}, Y, Y^{-1}]$ *and for it we have*

$$\deg_w J(A, B) = \deg_w(AB) - \deg_w(XY).$$

[*Note that* $\deg_w(AB) = \deg_w(A) + \deg_w(B)$].

(3.2.7) *Assume that* A *and* B *are nonzero members of* $k[X, Y]$ *such that* AB *is* w*-homogeneous. Then* A *and* B *are* w*-homogeneous.*

(3.2.8) *Assume that* A *and* B *are nonzero* w*-homogeneous members of* $k[X, Y]$ *such that* $\deg_w A \neq 0$ *and* A *is* w*-similar to* B. *Then we have* $A = \theta C^p$ *and* $B = \theta C^q$ *for some nonzero* w*-homogeneous member* C *of* $k[X, Y]$ *and some* p, q *in* $\mathbb{N}$ *with* $p > 0$.

(3.2.9) *Assume that* A *and* B *are nonzero* w*-homogeneous members of* $k[X, Y]$ *such that* $\deg_w A \neq 0 \leqslant (w_1 + w_2) \deg_w A$ *and* $J(A, B) = \theta A^u$ *with* $u \in \mathbb{N}_+$. *Then* $B/A^{u-1} \in k[X, Y]$.

Proof of (3.2.1). Use the product rule for derivatives. □

Proof of (3.2.2) to (3.2.4). (3.2.2) is a chain rule. The first equation of (3.2.3) follows from (3.2.2) by taking $\Gamma(X, Y) = X$ and $\Delta(X, Y) = \Theta(X, Y)$. The second equation of (3.2.3) is a special case of the first. The third equation of (3.2.3) follows from the second by noting that $J(B,A) = -J(A,B)$. (3.2.4) follows from (3.2.2) by taking $\Gamma(X,Y) = \gamma(X)$ and $\Delta(X,Y) = \delta(X)$. □

Proof of (3.2.5). Taking $(A_1, A_2) = (C^p, \widehat{A})$ and $(B_1, B_2) = (C^q, \widehat{B})$ with $n = m = 2$ in (3.2.1) we get

$$J(A,B) = C^{p+q} J(\widehat{A}, \widehat{B}) + C^p \widehat{B} J\big(\widehat{A}, C^q\big) + C^q \widehat{A} J\big(C^p, \widehat{B}\big) + \widehat{A}\widehat{B} J\big(C^p, C^q\big).$$

The last term is zero by (3.2.4). For the middle two terms, by (3.2.3) we have $J(\widehat{A}, C^q) = qC^{q-1} J(\widehat{A}, C)$ and $J(C^p, \widehat{B}) = pC^{p-1} J(C, \widehat{B})$. □

Proof of (3.2.6). This is obvious. □

Proof of (3.2.7). Let $N' = \min\{iw_1 + jw_2\colon (i,j) \in \operatorname{Supp}(A)\}$ and let $N = \deg_w A$. Let $M' = \min\{iw_1 + jw_2\colon (i,j) \in \operatorname{Supp}(B)\}$ and let $M = \deg_w B$. Let A_{ij}, B_{ij}, C_{ij} be the coefficients of $X^i Y^j$ in A, B, AB respectively. Let

$$A' = \sum_{(i,j)\in \operatorname{Supp}(A)\colon iw_1+jw_2=N'} A_{ij} X^i Y^j$$

and

$$B' = \sum_{(i,j)\in \operatorname{Supp}(B)\colon iw_1+jw_2=M'} B_{ij} X^i Y^j.$$

Finally let

$$(AB)' = \sum_{(i,j)\in \operatorname{Supp}(AB)\colon iw_1+jw_2=N'+M'} C_{ij} X^i Y^j.$$

Clearly $(AB)' = A'B' =$ a nonzero w-homogeneous member of $k[X,Y]$ of w-degree $M' + N'$. Also clearly AB is a nonzero member of $k[X,Y]$ of w-degree $M + N$. Since AB is w-homogeneous, we must have $M' + N' = M + N$ and hence $N' = N$ and $M' = M$, i.e., A and B are w-homogeneous. □

Proof of (3.2.8). Let $\deg_w A = N$ and $\deg_w B = M$. Since A and B are polynomials, we must have $MN \geqslant 0$ and $A^{|M|} = \theta B^{|N|}$. It follows that if $M = 0$ then $B \in k^\times$ and it suffices to take $C = A$ with $(p,q) = (1,0)$. Now assume that $M \neq 0$. Then by the UFD property of $k[X,Y]$ we can write $A = \theta C_1^{p_1} \ldots C_n^{p_n}$ and $B = \theta C_1^{q_1} \ldots C_n^{q_n}$ where $n, p_1, \ldots, p_n, q_1, \ldots, q_n$ are positive integers and $C_1, \ldots, C_n$ are pairwise coprime nonzero (nonconstant) irreducible members of $k[X,Y]$. By the UFD property we must also have $|M|p_i = |N|q_i$ for $1 \leqslant i \leqslant n$. Let $\operatorname{GCD}(|M|, |N|) = d$ and let $p = |N|/d$ and $q = |M|/d$. Then for $1 \leqslant i \leqslant n$, upon letting $\operatorname{GCD}(p_i, q_i) = d_i$ we get $p_i/d_i = p$ and $q_i/d_i = q$. Therefore upon letting $C = C_1^{d_1} \ldots C_n^{d_n}$ we get $A = \theta C^p$ and $B = \theta C^q$ with p, q in N_+. By (3.2.7) we see that $C_1, \ldots, C_n$ are w-homogeneous and hence so is C. □

Proof of (3.2.9). If $u = 1$ then we have nothing to show. So assume that $u > 1$. Then taking $F = A^{u-1}$ and $G = B$, by (3.2.3) we have $J(F, G) = \theta F^2$ and we are reduced to showing that $G/F \in k[X, Y]$. Moreover, upon letting $\deg_w F = N$ and $\deg_w G = M$, by (3.2.6) and by our assumption we get

(1) $$M = N + w_1 + w_2 \quad \text{and} \quad N \neq 0 \leqslant (M - N)/N.$$

As in the second display in the proof of (3.1) we have

$$w_1 X J(F, G) = \begin{vmatrix} NF & F_Y \\ MG & G_Y \end{vmatrix} = NFG_Y - MGF_Y$$

and hence we get

(2) $$NFG_Y - MGF_Y = \theta X F^2.$$

Given any nonconstant irreducible factor C of F in $k[X, Y]$ we can write

(3) $$F = C^p \widehat{F} \quad \text{and} \quad G = C^q \widehat{G} \quad \text{with } p \in \mathbb{N}_+ \text{ and } q \in \mathbb{N}$$

where $\widehat{F}$ and $\widehat{G}$ are members of $k[X, Y]$ which are nondivisible by C. By taking partial derivatives, the above equations tell us that

(4) $$F_Y = pC^{p-1}C_Y\widehat{F} + C^p\widehat{F}_Y \quad \text{and} \quad G_Y = qC^{q-1}C_Y\widehat{G} + C^q\widehat{G}_Y.$$

If $C \notin k[X]$ then C_Y is a nonzero polynomial of Y-degree smaller than the Y-degree of the irreducible polynomial C and hence C_Y is nondivisible by C. Thus

(5) $$\text{if } C \notin k[X] \quad \text{then } C_Y \text{ is nondivisible by } C.$$

We shall show that $q \geqslant p$ and this will complete the proof. □

(I) If $C \notin k[X]$ and $qN - pM \neq 0$ then, comparing the highest powers of C which divide the two sides of (2), by (3) to (5) we get $p + q - 1 = 2p$ and hence $q = p + 1 > p$.
If $qN - pM = 0$ then by (1) we get $q/p = M/N \geqslant 1$ and hence $q \geqslant p$. Therefore by (I) we see that:

(II) If $C \notin k[X]$ then $q \geqslant p$.
Because of symmetry in X, Y, by (II) we see that:

(III) If $C \notin k[Y]$ and $w_2 \neq 0$ then $q \geqslant p$.
By (3.2.7) we know that C is w-homogeneous. Consequently, if $C \in k[X]$ and $w_2 = 0$ then $C = \theta X$ and $(N, M) = (p, q)$ and hence by (1) we get $(q - p)/p \geqslant 0$ and hence $q \geqslant p$.
Therefore in view of (III) we conclude that:

(IV) If $C \in k[X]$ then $q \geqslant p$.

Lemma (3.3). *Assume that F and G belong to $k[X, Y]$ and $J(F, G) = \theta F$. Let $N = \deg_w F$ with $D = \deg_Y F$ and $M = \deg_w G$ with $E = \deg_Y G$. Then we have the following.*

(3.3.1) *Assume that* $M \neq 0$ *and* $G = \theta xy$ *where* (x, y) *is a* w*-automorphic pair. Then* $F = \theta x^i y^j$ *with* $i \neq j$ *in* $\mathbb{N}$.

(3.3.2) *For any* $H \in k[X, Y] \setminus k$ *we have* $G/H^2 \notin k[X, Y]$.

(3.3.3) *We have* $\deg_X G \geqslant 1$ *and* $E \geqslant 1$.

(3.3.4) *We have* $\deg_w(G) = \deg_w(XY)$*, i.e.,* $M = w_1 + w_2$.

(3.3.5) *Assume that* $EN - DM \neq 0$*. Then* $E = 1$.

[**Note (3.3.5*).** Concerning the above condition we observe that $EN - DM = 0$ iff the highest Y-degree term in F is w-similar to the highest Y-degree term in G. This is so because clearly the said terms are $X^{(N-w_2D)/w_1} Y^D$ and $X^{(M-w_2E)/w_1} Y^E$ respectively].

(3.3.6) *Assume that* $E = 1$*. Then* $G = \theta X(Y + \gamma X^{-w})$ *with* $\gamma \in k$ *such that* $1 - w \in \mathbb{N}$ *in case* $\gamma \neq 0$ [*cf. Remark* (3.13) *below*].

(3.3.7) *Assume that* $w < 0$ *and* $E > 2$*. Then* $G = \theta Y(X + \gamma Y^{E-1})$ *with* $0 \neq \gamma \in k$ *and we have* $\frac{-1}{w} = E - 1$.

(3.3.8) *Assume that* $w < 0$ *and* $E = 2$*. Then* $G = \theta(\alpha X + Y)(\beta X + Y)$ *with* $\alpha \neq \beta$ *in* k *and we have* $w = -1$.

Proof of (3.3.1). Now there is a unique

$$C = C(X, Y) = \sum C_{ij} X^i Y^j \in k[X, Y] \quad \text{with } C_{ij} \in k$$

such that $\mathrm{Supp}(C) \neq \emptyset$ and $C(x, y) = F$. Clearly $J(C, XY) = rC$ with $r \in k^\times$. Also $J(C, XY) = \sum (i - j) C_{ij} X^i Y^j$. Therefore for all (i, j) in $\mathrm{Supp}(C)$ we have $i - j = r$ and hence $X^i Y^j = X^r (XY)^j$. So for all $(i, j) \neq (i', j')$ in $\mathrm{Supp}(C)$ we have $j \neq j'$; since $\deg_w(C_{ij} x^i y^j) = r \deg_w(x) + j \deg_w(xy)$ with $\deg_w(xy) = \deg_w(G) \neq 0$ and similarly for (i', j'), we conclude that $\deg_w(C_{ij} x^i y^j) \neq \deg_w(C_{i'j'} x^{i'} y^{j'})$. Since F and $C_{ij} x^i y^j$ are w-homogeneous, the equation

$$F = C(x, y) = \sum_{(i,j) \in \mathrm{Supp}(C)} C_{ij} x^i y^j$$

tells us that $\mathrm{Supp}(C)$ consists of a unique element (i, j). It follows that $F = \theta x^i y^j$ with $i \neq j$ in $\mathbb{N}$. □

Proof of (3.3.2). Take an irreducible factor C of H in $k[X, Y] \setminus k$, and let C^p and C^q be the highest powers of C which divide F and G in $k[X, Y]$ respectively. If $q \geqslant 2$ then by (3.2.5) we get $F/C^{p+1} \in k[X, Y]$ which is a contradiction. Therefore $q \leqslant 1$ and hence $G/H^2 \notin k[X, Y]$.

Proof of (3.3.3). If $G \in k[Y]$ then $G_X = 0$ and hence $J(F, G) = F_X G_Y$ which, by comparing X-degrees, contradicts the equation $J(F, G) = \theta F$. Therefore $G \notin k[Y]$, i.e., $\deg_X G \geqslant 1$. Similarly $\deg_Y G \geqslant 1$, i.e., $E \geqslant 1$. □

Now (3.3.4) follows from (3.2.6). In proving (3.3.5) to (3.3.8) we can use the following common

Notation. We clearly have

$$G = \sum_{j \in S(G)} G_j X^{v(j)} Y^j \quad \text{with } 0 \neq G_j \in k$$

where

$$E \in S(G) \subset \{0, 1, \dots, E\} \quad \text{and} \quad v(j) \in \mathbb{N} \quad \text{with } v(j)w_1 + jw_2 = M.$$

Proof of (3.3.5). We also have

$$F = \sum_{j \in S(F)} F_j X^{u(j)} Y^j \quad \text{with } 0 \neq F_j \in k$$

where

$$D \in S(F) \subset \{0, 1, \dots, D\} \quad \text{and} \quad u(j) \in \mathbb{N} \quad \text{with } u(j)w_1 + jw_2 = N.$$

If $E > 1$ then equating the coefficients of $X^{u(D)+v(E)-1}Y^{D+E-1}$ on both sides of the equation $J(F, G) = \theta F$ we see that $(u(D)E - v(E)D)F_D G_E = 0$ and substituting $u(D) = (N/w_1) - D(w_2/w_1)$ and $v(E) = (M/w_1) - E(w_2/w_1)$ in this and then dividing out by $F_D G_E/w_1$ we get $EN - DM = 0$ which is a contradiction. Therefore $E = 1$ by (3.3.3). □

Proof of (3.3.6) and (3.3.7). By (3.3.4) and the above Notation, we see that for some ϵ, δ in k we have (i) $G = \epsilon XY + \delta X^{1-w}$ with $1 - w \in \mathbb{N}$ in case $\delta \neq 0$ or (ii) $G = \epsilon XY + \delta Y^E$ with $\delta \neq 0$, according as we are in (3.3.6) or (3.3.7). By (3.3.2) we get $\epsilon \neq 0$. The rest is clear. □

Proof of (3.3.8). By (3.3.2), (3.3.4) and the above Notation, we see that $w = -1$ and $G = \delta(\alpha X + Y)(\beta X + Y)$ with $0 \neq \delta \in k$ and $\alpha \neq \beta$ in an algebraic closure $\bar{k}$ of k. By (3.3.1) we get $F = \epsilon(\alpha X + Y)^i(\beta X + Y)^j$ with $0 \neq \epsilon \in k$ and $i \neq j$ in $\mathbb{N}$. It follows that [cf. Remark (3.7) below] α and β must belong to k. □

Lemma (3.4). *Assume that F and G belong to $k[X, Y]$ and $J(F, G) = \theta F$. Also assume that we have either* (i) *$w < 0$ or* (ii) *$w \geqslant 0 \neq w_1 + w_2 \neq 0 \neq EN - DM$ where $N = \deg_w F$ with $D = \deg_Y F$ and $M = \deg_w G$ with $E = \deg_Y G$. Then $G = \theta xy$ and $F = \theta x^i y^j$ with $i \neq j$ in $\mathbb{N}$ where (x, y) is the w-automorphic pair described below*; *see pictures in* (9.2).

(1) *If $w = -1 \neq -E$ then $x = \alpha X + Y$ and $y = \beta X + Y$ with $\alpha \neq \beta$ in k.*
(2) *If $w = -1 = -E$ then $x = X$ and $y = \beta X + Y$ with $\beta \in k$.*
(3) *If $w < -1$ then $x = X$ and $y = \gamma X^{-w} + Y$ with $\gamma \in k$ such that $-w \in \mathbb{N}_+$ in case $\gamma \neq 0$.*
(4) *If $0 > w > -1$ then $x = Y$ and $y = X + \gamma Y^{-1/w}$ with $\gamma \in k$ and $-1/w \in \mathbb{N}_+$ in case $\gamma \neq 0$.*
(5) *If $w \geqslant 0 \neq w_1 + w_2 \neq 0 \neq EN - DM$ then $x = X$ and $y = Y + \gamma X^{-w}$ with $\gamma \in k$ such that $w = 0$ in case $\gamma \neq 0$.*

Proof. Follows from (3.3). □

Lemma (3.5). *Assume that F and G belong to $k[X,Y]$ and $J(F,G)=\theta$. Also assume that we have either* (i) $w<0$ *or* (ii) $w\geqslant 0\neq w_1+w_2\neq 0\neq EN-DM$ *where* $N=\deg_w F$ *with* $D=\deg_Y F$ *and* $M=\deg_w G$ *with* $E=\deg_Y G$. *Then for the w-automorphic pair (x,y) described in the following* (*mutually exclusive*) *cases* (1) *to* (5) *we have* $(F,G)=(\theta x,\theta y)$ *or* $(\theta y,\theta x)$.

(1) *If* $w=-1\neq -(D+E)$ *then* $x=\alpha X+Y$ *and* $y=\beta X+Y$ *with* $\alpha\neq\beta$ *in* k.
(2) *If* $w=-1=-(D+E)$ *then* $x=X$ *and* $y=\beta X+Y$ *with* $\beta\in k$.
(3) *If* $w<-1$ *then* $x=X$ *and* $y=\gamma X^{-w}+Y$ *with* $\gamma\in k$ *such that* $-w\in\mathbb{N}_+$ *in case* $\gamma\neq 0$.
(4) *If* $0>w>-1$ *then* $x=Y$ *and* $y=X+\gamma Y^{-1/w}$ *with* $\gamma\in k$ *and* $-1/w\in\mathbb{N}_+$ *in case* $\gamma\neq 0$.
(5) *If* $w\geqslant 0\neq w_1+w_2\neq 0\neq EN-DM$ *then* $x=X$ *and* $y=Y+\gamma X^{-w}$ *with* $\gamma\in k$ *such that* $w=0$ *in case* $\gamma\neq 0$.

Proof. By (3.2.3) we have $J(F,FG)=FJ(F,G)$ and hence $J(F,FG)=\theta F$. Now apply (3.4) with G replaced by FG. □

Remark (3.6). The “Comparing terms” in the proof of (3.1) is literally correct only when $M\geqslant 0$ and $N\geqslant 0$. If $M<0$ and $N<0$ then compare the terms of weight $-MN$ in $A(X)G^{-N}=B(X)F^{-M}$ to get $A(0)G^{-N}=B(0)F^{-M}$ and then revert back to $A(0)F^{M}=B(0)G^{N}$. In the remaining two cases do similar things to avoid raising F and G to negative powers because the ring $k[X,X^{-1},Y,Y^{-1}]$ is not a field. For instance, if $M\geqslant 0$ and $N<0$ then compare the terms of weight zero in $A(X)F^{M}G^{-N}=B(X)$ to get $A(0)F^{M}G^{-N}=B(0)$ and from this deduce that $A(0)F^{M}=B(0)G^{N}$.

Here is an alternative proof of the part of (3.1) which says that if $J(F,G)=0$ then F is w-similar to G. Assuming $J(F,G)=0$, as in the present proof we see that $(\frac{F^M}{G^N})\in k(X)^\times$. If $w_2\neq 0$ then by symmetry in X,Y we get $(\frac{F^M}{G^N})\in k(Y)^\times$ and hence $(\frac{F^M}{G^N})\in k(X)^\times\cap k(Y)^\times=k^\times$. If $w_2=0$ then clearly $F=X^N\widetilde{F}$ and $G=X^M\widetilde{G}$ with $\widetilde{F}$ and $\widetilde{G}$ in $k(Y)^\times$; this gives $(\frac{F^M}{G^N})=(\frac{\widetilde{F}^M}{\widetilde{G}^N})$ with $(\frac{\widetilde{F}^M}{\widetilde{G}^N})_Y=0$, and hence $(\frac{F^M}{G^N})\in k^\times$.

Remark (3.7). Referring to the proof of (3.3.8), let $P=P(Y)=F(1,Y)/\epsilon$ and $Q=Q(Y)=G(1,Y)/\delta$. Then P,Q are monic polynomials in Y with coefficients in k. Let $R=P/Q^i$ or P/Q^j according as $i<j$ or $j<i$. Then R is also a monic polynomial in Y with coefficients in k. In $\bar{k}[Y]$ the “monic GCD” of Q and R in $\bar{k}[Y]$ is $\beta+Y$ or $\alpha+Y$. But the said GCD coincides with its version in $k[Y]$. Therefore one and hence both of the elements α,β belong to k.

Remark (3.8). Some of the calculations of this Section 3 seem to say that in some sense $J(F,G)=\frac{FG}{XY}$.

Remark (3.9). The proof of (3.3.1) appears to say that the expansion of $J(F,XY)$ is like a twisted Euler Theorem.

Remark (3.10). The proofs of (3.2.7) and (3.2.8) establish the truth of the corresponding assertions for any number of variables stated below. As notation for the new assertions let A and B be nonzero members of $k[X_1,\dots,X_r]$, where r is any positive integer and k is any field, and let us give any weight $W_i \in \mathbb{Z}$ to X_i for $1 \leqslant i \leqslant r$. Then in the terminology of (2.1) we have the following.

(1) If AB is isobaric then so are A and B.

(2) If A and B are isobaric of weights N and M respectively, and if $A^M = \theta B^N$ with $N \neq 0$ then there exists a nonzero isobaric member C of $k[X_1, \dots, X_r]$ such that $A = \theta C^p$ and $B = \theta C^q$ for some p, q in $\mathbb{N}$ with $p > 0$.

Remark–Definition (3.11). The proof (3.2.8) suggests the following sharper description of the quantities C, p, q introduced in (3.2.8) and (3.10)(2). Given any nonzero A in a Unique Factorization Domain R, we can write $A = C^* C_1^{p_1} \dots C_n^{p_n}$ where C^* is a unit in R, n is a nonnegative integer, $p_1, \dots, p_n$ are positive integers, and $C_1, \dots, C_n$ are pairwise coprime irreducible nonzero nonunits in R. Upon letting $p = \mathrm{GCD}(p_1, \dots, p_n) =$ the *nonnegative generator* of the ideal in $\mathbb{Z}$ generated by $p_1, \dots, p_n$, we define the *radical-number* of A by putting $\mathrm{radnum}(A) = p$, and we define the *radical-set* of A by putting $\mathrm{radset}(A) =$ the set of all members of R of the form $\widehat{C} C_1^{p_1/p} \dots C_n^{p_n/p}$ where $\widehat{C}$ varies over the set $U(R)$ of all units in R. Clearly

(1) for every C in $\mathrm{radset}(A)$ we have $A = C' C^p$ for some C' in $U(R)$, and

(2) we have $p \in \mathbb{N}$ with: $p \in \mathbb{N}_+ \Leftrightarrow A \notin U(R)$.

The proof of (3.2.8) yields the following result.

(3) For any nonzero A, B in a UFD R and any M, N in $\mathbb{Z}$ with $A \notin U(R)$ and $N \neq 0$, upon letting $\mathrm{radnum}(A) = p$ and $\mathrm{radnum}(B) = q$, we have that:

$$\begin{aligned} & B \notin U(R) \text{ and } A^M = B' B^N \text{ for some } B' \in U(R) \\ \Leftrightarrow\quad & M \neq 0 \text{ and } A^M = B' B^N \text{ for some } B' \in U(R) \\ \Leftrightarrow\quad & MN > 0 \text{ and } A^{|M|} = B^* B^{|N|} \text{ for some } B^* \in U(R) \\ \Leftrightarrow\quad & Mp = Nq \text{ and } \mathrm{radset}(A) = \mathrm{radset}(B). \end{aligned}$$

Finally note that $U(k[X_1, \dots, X_r]) = k^\times$.

Remark (3.12). Here is an application of the above sharper version (3.11) of (3.2.8). Given any $A \in k[X, Y] \setminus k$, take any $C \in \mathrm{radset}(A)$ and note that then $A = \theta C^p$ with $\mathrm{radnum}(A) = p \in \mathbb{N}_+$. Assuming $A_w^+ \notin k$, take any $\overline{C} \in \mathrm{radset}(A_w^+)$ and note that then $A_w^+ = \theta \overline{C}^{\overline{p}}$ with $\mathrm{radnum}(A_w^+) = \overline{p} \in \mathbb{N}_+$. Take any $\widetilde{C} \in \mathrm{radset}(C_w^+)$ and note that then $C_w^+ = \theta \widetilde{C}^e$ with $\mathrm{radnum}(C_w^+) = e \in \mathbb{N}_+$. By (3.11) we get the following.

(1) $\widetilde{C} \in \mathrm{radset}(A_w^+)$ and $pe = \overline{p}$.

(2) $\mathrm{radnum}(A) = \mathrm{radnum}(A_w^+) \Leftrightarrow e = 1 \Leftrightarrow C_w^+ \in \mathrm{radset}(A_w^+)$.

(3) If there exists $B \in k[X, Y] \setminus k$ such that $A^M = \theta B^N$ for some M, N in $\mathbb{Z}$ with $N \neq 0$ and $\mathrm{GCD}(\overline{p}, \overline{q}) = 1$ where we have put $\mathrm{radnum}(B_w^+) = \overline{q}$, then we have $B_w^+ \notin k$ and $\mathrm{radnum}(A) = \mathrm{radnum}(A_w^+)$.

Remark (3.13). Concerning the condition on $1 - w$ considered in (3.3.6) we note that $1 - w = (w_1 + w_2)/w_1$ and hence: (i) $1 - w = 0 \Leftrightarrow w = 1 \Leftrightarrow w_1 + w_2 = 0$, (ii) $1 - w = 1 \Leftrightarrow w = 0$, and (iii) $1 - w \in \mathbb{N} \setminus \{0, 1\} \Leftrightarrow w < 0$ with $-w \in \mathbb{N}_+$.

Remark–Problem (3.14). The following examples (1) and (2) show the necessity of conditions (i) and (ii) of (3.4).

(1) Take $F = X^2Y(Y+1)^3$ and $G = XY(Y+1)$ with $(w_1, w_2) = (1, 0)$. Then $J(F, G) = F$ but F is not a monomial in a w-automorphic pair.

(2) Take $F = X(XY^3+1)^2$ and $G = XY(XY^3+1)$ with $(w_1, w_2) = (3, -1)$. Then again $J(F, G) = F$ but F is not a monomial in a w-automorphic pair.

It would be interesting to further investigate what happens to (3.4) without conditions (i) and (ii).

4. Two points at infinity

Let

$$w = -w_2/w_1 \quad \text{where } w_1 \text{ and } w_2 \text{ are coprime integers with } w_1 > 0$$

and let

$$\begin{cases} f \text{ and } g \text{ be nonzero members of } k[X, Y] \\ \quad \text{where } k \text{ is a field of characteristic 0.} \end{cases}$$

We shall now prove a string of lemmas culminating in Main Lemma (4.7) which says that if $w < 0$ (or w satisfies some other conditions) and $J(f, g) = \theta$ then f as well as g has at most two points at infinity in the w-weighted sense. Here is a

Brief strategy. If $\mathrm{lag}_w(f, g) > 0$ then by the following generalized version (4.1) of (1.3) we know that f is w-similar to g, i.e.,

$$\left(f_w^+\right)^M = \theta \left(g_w^+\right)^N$$

for some positive integers M, N. Upon letting

$$g_2 = f^M - \theta g^N$$

we get $\mathrm{lag}_w(f, g_1) > \mathrm{lag}_w(f, g_2)$ with $g_1 = g$. Iterating this procedure we find a finite sequence $g_1, g_2, \ldots, g_e$ in $k[f, g]$ such that

$$\mathrm{lag}_w(f, g_1) > \mathrm{lag}_w(f, g_2) > \cdots > \mathrm{lag}_w(f, g_e) = 0$$

and such that f is w-similar to $J(f, g_i)$ for $1 \leqslant i \leqslant e$. If $\mathrm{lag}_w(f, g) = 0$ then we take $e = 1$.

Later on we shall exhibit a noniterative procedure by showing that g_e may be taken to be an "approximate root" which provides a good motivation for introducing the very versatile concept of such roots.

At any rate the w-similarity of f to $J(f, g_e)$ together with the zeroness of $\mathrm{lag}_w(f, g_e)$ leads to certain differential equations, and by solving them we get the desired result.

In greater detail, we have replaced the unstable condition that the Jacobian of f and g is a nonzero constant by the more stable condition that f is w-similar to the Jacobian of f and g. Now the conditions that f is w-similar to $J(f, g_e)$ and $\mathrm{lag}_w(f, g_e) = 0$, tell us that in some sense $J(f_w^+, (g_e)_w^+) = \theta (f_w^+)^{q/p}$ for some $q \in \mathbb{N}$ and $p \in \mathbb{N}_+$. Here the fractional power can be straightened out by finding w-homogeneous polynomials $\widetilde{F}$ and $\widetilde{G}$ such that $f_w^+ = \theta \widetilde{F}^p$

and $J(\widetilde{F}^p, \widetilde{G}) = \theta \widetilde{F}^u$. By subjecting the last equation to some clever manipulative trickery we show that either the differential equation $J(\widetilde{F}, \widetilde{G}) = \theta \widetilde{F}$ is satisfied or the differential equation $J(\widetilde{F}, \widetilde{G}) = \theta$ is satisfied. Finally by solving these differential equations we prove that $\widetilde{F}$, and hence f, has at most two points at infinity in the w-weighted sense.

Let us start off by proving the said generalization of (1.3).

Generalized Jacobian Lemma (4.1).

$$\mathrm{lag}_w(f, g) \geqslant 0$$

and

$$J\left(f_w^+, g_w^+\right) = \begin{cases} J(f, g)_w^+ \neq 0 & \text{if } \mathrm{lag}_w(f, g) = 0, \\ 0 & \text{if } \mathrm{lag}_w(f, g) > 0 \end{cases}$$

and

$$\mathrm{lag}_w(f, g) > 0 \quad \Leftrightarrow \quad J\left(f_w^+, g_w^+\right) = 0 \quad \Leftrightarrow \quad f \text{ is } w\text{-similar to } g.$$

Proof. Clearly

$$f = f_w^+ + \text{terms of } w\text{-degree} < \deg_w f$$

and

$$g = g_w^+ + \text{terms of } w\text{-degree} < \deg_w g$$

and hence

$$J(f, g) = J\left(f_w^+, g_w^+\right) + \text{terms of } w\text{-degree} < \deg_w(fg) - \deg_w(XY).$$

Consequently, by taking $F = f_w^+$ and $G = g_w^+$, we are done by (3.1) and (3.2.6). □

Lemma (4.2). *Assume that* $\deg_w f \neq 0$ *and let* $N = |\deg_w f|$. *Also assume that* $\mathrm{lag}_w(f, g) > 0$ *and* f *is* w*-similar to* $J(f, g)$. *Then there exists* $M \in \mathbb{N}$ *and* $\kappa \in k^\times$ *such that upon letting* $\bar{g} = f^M - \kappa g^N$ *we have that* $0 \neq \bar{g} \in k[X, Y]$ *with* $\mathrm{lag}_w(f, g) > \mathrm{lag}_w(f, \bar{g})$ *and* f *is* w*-similar to* $J(f, \bar{g})$.

Proof. By (4.1) we see that f is w-similar to g, i.e.,

$$\left(f_w^+\right)^{\deg_w g} = \kappa' \left(g_w^+\right)^{\deg_w f} \quad \text{with } \kappa' \in k^\times.$$

Let

$$(\kappa, M) = (\kappa', \deg_w g) \quad \text{or} \quad (1/\kappa', -\deg_w g)$$

according as $\deg_w f \geqslant 0$ or < 0. Then $\kappa \in k^\times$ with $M \in \mathbb{N}$ and upon letting $\bar{g} = f^M - \kappa g^N$ we have $0 \neq \bar{g} \in k[X,Y]$ with

$$\deg_w(f\bar{g}) < \deg_w(fg) + \deg_w\big(g^{N-1}\big).$$

By (3.2.3) we see that

$$J(f,\bar{g}) = \kappa N g^{N-1} J(f,g).$$

By the above two displays we get $\mathrm{lag}_w(f,g) > \mathrm{lag}_w(f,\bar{g})$. Since f is w-similar to g as well as to $J(f,g)$, by the last display we conclude that f is w-similar to $J(f,\bar{g})$. □

Lemma (4.3). *Assume that* $\deg_w f \neq 0$ *and let* $N = |\deg_w f|$. *Also assume that f is w-similar to $J(f,g)$. Then there exists a finite sequence $g_1, \ldots, g_e$ of nonzero elements in $k[X,Y]$ with $g_1 = g$ and* $\mathrm{lag}_w(f, g_e) = 0$ *such that for* $2 \leqslant i \leqslant e$ *we have* $\mathrm{lag}_w(f,g_i) < \mathrm{lag}_w(f,g_{i-1})$ *and* $g_i = f^{M_i} - \kappa_i g_{i-1}^N$ *for some* $M_i \in \mathbb{N}$ *and* $\kappa_i \in k^\times$*, and such that f is w-similar to $J(f,g_i)$ for* $1 \leqslant i \leqslant e$.

Proof. Since f is w-similar to $J(f,g)$, we must have $J(f,g) \neq 0$, and hence $\mathrm{lag}_w(f,g) \in \mathbb{N}$. Therefore, in view of (4.2), we are done by induction on $\mathrm{lag}_w(f,g)$. □

Lemma (4.4). *Assume that* $\deg_w f \neq 0$. *Also assume that* $\mathrm{lag}_w(f,g) = 0$ *and f is w-similar to $J(f,g)$. Then there exists a nonzero w-homogeneous $\widetilde{F}$ in $k[X,Y]$ and p,q in $\mathbb{N}$ with $p > 0$ such that we have* $f_w^+ = \theta \widetilde{F}^p$ *with* $J(f,g)_w^+ = \theta \widetilde{F}^q$ *and hence upon letting* $G = g_w^+$ *we have* $J(\widetilde{F}^p, G) = \theta \widetilde{F}^q$.

Proof. Upon letting $F = f_w^+$ and $G = g_w^+$ with $H = J(f,g)_w^+$, by (4.1) we see that F, G, H are nonzero homogeneous members of $k[X,Y]$ such that $J(F,G) = H$ and F is w-similar to H. By (3.2.8) we can find a nonzero w-homogeneous $\widetilde{F}$ in $k[X,Y]$ such that $F = \theta \widetilde{F}^p$ and $H = \theta \widetilde{F}^q$ for some p,q in $\mathbb{N}$ with $p > 0$. It follows that $J(\widetilde{F}^p, G) = \theta \widetilde{F}^q$. □

Lemma (4.5). *Let $\widetilde{F}$ and G be nonzero w-homogeneous members of $k[X,Y]$ such that* $\deg_w \widetilde{F} \neq 0 \leqslant (w_1 + w_2)\deg_w \widetilde{F}$ *and* $J(\widetilde{F}^p, G) = \theta \widetilde{F}^q$ *for some p,q in $\mathbb{N}$ with $p > 0$. Then there exists a nonzero w-homogeneous $\widetilde{G} \in k[X,Y]$ with $G = \widetilde{G}\widetilde{F}^v$ for some $v \in \mathbb{N}$ such that we have either* $J(\widetilde{F}, \widetilde{G}) = \theta \widetilde{F}$ *or* $J(\widetilde{F}, \widetilde{G}) = \theta$. *In greater detail*:

(1) *if* $q = p - 1$ *then* $v = 0$ *with* $J(\widetilde{F}, \widetilde{G}) = \theta$, *whereas*
(2) *if* $q \neq p - 1$ *then* $q \geqslant p$ *and* $v = q - p$ *with* $J(\widetilde{F}, \widetilde{G}) = \theta \widetilde{F}$.

Proof. Let $u = q - (p-1)$. By (3.2.3) we have $J(\widetilde{F}^p, G) = p\widetilde{F}^{p-1} J(\widetilde{F}, G)$ and hence we must have

$$J(\widetilde{F}, G) = \theta \widetilde{F}^u \quad \text{with } u \in \mathbb{N}.$$

If $u = 0$ then we are in case (1). If $u \in \mathbb{N}_+$ then by (3.2.3) we have $J(\widetilde{F}, G) = u\widetilde{F}^{u-1} J(\widetilde{F}, G/\widetilde{F}^{u-1})$ and hence by (3.2.9) we are in case (2). □

Lemma (4.6). *Assume that f is w-similar to $J(f,g)$, and also assume that either* (i) $w < 0$, *or* (ii) $w \geqslant 0 \neq N$ *with* $EN - DM \neq 0 < (w_1 + w_2)N$, *where* $N = \deg_w f$ *with* $D = \deg_Y f_w^+$ *and where* $M = \deg_w g$ *with* $E = \deg_Y g_w^+$. *Then f as well as g has at most two points at infinity in the w-weighted sense. More precisely* $f_w^+ = \not\theta x^i y^j$ *and* $g_w^+ = \not\theta x^{i^*} y^{j^*}$ *where* i, j, i^*, j^* *in* $\mathbb{N}$ *with* $i - j \neq 0 \neq i + j \neq 0 \neq i^* + j^*$ *and where* (x, y) *is the w-automorphic pair described below; see pictures in* (9.2).

(1) *If* $w = -1$ *then* $x = \alpha X + \alpha^* Y$ *and* $y = \beta X + \beta^* Y$ *where* $\alpha, \alpha^*, \beta, \beta^*$ *in k are such that* $\alpha\beta^* - \alpha^*\beta \neq 0$.
(2) *If* $w < -1$ *then* $x = X$ *and* $y = \gamma X^{-w} + Y$ *with* $\gamma \in k$ *such that* $-w \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(3) *If* $0 > w > -1$ *then* $x = Y$ *and* $y = X + \gamma Y^{-1/w}$ *with* $\gamma \in k$ *and* $-1/w \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(4) *If* $w \geqslant 0 \neq N$ *with* $EN - DM \neq 0 < (w_1 + w_2)N$ *then* $x = X$ *and* $y = Y + \gamma X^{-w}$ *with* $\gamma \in k$ *such that* $w = 0$ *in case* $\gamma \neq 0$.

Moreover we have $N \neq 0$ *and we have* (I) *and* (II) *stated below.*

(I) $\operatorname{lag}_w(f, g) \neq 0 \Leftrightarrow f$ *is w-similar to g.*
(II) $\operatorname{lag}_w(f, g) = 0 \Rightarrow (i^*, j^*) = (1 + ci, 1 + cj)$ *for some* $c \in \mathbb{Q}$.

Proof. Since $J(f,g) \neq 0$, in case of (i) we have $N \neq 0 < (w_1 + w_2)N$, and hence we always have $N \neq 0 < (w_1 + w_2)N$. Therefore, in the situation when $\operatorname{lag}_w(f,g) = 0$, our assertions follow from (3.4), (3.5), (4.4) and (4.5). Likewise, in the situation when $\operatorname{lag}_w(f,g) \neq 0$, our assertions follow from (3.4), (3.5), (4.1), (4.3), (4.4) and (4.5). □

Main Lemma (4.7). *Let us assume that* $J(f,g) = \not\theta$, *and let us also assume that we have either* (i) $w < 0$, *or* (ii) $w \geqslant 0 \neq N$ *with* $EN - DM \neq 0 < (w_1 + w_2)N$, *or* (iii) $w \geqslant 0 \neq M$ *with* $EN - DM \neq 0 < (w_1 + w_2)M$, *where* $N = \deg_w f$ *with* $D = \deg_Y f_w^+$ *and where* $M = \deg_w g$ *with* $E = \deg_Y g_w^+$. *Then f as well as g has at most two points at infinity in the w-weighted sense. More precisely* $f_w^+ = \not\theta x^i y^j$ *and* $g_w^+ = \not\theta x^{i^*} y^{j^*}$ *where* i, j, i^*, j^* *in* $\mathbb{N}$ *with* $i - j \neq 0 \neq i + j \neq 0 \neq i^* + j^* \neq 0 \neq i^* - j^*$ *and where* (x, y) *is the w-automorphic pair described below.*

(1) *If* $w = -1$ *then* $x = \alpha X + \alpha^* Y$ *and* $y = \beta X + \beta^* Y$ *where* $\alpha, \alpha^*, \beta, \beta^*$ *in k are such that* $\alpha\beta^* - \alpha^*\beta \neq 0$.
(2) *If* $w < -1$ *then* $x = X$ *and* $y = \gamma X^{-w} + Y$ *with* $\gamma \in k$ *such that* $-w \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(3) *If* $0 > w > -1$ *then* $x = Y$ *and* $y = X + \gamma Y^{-1/w}$ *with* $\gamma \in k$ *and* $-1/w \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(4) *If* $w \geqslant 0 \neq N$ *with* $EN - DM \neq 0 < (w_1 + w_2)N$ *then* $x = X$ *and* $y = Y + \gamma X^{-w}$ *with* $\gamma \in k$ *such that* $w = 0$ *in case* $\gamma \neq 0$.
(5) *If* $w \geqslant 0 \neq M$ *with* $EN - DM \neq 0 < (w_1 + w_2)M$ *then* $x = X$ *and* $y = Y + \gamma X^{-w}$ *with* $\gamma \in k$ *such that* $w = 0$ *in case* $\gamma \neq 0$.

Moreover we have $\operatorname{GCD}(N, M) \neq 0$ *and we have* (I) *and* (II) *stated below.*

(I) $\operatorname{lag}_w(f, g) \neq 0 \Leftrightarrow f$ *is w-similar to g.*
(II) $\operatorname{lag}_w(f, g) = 0 \Rightarrow$ *either* $(i^*, j^*) = (1 + ci, 1 + cj)$ *for some* $c \in \mathbb{Q}$ *or* $(i, j) = (1 + ci^*, 1 + cj^*)$ *for some* $c \in \mathbb{Q}$.

Proof. By symmetry in f, g, this follows from (4.6). □

Remark–Definition (4.8). We shall now give some VARIATIONS of the above lemmas. In particular the following Lemma (4.9) is a Variation of Lemma (4.5) as well as an illustration of Remark (3.8). Then the next Lemma (4.10) is a consequence of Lemmas (4.1), (4.3), (4.4), and (4.9). The Proof of (4.6) can be simplified by retaining the first sentence "Since $J(f,g)\neq 0\dots$" and replacing the remaining two sentences by the sentence "So we are done by (3.4) and (4.10)." Thus all the references to (3.5) may be dropped.

As a matter of definition we say that f is a *w-monomial* to mean that $\operatorname{Supp}(f_w^+)$ contains exactly one member. Integers $p_1, p_2, \dots$ are *coprime* means their GCD is 1.

Recall that for any member H of $k[X,Y]$ we have $H^+ = H_{-1}^+ =$ the *ordinary degree form* of H and $\deg H = \deg_{-1} H =$ the *ordinary* (*total*) *degree* of H.

Note that:

(1) If $w \geqslant -1$ then for any nonzero w-homogeneous members F and G of $k[X,Y]$, upon letting $N = \deg_w F$ with $D = \deg_Y F$ and $M = \deg_w G$ with $E = \deg_Y G$, we clearly have $F^+ = \theta X^{(N-w_2D)/w_1} Y^D$ and $G^+ = \theta X^{(M-w_2E)/w_1} Y^E$.

In view of (1), by (3.3) and (4.1) we get the following supplement to (3.4):

(2) Assume that $-1 \neq w \neq 1$ and let F and G be any nonzero w-homogeneous members of $k[X,Y]$ such that $J(F,G) = \theta F$ and $\deg G = 2$. Assume that F is not a w-monomial. Then $w = 0$ and there exists $\gamma \in k^\times$ such that $G = \theta X(Y+\gamma)$ and $F = \theta X^i (Y+\gamma)^j$ where $i \neq j$ in $\mathbb{N}$ with $j \neq 0$.

Lemma (4.9). *Let $\widetilde{F}$ and G be nonzero w-homogeneous members of $k[X,Y]$ such that $\deg_w \widetilde{F} \neq 0 \leqslant (w_1+w_2)\deg_w \widetilde{F}$ and $J(\widetilde{F}^p, G) = \theta \widetilde{F}^q$ for some p, q in $\mathbb{N}$ with $p > 0$. Then $q \geqslant p-1$ and there exists a nonzero w-homogeneous $\widetilde{G} \in k[X,Y]$ with $G = \widetilde{G}\widetilde{F}^{q-p}$ such that $J(\widetilde{F}, \widetilde{G}) = \theta \widetilde{F}$.*

Proof. Let $u = q-(p-1)$. By (3.2.3) $J(\widetilde{F}^p, G) = p\widetilde{F}^{p-1} J(\widetilde{F}, G)$ and hence $J(\widetilde{F}, G) = \theta \widetilde{F}^u$ with $u \in \mathbb{N}$. If $u = 0$ then clearly $\widetilde{G} = G\widetilde{F} \in k[X,Y]$ and by (3.2.3) we get $J(\widetilde{F}, \widetilde{G}) = \theta \widetilde{F}$. If $u \in \mathbb{N}_+$ then by (3.2.9) we have $\widetilde{G} = G/\widetilde{F}^{u-1} \in k[X,Y]$ and by (3.2.3) we get $J(\widetilde{F}, G) = u\widetilde{F}^{u-1} J(\widetilde{F}, G/\widetilde{F}^{u-1})$ and hence $J(\widetilde{F}, \widetilde{G}) = \theta \widetilde{F}$. □

Lemma (4.10). *Assume that f is w-similar to $J(f,g)$. Also assume that $\deg_w f \neq 0 \leqslant (w_1 + w_2)\deg_w f$. Then we have the following.*

(1) *There exist nonzero w-homogeneous members $\widetilde{F}, \widetilde{G}$ of $k[X,Y]$ such that $J(\widetilde{F}, \widetilde{G}) = \theta \widetilde{F}$ with $f_w^+ = \theta \widetilde{F}^p$ for some $p \in \mathbb{N}_+$.*

(2) *If $\operatorname{lag}_w(f,g) = 0$ then there exist nonzero w-homogeneous members $\widetilde{F}, \widetilde{G}$ of $k[X,Y]$ such that $J(\widetilde{F}, \widetilde{G}) = \theta \widetilde{F}$ with $f_w^+ = \theta \widetilde{F}^p$ for some $p \in \mathbb{N}_+$ and $g_w^+ = \widetilde{G}\widetilde{F}^v$ for some integer $v \geqslant -1$.*

(3) $\operatorname{lag}_w(f,g) \neq 0 \Leftrightarrow f$ *is w-similar to g.*

Proof. Follows from (4.1), (4.3), (4.4), and (4.9). □

Lemma (4.11). *Let $\overline{w} = -\overline{w}_2/\overline{w}_1$ where $\overline{w}_1$ and $\overline{w}_2$ are coprime integers with $\overline{w}_1 > 0$. Let F and G be nonzero $\overline{w}$-homogeneous members of $k[X,Y]$ and let $\deg_{\overline{w}} F = N$ and $\deg_{\overline{w}} G = M$. Assume that F is $\overline{w}$-similar to G and $N \neq 0$ with $G \notin k$. Let $\widehat{G} \in k[X,Y]$ be $\overline{w}$-homogeneous. Assume that $\operatorname{radnum}(F_w^+)$ and $\operatorname{radnum}(G_w^+)$ are coprime integers with $\deg_w F \neq 0$. Then we have the following.*

(1) *If* $J(F,\widehat{G}) \neq 0 < \text{lag}_w(F,\widehat{G})$ *then there exists* $\overline{w}$*-homogeneous* $\overline{G} \in k[X,Y]$ *such that* $J(F,\overline{G}) = J(F,\widehat{G})$ *and* $\text{lag}_w(F,\overline{G}) < \text{lag}_w(F,\widehat{G})$.
(2) *If* $J(F,\widehat{G}) \neq 0$ *then there exists* $\overline{w}$*-homogeneous* $\overline{G} \in k[X,Y]$ *such that* $J(F,\overline{G}) = J(F,\widehat{G})$ *and* $\text{lag}_w(F,\overline{G}) = 0$.
(3) *If* $J(F,\widehat{G}) = \theta F$ *then there exists* $\overline{w}$*-homogeneous* $\overline{G} \in k[X,Y]$ *such that* $J(F,\overline{G}) = \theta F$ *and* $\deg_w \overline{G} = w_1 + w_2$.

Proof. To prove (1), take $C \in \text{radset}(F)$ and $\overline{C} \in \text{radset}(F_w^+)$. Since F is $\overline{w}$-similar to G, we get $F^M = \theta G^N$; since $N \neq 0$ and $G \notin k$, by (3.11) we see that $M \neq 0$. Now clearly $(F_w^+)^M = \theta (G_w^+)^N$; since $\deg_w F \neq 0 \neq M$, again by (3.11) we see that $F_w^+ = \theta \overline{C}^{\overline{p}}$ and $G_w^+ = \theta \overline{C}^{\overline{q}}$ with $\text{radnum}(F_w^+) = \overline{p} \in \mathbb{N}_+$ and $\text{radnum}(G_w^+) = \overline{q} \in \mathbb{N}_+$. By taking $B = G$ in (3.12) we see that $C_w^+ = \theta \overline{C}$. Since $\text{lag}_w(F,\widehat{G}) > 0$ (if $\widehat{G}_w^+ \in k$ then trivially and if $\widehat{G}_w^+ \notin k$ then) by (4.1) and (3.11) we get $\widehat{G}_w^+ = \theta \overline{C}^q$ with $q \in \mathbb{N}$ and hence $\widehat{G}_w^+ = \kappa (C_w^+)^q$ with $\kappa \in k^\times$. Upon letting $\overline{G} = \widehat{G} - \kappa C^q$, by (3.2.3) we get $\overline{G} \in k[X,Y]$ with $J(F,\overline{G}) = J(F,\widehat{G})$. It follows that $\overline{G}$ is $\overline{w}$-homogeneous with $\deg_w(F\overline{G}) < \deg_w(F\widehat{G})$ and hence we get $\text{lag}_w(F,\overline{G}) < \text{lag}_w(F,\widehat{G})$.

(2) follows from (1) by induction on $\text{lag}_w(F,\widehat{G})$. (3) follows from (2). □

Lemma (4.12). *Let* $F = f_w^+$ *and* $G = g_w^+$ *with* $\deg_w F = N$ *and* $\deg_w G = M$. *Assume that* $N \neq 0$ *and* $\text{radnum}(F) = 1$. *Then we have the following.*

(1) *If* $J(f,g) \neq 0 < \text{lag}_w(f,g)$ *then there exists* $\overline{g} \in k[X,Y]$ *for which we have* $J(f,\overline{g}) = J(f,g)$ *and* $\text{lag}_w(f,\overline{g}) < \text{lag}_w(f,g)$.
(2) *If* $J(f,g) \neq 0$ *then there exists* $\overline{g} \in k[X,Y]$ *such that* $J(f,\overline{g}) = J(f,g)$ *and* $\text{lag}_w(f,\overline{g}) = 0$.
(3) *If* $J(f,g) = \theta$ *and* $w = -1$ *then* $N = 1$.
(4) *If* $J(f,g) = \theta$ *and* $w < 0$ *then* F *is a nonconstant irreducible member of* $k[X,Y]$.

Proof. To prove (1) note that, since $J(f,g) \neq 0 < \text{lag}_w(f,g)$, by (4.1) we have $F^M = \theta G^N$. Since $\text{radnum}(F) = 1$, we get $F \in \text{radset}(F)$. Therefore, obviously if $G \in k$ and by (3.11) if $G \notin k$, we see that $G = \kappa F^q$ with $\kappa \in k^\times$ and $q \in \mathbb{N}$. Upon letting $\overline{g} = g - \kappa f^q$, by (3.2.3) we get $\overline{g} \in k[X,Y]$ with $J(f,\overline{g}) = J(f,g)$. Clearly $\deg_w(f\overline{g}) < \deg_w(fg)$, and hence $\text{lag}_w(f,\overline{g}) < \text{lag}_w(f,g)$. This proves (1). (2) follows from (1) by induction on $\text{lag}_w(f,g)$. (3) follows from (2). (4) follows from (2) and (3.5). □

Note. By using (4.12)(2) and the full force of (3.5) we get the following sharper form (4.13) of (4.12). Actually this is not so significant because, in (4.16)(5) and (4.16)(6) below, we shall prove much sharper versions of (3.5).

Lemma (4.13). *Assume that* $\deg_w f \neq 0$ *and* $J(f,g) = \theta$. *Let* $F = f_w^+$ *and assume that* $\text{radnum}(F) = 1$. *Then there exists* $\overline{g} \in k[X,Y]$ *with* $J(f,\overline{g}) = J(f,g)$ *and* $\text{lag}_w(f,\overline{g}) = 0$. *Moreover, upon letting* $G = \overline{g}_w^+$ *for any such* $\overline{g}$*, and assuming* (i) *or* (ii) *of* (3.5)*, the conclusions of* (3.5) *hold.*

Remark–Definition (4.14). We shall now introduce the concepts of the *antecedent* and the *consequent*. As we shall later see, these correspond to the slopes of the *previous* newton line and the *next* newton line respectively. With any $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ we associate the unique pair $(\overline{w}_1, \overline{w}_2)$ of coprime integers with $\overline{w}_1 \geqslant 0$ such that: if $\overline{w} \in \mathbb{Q}$ then $\overline{w} = -\overline{w}_2/\overline{w}_1$ with $\overline{w}_1 > 0$, whereas if $\overline{w} = -\infty$ or ∞ then $(\overline{w}_1, \overline{w}_2) = (0,1)$ or $(0,-1)$ respectively. For pictures see (9.4) to (9.6).

So for a moment let $j^\dagger$ be the largest value of j with (i,j) varying over $\mathrm{Supp}(f_w^+)$, let $i^\dagger$ be the unique value of i with $(i,j^\dagger)\in\mathrm{Supp}(f_w^+)$, and let $S^\dagger$ be the set of all (i,j) in $\mathrm{Supp}(f)$ with $j>j^\dagger$. We define the *degreewise antecedent* $w^\dagger(f)$ of w relative to f by putting

$$w^\dagger(f)=\max_{(i,j)\in S^\dagger}\frac{i^\dagger-i}{j^\dagger-j}$$

with the understanding that if $S^\dagger=\emptyset$ then $w^\dagger(f)=-\infty$. Note that

$$w^\dagger(f)\in\mathbb{Q}\cup\{-\infty\}.$$

The coprime integer pair associated with $w^\dagger(f)$ is denoted by $(w_1^\dagger(f),w_2^\dagger(f))$.

Next for a moment let $j^\ddagger$ be the smallest value of j with (i,j) varying over $\mathrm{Supp}(f_w^+)$, let $i^\ddagger$ be the unique value of i with $(i,j^\ddagger)\in\mathrm{Supp}(f_w^+)$, and let $S^\ddagger$ be the set of all (i,j) in $\mathrm{Supp}(f)$ with $j<j^\ddagger$. We define the *degreewise consequent* $w^\ddagger(f)$ of w relative to f by putting

$$w^\ddagger(f)=\min_{(i,j)\in S^\ddagger}\frac{i^\ddagger-i}{j^\ddagger-j}$$

with the understanding that if $S^\ddagger=\emptyset$ then $w^\ddagger(f)=\infty$. Note that

$$w^\ddagger(f)\in\mathbb{Q}\cup\{\infty\}.$$

The coprime integer pair associated with $w^\ddagger(f)$ is denoted by $(w_1^\ddagger(f),w_2^\ddagger(f))$.

The proofs of the following assertions (1) to (6) are straightforward; details in (7.4.14)(3). Assertions (7) and (8) follow from the fact that if $f\in Y^dk[X,Y]$ and $g\in Y^ek[X,Y]$ where d,e in $\mathbb{N}$ are such that $d+e>0$ then $J(f,g)\in Y^{d+e-1}k[X,Y]$. In view of (4.12)(3), assertion (9) follows from assertion (8). *The definitions of* (4.14) *and assertions* (1) *to* (6) *continue to hold for all* $f\in k[X,X^{-1},Y,Y^{-1}]^\times$ *with any commutative ring* k. In assertions (7) to (9) and in (4.15) to (4.20) we revert to $f\in k[X,Y]^\times$ with characteristic zero field k.

(1) $w^\dagger(f)\in\mathbb{Q}\Leftrightarrow w^\dagger(f)\neq-\infty\Leftrightarrow$ there is some $(i,j)\in\mathrm{Supp}(f)$ such that $j>j^*$ for every $(i^*,j^*)\in\mathrm{Supp}(f_w^+)\Rightarrow f$ is not a $w^\dagger(f)$-monomial and upon letting $\overline{w}=w^\dagger(f)$ we have $\mathrm{Supp}(f_w^+)\cap\mathrm{Supp}(f_{\overline{w}}^+)=\{(i^\dagger,j^\dagger)\}$ where $(i^\dagger,j^\dagger)\in\mathrm{Supp}(f_w^+)$ maximizes $j^\dagger$ and for every $(i,j)\in\mathrm{Supp}(f_{\overline{w}}^+)\setminus\{(i^\dagger,j^\dagger)\}$ we have $\overline{w}=\frac{i^\dagger-i}{j^\dagger-j}$.

(2) $w^\ddagger(f)\in\mathbb{Q}\Leftrightarrow w^\ddagger(f)\neq\infty\Leftrightarrow$ there is some $(i,j)\in\mathrm{Supp}(f)$ such that $j<j^*$ for every $(i^*,j^*)\in\mathrm{Supp}(f_w^+)\Rightarrow f$ is not a $w^\ddagger(f)$-monomial and upon letting $\overline{w}=w^\ddagger(f)$ we have $\mathrm{Supp}(f_w^+)\cap\mathrm{Supp}(f_{\overline{w}}^+)=\{(i^\ddagger,j^\ddagger)\}$ where $(i^\ddagger,j^\ddagger)\in\mathrm{Supp}(f_w^+)$ minimizes $j^\ddagger$ and for every $(i,j)\in\mathrm{Supp}(f_{\overline{w}}^+)\setminus\{(i^\ddagger,j^\ddagger)\}$ we have $\overline{w}=\frac{i^\ddagger-i}{j^\ddagger-j}$.

(3) $w^\dagger(f)\in\mathbb{Q}\Rightarrow w>w^\dagger(f)$ and for all $\widehat{w}\in\mathbb{Q}$ with $w>\widehat{w}>w^\dagger(f)$ we have that f is a $\widehat{w}$-monomial with $\mathrm{Supp}(f_{\widehat{w}}^+)=\{(i^\dagger,j^\dagger)\}$ where $(i^\dagger,j^\dagger)$ is as in (1).

(4) $w^\ddagger(f)\in\mathbb{Q}\Rightarrow w<w^\ddagger(f)$ and for all $\widehat{w}\in\mathbb{Q}$ with $w<\widehat{w}<w^\ddagger(f)$ we have that f is a $\widehat{w}$-monomial with $\mathrm{Supp}(f_{\widehat{w}}^+)=\{(i^\ddagger,j^\ddagger)\}$ where $(i^\ddagger,j^\ddagger)$ is as in (2).

(5) $w^\dagger(f)\in\mathbb{Q}$ and f is not a w-monomial $\Rightarrow w^\dagger(f)^\ddagger(f)=w$.

(6) $w^\ddagger(f)\in\mathbb{Q}$ and f is not a w-monomial $\Rightarrow w^\ddagger(f)^\dagger(f)=w$.

(7) If $J(f,g) \notin Yk[X,Y]$ with $j^{\ddagger} - i^{\ddagger} \geqslant 1 \leqslant j^{*\ddagger} - i^{*\ddagger}$ where $(i^{\ddagger}, j^{\ddagger}) \in \mathrm{Supp}(f_w^+)$ minimizes $j^{\ddagger}$ and $(i^{*\ddagger}, j^{*\ddagger}) \in \mathrm{Supp}(g_w^+)$ minimizes $j^{*\ddagger}$, then either $1 > w^{\ddagger}(f) \in \mathbb{Q}$ or $1 > w^{\ddagger}(g) \in \mathbb{Q}$.
(8) If $J(f,g) \notin Yk[X,Y]$ with $j^{\ddagger} - i^{\ddagger} \geqslant 2$ where $(i^{\ddagger}, j^{\ddagger}) \in \mathrm{Supp}(f_w^+)$ minimizes $j^{\ddagger}$, then $1 > w^{\ddagger}(f) \in \mathbb{Q}$.
(9) If $J(f,g) = \theta$ and $w = -1$ with $j^{\ddagger} - i^{\ddagger} \geqslant 1$ and $i^{\ddagger} + j^{\ddagger} \geqslant 2$ where $(i^{\ddagger}, j^{\ddagger}) \in \mathrm{Supp}(f_w^+)$ minimizes $j^{\ddagger}$, then $1 > w^{\ddagger}(f) \in \mathbb{Q}$.

Remark–Definition (4.15). For any elements $\alpha, \beta, \gamma, \alpha^*, \beta^*, \gamma^*$ in k with $\alpha\beta^* - \alpha^*\beta \neq 0$, it is clear that

$$\begin{cases} H = H(X,Y) = \sum \widehat{H}_{ij}(\alpha X + \beta Y + \gamma)^i(\alpha^* X + \beta^* Y + \gamma^*)^j \\ \quad \mapsto \widehat{H} = \widehat{H}(X,Y) = \sum \widehat{H}_{ij} X^i Y^j \end{cases}$$

(where $\widehat{H}_{ij} \in k$ and the two summations are over a finite subset of $\mathbb{N}^2$) gives an automorphism of $k[X,Y]$. We call it a *k-linear automorphism*. Moreover we call

$$\begin{cases} H = H(X,Y) = \sum \widetilde{H}_{ij}(\alpha X + \beta Y)^i(\alpha^* X + \beta^* Y)^j \\ \quad \mapsto \widetilde{H} = \widetilde{H}(X,Y) = \sum \widetilde{H}_{ij} X^i Y^j \end{cases}$$

a *homogeneous k-linear automorphism* of $k[X,Y]$.

Concerning the above two automorphisms we observe the following fact (1). Then we proceed to observe some related facts (2) to (6).

(1) The set of k-linear automorphisms of $k[X,Y]$ is a group, and the set of all homogeneous k-linear automorphisms of $k[X,Y]$ is a subgroup of this group. For all $H = H(X,Y) \in k[X,Y] \setminus k$ we have $\widetilde{H}(X,Y) = \widehat{H}(X + \gamma, Y + \gamma^*)$, and if $H \notin k$ then we have $\deg(\widehat{H}) = \deg(\widetilde{H}) = \deg(H)$ and $\widehat{H}^+ = (\widehat{H^+})^+ = \widehat{H^+} = \widetilde{H}^+$.
(2) Note that the condition $\alpha\beta^* - \alpha^*\beta \neq 0$ is equivalent to the condition saying that $J(\alpha X + \beta Y + \gamma, \alpha^* X + \beta^* Y + \gamma^*) = \theta$.
(3) Also note that if $J(f,g) \neq 0$ then clearly $\deg(f) \geqslant 1$ and $\deg(g) \geqslant 1$.
(4) Finally note that if $J(f,g) = \theta$ and either $\deg(f) \leqslant 1$ or $\deg(g) \leqslant 1$, say if $\deg(f) = 1$, then writing $f = \alpha X + \beta Y + \gamma$ with α, β, γ in k and choosing suitable $\alpha^*, \beta^* \gamma^*$ in k with $\alpha\beta^* - \alpha^*\beta \neq 0$, we get $\widehat{f} = X$ and $g = \theta Y + \theta(X)$ with $\theta(X) \in k[X]$, and hence (f,g) is an automorphic pair. Thus we have (5) and (6):
(5) If $J(f,g) = \theta$ then $\deg(f) \geqslant 1$ and $\deg(g) \geqslant 1$.
(6) If $J(f,g) = \theta$ and either $\deg(f) \leqslant 1$ or $\deg(g) \leqslant 1$ then (f,g) is an automorphic pair.

Important Remark (4.16). To sharpen the above case (6) of the Jacobian Conjecture, we now prove *important fact* (1) stated below and from it deduce *important facts* (2) to (7) stated below; out of this, (5) and (6) are sharpenings of (3.5) (no conditions on the weight!!), while (7) is a sharpening of (4.14)(9).

So for a moment assume that $J(f,g) = \theta$ and $f = Y\overline{f}$ with $\overline{f} \in k[X,Y]^{\times}$. Now $g = g(X,Y) \in k[X,Y] \setminus k$, and subtracting $g(0,0)$ from $g(X,Y)$ we may assume that $g(0,0) = 0$. Then, as in the paragraph preceding (1) of (4.14), we see that $\overline{f} \notin Yk[X,Y]$ and $g \notin Yk[X,Y]$. Hence, taking $w_1 = 1$ and $w_2 \ll 0$, i.e., w_2 to be a very negative integer, say $w_2 < -\max\{\deg(f), \deg(g)\}$, we get $f_w^+ = \theta X^i Y$ and $g_w^+ = \theta X^{i^*}$ with $i \in \mathbb{N}$ and $i^* \in \mathbb{N}_+$. Now f is not w-similar to g and hence by (4.1) see that $\deg_w(fg) = \deg_w(XY)$ and therefore we must have $i^* = 1 = i + 1$, i.e., we get (i) stated below, and from it we easily deduce (ii) stated below. Moreover, from (i) and (ii) we deduce (iii) and (iv) stated below.

(i) $f_w^+ = \theta Y$ and $g_w^+ = \theta X$.
(ii) $f = \theta Y + Y^2 \widehat{f}$ and $g = \theta X + Y\widehat{g}$ with $\widehat{f}$ and $\widehat{g}$ in $k[X, Y]$.
(iii) If $w^\dagger(f) = -\infty$ then $f = \theta Y$. If $w^\dagger(f) \neq -\infty$ then $0 \leqslant w^\dagger(f) \in \mathbb{Q}$.
(iv) If $w^\dagger(g) = -\infty$ then $g = \theta X$. If $w^\dagger(g) \neq -\infty$ then $-1 \leqslant w^\dagger(g) \in \mathbb{Q}$. If $w^\dagger(g) < 0$ then $g = \theta X + P(Y)$ with $P(Y) \in k[Y]$.

In the rest of the proof we shall use properties (i) to (iv) tacitly.

We shall divide the remaining argument into two cases, the FIRST case having two subcases and the SECOND having four subcases. For the SECOND case let $w' = -1 = -w_2'/w_1'$ with $(w_1', w_2') = (1, 1)$.

Firstly suppose that either $w^\dagger(f) = -\infty = w^\dagger(g)$ or $w^\dagger(f) = -\infty \neq w^\dagger(g) < 0$. Then clearly $f = \theta Y$ and $g = \theta X + P(Y)$ with $P(Y) \in k[Y]$.

Secondly suppose that

$$\begin{cases} \text{either } w^\dagger(f) = -\infty \neq w^\dagger(g) \geqslant 0 \text{ and let } \overline{w} = w^\dagger(g), \\ \quad \text{or } w^\dagger(f) \neq -\infty = w^\dagger(g) \text{ and let } \overline{w} = w^\dagger(f), \\ \quad \text{or } w^\dagger(f) \neq -\infty \neq w^\dagger(g) > w^\dagger(f) \text{ and let } \overline{w} = w^\dagger(g), \\ \quad \text{or } w^\dagger(f) \neq -\infty \neq w^\dagger(g) \leqslant w^\dagger(f) \text{ and let } \overline{w} = w^\dagger(f). \end{cases}$$

Then in all the four subcases $(0, 1) \in \operatorname{Supp}(f_{\overline{w}}^+)$ and hence $\deg_{\overline{w}} f < 0$ with $N > 0$ where $N = \deg_{w'} f_{\overline{w}}^+$; likewise $(1, 0) \in \operatorname{Supp}(g_{\overline{w}}^+)$ and hence $\deg_{\overline{w}} g > 0$ with $M > 0$ where $M = \deg_{w'} g_{\overline{w}}^+$. Now $\deg_{w'}(f_{\overline{w}}^+ g_{\overline{w}}^+) > 2 = \deg_{w'}(XY)$ and hence by (4.1) we see that $f_{\overline{w}}^+$ is (w')-similar to $g_{\overline{w}}^+$, i.e.,

$$\left(\left(f_{\overline{w}}^+\right)_{w'}^+\right)^M = \theta \left(\left(g_{\overline{w}}^+\right)_{w'}^+\right)^N$$

and therefore, because N and M are positive integers, we get

$$\left(\deg_{\overline{w}}\left(f_{\overline{w}}^+\right)_{w'}^+\right)\left(\deg_{\overline{w}}\left(g_{\overline{w}}^+\right)_{w'}^+\right) \geqslant 0.$$

But clearly

$$\deg_{\overline{w}} f = \deg_{\overline{w}}\left(f_{\overline{w}}^+\right)_{w'}^+ \quad \text{and} \quad \deg_{\overline{w}} g = \deg_{\overline{w}}\left(g_{\overline{w}}^+\right)_{w'}^+$$

and hence

$$(\deg_{\overline{w}} f)(\deg_{\overline{w}} g) \geqslant 0.$$

This is a contradiction because $\deg_{\overline{w}} f < 0$ and $\deg_{\overline{w}} g > 0$.

Thus we have proved the following fact (1) and it clearly implies the following facts (2) to (4). Now we shall prove the following fact (5), and by symmetry it will also establish the following fact (6). So for a moment assume that $J(f, g) = \theta$ and f is w-homogeneous. We shall show that then either $\deg(f) = 1$ or $f = \theta X + P(Y)$ with $P(Y) \in k[Y]$ or $f = \theta Y + Q(X)$ with $Q(X) \in k[X]$; by (2) this will imply that (f, g) is an automorphic pair.

Upon replacing f and g by $f - f(0,0)$ and $g - g(0,0)$ respectively, without loss of generality we may assume that $f(0,0) = 0$ and $g(0,0) = 0$. We can uniquely write $g = \sum g_\lambda$ where $g_\lambda \in k[X,Y]^\times$ is w-homogeneous of w-degree λ and the summation is over a nonempty finite set of integers λ. Clearly $J(f,g) = \sum J(f, g_\lambda)$ and hence by (3.2.6) and (4.1) we get $J(f, g_l) = \theta$ where l is the unique integer with $\deg_w(f) + l = w_1 + w_2$. It follows that for every $(i,j) \in \mathrm{Supp}(fg_l)$ we must have: $(\bullet)$ $iw_1 + jw_2 = w_1 + w_2$. If $fg_l \in (Xk[X,Y]) \cup (Yk[X,Y])$ then we are done by (2) and (3). So assume that $fg_l \notin (Xk[X,Y]) \cup (Yk[X,Y])$. Then, since $(0,0) \notin \mathrm{Supp}(fg_l)$, we can find $(n,0), (0,m)$ in $\mathrm{Supp}(fg_l)$ with n, m in $\mathbb{N}_+$. By $(\bullet)$ we get $(n-1)w_1 = w_2$ and $(m-1)w_2 = w_1$. Consequently, since $w_1 \neq 0$, we get $m - 1 \neq 0 \neq w_2$; hence, because $\mathrm{GCD}(w_1, w_2) = 1$, we must have $w_1 = w_2 = 1$. Therefore $\deg(f) = 1$ and $\deg(g_l) = 1$.

Finally to prove the following fact (7), for a moment assume that $J(f,g) = \theta$ with $w = -1$ and $\deg(f) \geqslant 2$ with $f(0,0) = 0$ and $i^\ddagger < j^\ddagger$ where $(i^\ddagger, j^\ddagger) \in \mathrm{Supp}(f_w^+)$ minimizes $j^\ddagger$. We want to show that for $\overline{w} = w^\ddagger(f)$ we have $\deg_{\overline{w}} f > 0 < \overline{w}_1 + \overline{w}_2$ and $-1 < \overline{w} < i^\ddagger/j^\ddagger$.

First note that by (4.14)(4) we get $-1 < \overline{w}$ and by (4.14)(4) and (4.14)(9) we get $-1 < \overline{w} < 1$; consequently by (3.13) we get $0 < \overline{w}_1 + \overline{w}_2$. Next note that by (2) we have $f \notin Yk[X,Y]$ and hence, because $f(0,0) = 0$, we can find $(n, 0) \in \mathrm{Supp}(f)$ with $n \in \mathbb{N}_+$. Now by the definition of $w^\ddagger(f)$ we get $(i^\ddagger - n)/j^\ddagger \geqslant \overline{w} \in \mathbb{Q}$ and hence $\overline{w} < i^\ddagger/j^\ddagger$ and therefore $(i^\ddagger - j^\ddagger\overline{w})\overline{w}_1 > 0$. Since $(i^\ddagger, j^\ddagger) \in \mathrm{Supp}(f_w^+) \subset \mathrm{Supp}(f)$, we get $\deg_{\overline{w}} f \geqslant i^\ddagger\overline{w}_1 + j^\ddagger\overline{w}_2 = (i^\ddagger - j^\dagger\overline{w})\overline{w}_1$. Therefore $\deg_{\overline{w}} f > 0$.

(1) If $J(f,g) = \theta$ and $f = Y\overline{f}$ with $\overline{f} \in k[X,Y]^\times$ then we have $f = \theta Y$ and $g = \theta X + P(Y)$ with $P(Y) \in k[Y]$, and hence in particular (f,g) is an automorphic pair. For pictures see (9.4).
(2) If $J(f,g) = \theta$ and for some automorphic pair (x,y) we have $f = y\overline{f}$ with $\overline{f} \in k[X,Y]^\times$ then $f = \theta y$ and $g = \theta x + P(y)$ with $P(Y) \in k[Y]$, and hence in particular (f,g) is an automorphic pair.
(3) If $J(f,g) = \theta$ and for some automorphic pair (x,y) we have $g = x\overline{g}$ with $\overline{g} \in k[X,Y]^\times$ then $g = \theta x$ and $f = \theta y + Q(x)$ with $Q(X) \in k[X]$, and hence in particular (f,g) is an automorphic pair.
(4) If $J(f,g) = \theta$ and for some automorphic pair (x,y) we have $fg \in yk[X,Y]$ then (f,g) is an automorphic pair.
(5) If $J(f,g) = \theta$ and f is w-homogeneous then either $\deg(f) = 1$ or $f = \theta X + P(Y)$ with $P(Y) \in k[Y]$ or $f = \theta Y + Q(X)$ with $Q(X) \in k[X]$, and hence in particular (f,g) is an automorphic pair by (2).
(6) If $J(f,g) = \theta$ and g is w-homogeneous then either $\deg(g) = 1$ or $g = \theta X + P(Y)$ with $P(Y) \in k[Y]$ or $g = \theta Y + Q(X)$ with $Q(X) \in k[X]$, and hence in particular (f,g) is an automorphic pair by (3).
(7) If $J(f,g) = \theta$ with $w = -1$ and if we have $\deg(f) \geqslant 2$ with $f(0,0) = 0$ and $i^\ddagger < j^\ddagger$ where $(i^\ddagger, j^\ddagger) \in \mathrm{Supp}(f_w^+)$ minimizes $j^\ddagger$, then for $\overline{w} = w^\ddagger(f)$ we have $\deg_{\overline{w}} f > 0 < \overline{w}_1 + \overline{w}_2$ and $-1 < \overline{w} < i^\ddagger/j^\ddagger$.

Important Lemma (4.17). *Assume that* $J(f,g) = \theta$ *and* $\deg(f) \geqslant 2$ *with* $\deg(g) \geqslant 2$. *Let* $(i^\ddagger, j^\ddagger) \in \mathrm{Supp}(f_w^+)$ *minimize* $j^\ddagger$. *Then we have the following.*

(1) *Upon letting* $\deg_w f = N$ *and* $\deg_w g = M$ *we have* $\min(N, M) \geqslant \max(w_1, w_2)$ *and hence* $N > 0$ *and* $M > 0$. *Moreover* f *is* w*-similar to* g, *i.e.*, $(f_w^+)^M = \kappa(g_w^+)^N$ *for some* $\kappa \in k^\times$. *For pictures see* (9.5) *to* (9.7).

(2) *We have* $w < w^\ddagger(f) = w^\ddagger(g)$ *and we have*: $w^\ddagger(f) \in \mathbb{Q}$ *iff* $j^\ddagger \neq 0$. *Moreover, if* $\overline{w} \in \mathbb{Q}$ *with* $w < \overline{w} \leqslant w^\ddagger(f)$ *then* f *is* $\overline{w}$*-similar to* g *and, upon letting* $\deg_{\overline{w}} f = \overline{N}$ *with* $\deg_{\overline{w}} g = \overline{M}$, *we also have that*: $\overline{M}, \overline{N}$ *are in* $\mathbb{N}_+$ *with* $\overline{M}/\overline{N} = M/N$ *and for all* $\widehat{M}, \widehat{N}$ *in* $\mathbb{N}_+$ *with* $\widehat{M}/\widehat{N} = M/N$ *we obtain* $(f_w^+)^{\widehat{M}} = \widehat{\kappa}(g_w^+)^{\widehat{N}}$ *with* $(f_{\overline{w}}^+)^{\widehat{M}} = \widehat{\kappa}(g_{\overline{w}}^+)^{\widehat{N}}$ *for some* $\widehat{\kappa} \in k^\times$. *For pictures see* (9.5) *to* (9.7).

(3) *If* $w = -1$ *with* $f(0,0) = 0$ *and* $i^\ddagger < j^\ddagger$, *then* $-1 < w^\ddagger(f) = w^\ddagger(g) < 1$ *and upon letting* $\overline{w} = w^\ddagger(f)$ *we have that* f *is* w*-similar, as well as* $\overline{w}$*-similar, to* g, *and* $\deg_{\overline{w}} f > 0 < \overline{w}_1 + \overline{w}_2$ *with* $\overline{w} < i^\ddagger/j^\ddagger$.

Proof. If $w_1 = w_2$ then $w = -1$ and hence $\deg_w(f) = \deg(f) \geqslant 2$ and $\deg_w(g) = \deg(g) \geqslant 2$ with $\deg_w(XY) = \deg(XY) = 2$, and therefore we get $\deg_w(fg) > \deg_w(XY)$ with $N > 0$ and $M > 0$. Since $\deg(f) \geqslant 2$ with $\deg(g) \geqslant 2$, by (4.16)(1) we see that $fg \notin Yk[X,Y]$ and hence if $w_1 > w_2$ then $\deg_w(f) \geqslant w_1$ with $\deg_w(g) \geqslant w_1$ and hence $\deg_w(fg) > \deg_w(XY)$ with $N > 0$ and $M > 0$. Since $\deg(f) \geqslant 2$ with $\deg(g) \geqslant 2$, by (4.16)(1) we see that $fg \notin Xk[X,Y]$ and hence if $w_1 < w_2$ then $\deg_w(f) \geqslant w_2$ with $\deg_w(g) \geqslant w_2$ and hence $\deg_w(fg) > \deg_w(XY)$ with $N > 0$ and $M > 0$. Now (1) follows from (4.1).

In view of (3.11), (4.14)(2), (4.14)(4), and (4.16)(1), (2) follows from (1); details in (8.4.18). For pictures see (9.5).

In view of (4.16)(7), (3) follows from (1) and (2). □

Lemma (4.18). *Assume that* $J(f,g) = \theta$ *with* $w = -1$ *and assume that* $\deg(f) \geqslant 2$ *with* $\deg(g) \geqslant 2$. *Then we can find a* k*-linear automorphism* $H \mapsto \widehat{H}$ *of* $k[X,Y]$ *such that* $\overline{f} = \widehat{f} - \widehat{f}(0,0)$ *and* $\overline{g} = \widehat{g}$ *are* w*-monomials for which we have* $\overline{f}(0,0) = 0$ *and* $-1 < w^\ddagger(\overline{f}) < 1$ *with* $-1 < w^\ddagger(\overline{g}) < 1$, *and such that for* $\overline{w} = w^\ddagger(\overline{f})$ *we have* $\deg_{\overline{w}} \overline{f} > 0 < \overline{w}_1 + \overline{w}_2$ *and* $(\overline{f}_{\overline{w}}^+)^+ = \overline{f}^+$ *with* $(\overline{g}_{\overline{w}}^+)^+ = \overline{g}^+$ *and*: *if* $\overline{w} = 0$ *then* $\overline{f}_{\overline{w}}^+$ *is not of the form* $\theta X^i(Y+\gamma)^j$ *for any* $i \in \mathbb{N}$, $\gamma \in k^\times$, *and* $j \in \mathbb{N}_+$.

[Note that we automatically have $\deg(\overline{f}) = \deg(f)$ with $\deg(\overline{g}) = \deg(g)$, and $\operatorname{radnum}(\overline{f}^+) = \operatorname{radnum}(f^+)$ with $\operatorname{radnum}(\overline{g}^+) = \operatorname{radnum}(g^+)$. Likewise we have $J(\overline{f}, \overline{g}) = \theta J(f,g) = \theta$, and by (4.17)(3) we see that $\overline{f}$ is w-similar, as well as $\overline{w}$-similar, to $\overline{g}$].

Proof. By (4.7)(i)(1) we get a homogeneous k-linear automorphism $H \mapsto \widetilde{H}$ of $k[X,Y]$ such that $\widetilde{f}_w^+ = \theta X^i Y^j$ and $\widetilde{g}_w^+ = \theta X^{i^*} Y^{j^*}$ where i, j, i^*, j^* in $\mathbb{N}$ with $i - j \neq 0 \neq i^* - j^*$ are such that $\deg(\widetilde{f}) = \deg(f) = i + j \geqslant 2$ and $\deg(\widetilde{g}) = \deg(g) = i^* + j^* \geqslant 2$. Clearly $J(\widetilde{f}, \widetilde{g}) = \theta J(f,g) = \theta$ and hence by (4.1) we see that $\widetilde{f}$ is w-similar to $\widetilde{g}$. Consequently, permuting (X,Y) if necessary, we can arrange matters so that: (•) $(i,j) = (pi', pj')$ and $(i^*, j^*) = (qi', qj')$ where p, q in $\mathbb{N}_+$ and $i' < j'$ in $\mathbb{N}$ are such that $\mathrm{GCD}(i', j') = 1$.

If there is no $\lambda \in k^\times$ such that the terms of X-degree i in $\widetilde{f}$ are of the form $\theta X^i(Y+\lambda)^j$ then let the k-linear automorphism $H \mapsto \widehat{H}$ be the same as the automorphism $H \mapsto \widetilde{H}$; otherwise let $H \mapsto \widehat{H}$ be the k-linear automorphism of $k[X,Y]$ given by $\widehat{H}(X,Y) = \widetilde{H}(X, Y - \lambda)$. Then in both the cases, upon letting $\overline{f} = \widehat{f} - \widehat{f}(0,0)$ and $\overline{g} = \widehat{g}$, we see that $\overline{f}_w^+ = \theta X^i Y^j$ and $\overline{g}_w^+ = \theta X^{i^*} Y^{j^*}$ with (•), but now the terms of X-degree i in $\overline{f}$ are not of the form $\theta X^i(Y+\gamma)^j$ for any $\gamma \in k^\times$. Also clearly $\overline{f}(0,0) = 0$,

Now in view of (4.8)(1) we are done by (4.17)(3). □

Lemma (4.19). *Assume that* $J(f,g)=\theta$. *Also assume that* $\mathrm{radnum}(f^+)$ *and* $\mathrm{radnum}(g^+)$ *are coprime integers. Then either* $\deg(f)\leqslant 1$ *or* $\deg(g)\leqslant 1$ *and hence, by* (4.15)(6), (f,g) *is an automorphic pair.*

Proof. Suppose if possible that $\deg(f)\geqslant 2$ and $\deg(g)\geqslant 2$. Then upon taking $w=-1$, in view of (4.18) (after replacing f,g by $\overline{f},\overline{g}$) we may assume that for $\overline{w}=w^{\ddagger}(f)$ we have

$$-1<\overline{w}<1 \quad \text{with } \deg_{\overline{w}} f>0<\overline{w}_1+\overline{w}_2$$

and

$$\left(f^+_{\overline{w}}\right)^+=f^+ \quad \text{with } \left(g^+_{\overline{w}}\right)^+=g^+$$

and

$$\begin{cases} \text{if } \overline{w}=0 \text{ then } f^+_{\overline{w}} \text{ is not of the form } \theta X^i(Y+\gamma)^j \\ \quad \text{for some } i\in\mathbb{N},\ \gamma\in k^\times, \text{ and } j\in\mathbb{N}_+. \end{cases}$$

Note that still we have $J(f,g)=\theta$. Also the integers $\mathrm{radnum}(f^+)$ and $\mathrm{radnum}(g^+)$ have not changed and hence they are still coprime. Moreover f is w-similar, as well as $\overline{w}$-similar, to g.

Taking the above $\overline{w}$ for w in (4.10) we find nonzero $\overline{w}$-homogeneous members $F,\widehat{G}$ of $k[X,Y]$ such that $J(F,\widehat{G})=\theta F$ and $f^+_{\overline{w}}=\theta F^p$ for some $p\in\mathbb{N}_+$. Let $G=g^+_{\overline{w}}$. Then F,G are nonzero $\overline{w}$-homogeneous members of $k[X,Y]$ with $G\notin k$ and $\deg_w F\neq 0\neq\deg_{\overline{w}} F$ such that F is $\overline{w}$-similar to G and $\mathrm{radnum}(F^+)$ and $\mathrm{radnum}(G^+)$ are coprime integers. Consequently by (4.11)(3) we can find a nonzero $\overline{w}$-homogeneous member $\overline{G}$ of $k[X,Y]$ such that $J(F,\overline{G})=\theta F$ and $\deg_w\overline{G}=w_1+w_2$. By (4.14)(2) we know that F is not a $\overline{w}$-monomial. Therefore by taking $(\overline{w},\overline{G})$ for (w,G) in (4.8)(2) we get a contradiction. □

Important Note (4.20). The three important facts (4.16)(4), (4.16)(5), and (4.17)(2) correspond to Lemmas (3.1)(4), (3.1)(5), and (3.2)(2) of the 1971 Lecture Notes of my Purdue Lectures which were reproduced in [Ab4] with the same numbers. Special thanks are due to Sathaye for patiently reexplaining their mystery to me. The lesson learnt is that how things simplify by first proving (4.16.1) and then using it as much as possible.

5. One point at infinity

Let $f=f(X,Y)\in k[X,Y]$ with field k. Recall that f has *one point at infinity* means $f^+=\theta(\alpha X+\beta Y)^N$ with $(\alpha,\beta)\in k^2\setminus\{(0,0)\}$ and $N\in\mathbb{N}_+$. If moreover f is *analytically irreducible* at its unique point at infinity then we say that f has *one place at infinity*; in case $\alpha\neq 0$, upon letting $F(X,Y,Z)$ be the unique homogeneous polynomial of degree N with $F(X,Y,1)=f(X,Y)$, the analytic irreducibility says that $F(X,1,Z)$ is irreducible in the power series ring $k[[X,Z]]$; similarly for $\beta\neq 0$. After recording the following well-known criterion (5.1), whose proof can be found in [Ab3,Ab6], we shall prove the following equivalences (5.5). Before that we shall supplement (4.7) by (5.2) to (5.4).

To introduce some useful terminology, by a *k-special X-type automorphism* of $k[X,Y]$ we mean a k-automorphism of the form $(X,Y)\mapsto(X-\gamma Y^e,Y)$ with $\gamma\in k$ and $e\in\mathbb{N}$; if $\gamma\neq 0$ then instead of X-type we may say (X,e)-type. Similarly, by a *k-special Y-type automorphism* of $k[X,Y]$ we mean a k-automorphism of the form $(X,Y) \mapsto (X, Y - \gamma X^e)$ with $\gamma \in k$ and $e \in \mathbb{N}$; again if $\gamma \neq 0$ then instead of Y-type we may say (Y,e)-type. By a *k-elementary X-type automorphism* of $k[X,Y]$ we mean a k-automorphism of the form $(X,Y) \mapsto (X - P(Y), Y)$ with $P(Y) \in k[Y]$, and by a *k-elementary Y-type automorphism* of $k[X,Y]$ we mean a k-automorphism of the form $(X,Y) \mapsto (X, Y - P(X))$ with $P(X) \in k[X]$. As a combination, by a *k-special automorphism* we mean a k-special X-type or Y-type automorphism, and similarly by a *k-elementary automorphism* we mean a k-elementary X-type or Y-type automorphism. By a *tame automorphism* of $k[X,Y]$ we mean a composition of k-linear and k-special Y-type automorphisms; clearly in this we may replace k-special Y-type by k-special X-type or k-elementary Y-type or k-elementary X-type.

By the *k-flip automorphism* of $k[X,Y]$ we mean the k-linear automorphism given by $(X,Y) \mapsto (Y,X)$.

In 1942 Jung (see pp. 250, 407, 412 of [Ab2]) showed that if k is of characteristic 0 then every k-automorphism of $k[X,Y]$ is tame, and in 1953 van der Kulk (Nieuw Archief, vol. (3)1, pp. 33–41) removed the characteristic condition. In (5.6) we shall give a new simple proof of Jung's Theorem using (5.4) and indicate how a modification might also prove Kulk's Theorem. Recall that integer M divides integer N means $N\mathbb{Z} \subset M\mathbb{Z}$. Let us say *$M$ properly divides N* to mean that $N\mathbb{Z} \subset tM\mathbb{Z}$ for some integer $t \geqslant 2$. In a similar manner let us say *M very properly divides N* to mean that $N\mathbb{Z} \subset tM\mathbb{Z}$ for some integer $t \geqslant 3$. As more notation, f is *Y-submonic*, or submonic in Y, means $f = \theta Y^N + \sum_{j<N} a_{ij} X^i Y^j$ with $a_{ij} \in k$ and $\deg_Y f = N$; assuming this, if $f \notin k[Y]$ then clearly there exists a unique negative rational number $\breve{w}(f)$ such that $(0,N) \in \operatorname{Supp}(f^+_{\breve{w}})$ and f is not a $\breve{w}(f)$-monomial; $\operatorname{card}\operatorname{Supp}(f^+_{\breve{w}}) > 1$; we call $\breve{w}(f)$ the *initial degree weight* of f; if $f \in k[Y]$ then we put $\breve{w}(f) = 0$. If (f,g) is a Jacobian pair in $k[X,Y]$ with $\operatorname{ch}(k) = 0$, and either f or g is of the form $\theta Y +$ an element of k, then (f,g) is an automorphic pair by (4.15)(6); we call such (f,g) an *obvious automorphic pair*.

Lemma (5.1). *$f \in A = k[X,Y]$ has one place at infinity iff f is a nonzero nonunit irreducible member of A for which there is exactly one valuation ring V of the quotient field of $\overline{A} = A/fA$ such that V contains* (*the image of*) *k but does not contain $\overline{A}$, and this unique V is* residually rational *over k, i.e., the residue field of V coincides with* (*the image of*) *k.*

Lemma (5.2). *Let (f,g) be a Jacobian pair in $k[X,Y]$ where k is a field of characteristic* 0. *Then we have the following.*

(1) *If* $\deg(f) \geqslant 2$ *with* $\deg(g) \geqslant 2$ *and f has one point at infinity, then there is a homogeneous k-linear automorphism $\tau : k[X,Y] \to k[X,Y]$ together with a k-special automorphism $\sigma : k[X,Y] \to k[X,Y]$ such that* $\deg(\sigma\tau f) < \deg(f)$ *with* $\deg(\sigma\tau g) < \deg(g)$.
(2) *If f is Y-submonic with $\breve{w}(f) \leqslant -1$ and f has one point at infinity in the $\breve{w}(f)$-weighted sense, then there is a k-special (Y,e)-type automorphism $\sigma : k[X,Y] \to k[X,Y]$ with $e = -\breve{w}(f) \in \mathbb{N}_+$ such that σf is Y-submonic and*

$$\begin{cases} (\deg_Y(\sigma f), \deg_Y(\sigma g)) \\ \quad = (\deg_Y(f), \deg_Y(g)) \end{cases}$$

with $\breve{w}(\sigma f) > \breve{w}(f)$ *and* $\deg_X(\sigma f) < \deg_X(f)$.

(3) *If f is Y-submonic with $\breve{w}(f) \leqslant -1$ and f has one point at infinity in the $\breve{w}(f)$-weighted sense, then there is a k-elementary Y-type automorphism $\sigma : k[X,Y] \to k[X,Y]$ such that σf is Y-submonic with*

$$\begin{cases} (\deg_Y(\sigma f), \deg_Y(\sigma g)) \\ \quad = (\deg_Y(f), \deg_Y(g)) \end{cases}$$

and either (i) $\breve{w}(\sigma f) > -1$, *or* (ii) $\breve{w}(\sigma f) \leqslant -1$ *but σf does not have one point at infinity in the $\breve{w}(\sigma f)$-weighted sense.*

(4) *If f is Y-submonic with $\breve{w}(f) > -1$ and f has one point at infinity in the $\breve{w}(f)$-weighted sense, then either (f,g) is an obvious automorphic pair (which is certainly so when $\breve{w}(f) = 0$), or for the k-flip automorphism $\sigma : k[X,Y] \to k[X,Y]$ we have that σf is Y-submonic with*

$$\frac{\deg_Y(f)}{\deg_Y(\sigma f)} = \frac{\deg_Y(g)}{\deg_Y(\sigma g)} = -1/\breve{w}(f) = -\breve{w}(\sigma f) \in \mathbb{N}_+ \setminus \{1\}$$

where $f, g, \sigma f, \sigma g$ are all Y-submonic of positive Y-degrees and hence

$$\begin{cases} \mathrm{GCD}(\deg_Y(\sigma f), \deg_Y(\sigma g)) \\ \quad \textit{properly divides } \mathrm{GCD}(\deg_Y(f), \deg_Y(g)). \end{cases}$$

(5) *If f is Y-submonic and f does not have one point at infinity in the $\breve{w}(f)$-weighted sense, then $\breve{w}(f) \neq 0$ and*

$$\begin{cases} \mathrm{GCD}(\mathrm{radnum}((f)^+_{\breve{w}(f)}), \mathrm{radnum}((g)^+_{\breve{w}(f)})) \\ \quad \textit{very properly divides } \mathrm{GCD}(\deg_Y(f), \deg_Y(g)). \end{cases}$$

(6) *If f is Y-submonic and the polynomial f does not have one point at infinity in the $\breve{w}(f)$-weighted sense, then $\breve{w}(f) \geqslant -1$ and there is a k-automorphism $\sigma : k[X,Y] \to k[X,Y]$, which is k-linear or k-special (X,e)-type with $e = -1/\breve{w}(f) \in \mathbb{N}_+$ according as $\breve{w}(f) = -1$ or $\breve{w}(f) > -1$, such that*

$$\begin{cases} ((\sigma f)^+_{\breve{w}(f)}, (\sigma g)^+_{\breve{w}(f)}) = (\theta(X^{i'}Y^{j'})^p, \theta(X^{i'}Y^{j'})^q) \\ \quad \textit{where } i', j', p, q \textit{ are positive integers with } i' \neq j' \textit{ and } \mathrm{GCD}(i', j') = 1 \end{cases}$$

and such that either (i) *σf is Y-submonic and we have $pj' < \deg_Y(\sigma f) < \deg_Y(f)$ and $\breve{w}(\sigma f) < \breve{w}(f)$ with*

$$\begin{cases} (\mathrm{radnum}((f)^+_{\breve{w}(f)}), \mathrm{radnum}((g)^+_{\breve{w}(f)})) \\ \quad = (\mathrm{radnum}((\sigma f)^+_{\breve{w}(\sigma f)}), \mathrm{radnum}((\sigma g)^+_{\breve{w}(\sigma f)})) \end{cases}$$

or (ii) *σf is not Y-submonic and we have $pj' = \deg_Y(\sigma f) < \deg_Y(f)$ with*

$$\begin{cases} (\mathrm{radnum}((f)^+_{\breve{w}(f)}), \mathrm{radnum}((g)^+_{\breve{w}(f)})) \\ \quad = (\mathrm{radnum}((\sigma f)^+), \mathrm{radnum}((\sigma g)^+)) \end{cases}$$

and

$$\begin{cases} (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u,v) \in \mathbb{N}^2\colon\ v \leqslant pj'\} \\ \quad \textit{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u,v) \in \mathbb{N}^2\colon\ v \leqslant qj'\}. \end{cases}$$

(7) *If f is Y-submonic and the polynomial f does not have one point at infinity in the $\breve{w}(f)$-weighted sense, then there is a k-automorphism $\sigma : k[X,Y] \to k[X,Y]$, which consists of a k-linear automorphism followed by a k-elementary X-type automorphism, such that*

$$\begin{cases} (\mathrm{radnum}((f)^{+}_{\breve{w}(f)}), \mathrm{radnum}((g)^{+}_{\breve{w}(f)})) \\ \quad = (\mathrm{radnum}((\sigma f)^{+}), \mathrm{radnum}((\sigma g)^{+})) \end{cases}$$

and

$$\begin{cases} ((\sigma f)^{+}_{\breve{w}(f)}, (\sigma g)^{+}_{\breve{w}(f)}) = (-\!\!\!\!\theta(X^{i'}Y^{j'})^{p}, -\!\!\!\!\theta(X^{i'}Y^{j'})^{q}) \\ \quad \textit{where } i', j', p, q \textit{ are positive integers with } i' \neq j' \textit{ and } \mathrm{GCD}(i', j') = 1 \end{cases}$$

with

$$\begin{cases} (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u,v) \in \mathbb{N}^2\colon\ v \leqslant pj'\} \\ \quad \textit{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u,v) \in \mathbb{N}^2\colon\ v \leqslant qj'\}. \end{cases}$$

(8) *If f is Y-submonic and the polynomial f does not have one point at infinity in the $\breve{w}(f)$-weighted sense, then there is a tame automorphism $\sigma : k[X,Y] \to k[X,Y]$ such that*

$$\begin{cases} (\mathrm{radnum}((f)^{+}_{\breve{w}(f)}), \mathrm{radnum}((g)^{+}_{\breve{w}(f)})) \\ \quad = (\mathrm{radnum}((\sigma f)^{+}), \mathrm{radnum}((\sigma g)^{+})) \end{cases}$$

and

$$\begin{cases} ((\sigma f)^{+}_{\breve{w}(f)}, (\sigma g)^{+}_{\breve{w}(f)}) = (-\!\!\!\!\theta(X^{i'}Y^{j'})^{p}, -\!\!\!\!\theta(X^{i'}Y^{j'})^{q}) \\ \quad \textit{where } i', j', p, q \textit{ are positive integers with } i' \neq j' \textit{ and } \mathrm{GCD}(i', j') = 1 \end{cases}$$

with

$$\begin{cases} (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u,v) \in \mathbb{N}^2\colon\ u \leqslant pi' \textit{ and } v \leqslant pj'\} \\ \quad \textit{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u,v) \in \mathbb{N}^2\colon\ u \leqslant qi' \textit{ and } v \leqslant qj'\}. \end{cases}$$

(9) *If (f,g) is not an automorphic pair and f is Y-submonic, then there is a tame automorphism $\sigma : k[X,Y] \to k[X,Y]$ such that*

$$\begin{cases} \mathrm{GCD}(\mathrm{radnum}((\sigma f)^{+}), \mathrm{radnum}((\sigma g)^{+})) \\ \quad \textit{very properly divides } \mathrm{GCD}(\deg_Y(f), \deg_Y(g)) \end{cases}$$

and

$$\begin{cases} \textit{for some positive integers } i', j', p, q \textit{ with } i' \neq j' \textit{ and } \mathrm{GCD}(i', j') = 1 \\ \quad \textit{we have } (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u,v) \in \mathbb{N}^2\colon\ u \leqslant pi' \textit{ and } v \leqslant pj'\} \\ \quad \textit{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u,v) \in \mathbb{N}^2\colon\ u \leqslant qi' \textit{ and } v \leqslant qj'\}. \end{cases}$$

(10) *If* (f, g) *is not an automorphic pair, then there is a tame automorphism* $\sigma : k[X, Y] \to k[X, Y]$ *such that*

$$\begin{cases} \mathrm{GCD}(\mathrm{radnum}((\sigma f)^+), \mathrm{radnum}((\sigma g)^+)) \\ \quad \textit{very properly divides } \mathrm{GCD}(\deg(f), \deg(g)) \end{cases}$$

and

$$\begin{cases} \textit{for some positive integers } i', j', p, q \textit{ with } i' \neq j' \textit{ and } \mathrm{GCD}(i', j') = 1 \\ \quad \textit{we have } (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u, v) \in \mathbb{N}^2 : u \leqslant pi' \textit{ and } v \leqslant pj'\} \\ \quad \textit{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u, v) \in \mathbb{N}^2 : u \leqslant qi' \textit{ and } v \leqslant qj'\}. \end{cases}$$

(11) *If* $\mathrm{GCD}(\deg(f), \deg(g)) = 1$, *or a prime number, or* 4, *then either* (f, g) *is an automorphic pair, or there is a tame automorphism* $\sigma : k[X, Y] \to k[X, Y]$ *such that* $\mathrm{radnum}((\sigma f)^+)$ *and* $\mathrm{radnum}((\sigma g)^+)$ *are coprime integers.*

Proof. To prove (1) assume that $\deg(f) \geqslant 2$ with $\deg(g) \geqslant 2$ and f has one point at infinity. Let $\widetilde{w} = -1$. By (4.17)(1) fg has one point at infinity, and hence we can find a homogeneous k-linear automorphism $\tau : k[X, Y] \to k[X, Y]$ such that $(\tau f)^+_{\widetilde{w}} = \theta Y^N$ and $(\tau g)^+_{\widetilde{w}} = \theta Y^M$ where $\deg(\tau f) = \deg(f) = N \geqslant 2$ and $\deg(\tau g) = \deg(g) = M \geqslant 2$. Upon letting $w = \widetilde{w}^\ddagger(\tau f)$, by (4.17)(3) we see that $-1 < w = \widetilde{w}^\ddagger(\tau g) < 0$ and τf is w-similar to τg, and by (4.14)(2) we know that τf is not a w-monomial and neither is τg; therefore by (4.7)(3) we get

$$(\tau f)^+_w = \theta Y^{pi'} \big(X + \gamma Y^e\big)^{pj'} \quad \text{and} \quad (\tau g)^+_w = \theta Y^{qi'} \big(X + \gamma Y^e\big)^{qj'}$$

where $i' \neq j' \neq 0 \neq p \neq 0 \neq q$ in $\mathbb{N}$ with $\mathrm{GCD}(i', j') = 1$ and $\gamma \in k^\times$ and $e = -1/w \in \mathbb{N}_+ \setminus \{1\}$. Taking $\sigma : k[X, Y] \to k[X, Y]$ to be the k-special automorphism given by $(X, Y) \mapsto (X - \gamma Y^e, Y)$, we get $(\sigma\tau f)^+_w = \theta Y^{pi'} X^{pj'}$ and $(\sigma\tau g)^+_w = \theta Y^{qi'} X^{qj'}$. Now by (5.3) below we see that $\deg(\sigma\tau f) < \deg(f)$ with $\deg(\sigma\tau g) < \deg(g)$.

To prove (2) assume that f is Y-submonic with $\breve{w}(f) \leqslant -1$ and f has one point at infinity in the $\breve{w}(f)$-weighted sense. Then by (4.7)(i)(1) and (4.7)(i)(2) we get $f^+_{\breve{w}(f)} = \theta (Y + \gamma X^e)^i$ where $\gamma \in k^\times$ and $e = -\breve{w}(f) \in \mathbb{N}_+$ with $i = \deg_Y f \in \mathbb{N}_+$, and it suffices to take $\sigma : k[X, Y] \to k[X, Y]$ to be the k-special (Y, e)-type automorphism given by $(X, Y) \mapsto (X, Y - \gamma X^e)$.

(3) follows from (2) by decreasing induction on the X-degree of f.

In proving (4) to (7), the following observations ($1^\bullet$) to ($3^\bullet$) may be used tacitly.

($1^\bullet$) If f is Y-submonic with $\breve{w}(f) = 0$ then by (4.16) we see that f has one point at infinity in the $\breve{w}(f)$-weighted sense, and (f, g) is an obvious automorphic pair.

($2^\bullet$) If f is Y-submonic with $\breve{w}(f) \neq 0$ then by parts (1) to (3) of (4.7)(i) we see that

$$f^+_{\breve{w}(f)} = \theta x^i y^j \quad \text{and} \quad g^+_{\breve{w}(f)} = \theta x^{i^*} y^{j^*}$$

where i, j, i^*, j^* are in $\mathbb{N}$ with $i - j \neq 0 \neq i + j \neq 0 \neq i^* + j^* \neq 0 \neq i^* - j^*$ and (x, y) is the $\breve{w}(f)$-automorphic pair described below.

(1^*) If $\breve{w}(f) = -1$ then $x = \alpha X + \alpha^* Y$ and $y = \beta X + \beta^* Y$ where $\alpha, \alpha^*, \beta, \beta^*$ in $k^\times$ are such that $\alpha\beta^* - \alpha^*\beta \neq 0$.

(2^*) If $\breve{w}(f) < -1$ then $x = X$ and $y = Y + \gamma X^e$ with $\gamma \in k^\times$ and $e = -\breve{w}(f) \in \mathbb{N}_+ \setminus \{1\}$. [Since f is Y-submonic, we see that if $\breve{w}(f) < -1$ then f has one point at infinity in the $\breve{w}(f)$-weighted sense.]

(3*) If $\breve{w}(f) > -1$ then $x = Y$ and $Y = X + \gamma Y^e$ with $\gamma \in k^\times$ and $e = -1/\breve{w}(f) \in \mathbb{N}_+ \setminus \{1\}$. [In (2*) and (3*) we have $\gamma \neq 0$ because f is not a $\breve{w}(f)$-monomial.]

($3^\bullet$) In ($2^\bullet$), clearly f is $\breve{w}(f)$-similar to g iff $ij^* - ji^* = 0$, and by (4.1) we see that f is not $\breve{w}(f)$-similar to g iff $J(x^i y^j, x^{i^*} y^{j^*}) = \theta$. By a straightforward calculation we get $J(x^i y^j, x^{i^*} y^{j^*}) = \theta(ij^* - ji^*) x^{i+i^*-1} y^{j+j^*-1}$, where we note that (x, y) being an automorphic pair, Jacobians relative to (X, Y) and (x, y) differ by a nonzero constant. Hence if f is not $\breve{w}(f)$-similar to g then $(i, j, i^*, j^*) = (1, 0, 0, 1)$ or $(0, 1, 1, 0)$.

To prove (4) assume that f is Y-submonic with $\breve{w}(f) > -1$ and f has one point at infinity in the $\breve{w}(f)$-weighted sense. If $\breve{w}(f) = 0$ then (f, g) is an obvious automorphic pair by ($1^\bullet$). So now assume that $\breve{w}(f) \neq 0$ and let $\sigma : k[X, Y] \to k[X, Y]$ be the k-flip automorphism. Then by (3*) we see that $i = 0$ with $j = (\deg_Y f)/e \in \mathbb{N}_+$, and σf is Y-submonic with $\deg_Y(\sigma f) = j$ and $-1/\breve{w}(f) = -\breve{w}(\sigma f) = e \in \mathbb{N}_+ \setminus \{1\}$. If $i^* = 0$ then clearly g and σg are Y-submonic with $\deg_Y(g)/e = \deg_Y(\sigma g) = j^* \in \mathbb{N}_+$ and $-1/\breve{w}(f) = -\breve{w}(\sigma f) = e$, and so we are done. If $i^* \neq 0$ then f is not $\breve{w}$-similar to g and hence by ($3^\bullet$) we get $(j, i^*, j^*) = (1, 1, 0)$ showing that (f, g) is an obvious automorphic pair, and so again we are done.

(5) follows from ($1^\bullet$) to ($3^\bullet$).

To prove (6) assume that f is Y-submonic and the polynomial f does not have one point at infinity in the $\breve{w}(f)$-weighted sense. By (2*) we must have $\breve{w}(f) \geqslant -1$. In view of ($3^\bullet$) we see that i, j, i^*, j^* are positive integers with $ij^* = ji^*$. In case of (1*) let σ be the k-linear automorphism $k[X, Y] \to k[X, Y]$ given by $(x, y) \mapsto (X, Y)$, and in case of (3*) let σ be the k-special (X, e)-type automorphism $k[X, Y] \to k[X, Y]$ given by $(X, Y) \mapsto (X - \gamma Y^e, Y)$ with $e = -1/\breve{w}(f) \in \mathbb{N}_+$. Then $\breve{w}(\sigma f) < \breve{w}(f)$ and $\deg_Y(\sigma f) < \deg_Y(f)$ with $\deg_Y(\sigma g) < \deg_Y(g)$ and

$$\begin{cases} ((\sigma f)^+_{\breve{w}(f)}, (\sigma g)^+_{\breve{w}(f)}) = (\theta(X^{i'} Y^{j'})^p, \theta(X^{i'} Y^{j'})^q) \\ \quad \text{where } i', j', p, q \text{ are positive integers with } i' \neq j' \text{ and } \mathrm{GCD}(i', j') = 1. \end{cases}$$

Considering antecedents let $w = \max(\breve{w}(f)^\dagger(\sigma f), \breve{w}(f)^\dagger(\sigma g))$. If $w \neq -\infty$ then taking $(\sigma f, \sigma g)$ for (f, g) in (4.7) we see that σf and σg are Y-submonic and $\breve{w}(\sigma f) = w = \breve{w}(\sigma g)$ with $pj' < \deg_Y(\sigma f) < \deg_Y(f)$ and

$$\begin{cases} (\mathrm{radnum}((f)^+_{\breve{w}(f)}), \mathrm{radnum}((g)^+_{\breve{w}(f)})) \\ \quad = (\mathrm{radnum}((\sigma f)^+_{\breve{w}(\sigma f)}), \mathrm{radnum}((\sigma g)^+_{\breve{w}(\sigma f)})) \end{cases}$$

and hence we are done. If $w = -\infty$ then clearly σf is not Y-submonic and we have $pj' = \deg_Y(\sigma f) < \deg_Y(f)$ with

$$\begin{cases} (\mathrm{radnum}((f)^+_{\breve{w}(f)}), \mathrm{radnum}((g)^+_{\breve{w}(f)})) \\ \quad = (\mathrm{radnum}((\sigma f)^+), \mathrm{radnum}((\sigma g)^+)) \end{cases}$$

and

$$\begin{cases} (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u, v) \in \mathbb{N}^2 \colon v \leqslant pj'\} \\ \quad \text{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u, v) \in \mathbb{N}^2 \colon v \leqslant qj'\} \end{cases}$$

and hence again we are done.

(7) follows from (6) by decreasing induction on the Y-degree of f.

To prove (8) assume that f is Y-submonic and the polynomial f does not have one point at infinity in the $\breve{w}(f)$-weighted sense. Then by (7) we see that there is a k-automorphism $\sigma' : k[X, Y] \to k[X, Y]$, which consists of a k-linear automorphism followed by a k-elementary X-type automorphism, and which has the properties stated in (7) with σ replaced by σ'. Let $\sigma'' : k[X, Y] \to k[X, Y]$ be the k-flip automorphism. Let $f^* = \sigma''\sigma' f$ and $g^* = \sigma''\sigma' g$. Applying (7) with (f^*, g^*) replacing (f, g) we find a k-automorphism $\sigma^* : k[X, Y] \to k[X, Y]$, which consists of a k-linear automorphism followed by a k-elementary X-type automorphism, and which has the properties stated in (7) with (f^*, g^*, σ^*) replacing (f, g, σ). Now it suffices to take $\sigma = \sigma^*\sigma''\sigma'$.

To prove (9) assume that (f, g) is not an automorphic pair and f is Y-submonic. Briefly speaking: by repeatedly applying (3) and (4) we can arrange f to have two points at infinity in a suitable weighted sense, and then by (5) we get a very proper reduction in the GCD of radnums in a weighted sense which, by applying (6) and (8), gets converted into the GCD of ordinary (i.e., weight -1) radnums. Here the application of (3) and (4) may be compared with the game of MONOPOLY in which each time you pass GO you collect a dividend, which cannot be repeated indefinitely lest it may break the bank. In our case (4) plays the role of GO. In greater detail we may proceed thus.

By induction on $\mathrm{GCD}(\deg_Y f, \deg_Y g)$ we define a sequence $(f_1, g_1), \ldots, (f_n, g_n)$ of Jacobian pairs in $k[X, Y]$ where f_i is Y-submonic for $1 \leqslant i \leqslant n \in \mathbb{N}_+$ together with a k-automorphism $\sigma_i : k[X, Y] \to k[X, Y]$ for $1 \leqslant i \leqslant n - 1$ such that: $(f_1, g_1) = (f, g)$; for $1 \leqslant i \leqslant n - 1$ we have $f_{i+1} = \sigma_i f_i$ and $g_{i+1} = \sigma_i g_i$; for $1 \leqslant i \leqslant n-1$ with i odd we have that σ_i is k-elementary Y-type and $\mathrm{GCD}(\deg_Y f_{i+1}, \deg_Y g_{i+1})$ equals $\mathrm{GCD}(\deg_Y f_i, \deg_Y g_i)$; for $1 \leqslant i \leqslant n-1$ with i even we have that σ_i is k-flip and $\mathrm{GCD}(\deg_Y f_{i+1}, \deg_Y g_{i+1})$ properly divides $\mathrm{GCD}(\deg_Y f_i, \deg_Y g_i)$; and f_n does not have one point at infinity in the $\breve{w}(f_n)$-weighted sense. If f does not have one point at infinity in the $\breve{w}(f)$-weighted sense then we are done by taking $n = 1$ and $(f_1, g_1) = (f, g)$.

If f has one point at infinity in the $\breve{w}(f)$-weighted sense and $\breve{w}(f) \leqslant -1$ then, taking $(f_1, g_1) = (f, g)$, by (3) we find a k-elementary Y-type automorphism $\sigma_1 : k[X, Y] \to k[X, Y]$ such that, upon letting $f_2 = \sigma_1 f_1$ and $g_2 = \sigma_1 g_1$, we have that f_2 is Y-submonic with

$$\begin{cases} (\deg_Y(f_2), \deg_Y(g_2)) \\ \quad = (\deg_Y(f_1), \deg_Y(g_1)) \end{cases}$$

and either (i) $\breve{w}(f_2) > -1$, or (ii) $\breve{w}(f_2) \leqslant -1$ but f_2 does not have one point at infinity in the $\breve{w}(f_2)$-weighted sense; if f_2 does not have one point at infinity in the $\breve{w}(f_2)$-weighted sense then we are done by taking $n = 2$; if f_2 has one point at infinity in the $\breve{w}(f_2)$-weighted sense then, taking $\sigma_2 : k[X, Y] \to k[X, Y]$ to be the k-flip with $f_3 = \sigma_2 f_2$ and $g_3 = \sigma_2 g_2$, by (4) we see that f_3 is Y-submonic with $\breve{w}(f_3) < -1$ and

$$\begin{cases} \mathrm{GCD}(\deg_Y(f_3), \deg_Y(g_3)) \\ \quad \text{properly divides } \mathrm{GCD}(\deg_Y(f_2), \deg_Y(g_2)) \end{cases}$$

and so now the induction takes over.

If f has one point at infinity in the $\breve{w}(f)$-weighted sense and $\breve{w}(f) > -1$ then, taking $\sigma_1 : k[X, Y] \to k[X, Y]$ to be the identity map with $(f_2, g_2) = (f_1, g_1) = (f, g)$, and taking

$\sigma_2 : k[X,Y] \to k[X,Y]$ to be the k-flip with $f_3 = \sigma_2 f_2$ and $g_3 = \sigma_2 g_2$, by (4) we see that f_3 is Y-submonic with $\breve{w}(f_3) < -1$ and

$$\begin{cases} \mathrm{GCD}(\deg_Y(f_3), \deg_Y(g_3)) \\ \quad \text{properly divides } \mathrm{GCD}(\deg_Y(f_2), \deg_Y(g_2)) \end{cases}$$

and so again the induction takes over.

This completes the definition of the sequence $(f_1, g_1), \dots, (f_n, g_n)$. Since f_n does not have one point at infinity in the $\breve{w}(f_n)$-weighted sense, by (5) we see that $\breve{w}(f_n) \neq 0$ and

$$\begin{cases} \mathrm{GCD}(\mathrm{radnum}((f_n)^+_{\breve{w}(f_n)}), \mathrm{radnum}((g_n)^+_{\breve{w}(f_n)})) \\ \quad \text{very properly divides } \mathrm{GCD}(\deg_Y(f_n), \deg_Y(g_n)). \end{cases}$$

By (8) there is a tame automorphism $\sigma_n : k[X,Y] \to k[X,Y]$ such that upon letting $f_{n+1} = \sigma_n f_n$ and $g_{n+1} = \sigma_n g_n$ we see that

$$\begin{cases} (\mathrm{radnum}((f_n)^+_{\breve{w}(f_n)}), \mathrm{radnum}((g_n)^+_{\breve{w}(f_n)})) \\ \quad = (\mathrm{radnum}(f^+_{n+1}), \mathrm{radnum}(g^+_{n+1})) \end{cases}$$

and

$$\begin{cases} \text{for some positive integers } i', j', p, q \text{ with } i' \neq j' \text{ and } \mathrm{GCD}(i', j') = 1 \\ \quad \text{we have } (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u,v) \in \mathbb{N}^2\colon\ u \leqslant pi' \text{ and } v \leqslant pj'\} \\ \quad \text{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u,v) \in \mathbb{N}^2\colon\ u \leqslant qi' \text{ and } v \leqslant qj'\}. \end{cases}$$

The proof of (9) is completed by taking $\sigma = \sigma_n \sigma_{n-1} \dots \sigma_1$.

To prove (10) assume that (f, g) is not an automorphic pair. Now the degrees as well radnums are unchanged by a homogeneous k-linear transformation, and hence without loss of generality, by making such a transformation we may assume that f and g are Y-regular. Then f and g are Y-submonic and their (total) degrees coincide with their Y-degrees. Therefore (10) follows from (9).

(11) follows from (10). □

Lemma (5.3). *Given $\gamma \in k^\times$ and integer $e \geqslant 2$, let $w = -w_2/w_1$ where $(w_1, w_2) = (e, 1)$ or $(1, e)$ and consider the k automorphism $\sigma : k[X,Y] \to k[X,Y]$ given by $(X,Y) \mapsto (X - \gamma Y^e, Y)$ or $(X, Y - \gamma X^e)$ respectively. Then for every $f \in k[X,Y]$ we have* $\deg(f) \leqslant \deg_w(f)$ *and* $\deg(\sigma f) \leqslant \deg_w(f)$. *Moreover, if respectively $f^+_w = \theta Y^i (X + \gamma Y^e)^j$ or $\theta X^i (Y + \gamma X^e)^j$, with $i \in \mathbb{N}$ and $j \in \mathbb{N}_+$, then we have* $\deg(\sigma f) < \deg(f) = \deg_w(f)$.

Proof. By symmetry it suffices to consider the case of $(w_1, w_2) = (e, 1)$. For $(j, i) \in \mathbb{N}^2$ we clearly have

$$\deg\big(X^j Y^i\big) = j + i \leqslant ej + i = \deg_w\big(X^j Y^i\big) = ej + i = \deg\big(\sigma\big(X^j Y^i\big)\big).$$

Since by definition $\deg_w(f) = \max\{\deg_w(X^j Y^i)\colon\ (j,i) \in \mathrm{Supp}(f)\}$ and clearly the degree of a sum is less equal the degree of each summand of maximal degree, we get $\deg(f) \leqslant \deg_w(f)$ and $\deg(\sigma f) \leqslant \deg_w(f)$. It only remains to note that if $f^+_w = \theta Y^i (X + \gamma Y^e)^j$ with $i \in \mathbb{N}$ and $j \in \mathbb{N}_+$, then we have $\deg(f^+_w) = \deg_w(f^+_w) > \deg(\sigma f^+_w)$ with $\deg_w(f - f^+_w) < \deg_w(f^+_w)$. □

Lemma (5.4). *Let (f, g) be a Jacobian pair in $k[X, Y]$ where k is a field of characteristic 0. Assume that f has one point at infinity. Then there is a homogeneous k-linear automorphism $\tau : k[X, Y] \to k[X, Y]$ and a k-special automorphism $\sigma : k[X, Y] \to k[X, Y]$ such that*:

(1) *If* $\deg(f) \geqslant 2$ *and* $\deg(g) \geqslant 2$ *then we have* $\deg(\sigma\tau f) < \deg(f)$ *as well as* $\deg(\sigma\tau g) < \deg(g)$.
(2) *If* $\deg(f) + \deg(g) \geqslant 3$ *then* $\deg(\sigma\tau f) + \deg(\sigma\tau g) < \deg(f) + \deg(g)$.
(3) *If* $\deg(f) + \deg(g) < 3$ *then* $\deg(\sigma\tau f) + \deg(\sigma\tau g) = \deg(f) + \deg(g)$.

Proof. (1) follows by (5.2)(1), and (3) follows by taking $\sigma = \tau =$ identity. Thus, disregarding the condition for f to have one point at infinity, we only have to deal with the two cases $\deg(f) = 1 < \deg(g)$ or $\deg(g) = 1 < \deg(f)$, and by symmetry we may suppose we are in the first case. Then by a homogeneous k-linear automorphism τ we can arrange matters so that $\tau f = Y + \alpha$ with $\alpha \in k$. Now we must have $\tau g = \theta(X + \gamma Y^e) + \theta(Y)$ where $\gamma \in k^\times$ with integer $e \geqslant 2$ and $\theta(Y) \in k[Y]$ of degree smaller than e. It suffices to take $\sigma : k[X, Y] \to k[X, Y]$ to be the k-special automorphism given by $(X, Y) \mapsto (X - \gamma Y^e, Y)$. □

Lemma (5.5). *The following three implications on pairs (f, g) in $k[X, Y]$, where k is a field of characteristic 0, are equivalent.*

(i) *(f, g) is a Jacobian pair $\Rightarrow$ (f, g) is an automorphic pair.*
(ii) *(f, g) is a Jacobian pair $\Rightarrow$ f has one place at infinity.*
(iii) *(f, g) is a Jacobian pair $\Rightarrow$ f has one point at infinity.*

Proof. (i) $\Rightarrow$ (ii) follows from the "obvious" implication (f, g) automorphic pair $\Rightarrow f$ has one place at infinity, for which see [Ab2] or [Ab3]. (ii) $\Rightarrow$ (iii) follows from (5.1). To prove (iii) $\Rightarrow$ (i), assume (iii) and let there be given any Jacobian pair (f, g). If $\deg(f) \leqslant 1$ or $\deg(g) \leqslant 1$ then (f, g) is an automorphic pair by (4.15)(6). If $\deg(f) \geqslant 2$ and $\deg(g) \geqslant 2$ then by (iii) f has one point at infinity, and hence in view of (5.4) we are done by induction on $\deg(f) + \deg(g)$. □

Remark (5.6). Given any k-automorphism $\lambda : k[X, Y] \to k[X, Y]$ where k is a field of characteristic 0, let $f = \lambda(X)$ and $g = \lambda(Y)$. Then (f, g) is an automorphic pair and hence (f, g) is a Jacobian pair and, as in the proof of (5.5), f has one point at infinity. If $\deg(f) \leqslant 1$ and $\deg(g) \leqslant 1$ then clearly λ is a k-linear automorphism. Therefore, in view of (5.4), by induction on $\deg(f) + \deg(g)$ we see that λ is a tame automorphism. This is the new simple proof of Jung's Theorem we spoke of in the preamble to this section. Elsewhere we shall handle the task of modifying this to include Kulk's Theorem.

Remark (5.7). Thanks to Nick Inglis for help with the following observations about a Jacobian pair (f, g) in $k[X, Y]$ over a field k of characteristic zero.

(1) If $w = -1$ and f is not a w-monomial then f is Y-submonic or X-submonic.
(2) If $w \in \mathbb{Q}$ with $w < -1$ and f is not a w-monomial then f is X-submonic.
(3) If $w \in \mathbb{Q}$ with $0 > w > -1$ and f is not a w-monomial then f is Y-submonic.
(4) There is at most one negative $w \in \mathbb{Q}$ such that f is not a w-monomial.
(5) If f is Y-submonic and $\breve{w}(f) \geqslant -1$ then $\breve{w}(f)^{\ddagger}(f) \geqslant 0$.

Namely, (4.7) yields (1) to (3). Also (1) to (3) and (4.14) yield (4). Moreover, (3) and (4.14) yield (5). Applying (5) to the $w = -\infty$ case of the proof of (5.2)(6) we see that the last displayed claim of that proof which now reads

$$\begin{cases} (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u, v) \in \mathbb{N}^2\colon\ v \leqslant pj'\} \\ \quad \text{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u, v) \in \mathbb{N}^2\colon\ v \leqslant qj'\} \end{cases}$$

can be replaced by the stronger claim

$$\begin{cases} (pi', pj') \in \mathrm{Supp}(\sigma f) \subset \{(u, v) \in \mathbb{N}^2\colon\ u \leqslant pi' \text{ and } v \leqslant pj'\} \\ \quad \text{with } (qi', qj') \in \mathrm{Supp}(\sigma g) \subset \{(u, v) \in \mathbb{N}^2 : u \leqslant qi' \text{ and } v \leqslant qj'\}. \end{cases}$$

This proves the stronger version of (5.2)(6) with the same replacement of its last displayed claim. Now disregarding (5.2)(7), we immediately deduce (5.2)(8) from the said stronger version (5.2)(6) by making decreasing induction on the Y-degree of f. The previous proof of (5.2)(8) exhibits another use of the k-flip automorphism.

6. Coprime degrees and principal degree pairs

In (6.1) we shall prove the Jacobian Conjecture if the degrees of a Jacobian pair are coprime; the proof will also include the case when the GCD of the degrees is either a prime number or 4. In (6.2) we supplement (5.5) by showing that the Jacobian Conjecture is equivalent to saying that the degrees of a Jacobian pair always form a principal pair where we recall that a pair of integers (N, M) is a *principal pair* means either N divides M, or M divides N.

In (6.2) we also show that the Jacobian Conjecture is equivalent to saying that for any Jacobian pair (f, g) the *Newton polygon of f is a triangle*, i.e., either $\deg(f) = 1$ or there exist positive integers ν, μ such that $(0, \nu)$, $(\mu, 0)$ belong to $\mathrm{Supp}(f)$ and for every (i, j) in $\mathrm{Supp}(f)$ we have $i\nu + j\mu \leqslant \nu\mu$.

Moreover, in (6.2) we show that the Jacobian Conjecture is equivalent to saying that for any Jacobian pair (f, g) the Newton polygon of f is not a rectangle according to the following definition. We say that *the Newton polygon of f is a rectangle* if

$$\begin{cases} \text{for some positive integers } i', j', p \text{ with } i' \neq j' \text{ and } \mathrm{GCD}(i', j') = 1 \\ \quad \text{we have } (pi', pj') \in \mathrm{Supp}(f) \subset \{(u, v) \in \mathbb{N}^2\colon\ u \leqslant pi' \text{ and } v \leqslant pj'\}. \end{cases}$$

We call (i', j') *the shape* of the said rectangle. Note that the second displays in (5.2)(9) and (5.2)(10) say that under certain conditions the Newton polygons of f and g are rectangles of same shape. More details about Newton polygons will be given in the next section.

Lemma (6.1). *Let (f, g) be a Jacobian pair in $k[X, Y]$ where k is a field of characteristic* 0*, and let* $\deg(f) = N$ *with* $\deg(g) = M$*. Assume that we have* $\mathrm{GCD}(N, M) = 1$*, or a prime number, or* 4*. Then (f, g) is an automorphic pair.*

Proof. Follows from (4.19) and (5.2)(11). □

Lemma (6.2). *The following six implications on pairs (f, g) in $k[X, Y]$, where k is a field of characteristic* 0*, are equivalent.*

(i) (f,g) *is a Jacobian pair* $\Rightarrow$ (f,g) *is an automorphic pair.*
(ii) (f,g) *is a Jacobian pair* $\Rightarrow$ f *has one place at infinity.*
(iii) (f,g) *is a Jacobian pair* $\Rightarrow$ f *has one point at infinity.*
(iv) (f,g) *is a Jacobian pair* $\Rightarrow$ $(\deg(f),\deg(g))$ *is a principal pair.*
(v) (f,g) *is a Jacobian pair* $\Rightarrow$ *the Newton polygon of* f *is a triangle.*
(vi) (f,g) *is a Jacobian pair* $\Rightarrow$ *the Newton polygon of* f *is not a rectangle.*

Proof. In (5.5) we have shown that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

The implication (i) $\Rightarrow$ (iv) follows from the implication (f,g) is an automorphic pair $\Rightarrow$ $(\deg(f),\deg(g))$ is a principal pair proved in [Ab2]. To prove the implication (iv) $\Rightarrow$ (i), assume (iv) and suppose that (f,g) is a Jacobian pair. To show that (f,g) is an automorphic pair, in view of (4.15)(5), without loss of generality, we may suppose that $\deg(g)=M=eN$ where e,N are in $\mathbb{N}_+$ with $e\geqslant 2$ and $\deg(f)=N$. By (3.11) and (4.1) we can find $\kappa\in k^\times$ such that $\deg(g-\kappa f^e)<M$. Clearly $(f,g-\kappa f^e)$ is a Jacobian pair and hence we are done by induction on $N+M$.

The implication (i) $\Rightarrow$ (v) follows from the implication (f,g) is an automorphic pair $\Rightarrow$ the Newton polygon of f is a triangle proved in [Ab2]. Moreover, the implication (v) $\Rightarrow$ (ii) follows from (4.7).

The implication (iii) $\Rightarrow$ (vi) is obvious. The implication (vi) $\Rightarrow$ (iii) follows from (5.2)(10). □

Note (6.3). The reference to the Main Lemma (4.7) in the proofs of (6.2) and elsewhere may basically be reduced to a reference to (3.3) which indeed is the king-pin of this entire paper.

7. More general weight systems

To generalize the concept of weight systems we proceed thus.

Definition (7.1). Let $\theta=\theta(X,Y)\in k[X,X^{-1},Y,Y^{-1}]$ where k is a field.

For any $\omega=(\omega_1,\omega_2)\in\mathbb{Z}^2$ we define the *ω-degree* $\deg_\omega\theta$, the *ω-degree form* θ_ω^+, the *ω-homogeneity*, and the *ω-version* θ_ω^0 of θ exactly as in Section 2 with (ω_1,ω_2) replacing (w_1,w_2). For instance

$$\deg_\omega\theta=\max\bigl\{i\omega_1+j\omega_2\colon (i,j)\in\operatorname{Supp}(\theta)\bigr\}$$

with the understanding that if $\theta=0$ then $\deg_\omega\theta=-\infty$. Likewise, with (ω_1,ω_2) replacing (w_1,w_2), we also define the concepts of a *ω-automorphic pair*, having one or two (respectively at most two) *points at infinity* in the ω-weighted sense, the *ω-similarity* of $f\in k[X,Y]^\times$ with $g\in k[X,Y]$, and the *ω-lag* $\operatorname{lag}_\omega(f,g)$ of (f,g) in $k[X,Y]^\times$ exactly as in Section 2, and we define the concept of a *ω-monomial* exactly as in (4.8). Pictures in (9.4) to (9.6).

Just as $w=0$ corresponds to $(w_1,w_2)=(1,0)$, letting $w=-\infty$ or ∞ respectively corresponds to $(w_1,w_2)=(0,1)$ or $(0,-1)$. Now all of the above concepts from *w-degree* to *w-monomial* have obvious definitions for $w\in\{\pm\infty\}$. For instance $\deg_{-\infty}\theta=\deg_Y\theta$ and $\deg_\infty\theta=-\operatorname{ord}_Y\theta$, where we recall that *ord* is defined by putting

$$\operatorname{ord}_Y\theta=\min\bigl\{j\colon (i,j)\in\operatorname{Supp}(\theta)\bigr\}\quad\text{and}\quad\operatorname{ord}_X\theta=\min\bigl\{i\colon (i,j)\in\operatorname{Supp}(\theta)\bigr\}.$$

For any $\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2$ and $e \in \mathbb{Z}$, as usual we put $e\omega = (e\omega_1, e\omega_2) \in \mathbb{Z}^2$; in particular, $1\omega = \omega$, and $-1\omega = -\omega = (-\omega_1, -\omega_2)$. For any $\omega \in \mathbb{Z}^2$ and $e \in \mathbb{N}_+$, we clearly have $\deg_{e\omega} \theta = e \deg_\omega \theta$ and hence $\theta^+_{e\omega} = \theta^+_\omega$, and therefore sometimes it is more appropriate to let ω vary over the *coprime integer pairs set*

$$\mathbb{Z}^2_c = \{\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2\colon \operatorname{GCD}(\omega_1, \omega_2) = 1\}.$$

The weight systems with $\omega_1 < 0$ and $\omega_2 < 0$ are not very interesting. So we consider the *zerocomplement*, the *x-positive,* the *y-positive*, the *x-negative*, and the *y-negative integer pairs set* by putting

$$\begin{cases} \mathbb{Z}^2_z = \mathbb{Z}^2 \setminus \{(0,0)\}, \\ \mathbb{Z}^2_x = \{\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2\colon \omega_1 > 0\}, \\ \mathbb{Z}^2_y = \{\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2\colon \omega_2 > 0\}, \\ \mathbb{Z}^2_{x-} = \{\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2\colon \omega_1 < 0\}, \\ \mathbb{Z}^2_{y-} = \{\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2\colon \omega_2 < 0\}. \end{cases}$$

Note that then $\mathbb{Z}^2_c \subset \mathbb{Z}^2_z$ and $\mathbb{Z}^2_x \cup \mathbb{Z}^2_y \cup \mathbb{Z}^2_{x-} \cup \mathbb{Z}^2_{y-} \subset \mathbb{Z}^2_z$.

To take care of the above mentioned cases of $w = -\infty$ or ∞ which respectively correspond to $(0, 1)$ or $(0, -1)$, let us also consider the *negative-infinite* and the *positive-infinite integer pairs set* by putting

$$\mathbb{Z}^2_{-\infty} = \{\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2\colon \omega_1 = 0 < \omega_2\}$$

and

$$\mathbb{Z}^2_{\infty} = \{\omega = (\omega_1, \omega_2) \in \mathbb{Z}^2\colon \omega_1 = 0 > \omega_2\}$$

and note that as a disjoint partition of nonempty subsets we have

$$\mathbb{Z}^2_z = \mathbb{Z}^2_{x-} \amalg \mathbb{Z}^2_{-\infty} \amalg \mathbb{Z}^2_x \amalg \mathbb{Z}^2_\infty.$$

Flipping (7.2). Flipping X and Y has the effect that several items proved in Sections 3 and 4 have a counterpart which follows by symmetry. In (7.3.u) and (7.4.v) we shall state a complementary versions of items (3.u) and (4.v) respectively. Sometimes, such as in (3.1), the proof only uses $w_1 \neq 0$ and by symmetry we get it also when $w_2 \neq 0$; thus (7.3.1) can say that (3.1) remains true when $w \in \mathbb{Q} \cup \{\pm\infty\}$ or w is replaced by $\omega \in \mathbb{Z}^2_z$. Similarly for (7.3.2), (7.3.3), (7.4.1) to (7.4.5), (7.4.9), (7.4.10), and so on. On the other hand, (7.3.4), (7.4.6), (7.4.7) are only paraphrases of (3.4), (4.6), (4.7) respectively, when we replace $w \in \mathbb{Q}$ by $\omega \in \mathbb{Z}^2_x$.

The *reader is advised* to recheck that (3.1) to (3.3) (excluding (3.3.5) to (3.3.8)), (4.1) to (4.5), and (4.9) to (4.10), need only the weaker assumption $w_1 \neq 0$ and not the stronger assumption $w_1 > 0$.

As common notation for (7.3.u) and (7.4.v) let k be a field of characteristic 0 and let $w \in \mathbb{Q} \cup \{\pm\infty\}$ or replace it by $\omega \in \mathbb{Z}^2_z$; in Paraphrased Lemmas (7.3.4), (7.4.6), (7.4.7), and Remark–Definition (7.4.14), we specify the range of w or ω more explicitly. As common notation for (7.3.u) let F, G be nonzero w-homogeneous or ω-homogeneous members of $k[X, X^{-1}, Y, Y^{-1}]$.

As common notation for (7.4.v) let f, g be nonzero members of $k[X, Y]$. Note that for $w \in \mathbb{Q} \cup \{\pm\infty\}$, the pair (w_1, w_2) consists of coprime integers such that if $w \in \mathbb{Q}$ then $w = -w_2/w_1$ with $w_1 > 0$, whereas if $w = -\infty$ or ∞ then $(w_1, w_2) = (0, 1)$ or $(0, -1)$ respectively. This is particularly significant in (7.3.2.9), (7.3.3.4), (7.4.5), (7.4.9), and (7.4.10).

Generalized Eulerian Lemma with consequences (7.3.1) and (7.3.2.6) to (7.3.2.9) and (7.3.3.1) to (7.3.3.4). *Respectively same as* (3.1) *and* (3.2.6) *to* (3.2.9) *and* (3.3.1) *to* (3.3.4), *with* $w \in \mathbb{Q} \cup \{\pm\infty\}$ *or replaced by* $\omega \in \mathbb{Z}_z^2$.

Paraphrased Lemma (7.3.4). *Suppose that* $\omega \in \mathbb{Z}_x^2$. *Assume that* F *and* G *belong to* $k[X, Y]$ *and* $J(F, G) = \theta F$. *Also assume that we have either* (i) $\omega_2 > 0$ *or* (ii) $\omega_2 \leqslant 0 \neq \omega_1 + \omega_2 \neq 0 \neq EN - DM$ *where* $N = \deg_\omega F$ *with* $D = \deg_Y F$ *and* $M = \deg_\omega G$ *with* $E = \deg_Y G$. *Then* $G = \theta xy$ *and* $F = \theta x^i y^j$ *with* $i \neq j$ *in* $\mathbb{N}$ *where* (x, y) *is the* ω*-automorphic pair described below.*

(1) *If* $\omega_2 = \omega_1 = 1 \neq E$ *then* $x = \alpha X + Y$ *and* $y = \beta X + Y$ *with* $\alpha \neq \beta$ *in* k.
(2) *If* $\omega_2 = \omega_1 = 1 = E$ *then* $x = X$ *and* $y = \beta X + Y$ *with* $\beta \in k$.
(3) *If* $\omega_2 > \omega_1$ *then* $x = X$ *and* $y = \gamma X^{\omega_2/\omega_1} + Y$ *with* $\gamma \in k$ *such that* $\omega_2/\omega_1 \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(4) *If* $0 < \omega_2 < \omega_1$ *then* $x = Y$ *and* $y = X + \gamma Y^{\omega_1/\omega_2}$ *with* $\gamma \in k$ *and* $\omega_1/\omega_2 \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(5) *If* $\omega_2 \leqslant 0 \neq \omega_1 + \omega_2 \neq 0 \neq EN - DM$ *then* $x = X$ *and* $y = Y + \gamma X^{\omega_2/\omega_1}$ *with* $\gamma \in k$ *such that* $\omega_2 = 0$ *in case* $\gamma \neq 0$.

Generalized Jacobian Lemma with consequences (7.4.1) to (7.4.5). *Respectively same as* (4.1) *to* (4.5), with $w \in \mathbb{Q} \cup \{\pm\infty\}$ *or replaced by* $\omega \in \mathbb{Z}_z^2$.

Paraphrased Lemma (7.4.6). *Suppose that* $\omega \in \mathbb{Z}_x^2$. *Assume that* f *is* ω*-similar to* $J(f, g)$, *and also assume that either* (i) $\omega_2 > 0$, *or* (ii) $\omega_2 \leqslant 0 \neq N$ *with* $EN - DM \neq 0 < (\omega_1 + \omega_2)N$, *where* $N = \deg_\omega f$ *with* $D = \deg_Y f_\omega^+$ *and where* $M = \deg_\omega g$ *with* $E = \deg_Y g_\omega^+$. *Then* f *as well as* g *has at most two points at infinity in the* ω*-weighted sense. More precisely* $f_\omega^+ = \theta x^i y^j$ *and* $g_\omega^+ = \theta x^{i^*} y^{j^*}$ *where* i, j, i^*, j^* *in* $\mathbb{N}$ *with* $i - j \neq 0 \neq i + j \neq 0 \neq i^* + j^*$ *and where* (x, y) *is the* ω*-automorphic pair described below.*

(1) *If* $\omega_2 = \omega_1 = 1$ *then* $x = \alpha X + \alpha^* Y$ *and* $y = \beta X + \beta^* Y$ *where* $\alpha, \alpha^*, \beta, \beta^*$ *in* k *are such that* $\alpha\beta^* - \alpha^*\beta \neq 0$.
(2) *If* $\omega_2 > \omega_1$ *then* $x = X$ *and* $y = \gamma X^{\omega_2/\omega_1} + Y$ *with* $\gamma \in k$ *such that* $\omega_2/\omega_1 \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(3) *If* $0 < \omega_2 < \omega_1$ *then* $x = Y$ *and* $y = X + \gamma Y^{\omega_1/\omega_2}$ *with* $\gamma \in k$ *and* $\omega_1/\omega_2 \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(4) *If* $\omega_2 \leqslant 0 \neq N$ *with* $EN - DM \neq 0 < (\omega_1 + \omega_2)N$ *then* $x = X$ *and* $y = Y + \gamma X^{\omega_2/\omega_1}$ *with* $\gamma \in k$ *such that* $\omega_2 = 0$ *in case* $\gamma \neq 0$.

Moreover we have $N \neq 0$ *and we have* (I) *and* (II) *stated below.*

(I) $\mathrm{lag}_\omega(f, g) \neq 0 \Leftrightarrow f$ *is* ω*-similar to* g.
(II) $\mathrm{lag}_\omega(f, g) = 0 \Rightarrow (i^*, j^*) = (1 + ci, 1 + cj)$ *for some* $c \in \mathbb{Q}$.

Paraphrased Main Lemma (7.4.7). *Suppose that $\omega \in \mathbb{Z}_x^2$. Let us assume that $J(f,g) = \theta$, and let us also assume that we have either* (i) $\omega_2 > 0$, *or* (ii) $\omega_2 \leqslant 0 \neq N$ *with* $EN - DM \neq 0 < (\omega_1 + \omega_2)N$, *or* (iii) $\omega_2 \leqslant 0 \neq M$ *with* $EN - DM \neq 0 < (\omega_1 + \omega_2)M$, *where* $N = \deg_\omega f$ *with* $D = \deg_Y f_\omega^+$ *and where* $M = \deg_\omega g$ *with* $E = \deg_Y g_w^+$. *Then* f *as well as* g *has at most two points at infinity in the* ω*-weighted sense. More precisely* $f_\omega^+ = \theta x^i y^j$ *and* $g_\omega^+ = \theta x^{i^*} y^{j^*}$ *where* i, j, i^*, j^* *in* $\mathbb{N}$ *with* $i - j \neq 0 \neq i + j \neq 0 \neq i^* + j^* \neq 0 \neq i^* - j^*$ *and where* (x, y) *is the* ω*-automorphic pair described below.*

(1) *If* $\omega_2 = \omega_1 = 1$ *then* $x = \alpha X + \alpha^* Y$ *and* $y = \beta X + \beta^* Y$ *where* $\alpha, \alpha^*, \beta, \beta^*$ *in* k *are such that* $\alpha\beta^* - \alpha^*\beta \neq 0$.
(2) *If* $\omega_2 > \omega_1$ *then* $x = X$ *and* $y = \gamma X^{\omega_2/\omega_1} + Y$ *with* $\gamma \in k$ *such that* $\omega_2/\omega_1 \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(3) *If* $0 < \omega_2 < \omega_1$ *then* $x = Y$ *and* $y = X + \gamma Y^{\omega_1/\omega_2}$ *with* $\gamma \in k$ *and* $\omega_1/\omega_2 \in \mathbb{N}_+$ *in case* $\gamma \neq 0$.
(4) *If* $\omega_2 \leqslant 0 \neq N$ *with* $EN - DM \neq 0 < (\omega_1 + \omega_2)N$ *then* $x = X$ *and* $y = Y + \gamma X^{\omega_2/\omega_1}$ *with* $\gamma \in k$ *such that* $\omega_2 = 0$ *in case* $\gamma \neq 0$.
(5) *If* $\omega_2 \leqslant 0 \neq M$ *with* $EN - DM \neq 0 < (\omega_1 + \omega_2)M$ *then* $x = X$ *and* $y = Y + \gamma X^{\omega_2/\omega_1}$ *with* $\gamma \in k$ *such that* $\omega_2 = 0$ *in case* $\gamma \neq 0$.

Moreover we have $\mathrm{GCD}(N, M) \neq 0$ *and we have* (I) *and* (II) *stated below.*

(I) $\mathrm{lag}_\omega(f, g) \neq 0 \Leftrightarrow f$ *is* ω*-similar to* g.
(II) $\mathrm{lag}_\omega(f, g) = 0 \Rightarrow$ *either* $(i^*, j^*) = (1 + ci, 1 + cj)$ *for some* $c \in \mathbb{Q}$ *or* $(i, j) = (1 + ci^*, 1 + cj^*)$ *for some* $c \in \mathbb{Q}$.

Lemmas (7.4.9) and (7.4.10). *Respectively same as* (4.9) *and* (4.10), *with* $w \in \mathbb{Q} \cup \{\pm\infty\}$ *or replaced by* $\omega \in \mathbb{Z}_z^2$.

Remark–Definition (7.4.14). Noting that items (7.4.11) to (7.4.13) are nonexistent, we proceed to generalize the concepts of antecedent and consequent introduced in (4.14) to $\omega \in \mathbb{Z}_z^2$. For pictures see (9.4) to (9.6).

In (7.4.14) *we let* f *belong to* $k[X, X^{-1}, Y, Y^{-1}]^\times$ *with any commutative ring* k; taking k to be the prime field of characteristic two, this becomes a theory of nonempty finite subsets of $\mathbb{Z}^2$.

Let card denote *cardinality* and note that then for any $w \in \mathbb{Q} \cup \{\pm\infty\}$ or $\omega \in \mathbb{Z}^2$: f is a w-monomial or f is a ω-monomial iff card $\mathrm{Supp}(f_w^+) = 1$ or card $\mathrm{Supp}(f_\omega^+) = 1$ respectively, and hence in particular: f is a monomial (i.e., equivalently, f is a $(0,0)$-monomial) iff card $\mathrm{Supp}(f) = 1$.

Recall that *defo* and *info* are defined by putting

$$\mathrm{defo}_Y f = \text{sum of the terms in } f \text{ whose } Y\text{-degree equals } \deg_Y f$$

and

$$\mathrm{defo}_X f = \text{sum of the terms in } f \text{ whose } X\text{-degree equals } \deg_X f$$

and we define *info* by putting

$$\mathrm{info}_Y f = \text{sum of the terms in } f \text{ whose } Y\text{-order equals } \mathrm{ord}_Y f$$

and

$$\mathrm{info}_X f = \text{sum of the terms in } f \text{ whose } X\text{-order equals } \mathrm{ord}_X f.$$

We define the *absolute value* of any $\omega = (\omega_1, \omega_2) \in \mathbb{Z}_z^2$ by putting

$$|\omega| = \begin{cases} -\omega_2/\omega_1 & \text{if } \omega_1 \neq 0, \\ -\infty & \text{if } \omega_1 = 0 < \omega_2, \\ \infty & \text{if } \omega_1 = 0 > \omega_2. \end{cases}$$

Note that $\omega \mapsto |\omega|$ gives a surjection $\mathbb{Z}_z^2 \to \mathbb{Q} \cup \{\pm\infty\}$. By restricting this to $\mathbb{Z}_z^2 \setminus \mathbb{Z}_{x-}^2$ we again get a surjection $\mathbb{Z}_z^2 \setminus \mathbb{Z}_{x-}^2 \to \mathbb{Q} \cup \{\pm\infty\}$. Further restricting it to $\mathbb{Z}_c^2 \setminus \mathbb{Z}_{x-}^2$ we get a bijection $\mathbb{Z}_c^2 \setminus \mathbb{Z}_{x-}^2 \to \mathbb{Q} \cup \{\pm\infty\}$ whose inverse was described in the last paragraph of (7.2). Moreover

$$\begin{cases} \text{for all } \omega \in \mathbb{Z}_z^2 \setminus \mathbb{Z}_{x-}^2 \text{ we have:} \\ \quad f_{|\omega|}^+ = f_\omega^+. \end{cases}$$

For any $\omega \in \mathbb{Z}_z^2$ we also define its *norm* $\|\omega\| \in \mathbb{Z}_c^2$ by putting

$$\|\omega\| = \left(\frac{\omega_1}{\mathrm{GCD}(\omega_1, \omega_2)}, \frac{\omega_2}{\mathrm{GCD}(\omega_1, \omega_2)} \right).$$

Note that

$$\omega \mapsto \|\omega\| \text{ gives a surjection } \mathbb{Z}_z^2 \to \mathbb{Z}_c^2$$

and

$$\begin{cases} \text{for all } \omega, \omega' \text{ in } \mathbb{Z}_x^2 \text{ we have:} \\ \quad \|\omega\| = \|\omega'\| \Leftrightarrow e\omega = e'\omega' \text{ for some } e, e' \text{ in } \mathbb{N}_+. \end{cases}$$

For any $\omega \in \mathbb{Z}_z^2$ we define the *dotted ω-degree form* $f_\omega^\odot$ of f and the *double dotted ω-degree form* $f_\omega^{\odot\odot}$ of f by putting

$$f_\omega^\odot = \begin{cases} \mathrm{defo}_Y f_\omega^+ & \text{if } \omega_1 > 0, \\ \mathrm{info}_X f_\omega^+ & \text{if } \omega_1 = 0 < \omega_2, \\ \mathrm{defo}_X f_\omega^+ & \text{if } \omega_1 = 0 > \omega_2, \\ \mathrm{info}_Y f_\omega^+ & \text{if } \omega_1 < 0 \end{cases}$$

and

$$f_\omega^{\odot\odot} = \begin{cases} \mathrm{info}_Y f_\omega^+ & \text{if } \omega_1 > 0, \\ \mathrm{defo}_X f_\omega^+ & \text{if } \omega_1 = 0 < \omega_2, \\ \mathrm{info}_X f_\omega^+ & \text{if } \omega_1 = 0 > \omega_2, \\ \mathrm{defo}_Y f_\omega^+ & \text{if } \omega_1 < 0. \end{cases}$$

Clearly these are monomials and hence we have unique $\kappa^{\odot}$ and $\kappa^{\odot\odot}$ in $k^{\times}$ together with $(i^{\odot}, j^{\odot})$ and $(i^{\odot\odot}, j^{\odot\odot})$ in $\mathbb{Z}^2$ such that

$$f_{\omega}^{\odot} = \kappa^{\odot} X^{i^{\odot}} Y^{j^{\odot}} \quad \text{and} \quad f_{\omega}^{\odot\odot} = \kappa^{\odot\odot} X^{i^{\odot\odot}} Y^{j^{\odot\odot}}.$$

We put

$$f_{\omega}^{?} = \kappa^{\odot} \in k^{\times} \quad \text{and} \quad f_{\omega}^{??} = \kappa^{\odot\odot} \in k^{\times}$$

and we call these the *questioned ω-constant* of f and the *double questioned ω-constant* of f respectively. Likewise we put

$$f_{\omega}^{!} = \left((f_{\omega}^{!})_1, (f_{\omega}^{!})_2\right) = \left(i^{\odot}, j^{\odot}\right) \in \mathbb{Z}^2$$

and

$$f_{\omega}^{!!} = \left((f_{\omega}^{!!})_1, (f_{\omega}^{!!})_2\right) = \left(i^{\odot\odot}, j^{\odot\odot}\right) \in \mathbb{Z}^2$$

and we call these the *banged ω-point* of f and the *double banged ω-point* of f respectively.

Instead of dotted (respectively questioned, banged) and double dotted (respectively double questioned, double banged) we may say *top* and *bottom* respectively. Calling $f_{\omega}^{!}$ and $f_{\omega}^{!!}$ the *top* and *bottom* ω-points of f is quite picturesque, with the caution that in case of negative x-weight the top and bottom may look upside down; just remember that from top to bottom is always *clockwise*. All this applies with ω replaced by $w \in \mathbb{Q} \cup \{\pm\infty\}$ or $w \in \mathbb{Q}_d$ as in (8.4.14) below.

We say that f is *pseudolinear* to mean that $f = f_{\omega}^{+}$ for some $\omega \in \mathbb{Z}_z^2$. Note that if f is a monomial then $f = f_{\omega}^{+}$ for every $\omega \in \mathbb{Z}_z^2$. We say that f is *skew* to mean that $f = f_{\omega}^{+}$ for some $\omega \in \mathbb{Z}_z^2$ with $\omega_1 \neq 0$. We say that f is *horizontal* to mean that $f = f_{\omega}^{+}$ for some $\omega \in \mathbb{Z}_z^2$ with $\omega_1 = 0$. We say that f is *vertical* to mean that $f = f_{\omega}^{+}$ for some $\omega \in \mathbb{Z}_z^2$ with $\omega_2 = 0$. To stress the specific ω we may say that f is *ω-pseudolinear* or *ω-skew* or *ω-horizontal* or *ω-vertical* respectively.

We shall now state and prove Facts (1) to (3) concerning the above concepts.

In Fact (1) we characterize pseudolinearity.

In Fact (3) we prepare the algebraic groundwork for the Degreewise Newton Polygon to be discussed in Section 9. The said Fact (3) may sound a bit *convoluted* because it is like trying to put a square peg through a round hole, or rather like trying to rotate a rectangular lid as if it were a round one. After all, as will be illustrated in Section 9, the Degreewise Newton Polygon may very well have horizontal sides, and it is certainly not a circle.

Fact (1). Concerning the absolute values and norms of members of $\mathbb{Z}_z^2$ we have the following.

(I) For any $\omega \in \mathbb{Z}_z^2$ and $(i, j) \neq (i', j')$ in $\mathrm{Supp}(f_{\omega}^{+})$ we have

$$|\omega| \begin{cases} = \frac{i'-i}{j'-j} \in \mathbb{Q} & \text{if } j' \neq j, \\ \in \{\pm\infty\} & \text{if } j' = j, \\ = -\infty & \text{if } j' = j = \deg_Y f \neq \mathrm{ord}_Y f, \\ = \infty & \text{if } j' = j = \mathrm{ord}_Y f \neq \deg_Y f. \end{cases}$$

(II) For any $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ with $\overline{\omega} = -\omega$ we have:

$$\begin{aligned}\operatorname{Supp}\bigl(f_\omega^+\bigr) \cap \operatorname{Supp}\bigl(f_{\overline{\omega}}^+\bigr) \neq \emptyset \quad &\Leftrightarrow \quad f_\omega^+ = f = f_{\overline{\omega}}^+ \quad \Leftrightarrow \quad f = f_\omega^+ \\ \Leftrightarrow \quad f = f_{\overline{\omega}}^+ \quad &\Leftrightarrow \quad f_\omega^+ = f_{\overline{\omega}}^+.\end{aligned}$$

(III) For any $\omega \in \mathbb{Z}_z^2$ we have $f_\omega^+ = f_{\|\omega\|}^+$, and the absolute value $|\omega|$ of ω coincides with the absolute value $|(\|\omega\|)|$ of its norm $\|\omega\|$. For any $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ we have

$$|\overline{\omega}| = |\omega| \quad \Leftrightarrow \quad \begin{cases}\text{either (i) } \|\overline{\omega}\| = \|\omega\| \\ \quad \text{or (ii) } \|\overline{\omega}\| = \|-\omega\|\end{cases}$$

and

$$\|\overline{\omega}\| = \|-\omega\| \quad \Leftrightarrow \quad \begin{cases}\text{either (i) } \omega_1\overline{\omega}_1 < 0 \text{ with } |\omega| = |\overline{\omega}| \\ \quad \text{or (ii) } \{|\omega|, |\overline{\omega}|\} = \{\pm\infty\}.\end{cases}$$

(IV) If $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ are such that $\operatorname{Supp}(f_\omega^+) \cap \operatorname{Supp}(f_{\overline{\omega}}^+) \neq \emptyset$ and $\omega_1\overline{\omega}_1 < 0$ with $|\omega| = |\overline{\omega}|$, then $f_\omega^+ = f = f_{\overline{\omega}}^+$ and hence in particular f is skew.

(V) If $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ are such that $\operatorname{Supp}(f_\omega^+) \cap \operatorname{Supp}(f_{\overline{\omega}}^+) \neq \emptyset$ and $\{|\omega|, |\overline{\omega}|\} = \{\pm\infty\}$, then $f_\omega^+ = f = f_{\overline{\omega}}^+$ and hence in particular f is horizontal.

(VI) If $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ are such that $\operatorname{Supp}(f_\omega^+) \cap \operatorname{Supp}(f_{\overline{\omega}}^+) \neq \emptyset$ and $\omega_1\overline{\omega}_1 < 0$ with $|\omega| = 0 = |\overline{\omega}|$, then $f_\omega^+ = f = f_{\overline{\omega}}^+$ and hence in particular f is vertical.

(VII) f is skew iff $f_\omega^+ = f_{\overline{\omega}}^+$ for some $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ with $\omega_1\overline{\omega}_1 < 0$ and $|\omega| = |\overline{\omega}|$.

(VIII) f is horizontal iff $f_\omega^+ = f_{\overline{\omega}}^+$ for some $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ with $\{|\omega|, |\overline{\omega}|\} = \{\pm\infty\}$.

(IX) f is vertical iff $f_\omega^+ = f_{\overline{\omega}}^+$ for some $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ with $\omega_1\overline{\omega}_1 < 0$ and $|\omega| = 0 = |\overline{\omega}|$.

(X) If f is *nonmonomially* horizontal then f is not skew (i.e., if f is horizontal but not a monomial then f is not skew). Equivalently, if f is *nonmonomially* skew then f is not horizontal.

(XI) If $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ are such that $|\omega| = |\overline{\omega}| \in \mathbb{Q}$ then either $\omega_1\overline{\omega}_1 < 0$ or $\|\omega\| = \|\overline{\omega}\|$. If $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ are such that $\{|\omega|, |\overline{\omega}|\} \subset \{\pm\infty\}$ then either $\{|\omega|, |\overline{\omega}|\} = \{\pm\infty\}$ or $\|\omega\| = \|\overline{\omega}\|$. If $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ are such that $\|\omega\| = \|\overline{\omega}\|$ then $f_\omega^+ = f_{\overline{\omega}}^+$.

(XII) If $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ are such that $\operatorname{card}(\operatorname{Supp}(f_\omega^+) \cap \operatorname{Supp}(f_{\overline{\omega}}^+)) \geqslant 2$ then we have $f_\omega^+ = f_{\overline{\omega}}^+$ and either $|\omega| = |\overline{\omega}| \in \mathbb{Q}$ or $\{|\omega|, |\overline{\omega}|\} \subset \{\pm\infty\}$.

Fact (2). Let $\omega, \overline{\omega}$ in $\mathbb{Z}_z^2$ and $(i', j'), (i, j), (i'', j'')$ in $\mathbb{Z}^2$ be such that

(I) $$\deg_\omega\bigl(X^{i'}Y^{j'}\bigr) = \deg_\omega\bigl(X^iY^j\bigr) = \deg_\omega\bigl(X^{i''}Y^{j''}\bigr)$$

and

(II) $$\deg_{\overline{\omega}}\bigl(X^{i'}Y^{j'}\bigr) \neq \deg_{\overline{\omega}}\bigl(X^iY^j\bigr) \neq \deg_{\overline{\omega}}\bigl(X^{i''}Y^{j''}\bigr).$$

Assume that

(III) $$\text{either } j' < j < j'' \text{ or } j' > j > j'' \text{ or } i' < i < i'' \text{ or } i' > i > i''.$$

Then we have

$$\text{(IV)}\qquad \begin{cases} \text{either } \deg_{\overline{\omega}}(X^{i'}Y^{j'}) < \deg_{\overline{\omega}}(X^{i}Y^{j}) < \deg_{\overline{\omega}}(X^{i''}Y^{j''}) \\ \quad \text{or } \deg_{\overline{\omega}}(X^{i'}Y^{j'}) > \deg_{\overline{\omega}}(X^{i}Y^{j}) > \deg_{\overline{\omega}}(X^{i''}Y^{j''}). \end{cases}$$

In proving this we shall use the obvious *monotonicity* of linear functions which says that

$$\text{(V)}\qquad \begin{cases} \text{for any } W, \overline{W}, I, J \text{ in } \mathbb{R} \text{ we have:} \\ \text{(i) } J > 0 \text{ and } \overline{W} > W \Rightarrow J\overline{W} + I > JW + I, \\ \text{(ii) } J > 0 \text{ and } \overline{W} < W \Rightarrow J\overline{W} + I < JW + I, \\ \text{(iii) } J < 0 \text{ and } \overline{W} > W \Rightarrow J\overline{W} + I < JW + I, \\ \text{(iv) } J < 0 \text{ and } \overline{W} < W \Rightarrow J\overline{W} + I > JW + I, \\ \text{(v) } J = 0 \Rightarrow J\overline{W} + I = JW + I. \end{cases}$$

[Actually we shall use this only with $\mathbb{R}$ replaced by $\mathbb{Q}$.]

Fact (3). Let $\omega \in \mathbb{Z}_z^2$ and $(i', j'), (i'', j'')$ in $\mathrm{Supp}(f_\omega^+)$ be such that

$$\begin{cases} \text{if } \omega_1 \neq 0 \text{ then} \\ \quad j' = \max\{j\colon (i,j) \in \mathrm{Supp}(f_\omega^+)\} \text{ and } j'' = \min\{j\colon (i,j) \in \mathrm{Supp}(f_\omega^+)\} \end{cases}$$

whereas

$$\begin{cases} \text{if } \omega_1 = 0 \text{ then} \\ \quad i' = \min\{i\colon (i,j) \in \mathrm{Supp}(f_\omega^+)\} \text{ and } i'' = \max\{i\colon (i,j) \in \mathrm{Supp}(f_\omega^+)\} \end{cases}$$

and note that then

$$f_\omega^! = \begin{cases} \{(i', j')\} & \text{if either } \omega_1 > 0 \text{ or } \omega_1 = 0 < \omega_2, \\ \{(i'', j'')\} & \text{if either } \omega_1 < 0 \text{ or } \omega_1 = 0 > \omega_2 \end{cases}$$

and

$$f_\omega^{!!} = \begin{cases} \{(i'', j'')\} & \text{if either } \omega_1 > 0 \text{ or } \omega_1 = 0 < \omega_2, \\ \{(i', j')\} & \text{if either } \omega_1 < 0 \text{ or } \omega_1 = 0 > \omega_2. \end{cases}$$

Let

$$S' = \{(i,j) \in \mathrm{Supp}(f)\colon j > j'\} \quad \text{and} \quad S'' = \{(i,j) \in \mathrm{Supp}(f)\colon j < j''\}$$

and

$$\Omega = \begin{cases} \{\overline{w} \in \mathbb{Z}_z^2\colon |\overline{\omega}| = |\omega| \text{ and } \overline{\omega}_1\omega_1 > 0\} & \text{if } \omega_1 \neq 0, \\ \{\overline{\omega} \in \mathbb{Z}_z^2\colon |\overline{\omega}| = |\omega| \text{ and } \overline{\omega}_1 = 0\} & \text{if } \omega_1 = 0. \end{cases}$$

Let

$$\widehat{\Omega}' = \begin{cases} \{\overline{\omega} \in \mathbb{Z}^2_x\colon\ |\overline{\omega}| = \max_{(i,j)\in S'} \frac{i'-i}{j'-j}\} & \text{if } \omega_1 > 0 \text{ and } S' \neq \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_z\colon\ \overline{\omega}_1 = 0 < \overline{\omega}_2\} & \text{if } \omega_1 > 0 \text{ and } S' = \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_x\colon\ |\overline{\omega}| = \max_{(i,j)\in S'} \frac{i''-i}{j''-j}\} & \text{if } \omega_1 = 0 > \omega_2 \text{ and } S' \neq \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_z\colon\ \overline{\omega}_1 = 0 < \overline{\omega}_2\} & \text{if } \omega_1 = 0 > \omega_2 \text{ and } S' = \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\overline{\omega}| = \max_{(i,j)\in S''} \frac{i''-i}{j''-j}\} & \text{if } \omega_1 < 0 \text{ and } S'' \neq \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_z\colon\ \overline{\omega}_1 = 0 > \overline{\omega}_2\} & \text{if } \omega_1 < 0 \text{ and } S'' = \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\overline{\omega}| = \max_{(i,j)\in S''} \frac{i'-i}{j'-j}\} & \text{if } \omega_1 = 0 < \omega_2 \text{ and } S'' \neq \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_z\colon\ \overline{\omega}_1 = 0 > \overline{\omega}_2\} & \text{if } \omega_1 = 0 < \omega_2 \text{ and } S'' = \emptyset \end{cases}$$

and

$$\widehat{\Omega}'' = \begin{cases} \{\overline{\omega} \in \mathbb{Z}^2_x\colon\ |\overline{\omega}| = \min_{(i,j)\in S''} \frac{i''-i}{j''-j}\} & \text{if } \omega_1 > 0 \text{ and } S'' \neq \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_z\colon\ \overline{\omega}_1 = 0 > \overline{\omega}_2\} & \text{if } \omega_1 > 0 \text{ and } S'' = \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_x\colon\ |\overline{\omega}| = \min_{(i,j)\in S''} \frac{i''-i}{j''-j}\} & \text{if } \omega_1 = 0 < \omega_2 \text{ and } S'' \neq \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_z\colon\ \overline{\omega}_1 = 0 > \overline{\omega}_2\} & \text{if } \omega_1 = 0 < \omega_2 \text{ and } S'' = \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\overline{\omega}| = \min_{(i,j)\in S'} \frac{i'-i}{j'-j}\} & \text{if } \omega_1 < 0 \text{ and } S' \neq \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_z\colon\ \overline{\omega}_1 = 0 < \overline{\omega}_2\} & \text{if } \omega_1 < 0 \text{ and } S' = \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\overline{\omega}| = \min_{(i,j)\in S'} \frac{i'-i}{j'-j}\} & \text{if } \omega_1 = 0 > \omega_2 \text{ and } S' \neq \emptyset, \\ \{\overline{\omega} \in \mathbb{Z}^2_z\colon\ \overline{\omega}_1 = 0 < \overline{\omega}_2\} & \text{if } \omega_1 = 0 > \omega_2 \text{ and } S' = \emptyset \end{cases}$$

and call $\widehat{\Omega}'$ (respectively $\widehat{\Omega}''$) the *preantecedental set* (respectively the *preconsequental set*) of ω relative to f.

Let (where we condense the top 4 cases of $\widehat{\Omega}'$ into one case)

$$\widehat{\Omega}^* = \begin{cases} \{\widehat{\omega} \in \mathbb{Z}^2_x\colon\ |\omega| > |\widehat{\omega}| > |\overline{\omega}| \text{ for all } \overline{\omega} \in \widehat{\Omega}'\} & \text{if } \omega_1 > 0 \text{ or } \omega_1 = 0 > \omega_2, \\ \{\widehat{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\omega| > |\widehat{\omega}| > |\overline{\omega}| \text{ for all } \overline{\omega} \in \widehat{\Omega}'\} & \text{if } \omega_1 < 0 \text{ and } S'' \neq \emptyset, \\ \{\widehat{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\omega| > |\widehat{\omega}|\} & \text{if } \omega_1 < 0 \text{ and } S'' = \emptyset, \\ \{\widehat{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\widehat{\omega}| > |\overline{\omega}| \text{ for all } \overline{\omega} \in \widehat{\Omega}'\} & \text{if } \omega_1 = 0 < \omega_2 \text{ and } S'' \neq \emptyset, \\ \mathbb{Z}^2_{x-} & \text{if } \omega_1 = 0 < \omega_2 \text{ and } S'' = \emptyset \end{cases}$$

and (where we condense the top 4 cases of $\widehat{\Omega}''$ into one case)

$$\widehat{\Omega}^{**} = \begin{cases} \{\widehat{\omega} \in \mathbb{Z}^2_x\colon\ |\omega| < |\widehat{\omega}| < |\overline{\omega}| \text{ for all } \overline{\omega} \in \widehat{\Omega}''\} & \text{if } \omega_1 > 0 \text{ or } \omega_1 = 0 < \omega_2, \\ \{\widehat{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\omega| < |\widehat{\omega}| < |\overline{\omega}| \text{ for all } \overline{\omega} \in \widehat{\Omega}''\} & \text{if } \omega_1 < 0 \text{ and } S' \neq \emptyset, \\ \{\widehat{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\omega| < |\widehat{\omega}|\} & \text{if } \omega_1 < 0 \text{ and } S' = \emptyset, \\ \{\widehat{\omega} \in \mathbb{Z}^2_{x-}\colon\ |\widehat{\omega}| < |\overline{\omega}| \text{ for all } \overline{\omega} \in \widehat{\Omega}''\} & \text{if } \omega_1 = 0 > \omega_2 \text{ and } S' \neq \emptyset, \\ \mathbb{Z}^2_{x-} & \text{if } \omega_1 = 0 > \omega_2 \text{ and } S' = \emptyset \end{cases}$$

and call $\widehat{\Omega}^*$ (respectively $\widehat{\Omega}^{**}$) the *preantecedental segment* (respectively the *preconsequental segment*) of ω relative to f.

Let

$$\widehat{\Omega}'^* = \begin{cases} \mathbb{Z}_x^2 \setminus (\Omega \cup \widehat{\Omega}' \cup \widehat{\Omega}'' \cup \widehat{\Omega}^* \cup \widehat{\Omega}^{**}) & \text{if } \omega \in \mathbb{Z}_x^2, \\ (\mathbb{Z}_{x-}^2 \setminus (\widehat{\Omega}' \cup \widehat{\Omega}^*)) \cup (\mathbb{Z}_x^2 \setminus (\widehat{\Omega}'' \cup \widehat{\Omega}^{**})) & \text{if } \omega \in \mathbb{Z}_{-\infty}^2, \\ (\mathbb{Z}_x^2 \setminus (\widehat{\Omega}' \cup \widehat{\Omega}^*)) \cup (\mathbb{Z}_{x-}^2 \setminus (\widehat{\Omega}'' \cup \widehat{\Omega}^{**})) & \text{if } \omega \in \mathbb{Z}_{\infty}^2, \\ \mathbb{Z}_{x-}^2 \setminus (\Omega \cup \widehat{\Omega}' \cup \widehat{\Omega}'' \cup \widehat{\Omega}^* \cup \widehat{\Omega}^{**}) & \text{if } \omega \in \mathbb{Z}_{x-}^2 \end{cases}$$

and call $\widehat{\Omega}'^*$ the *prenonadjacency set* of ω relative to f.

Here the prefix "pre" is meant to suggest that to take care of some exceptional cases, we want to introduce an improved version of these concepts. To prepare for this, by $\widehat{\Omega}'_{-\infty}$ (respectively $\widehat{\Omega}''_{-\infty}$) we denote the common preantecendental set (respectively the common preconsequental set) of all $\overline{\omega} \in \mathbb{Z}_{-\infty}^2$, and call this the *preantecedental set* (respectively the *preconsequental set*) of $\mathbb{Z}_{-\infty}^2$. Moreover, by $\widehat{\Omega}^*_{-\infty}$ (respectively $\widehat{\Omega}^{**}_{-\infty}$) we denote the common preantecendental segment (respectively the common preconsequental segment) of all $\overline{\omega} \in \mathbb{Z}_{-\infty}^2$, and call this the *preantecedental segment* (respectively the *preconsequental segment*) of $\mathbb{Z}_{-\infty}^2$. Likewise, by $\widehat{\Omega}'_{\infty}$ (respectively $\widehat{\Omega}''_{\infty}$) we denote the common preantecendental set (respectively the common preconsequental set) of all $\overline{\omega} \in \mathbb{Z}_{\infty}^2$, and call this the *preantecedental set* (respectively the *preconsequental set*) of $\mathbb{Z}_{-\infty}^2$. Moreover, by $\widehat{\Omega}^*_{\infty}$ (respectively $\widehat{\Omega}^{**}_{\infty}$) we denote the common preantecendental segment (respectively the common preconsequental segment) of all $\overline{\omega} \in \mathbb{Z}_{\infty}^2$, and call this the *preantecedental segment* (respectively the *preconsequental segment*) of $\mathbb{Z}_{\infty}^2$.

Now let

$$\Omega' = \begin{cases} \widehat{\Omega}'_{-\infty} & \text{if } \omega_1 > 0 \ \&\ S' = \emptyset \ \&\ \text{defo}_Y f \text{ is a monomial}, \\ \widehat{\Omega}'_{-\infty} & \text{if } \omega_1 = 0 > \omega_2 \ \&\ S' = \emptyset \ \&\ \text{defo}_Y f \text{ is a monomial}, \\ \widehat{\Omega}'_{\infty} & \text{if } \omega_1 < 0 \ \&\ S'' = \emptyset \ \&\ \text{info}_Y f \text{ is a monomial}, \\ \widehat{\Omega}'_{\infty} & \text{if } \omega_1 = 0 < \omega_2 \ \&\ S'' = \emptyset \ \&\ \text{info}_Y f \text{ is a monomial}, \\ \widehat{\Omega}' & \text{if none of the above} \end{cases}$$

and let

$$\Omega'' = \begin{cases} \widehat{\Omega}''_{\infty} & \text{if } \omega_1 > 0 \ \&\ S'' = \emptyset \ \&\ \text{info}_Y f \text{ is a monomial}, \\ \widehat{\Omega}''_{\infty} & \text{if } \omega_1 = 0 < \omega_2 \ \&\ S'' = \emptyset \ \&\ \text{info}_Y f \text{ is a monomial}, \\ \widehat{\Omega}''_{-\infty} & \text{if } \omega_1 < 0 \ \&\ S' = \emptyset \ \&\ \text{defo}_Y f \text{ is a monomial}, \\ \widehat{\Omega}''_{-\infty} & \text{if } \omega_1 = 0 > \omega_2 \ \&\ S' = \emptyset \ \&\ \text{defo}_Y f \text{ is a monomial}, \\ \widehat{\Omega}'' & \text{if none of the above.} \end{cases}$$

Also let

$$\Omega^* = \begin{cases} \widehat{\Omega}^* \cup \widehat{\Omega}' \cup \widehat{\Omega}^*_{-\infty} & \text{if } \omega_1 > 0 \ \&\ S' = \emptyset \ \&\ \text{defo}_Y f \text{ is a monomial}, \\ \widehat{\Omega}^* \cup \widehat{\Omega}' \cup \widehat{\Omega}^*_{-\infty} & \text{if } \omega_1 = 0 > \omega_2 \ \&\ S' = \emptyset \ \&\ \text{defo}_Y f \text{ is a monomial}, \\ \widehat{\Omega}^* \cup \widehat{\Omega}' \cup \widehat{\Omega}^*_{\infty} & \text{if } \omega_1 < 0 \ \&\ S'' = \emptyset \ \&\ \text{info}_Y f \text{ is a monomial}, \\ \widehat{\Omega}^* \cup \widehat{\Omega}' \cup \widehat{\Omega}^*_{\infty} & \text{if } \omega_1 = 0 < \omega_2 \ \&\ S'' = \emptyset \ \&\ \text{info}_Y f \text{ is a monomial}, \\ \widehat{\Omega}^* & \text{if none of the above} \end{cases}$$

and let

$$\Omega^{**} = \begin{cases} \widehat{\Omega}^{**} \cup \widehat{\Omega}'' \cup \widehat{\Omega}^{**}_{\infty} & \text{if } \omega_1 > 0 \ \& \ S'' = \emptyset \ \& \ \text{info}_Y f \text{ is a monomial,} \\ \widehat{\Omega}^{**} \cup \widehat{\Omega}'' \cup \widehat{\Omega}^{**}_{\infty} & \text{if } \omega_1 = 0 < \omega_2 \ \& \ S'' = \emptyset \ \& \ \text{info}_Y f \text{ is a monomial,} \\ \widehat{\Omega}^{**} \cup \widehat{\Omega}'' \cup \widehat{\Omega}^{**}_{-\infty} & \text{if } \omega_1 < 0 \ \& \ S' = \emptyset \ \& \ \text{defo}_Y f \text{ is a monomial,} \\ \widehat{\Omega}^{**} \cup \widehat{\Omega}'' \cup \widehat{\Omega}^{**}_{-\infty} & \text{if } \omega_1 = 0 > \omega_2 \ \& \ S' = \emptyset \ \& \ \text{defo}_Y f \text{ is a monomial,} \\ \widehat{\Omega}^{**} & \text{if none of the above} \end{cases}$$

and call Ω^* (respectively Ω^{**}) the *antecedental segment* (respectively the *consequental segment*) of ω relative to f.

Finally let

$$\Omega'^* = \mathbb{Z}^2_z \setminus \left(\Omega \cup \Omega' \cup \Omega'' \cup \Omega^* \cup \Omega^{**}\right)$$

and call Ω'^* the *nonadjacency set* of ω relative to f.

Then we have the following.

(I) The nine sets $\Omega, \widehat{\Omega}', \Omega', \widehat{\Omega}'', \Omega'', \widehat{\Omega}^* \subset \Omega^*, \widehat{\Omega}^{**} \subset \Omega^{**}$ are nonempty subsets of $\mathbb{Z}^2_z$ with $\omega \in \Omega$ such that for all $\overline{\omega} \in \Omega$ we have $\|\overline{\omega}\| = \|\omega\|$, for all $\overline{\omega}, \widehat{\omega}$ in $\widehat{\Omega}'$ we have $\|\overline{\omega}\| = \|\widehat{\omega}\|$, for all $\overline{\omega}, \widehat{\omega}$ in Ω' we have $\|\overline{\omega}\| = \|\widehat{\omega}\|$, for all $\overline{\omega}, \widehat{\omega}$ in $\widehat{\Omega}''$ we have $\|\overline{\omega}\| = \|\widehat{\omega}\|$, and for all $\overline{\omega}, \widehat{\omega}$ in Ω'' we have $\|\overline{\omega}\| = \|\widehat{\omega}\|$. Out of these the five sets $\Omega, \widehat{\Omega}', \widehat{\Omega}'', \widehat{\Omega}^*, \widehat{\Omega}^{**}$ are pairwise disjoint with the following exception $(\widehat{E})$, and the five sets $\Omega, \Omega', \Omega'', \Omega^*, \Omega^{**}$ are pairwise disjoint, with the following exceptions (E_1) and (E_2).

$(\widehat{E})$ *If f is ω-horizontal* (which means if $f = f^+_\omega$ with $\omega_1 = 0$) then $\widehat{\Omega}' = \widehat{\Omega}''$ and for all $\overline{\omega} \in \widehat{\Omega}' = \widehat{\Omega}''$ we have $\|\overline{\omega}\| = \|-\omega\|$ with $f^+_{\overline{\omega}} = f^+_{-\omega} = f$; note that now $\text{Supp}(f^+_\omega) = \text{Supp}(f^+_{\overline{\omega}})$ for all $\overline{\omega} \in \widehat{\Omega}' = \widehat{\Omega}''$ and hence: f is a monomial $\Leftrightarrow$ f is a ω-monomial $\Leftrightarrow$ f is a $\overline{\omega}$-monomial for all $\overline{\omega} \in \widehat{\Omega}' = \widehat{\Omega}''$.

(E_1) *If f is ω-pseudolinear and f is a nonmonomial* (which means if $f = f^+_\omega$ and card $\text{Supp}(f) \neq 1$) then $\Omega' = \Omega''$ and for all $\overline{\omega} \in \Omega' = \Omega''$ we have $\|\overline{\omega}\| = \|-\omega\|$ with $f^+_{\overline{\omega}} = f^+_{-\omega} = f$, and hence in particular $\text{Supp}(f^+_\omega) = \text{Supp}(f^+_{\overline{\omega}})$ and f is not a $\overline{w}$-monomial.

(E_2) *If f is a monomial* (i.e., if card $\text{Supp}(f) = 1$) then $\Omega^* \cap \Omega^{**} \neq \emptyset$ and for all $\overline{\omega} \in \mathbb{Z}^2_z$ we have $f^+_{\overline{\omega}} = f$ and hence in particular $\text{Supp}(f^+_\omega) = \text{Supp}(f^+_{\overline{\omega}})$ and f is a $\overline{w}$-monomial.

(II) If $\omega_1 > 0$ then for all $\overline{\omega} \in \widehat{\Omega}'$ we have $|\overline{\omega}| < |\omega|$, and for all $\overline{\omega} \in \widehat{\Omega}''$ we have $|\overline{\omega}| > |\omega|$. If $\omega_1 < 0$ then for all $\overline{\omega} \in \widehat{\Omega}'$ we have either $|\overline{\omega}| < |\omega|$ or $|\overline{\omega}| = \infty$, and for all $\overline{\omega} \in \widehat{\Omega}''$ we have either $|\overline{\omega}| > |\omega|$ or $|\overline{\omega}| = -\infty$. If $\omega_1 = 0 < \omega_2$ then for all $\overline{\omega} \in \widehat{\Omega}'$ we have $|\overline{\omega}| \in \mathbb{Q} \cup \{\infty\}$, and for all $\overline{\omega} \in \widehat{\Omega}''$ we have $|\overline{\omega}| \in \mathbb{Q} \cup \{\infty\}$. If $\omega_1 = 0 > \omega_2$ then for all $\overline{\omega} \in \widehat{\Omega}'$ we have $|\overline{\omega}| \in \mathbb{Q} \cup \{-\infty\}$, and for all $\overline{\omega} \in \widehat{\Omega}''$ we have $|\overline{\omega}| \in \mathbb{Q} \cup \{-\infty\}$.

(III) For all $\overline{\omega} \in \widehat{\Omega}^*$ we have $\text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \{f^!_\omega\} = \{f^{!!}_{\overline{\omega}}\}$, for all $\overline{\omega} \in \widehat{\Omega}^{**}$ we have $\text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \{f^{!!}_\omega\} = \{f^!_{\overline{\omega}}\}$, for all $\overline{\omega} \in \widehat{\Omega}^* \cup \widehat{\Omega}^{**}$ we have that f is a $\overline{\omega}$-monomial. If $\omega \in \mathbb{Z}^2_x$ then for any $\overline{\omega} \in \mathbb{Z}^2_x$ we have: $\overline{\omega} \in \widehat{\Omega}'^* \Leftrightarrow \text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \emptyset$. If $\omega \in \mathbb{Z}^2_{-\infty} \cup \mathbb{Z}^2_\infty$ then for any $\overline{\omega} \in \mathbb{Z}^2_x \cup \mathbb{Z}^2_{x-}$ we have: $\overline{\omega} \in \widehat{\Omega}'^* \Leftrightarrow \text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \emptyset$. If $\omega \in \mathbb{Z}^2_{x-}$ then for any $\overline{\omega} \in \mathbb{Z}^2_{x-}$ we have: $\overline{\omega} \in \widehat{\Omega}'^* \Leftrightarrow \text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \emptyset$.

If f is not ω-horizontal then: for all $\overline{\omega} \in \widehat{\Omega}'$ we have $\text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \{f^!_\omega\} = \{f^{!!}_{\overline{\omega}}\}$, for all $\overline{\omega} \in \widehat{\Omega}''$ we have $\text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \{f^{!!}_\omega\} = \{f^!_{\overline{\omega}}\}$, and for all $\overline{\omega} \in (\widehat{\Omega}' \cup \widehat{\Omega}'') \cap (\mathbb{Z}^2_x \cup \mathbb{Z}^2_{x-})$ we have that f is not a $\overline{\omega}$-monomial.

(IV) For all $\overline{\omega} \in \Omega^*$ we have $\text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \{f^!_\omega\} = \{f^{!!}_{\overline{\omega}}\}$, for all $\overline{\omega} \in \Omega^{**}$ we have $\text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \{f^{!!}_\omega\} = \{f^!_{\overline{\omega}}\}$, for all $\overline{\omega} \in \Omega^* \cup \Omega^{**}$ we have that f is a $\overline{\omega}$-monomial, and for any $\overline{\omega} \in \mathbb{Z}^2_z$ we have: $\overline{\omega} \in \Omega'^* \Leftrightarrow \text{Supp}(f^+_\omega) \cap \text{Supp}(f^+_{\overline{\omega}}) = \emptyset$.

If f is not ω-pseudolinear then: for all $\overline{\omega} \in \Omega'$ we have $\mathrm{Supp}(f_\omega^+) \cap \mathrm{Supp}(f_{\overline{\omega}}^+) = \{f_\omega^!\} = \{f_{\overline{\omega}}^{!!}\}$, for all $\overline{\omega} \in \Omega''$ we have $\mathrm{Supp}(f_\omega^+) \cap \mathrm{Supp}(f_{\overline{\omega}}^+) = \{f_\omega^{!!}\} = \{f_{\overline{\omega}}^!\}$, and for all $\overline{\omega} \in \Omega' \cup \Omega''$ we have that f is not a $\overline{\omega}$-monomial.

(V) Assuming $f \in k[X,Y]^\times$ is not a ω-monomial we have the following.

(V.1) If $\omega_1 \neq 0$ then $j' > j''$ and $|\omega| = \frac{i'-i''}{j'-j''}$ with $\deg_\omega f = \frac{(i''j'-i'j'')\omega_1}{j'-j''}$.

(V.2) If $\omega_1 = 0$ then $j' = j''$ with $i' < i''$ and $|\omega| = \pm\infty$ (according as ω_2 is negative or positive) with $\deg_\omega f = j'\omega_2$.

(V.3) If $\{f_\omega^!, f_\omega^{!!}\} \subset \mathbb{Z}_z^2$ then $|f_\omega^!| < |f_\omega^{!!}|$ or $|f_\omega^!| = |f_\omega^{!!}|$ or $|f_\omega^!| > |f_\omega^{!!}|$ according as $\deg_\omega f > 0$ or $\deg_\omega f = 0$ or $\deg_\omega f < 0$. [*Note that* if $\deg_\omega f \neq 0$ then obviously $\{f_\omega^!, f_\omega^{!!}\} \subset \mathbb{Z}_z^2$.] For pictures see (9.7).

(V.4) If either (i) $\omega_1 > 0 < \omega_2$, or (ii) $\omega_1 > 0 \geqslant \omega_2$ with $f - f(0,0) \notin Yk[X,Y]$, or (iii) $\omega_2 > 0 \geqslant \omega_1$ with $f - f(0,0) \notin Xk[X,Y]$, then $\deg_\omega f > 0$ and hence by (V.3) we get $\{f_\omega^!, f_\omega^{!!}\} \subset \mathbb{Z}_z^2$ with $|f_\omega^!| < |f_\omega^{!!}|$.

Proof of Fact (1). To prove (I), since (i,j) and (i',j') both belong to $\mathrm{Supp}(f_\omega^+)$, we see that $i\omega_1 + j\omega_2 = i'\omega_1 + j'\omega_2$; consequently, if $j' \neq j$ then we must have $\omega_1 \neq 0$ and dividing by it we get $|\omega| = \frac{i'-i}{j'-j} \in \mathbb{Q}$. By the relations $i\omega_1 + j\omega_2 = i'\omega_1 + j'\omega_2$ and $(i,j) \neq (i',j')$, we also see that if $j' = j$ then we must have $\omega_1 = 0$. The rest of (I) is now clear.

To prove (II), clearly it suffices to establish the three implications:

(i) $\mathrm{Supp}(f_\omega^+) \cap \mathrm{Supp}(f_{\overline{\omega}}^+) \neq \emptyset \Rightarrow f_\omega^+ = f = f_{\overline{\omega}}^+$,
(ii) $f_\omega^+ = f_{\overline{\omega}}^+ \Rightarrow \mathrm{Supp}(f_\omega^+) \cap \mathrm{Supp}(f_{\overline{\omega}}^+) \neq \emptyset$, and
(iii) $f = f_\omega^+ \Rightarrow f = f_{\overline{\omega}}^+$.

To establish (i), taking some $(i^\flat, j^\flat)$ in $\mathrm{Supp}(f_\omega^+) \cap \mathrm{Supp}(f_{\overline{\omega}}^+)$ we get

$$\begin{cases} \max\{i\omega_1 + j\omega_2\colon (i,j) \in \mathrm{Supp}(f)\} \\ \quad = i^\flat\omega_1 + j^\flat\omega_2 \\ \quad = -(i^\flat\overline{\omega}_1 + j^\flat\overline{\omega}_2) \\ \quad = -\max\{i\overline{\omega}_1 + j\overline{\omega}_2\colon (i,j) \in \mathrm{Supp}(f)\} \\ \quad = -\max\{-i\omega_1 - j\omega_2\colon (i,j) \in \mathrm{Supp}(f)\} \\ \quad = \min\{i\omega_1 + j\omega_2\colon (i,j) \in \mathrm{Supp}(f)\} \end{cases}$$

and hence for all $(i,j) \in \mathrm{Supp}(f)$ we must have $i\omega_1 + j\omega_2 = i^\flat\omega_1 + j^\flat\omega_2$ and therefore also $i\overline{\omega}_1 + j\overline{\omega}_2 = i^\flat\overline{\omega}_1 + j^\flat\overline{\omega}_2$; consequently $f_\omega^+ = f = f_{\overline{\omega}}^+$. (ii) is obvious. To establish (iii), taking some $(i^\flat, j^\flat) \in \mathrm{Supp}(f_\omega^+)$ we see that for all $(i,j) \in \mathrm{Supp}(f)$ we have $i\omega_1 + j\omega_2 = i^\flat\omega_1 + j^\flat\omega_2$, and hence for all $(i,j) \in \mathrm{Supp}(f)$ we have $i\overline{\omega}_1 + j\overline{\omega}_2 = i^\flat\overline{\omega}_1 + j^\flat\overline{\omega}_2$, and therefore $f = f_{\overline{\omega}}^+$.

(III) is obvious and in view of it, while proving (IV) to (IX), by replacing $\omega, \overline{\omega}$ by their norms we may assume that they coincide with their norms, and then (IV) to (IX) follow from (II). By (I) to (III) we get (X). (XI) is obvious and in view of it (XII) follows from (I), (IV), and (V). □

Proof of Fact (2). By symmetry, first in X, Y and then in $(i',j'), (i'',j'')$, without loss of generality, we may assume that

(III*) $$j' < j < j''.$$

Then by (I) we get $\omega_1 \neq 0$, and hence dividing (I) by ω_1 we see that $|\omega| \in \mathbb{Q}$ with

$$(\mathrm{I}^*) \qquad i' - j'|\omega| = i - |\omega| j = i'' - |\omega| j''.$$

If $\bar{\omega}_1 = 0$ then (IV) follows from (III*). So assume that $\bar{\omega}_1 \neq 0$. Then dividing (II) by $\bar{\omega}_1$ we see that $|\bar{\omega}| \in \mathbb{Q}$ with

$$(\mathrm{II}^*) \qquad i' - j'|\bar{\omega}| \neq i - |\bar{\omega}| j \neq i'' - |\bar{\omega}| j''.$$

By (I*) and (II*) we get $|\bar{\omega}| \neq |\omega|$ and hence

$$(\mathrm{III}^{**}) \qquad \text{either } |\bar{\omega}| < |\omega| \text{ or } |\bar{\omega}| > |\omega|.$$

Dividing by $\bar{\omega}_1$ we see that (IV) is equivalent to saying that

$$(\mathrm{IV}^*) \qquad \begin{cases} \text{either } i' - j'|\bar{\omega}| < i - |\bar{\omega}| j < i'' - |\bar{\omega}| j'' \\ \quad \text{or } i' - j'|\bar{\omega}| > i - |\bar{\omega}| j > i'' - |\bar{\omega}| j'' \end{cases}$$

and hence it is also equivalent to saying that

$$(\mathrm{IV}^{**}) \qquad \begin{cases} \text{either } (j - j')|\bar{\omega}| + (i' - i) < 0 \text{ and } (j'' - j)|\bar{\omega}| + (i - i'') < 0 \\ \quad \text{or } (j - j')|\bar{\omega}| + (i' - i) > 0 \text{ and } (j'' - j)|\bar{\omega}| + (i - i'') > 0. \end{cases}$$

By (I*) we see that

$$(\mathrm{I}^{**}) \qquad (j - j')|\omega| + (i' - i) = 0 \quad \text{and} \quad (j'' - j)|\omega| + (i - i'') = 0.$$

Now (IV**) follows from (III**) and (I**) by using (V). □

Proof of Fact (3). The proof of (3) follows from the thirty-five parts (10), (11), (12), (13′), ..., (35) which are established below and are collated in the last part (35). Thus the proof of (3) is long, but it is internally repetitive. For instance parts (20) to (28″) are mostly obtained from parts (10) to (18″) by selectively flipping the signs $<$ and $>$ (also the signs $\leqslant$ and $\geqslant$); same applies to parts (13″) to (18″) and (13′) to (18′); likewise parts (23″) to (28″) and (23′) to (28′); similarly the material between (10) and (12) has an internal repetition of this type, and so does the material between (20) and (22).

Now

$$\text{for any } \omega \in \mathbb{Z}_x^2 \quad \text{we have } |\omega| = -\omega_2/\omega_1 \quad \text{with } \omega_1 > 0$$

and therefore

$$\begin{cases} (i, j) \in \operatorname{Supp}(f) \text{ and } (i^\flat, j^\flat) \in \operatorname{Supp}(f_\omega^+) \text{ with } \omega \in \mathbb{Z}_x^2 \\ \quad \Rightarrow i\omega_1 + j\omega_2 \leqslant i^\flat \omega_1 + j^\flat \omega_2 \\ \quad \Rightarrow i - j|\omega| \leqslant i^\flat - j^\flat |\omega| \\ \quad \Rightarrow (j - j^\flat)|\omega| + (i^\flat - i) \geqslant 0 \end{cases}$$

(where for the second implication divide by $\omega_1 > 0$) and hence

$$(10)\qquad \begin{cases} \text{for any } (i,j) \in \mathrm{Supp}(f) \text{ and } (i^\flat, j^\flat) \in \mathrm{Supp}(f_\omega^+) \text{ with } \omega \in \mathbb{Z}_x^2 \\ \quad \text{we have } (j - j^\flat)|\omega| + (i^\flat - i) \geqslant 0 \end{cases}$$

and therefore by (2)(V) we see that

$$(11)\qquad \begin{cases} \text{for any } \omega \in \mathbb{Z}_x^2 \;\&\; \widehat{\omega} \in \mathbb{Z}_z^2 \text{ with } (i^\flat, j^\flat) \in \mathrm{Supp}(f_\omega^+) \text{ and } (i,j) \in \mathrm{Supp}(f) \\ \quad \text{we have that if } |\widehat{\omega}| < |\omega| \text{ and } j \leqslant j^\flat \text{ then } (j - j^\flat)|\widehat{\omega}| + (i^\flat - i) \geqslant 0 \\ \quad \text{whereas if } |\widehat{\omega}| < |\omega| \text{ and } j < j^\flat \text{ then } (j - j^\flat)|\widehat{\omega}| + (i^\flat - i) > 0 \\ \quad \& \text{ if } |\widehat{\omega}| < |\omega| \text{ and } j = j^\flat \text{ with } i \neq i^\flat \text{ then } (j^\flat - j)\widehat{\omega} + (i^\flat - i) > 0 \\ \quad \text{and similarly if } |\widehat{\omega}| > |\omega| \text{ and } j \geqslant j^\flat \text{ then } (j - j^\flat)|\widehat{\omega}| + (i^\flat - i) \geqslant 0 \\ \quad \text{whereas if } |\widehat{\omega}| > |\omega| \text{ and } j > j^\flat \text{ then } (j - j^\flat)|\widehat{\omega}| + (i^\flat - i) > 0 \\ \quad \& \text{ if } |\widehat{\omega}| > |\omega| \text{ and } j = j^\flat \text{ with } i \neq i^\flat \text{ then } (j^\flat - j)\widehat{\omega} + (i^\flat - i) > 0 \end{cases}$$

where the $j = j^\flat$ assertions follow from the $j \geqslant j^\flat$ and $j \leqslant j^\flat$ assertions, and hence upon multiplying by $\widehat{\omega}_1 > 0$ we see that

$$\begin{cases} \text{for any } \omega, \widehat{\omega} \text{ in } \mathbb{Z}_x^2 \text{ with } (i^\flat, j^\flat) \in \mathrm{Supp}(f_\omega^+) \text{ and } (i,j) \in \mathrm{Supp}(f) \\ \quad \text{we have that if } |\widehat{\omega}| < |\omega| \text{ and } j \leqslant j^\flat \text{ then } (j^\flat - j)\widehat{\omega}_2 + (i^\flat - i)\widehat{\omega}_1 \geqslant 0 \\ \quad \text{whereas if } |\widehat{\omega}| < |\omega| \text{ and } j < j^\flat \text{ then } (j^\flat - j)\widehat{\omega}_2 + (i^\flat - i)\widehat{\omega}_1 > 0 \\ \quad \& \text{ if } |\widehat{\omega}| < |\omega| \text{ and } j = j^\flat \text{ with } i \neq i^\flat \text{ then } (j^\flat - j)\widehat{\omega}_2 + (i^\flat - i)\widehat{\omega}_1 > 0 \\ \quad \text{and similarly if } |\widehat{\omega}| > |\omega| \text{ and } j \geqslant j^\flat \text{ then } (j^\flat - j)\widehat{\omega}_2 + (i^\flat - i)\widehat{\omega}_1 \geqslant 0 \\ \quad \text{whereas if } |\widehat{\omega}| > |\omega| \text{ and } j > j^\flat \text{ then } (j^\flat - j)\widehat{\omega}_2 + (i^\flat - i)\widehat{\omega}_1 > 0 \\ \quad \& \text{ if } |\widehat{\omega}| > |\omega| \text{ and } j = j^\flat \text{ with } i \neq i^\flat \text{ then } (j^\flat - j)\widehat{\omega}_2 + (i^\flat - i)\widehat{\omega}_1 > 0 \end{cases}$$

and therefore by transferring terms we see that

$$(12)\qquad \begin{cases} \text{for any } \omega, \widehat{\omega} \text{ in } \mathbb{Z}_x^2 \text{ with } (i^\flat, j^\flat) \in \mathrm{Supp}(f_\omega^+) \text{ and } (i,j) \in \mathrm{Supp}(f) \\ \quad \text{we have that if } |\widehat{\omega}| < |\omega| \text{ and } j \leqslant j^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 \geqslant i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \text{whereas if } |\widehat{\omega}| < |\omega| \text{ and } j < j^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 > i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \& \text{ if } |\widehat{\omega}| < |\omega| \text{ and } j = j^\flat \text{ with } i \neq i^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 > i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \text{and similarly if } |\widehat{\omega}| > |\omega| \text{ and } j \geqslant j^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 \geqslant i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \text{whereas if } |\widehat{\omega}| > |\omega| \text{ and } j > j^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 > i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \& \text{ if } |\widehat{\omega}| > |\omega| \text{ and } j = j^\flat \text{ with } i \neq i^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 > i\widehat{\omega}_1 + j\widehat{\omega}_2 \end{cases}$$

(13′) *From here until* (18′) suppose that $\overline{\omega} \in \widehat{\Omega}'$ with $S' \neq \emptyset$ and either (i) $\omega_1 > 0$ or (ii) $\omega_1 = 0 > \omega_2$ with i' changed to $\max\{i\colon\ (i,j) \in \mathrm{Supp}(f_\omega^+)\}$.

Now

$$(14')\qquad |\overline{\omega}| = -\overline{\omega}_2/\overline{\omega}_1 \quad \text{with } \overline{\omega}_1 > 0$$

and we can take

$$(15') \qquad \big(i^*, j^*\big) \in S' \quad \text{with } |\overline{\omega}| = \frac{i' - i^*}{j' - j^*} = \max_{(i,j)\in S'} \frac{i' - i}{j' - j}.$$

If $\omega_1 = 0 > \omega_2$ then $|\omega| = \infty$ and hence $|\overline{\omega}| < |\omega|$ by (14′), whereas if $\omega_1 > 0$ then

$$\begin{cases} (i^*, j^*) \in S' \\ \quad \Rightarrow i^*\omega_1 + j^*\omega_2 < i'\omega_1 + j'\omega_2 \\ \quad \Rightarrow i^* - j^*|\omega| < i' - j'|\omega| & \text{dividing by } \omega_1 > 0 \\ \quad \Rightarrow (j' - j^*)|\omega| < i' - i^* \\ \quad \Rightarrow |\omega| > \frac{i'-i^*}{j'-j^*} & \text{dividing by } j' - j^* < 0 \end{cases}$$

and so $|\overline{\omega}| < |\omega|$ by (15′). Thus in both the cases we have

$$(16') \qquad |\overline{\omega}| < |\omega|.$$

By (15′) we also see that

$$\begin{cases} \text{for all } (i, j) \in S' \text{ we have} \\ \quad i - j|\overline{\omega}| \leqslant i' - j'|\overline{\omega}| = i^* - j^*|\overline{\omega}| \end{cases}$$

and hence, in view of (14′), multiplying by $\overline{\omega}_1 > 0$ we conclude that

$$(17') \qquad \begin{cases} \text{for all } (i, j) \in S' \text{ we have} \\ \quad i\overline{\omega}_1 + j\overline{\omega}_2 \leqslant i'\overline{\omega}_1 + j'\overline{\omega}_2 = i^*\overline{\omega}_1 + j^*\overline{\omega}_2. \end{cases}$$

Transferring terms in (17′) and dividing by $\overline{\omega}_1 > 0$ we see that

$$\begin{cases} \text{for all } (i, j) \in S' \text{ we have } (j' - j)|\overline{\omega}| + (i - i') \geqslant 0 \\ \quad \text{and we always have } (j' - j^*)|\overline{\omega}| + (i^* - i') = 0 \end{cases}$$

and hence by (2)(V) we see that

$$\begin{cases} \text{for all } \widehat{\omega} \in \mathbb{Z}_x^2 \text{ we have that} \\ \quad \text{if } |\widehat{\omega}| > |\overline{\omega}| \text{ then } (j' - j)|\widehat{\omega}| + (i - i') < 0 \text{ for all } (i, j) \in S', \\ \quad \text{whereas if } |\widehat{\omega}| < |\overline{\omega}| \text{ then } (j' - j^*)|\widehat{\omega}| + (i^* - i') > 0 \end{cases}$$

and therefore transferring terms and multiplying by $\widehat{\omega}_1 > 0$ we conclude that

$$(18') \qquad \begin{cases} \text{for all } \widehat{\omega} \in \mathbb{Z}_x^2 \text{ we have that} \\ \quad \text{if } |\widehat{\omega}| > |\overline{\omega}| \text{ then } i\widehat{\omega}_1 + j\widehat{\omega}_2 < i'\widehat{\omega}_1 + j'\widehat{\omega}_2 \text{ for all } (i, j) \in S', \\ \quad \text{whereas if } |\widehat{\omega}| < |\overline{\omega}| \text{ then } i^*\widehat{\omega}_1 + j^*\widehat{\omega}_2 > i'\widehat{\omega}_1 + j'\widehat{\omega}_2. \end{cases}$$

(13″) *From here until* (18″) suppose that $\overline{\omega} \in \widehat{\Omega}''$ with $S'' \neq \emptyset$ and either $\omega_1 > 0$ or $\omega_1 = 0 < \omega_2$.

Now

$$|\overline{\omega}| = -\overline{\omega}_2/\overline{\omega}_1 \quad \text{with } \overline{\omega}_1 > 0 \tag{14''}$$

and we can take

$$\left(i^*, j^*\right) \in S'' \quad \text{with } |\overline{\omega}| = \frac{i'' - i^*}{j'' - j^*} = \min_{(i,j)\in S''} \frac{i'' - i}{j'' - j}. \tag{15''}$$

If $\omega_1 = 0 < \omega_2$ then $|\omega| = -\infty$ and hence $|\overline{\omega}| > |\omega|$ by (14″), whereas if $\omega_1 > 0$ then

$$\begin{cases} (i^*, j^*) \in S'' \\ \quad \Rightarrow i^*\omega_1 + j^*\omega_2 < i''\omega_1 + j''\omega_2 \\ \quad \Rightarrow i^* - j^*|\omega| < i'' - j''|\omega| & \text{dividing by } \omega_1 > 0 \\ \quad \Rightarrow (j'' - j^*)|\omega| < i'' - i^* \\ \quad \Rightarrow |\omega| < \frac{i''-i^*}{j''-j^*} & \text{dividing by } j'' - j^* > 0 \end{cases}$$

and so $|\overline{\omega}| > |\omega|$ by (15″). Thus in both the cases we have

$$|\overline{\omega}| > |\omega|. \tag{16''}$$

By (15″) we also see that

$$\begin{cases} \text{for all } (i, j) \in S'' \text{ we have} \\ \quad i - j|\overline{\omega}| \leqslant i'' - j''|\overline{\omega}| = i^* - j^*|\overline{\omega}| \end{cases}$$

and hence, in view of (14″), multiplying by $\overline{\omega}_1 > 0$ we conclude that

$$\begin{cases} \text{for all } (i, j) \in S'' \text{ we have} \\ \quad i\overline{\omega}_1 + j\overline{\omega}_2 \leqslant i''\overline{\omega}_1 + j''\overline{\omega}_2 = i^*\overline{\omega}_1 + j^*\overline{\omega}_2. \end{cases} \tag{17''}$$

Transferring terms in (17″) and dividing by $\overline{\omega}_1 > 0$ we see that

$$\begin{cases} \text{for all } (i, j) \in S'' \text{ we have } (j'' - j)|\overline{\omega}| + (i - i'') \leqslant 0 \\ \quad \text{and we always have } (j'' - j^*)|\overline{\omega}| + (i^* - i'') = 0 \end{cases}$$

and hence by (2)(V) we see that

$$\begin{cases} \text{for all } \widehat{\omega} \in \mathbb{Z}_x^2 \text{ we have that} \\ \quad \text{if } |\widehat{\omega}| < |\overline{\omega}| \text{ then } (j'' - j)|\widehat{\omega}| + (i - i'') < 0 \text{ for all } (i, j) \in S'', \\ \quad \text{whereas if } |\widehat{\omega}| > |\overline{\omega}| \text{ then } (j'' - j^*)|\widehat{\omega}| + (i^* - i'') > 0 \end{cases}$$

and therefore transferring terms and multiplying by $\widehat{\omega}_1 > 0$ we conclude that

$$\begin{cases} \text{for all } \widehat{\omega} \in \mathbb{Z}_x^2 \text{ we have that} \\ \quad \text{if } |\widehat{\omega}| < |\overline{\omega}| \text{ then } i\widehat{\omega}_1 + j\widehat{\omega}_2 < i''\widehat{\omega}_1 + j''\widehat{\omega}_2 \text{ for all } (i, j) \in S'', \\ \quad \text{whereas if } |\widehat{\omega}| > |\overline{\omega}| \text{ then } i^*\widehat{\omega}_1 + j^*\widehat{\omega}_2 > i''\widehat{\omega}_1 + j''\widehat{\omega}_2. \end{cases} \tag{18''}$$

Turning to negative x-weight

$$\text{for any } \omega\in\mathbb{Z}^2_{x-}\quad \text{we have } |\omega| = -\omega_2/\omega_1 \quad \text{with } \omega_1<0$$

and therefore

$$\begin{cases} (i,j)\in\operatorname{Supp}(f) \text{ and } (i^\flat,j^\flat)\in\operatorname{Supp}(f^+_\omega) \text{ with } \omega\in\mathbb{Z}^2_{x-} \\ \quad \Rightarrow i\omega_1 + j\omega_2 \leqslant i^\flat\omega_1 + j^\flat\omega_2 \\ \quad \Rightarrow i - j|\omega| \geqslant i^\flat - j^\flat|\omega| \\ \quad \Rightarrow (j-j^\flat)|\omega| + (i^\flat - i) \leqslant 0 \end{cases}$$

(where for the second implication divide by $\omega_1<0$) and hence

(20) $$\begin{cases} \text{for any } (i,j)\in\operatorname{Supp}(f) \text{ and } (i^\flat,j^\flat)\in\operatorname{Supp}(f^+_\omega) \text{ with } \omega\in\mathbb{Z}^2_{x-} \\ \quad \text{we have } (j-j^\flat)|\omega| + (i^\flat - i) \leqslant 0 \end{cases}$$

and therefore by (2)(V) we see that

(21) $$\begin{cases} \text{for any } \omega\in\mathbb{Z}^2_{x-}\ \&\ \widehat{\omega}\in\mathbb{Z}^2_{z} \text{ with } (i^\flat,j^\flat)\in\operatorname{Supp}(f^+_\omega) \text{ and } (i,j)\in\operatorname{Supp}(f) \\ \quad \text{we have that if } |\widehat{\omega}|>|\omega| \text{ and } j\leqslant j^\flat \text{ then } (j-j^\flat)|\widehat{\omega}| + (i^\flat-i)\leqslant 0 \\ \quad \text{whereas if } |\widehat{\omega}|>|\omega| \text{ and } j< j^\flat \text{ then } (j-j^\flat)|\widehat{\omega}| + (i^\flat-i)< 0 \\ \quad \&\text{ if } |\widehat{\omega}|>|\omega| \text{ and } j= j^\flat \text{ with } i\neq i^\flat \text{ then } (j^\flat-j)\widehat{\omega} + (i^\flat-i)> 0 \\ \quad \text{and similarly if } |\widehat{\omega}|<|\omega| \text{ and } j\geqslant j^\flat \text{ then } (j-j^\flat)|\widehat{\omega}| + (i^\flat-i)\leqslant 0 \\ \quad \text{whereas if } |\widehat{\omega}|<|\omega| \text{ and } j> j^\flat \text{ then } (j-j^\flat)|\widehat{\omega}| + (i^\flat-i)< 0 \\ \quad \&\text{ if } |\widehat{\omega}|<|\omega| \text{ and } j= j^\flat \text{ with } i\neq i^\flat \text{ then } (j^\flat-j)\widehat{\omega} + (i^\flat-i)> 0 \end{cases}$$

where the $j=j^\flat$ assertions follow from the $j\geqslant j^\flat$ and $j\leqslant j^\flat$ assertions, and hence upon multiplying by $\widehat{\omega}_1<0$ we see that

$$\begin{cases} \text{for any } \omega,\widehat{\omega} \text{ in } \mathbb{Z}^2_{x-} \text{ with } (i^\flat,j^\flat)\in\operatorname{Supp}(f^+_\omega) \text{ and } (i,j)\in\operatorname{Supp}(f) \\ \quad \text{we have that if } |\widehat{\omega}|>|\omega| \text{ and } j\leqslant j^\flat \text{ then } (j^\flat-j)\widehat{\omega}_2 + (i^\flat-i)\widehat{\omega}_1\geqslant 0 \\ \quad \text{whereas if } |\widehat{\omega}|>|\omega| \text{ and } j< j^\flat \text{ then } (j^\flat-j)\widehat{\omega}_2 + (i^\flat-i)\widehat{\omega}_1> 0 \\ \quad \&\text{ if } |\widehat{\omega}|>|\omega| \text{ and } j= j^\flat \text{ with } i\neq i^\flat \text{ then } (j^\flat-j)\widehat{\omega}_2 + (i^\flat-i)\widehat{\omega}_1> 0 \\ \quad \text{and similarly if } |\widehat{\omega}|<|\omega| \text{ and } j\geqslant j^\flat \text{ then } (j^\flat-j)\widehat{\omega}_2 + (i^\flat-i)\widehat{\omega}_1\geqslant 0 \\ \quad \text{whereas if } |\widehat{\omega}|<|\omega| \text{ and } j> j^\flat \text{ then } (j^\flat-j)\widehat{\omega}_2 + (i^\flat-i)\widehat{\omega}_1> 0 \\ \quad \&\text{ if } |\widehat{\omega}|<|\omega| \text{ and } j= j^\flat \text{ with } i\neq i^\flat \text{ then } (j^\flat-j)\widehat{\omega}_2 + (i^\flat-i)\widehat{\omega}_1> 0. \end{cases}$$

and therefore by transferring terms we see that

(22) $$\begin{cases} \text{for any } \omega, \widehat{\omega} \text{ in } \mathbb{Z}^2_{x-} \text{ with } (i^\flat, j^\flat) \in \mathrm{Supp}(f^+_\omega) \text{ and } (i,j) \in \mathrm{Supp}(f) \\ \quad \text{we have that if } |\widehat{\omega}| > |\omega| \text{ and } j \leqslant j^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 \geqslant i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \text{whereas if } |\widehat{\omega}| > |\omega| \text{ and } j < j^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 > i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \text{\& if } |\widehat{\omega}| > |\omega| \text{ and } j = j^\flat \text{ with } i \neq i^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 > i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \text{and similarly if } |\widehat{\omega}| < |\omega| \text{ and } j \geqslant j^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 \geqslant i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \text{whereas if } |\widehat{\omega}| < |\omega| \text{ and } j > j^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 > i\widehat{\omega}_1 + j\widehat{\omega}_2 \\ \quad \text{\& if } |\widehat{\omega}| < |\omega| \text{ and } j = j^\flat \text{ with } i \neq i^\flat \text{ then } i^\flat\widehat{\omega}_1 + j^\flat\widehat{\omega}_2 > i\widehat{\omega}_1 + j\widehat{\omega}_2. \end{cases}$$

(23′) *From here until* (28′) suppose that $\overline{\omega} \in \widehat{\Omega}''$ with $S' \neq \emptyset$ and either $\omega_1 < 0$ or $\omega_1 = 0 > \omega_2$.

Now

(24′) $$|\overline{\omega}| = -\overline{\omega}_2/\overline{\omega}_1 \quad \text{with } \overline{\omega}_1 < 0$$

and we can take

(25′) $$(i^*, j^*) \in S' \quad \text{with } |\overline{\omega}| = \frac{i'-i^*}{j'-j^*} = \min_{(i,j)\in S'} \frac{i'-i}{j'-j}.$$

If $\omega_1 = 0 > \omega_2$ then $|\omega| = \infty$ and hence $|\overline{\omega}| < |\omega|$ by (24′), whereas if $\omega_1 < 0$ then

$$\begin{cases} (i^*, j^*) \in S' \\ \quad \Rightarrow i^*\omega_1 + j^*\omega_2 < i'\omega_1 + j'\omega_2 \\ \quad \Rightarrow i^* - j^*|\omega| > i' - j'|\omega| & \text{dividing by } \omega_1 < 0 \\ \quad \Rightarrow (j'-j^*)|\omega| > i' - i^* \\ \quad \Rightarrow |\omega| < \frac{i'-i^*}{j'-j^*} & \text{dividing by } j'-j^* < 0 \end{cases}$$

and so $|\overline{\omega}| > |\omega|$ by (25′). Thus

(26′) $$|\overline{\omega}| > |\omega| \text{ or } |\overline{\omega}| < |\omega| \text{ according as } \omega_1 < 0 \text{ or } \omega_1 = 0 > \omega_2.$$

By (25′) we also see that

$$\begin{cases} \text{for all } (i,j) \in S' \text{ we have} \\ \quad i - j|\overline{\omega}| \geqslant i' - j'|\overline{\omega}| = i^* - j^*|\overline{\omega}| \end{cases}$$

and hence, in view of (24′), multiplying by $\overline{\omega}_1 < 0$ we conclude that

(27′) $$\begin{cases} \text{for all } (i,j) \in S' \text{ we have} \\ \quad i\overline{\omega}_1 + j\overline{\omega}_2 \leqslant i'\overline{\omega}_1 + j'\overline{\omega}_2 = i^*\overline{\omega}_1 + j^*\overline{\omega}_2. \end{cases}$$

Transferring terms in (27′) and dividing by $\overline{\omega}_1 < 0$ we see that

$$\begin{cases} \text{for all } (i,j) \in S' \text{ we have } (j'-j)|\overline{\omega}| + (i-i') \geqslant 0 \\ \quad \text{and we always have } (j'-j^*)|\overline{\omega}| + (i^*-i') = 0 \end{cases}$$

and hence by (2)(V) we see that

$$\begin{cases} \text{for all } \widehat{\omega} \in \mathbb{Z}^2_{x-} \text{ we have that} \\ \quad \text{if } |\widehat{\omega}| < |\overline{\omega}| \text{ then } (j'-j)|\widehat{\omega}| + (i-i') > 0 \text{ for all } (i,j) \in S', \\ \quad \text{whereas if } |\widehat{\omega}| > |\overline{\omega}| \text{ then } (j'-j^*)|\widehat{\omega}| + (i^*-i') < 0 \end{cases}$$

and therefore transferring terms and multiplying by $\widehat{\omega}_1 < 0$ we conclude that

$$(28') \qquad \begin{cases} \text{for all } \widehat{\omega} \in \mathbb{Z}^2_{x-} \text{ we have that} \\ \quad \text{if } |\widehat{\omega}| < |\overline{\omega}| \text{ then } i\widehat{\omega}_1 + j\widehat{\omega}_2 < i'\widehat{\omega}_1 + j'\widehat{\omega}_2 \text{ for all } (i,j) \in S', \\ \quad \text{whereas if } |\widehat{\omega}| > |\overline{\omega}| \text{ then } i^*\widehat{\omega}_1 + j^*\widehat{\omega}_2 > i'\widehat{\omega}_1 + j'\widehat{\omega}_2. \end{cases}$$

(23″) *From here until* (28″) suppose that $\overline{\omega} \in \widehat{\Omega}'$ with $S'' \neq \emptyset$ and either (i) $\omega_1 < 0$ or (ii) $\omega_1 = 0 < \omega_2$ with i'' changed to $\min\{i\colon (i,j) \in \mathrm{Supp}(f^+_\omega)\}$.

Now

$$(24'') \qquad |\overline{\omega}| = -\overline{\omega}_2/\overline{\omega}_1 \quad \text{with } \overline{\omega}_1 < 0$$

and we can take

$$(25'') \qquad (i^*, j^*) \in S'' \quad \text{with } |\overline{\omega}| = \frac{i''-i^*}{j''-j^*} = \max_{(i,j)\in S''} \frac{i''-i}{j''-j}.$$

If $\omega_1 = 0 < \omega_2$ then $|\omega| = -\infty$ and hence $|\overline{\omega}| > |\omega|$ by (24″), whereas if $\omega_1 < 0$ then

$$\begin{cases} (i^*, j^*) \in S'' \\ \quad \Rightarrow i^*\omega_1 + j^*\omega_2 < i''\omega_1 + j''\omega_2 \\ \quad \Rightarrow i^* - j^*|\omega| > i'' - j''|\omega| & \text{dividing by } \omega_1 < 0 \\ \quad \Rightarrow (j''-j^*)|\omega| > i'' - i^* \\ \quad \Rightarrow |\omega| > \frac{i''-i^*}{j''-j^*} & \text{dividing by } j''-j^* > 0 \end{cases}$$

and so $|\overline{\omega}| < |\omega|$ by (25″). Thus

$$(26'') \qquad |\overline{\omega}| < |\omega| \text{ or } |\overline{\omega}| > |\omega| \text{ according as } \omega_1 < 0 \text{ or } \omega_1 = 0 < \omega_2.$$

By (25″) we also see that

$$\begin{cases} \text{for all } (i,j) \in S'' \text{ we have} \\ \quad i - j|\overline{\omega}| \geqslant i'' - j''|\overline{\omega}| = i^* - j^*|\overline{\omega}| \end{cases}$$

and hence, in view of (24″), multiplying by $\overline{\omega}_1 < 0$ we conclude that

$$(27'') \qquad \begin{cases} \text{for all } (i,j) \in S'' \text{ we have} \\ \quad i\overline{\omega}_1 + j\overline{\omega}_2 \leqslant i''\overline{\omega}_1 + j''\overline{\omega}_2 = i^*\overline{\omega}_1 + j^*\overline{\omega}_2. \end{cases}$$

Transferring terms in (27″) and dividing by $\overline{\omega}_1 < 0$ we see that

$$\begin{cases} \text{for all } (i,j) \in S'' \text{ we have } (j''-j)|\overline{\omega}| + (i-i'') \geqslant 0 \\ \quad \text{and we always have } (j''-j^*)|\overline{\omega}| + (i^*-i'') = 0 \end{cases}$$

and hence by (2)(V) we see that

$$\begin{cases} \text{for all } \widehat{\omega} \in \mathbb{Z}^2_{x-} \text{ we have that} \\ \quad \text{if } |\widehat{\omega}| > |\overline{\omega}| \text{ then } (j''-j)|\widehat{\omega}| + (i-i'') > 0 \text{ for all } (i,j) \in S'', \\ \quad \text{whereas if } |\widehat{\omega}| < |\overline{\omega}| \text{ then } (j''-j^*)|\widehat{\omega}| + (i^*-i'') < 0 \end{cases}$$

and therefore transferring terms and multiplying by $\widehat{\omega}_1 < 0$ we conclude that

$$(28'')\qquad \begin{cases} \text{for all } \widehat{\omega} \in \mathbb{Z}^2_{x-} \text{ we have that} \\ \quad \text{if } |\widehat{\omega}| > |\overline{\omega}| \text{ then } i\widehat{\omega}_1 + j\widehat{\omega}_2 < i''\widehat{\omega}_1 + j''\widehat{\omega}_2 \text{ for all } (i,j) \in S'', \\ \quad \text{whereas if } |\widehat{\omega}| < |\overline{\omega}| \text{ then } i^*\widehat{\omega}_1 + j^*\widehat{\omega}_2 > i''\widehat{\omega}_1 + j''\widehat{\omega}_2. \end{cases}$$

Putting together parts (10) to (28″) we claim that

(31) assertions (I) to (III) are true.

Namely, paying special attention to the easy case of $S' = \emptyset$, by (10), (12) and (13′) to (18′) we get (1′) stated below. Similarly, paying special attention to the easy case of $S'' = \emptyset$, by (10), (12) and (13″) to (18″) we get (1″) stated below. Moreover, paying special attention to the easy case of $S' = \emptyset$, by (20), (22) and (23′) to (28′) we get (2′) stated below. Likewise, paying special attention to the easy case of $S'' = \emptyset$, by (20), (22) and (23″) to (28″) we get (2″) stated below.

(1′) If $\omega_1 > 0$ or $f - f^+_\omega \neq 0 = \omega_1 = 0 > \omega_2$ then: For all $\overline{\omega} \in \widehat{\Omega}'$ we have that $|\overline{\omega}| < |\omega|$. For all $\overline{\omega} \in \widehat{\Omega}' \cup \widehat{\Omega}^*$ we have that $\mathrm{Supp}(f^+_\omega) \cap \mathrm{Supp}(f^+_{\overline{\omega}}) = \{f^!_\omega\} = \{f^{!!}_{\overline{\omega}}\}$. For all $\widehat{\omega} \in \mathbb{Z}^2_x$ and $\overline{\omega} \in \widehat{\Omega}'$ with $|\widehat{\omega}| < |\overline{\omega}|$ we have that $\mathrm{Supp}(f^+_\omega) \cap \mathrm{Supp}(f^+_{\widehat{\omega}}) = \emptyset$. For all $\overline{\omega} \in \widehat{\Omega}^*$ we have that f is a $\overline{\omega}$-monomial. For all $\overline{\omega} \in \widehat{\Omega}' \cap \mathbb{Z}^2_x$ we have that f is not a $\overline{\omega}$-monomial.

(1″) If $\omega_1 > 0$ or $f - f^+_\omega \neq 0 = \omega_1 = 0 < \omega_2$ then: For all $\overline{\omega} \in \widehat{\Omega}''$ we have that $|\overline{\omega}| > |\omega|$. For all $\overline{\omega} \in \widehat{\Omega}'' \cup \widehat{\Omega}^{**}$ we have that $\mathrm{Supp}(f^+_\omega) \cap \mathrm{Supp}(f^+_{\overline{\omega}}) = \{f^{!!}_\omega\} = \{f^!_{\overline{\omega}}\}$. For all $\widehat{\omega} \in \mathbb{Z}^2_x$ and $\overline{\omega} \in \widehat{\Omega}''$ with $|\widehat{\omega}| > |\overline{\omega}|$ we have that $\mathrm{Supp}(f^+_\omega) \cap \mathrm{Supp}(f^+_{\widehat{\omega}}) = \emptyset$. For all $\overline{\omega} \in \widehat{\Omega}^{**}$ we have that f is a $\overline{\omega}$-monomial. For all $\overline{\omega} \in \widehat{\Omega}'' \cap \mathbb{Z}^2_x$ we have that f is not a $\overline{\omega}$-monomial.

(2′) If $\omega_1 < 0$ or $f - f^+_\omega \neq 0 = \omega_1 = 0 < \omega_2$ then: For all $\overline{\omega} \in \widehat{\Omega}'$ we have $|\overline{\omega}| > |\omega|$ or $|\overline{\omega}| < |\omega|$ according as $\omega_1(\mathrm{ord}_Y f^+_w - \mathrm{ord}_Y f) = 0$ or not. For all $\overline{\omega} \in \widehat{\Omega}' \cup \widehat{\Omega}^*$ we have that $\mathrm{Supp}(f^+_\omega) \cap \mathrm{Supp}(f^+_{\overline{\omega}}) = \{f^!_\omega\} = \{f^{!!}_{\overline{\omega}}\}$. For all $\widehat{\omega} \in \mathbb{Z}^2_{x-}$ and $\overline{\omega} \in \widehat{\Omega}'$ with $|\widehat{\omega}| < |\overline{\omega}|$ we have that $\mathrm{Supp}(f^+_\omega) \cap \mathrm{Supp}(f^+_{\widehat{\omega}}) = \emptyset$. For all $\overline{\omega} \in \widehat{\Omega}^*$ we have that f is a $\overline{\omega}$-monomial. For all $\overline{\omega} \in \widehat{\Omega}' \cap \mathbb{Z}^2_{x-}$ we have that f is not a $\overline{\omega}$-monomial.

(2″) If $\omega_1 < 0$ or $f - f^+_\omega \neq 0 = \omega_1 = 0 > \omega_2$ then: For all $\overline{\omega} \in \widehat{\Omega}''$ we have $|\overline{\omega}| < |\omega|$ or $|\overline{\omega}| > |\omega|$ according as $\omega_1(\deg_Y f - \deg_Y f^+_w) = 0$ or not. For all $\overline{\omega} \in \widehat{\Omega}'' \cup \widehat{\Omega}^{**}$ we have that $\mathrm{Supp}(f^+_\omega) \cap \mathrm{Supp}(f^+_{\overline{\omega}}) = \{f^{!!}_\omega\} = \{f^!_{\overline{\omega}}\}$. For all $\widehat{\omega} \in \mathbb{Z}^2_{x-}$ and $\overline{\omega} \in \widehat{\Omega}''$ with $|\widehat{\omega}| > |\overline{\omega}|$ we have that $\mathrm{Supp}(f^+_\omega) \cap \mathrm{Supp}(f^+_{\widehat{\omega}}) = \emptyset$. For all $\overline{\omega} \in \widehat{\Omega}^{**}$ we have that f is a $\overline{\omega}$-monomial. For all $\overline{\omega} \in \widehat{\Omega}'' \cap \mathbb{Z}^2_{x-}$ we have that f is not a $\overline{\omega}$-monomial.

Now, tacitly using Fact (I) and paying special attention to the easy case of $f = f_\omega^+$ with $\omega_1 = 0$. Assertions (II) and (III) follow from (1′), (1″), (2′), and (2″). Assertion (I) is obvious except for the pairwise disjointness claims which use Assertion (II).

By (11) and (20) (or alternatively by (10) and (21)) we get the following.

(32) $$\begin{cases} \text{If } \overline{\omega} \in \mathbb{Z}_z^2 \text{ with } \omega_1\overline{\omega}_1 < 0 \text{ is such that} \\ \quad \text{for some } (i^\flat, j^\flat) \text{ and } (i^\sharp, j^\sharp) \text{ in } \mathrm{Supp}(f_\omega^+) \cap \mathrm{Supp}(f_{\overline{\omega}}^+) \\ \quad \text{we have } \deg_Y f \neq j^\flat \text{ and } \mathrm{ord}_Y f \neq j^\sharp, \text{ then } |\omega| = |\overline{\omega}|. \\ \quad \text{Likewise if } \overline{\omega} \in \mathbb{Z}_z^2 \text{ with } \omega_1\overline{\omega}_1 < 0 \text{ is such that} \\ \quad \text{for some } (i^\flat, j^\flat) \in \mathrm{Supp}(f_\omega^+) \cap \mathrm{Supp}(f_{\overline{\omega}}^+) \text{ and } (i, j) \in \mathrm{Supp}(f) \\ \quad \text{we have } j = j^\flat \text{ and } i \neq i^\flat, \text{ then } |\omega| = |\overline{\omega}|. \end{cases}$$

By (31) and (32) we see that

(33) assertion (IV) is true.

Next we claim that

(34) assertion (V) is true.

To see this note that

(1*) $$i'\omega_1 + j'\omega_2 = i''\omega_1 + j''\omega_2 = \deg_\omega f \geqslant i\omega_1 + j\omega_2 \quad \text{for all } (i, j) \in \mathrm{Supp}(f).$$

For a moment assume that $\omega_1 \neq 0$. Then upon letting $\lambda = (\deg_w f)/\omega_1$ and dividing (1*) by ω_1, we get

(2*) $$i' - j'|\omega| = i'' - j''|\omega| = \lambda$$

and

(3*) $$\text{for all } (i, j) \in \mathrm{Supp}(f) \quad \text{we have } \lambda \begin{cases} \geqslant i - j|\omega| & \text{if } \omega_1 > 0, \\ \leqslant i - j|\omega| & \text{if } \omega_1 < 0. \end{cases}$$

By (2*) we see that

(4*) $$j' > j''.$$

Subtracting the RHS of the first equation of (2*) from the LHS of that equation we see that $(i' - i'') - (j' - j'')|\omega| = 0$ and hence

(5*) $$|\omega| = \frac{i' - i''}{j' - j''}$$

and substituting this in the second equation of (2*) we get

$$\lambda = i'' - j''\frac{i'-i''}{j'-j''} = \frac{i''j'-i'j''}{j'-j''}$$

and therefore

(6*) $$i''j'-i'j'' = (j'-j'')\lambda$$

which proves (V.1). By (1*) we get (V.2). So let us prove (V.3).

Now assuming $\omega_1 > 0$, we have the following. If $\deg_\omega f > 0$ then by (4*) and (6*) we get $i''j'-i'j'' > 0 \neq i''$; therefore if $i' \neq 0$ then $j'/i' > j''/i''$ and hence $|f^!_\omega| = -j'/i' < -j''/i'' = |f^{!!}_\omega|$; moreover if $i'=0$ then $|f^!_\omega| = -\infty < -j''/i'' = |f^{!!}_\omega|$. If $\deg_\omega f < 0$ then by (4*) and (6*) we get $i''j'-i'j'' < 0 \neq i'$; therefore if $i'' \neq 0$ then $j'/i' < j''/i''$ and hence $|f^!_\omega| = -j'/i' > -j''/i'' = |f^{!!}_\omega|$; moreover if $i''=0$ then $|f^!_\omega| = -j'/i' > -\infty = |f^{!!}_\omega|$. If $\deg_\omega f = 0$ then by (4*) and (6*) we get $i''j'-i'j'' = 0 < j'-j''$; therefore if $i'' \neq 0$ then $i' \neq 0$ and hence $|f^!_\omega| = -j'/i' = -j''/i'' = |f^{!!}_\omega|$; moreover if $i'' = 0 \neq j''$ then $i'=0$ and hence $|f^!_\omega| = -\infty = |f^{!!}_\omega|$. So far $\{f^!_\omega, f^{!!}_\omega\} \subset \mathbb{Z}^2_z$. In the left over case $\{f^!_\omega, f^{!!}_\omega\} \not\subset \mathbb{Z}^2_z$.

Next assuming $\omega_1 < 0$, we have the following. If $\deg_\omega f < 0$ then by (4*) and (6*) we get $i''j'-i'j'' > 0 \neq i''$; therefore if $i' \neq 0$ then $j'/i' > j''/i''$ and hence $|f^{!!}_\omega| = -j'/i' < -j''/i'' = |f^!_\omega|$; moreover if $i'=0$ then $|f^{!!}_\omega| = -\infty < -j''/i'' = |f^!_\omega|$. If $\deg_\omega f > 0$ then by (4*) and (6*) we get $i''j'-i'j'' < 0 \neq i'$; therefore if $i'' \neq 0$ then $j'/i' < j''/i''$ and hence $|f^{!!}_\omega| = -j'/i' > -j''/i'' = |f^!_\omega|$; moreover if $i''=0$ then $|f^{!!}_\omega| = j'/i' > -\infty = |f^!_\omega|$. If $\deg_\omega f = 0$ then by (4*) and (6*) we get $i''j'-i'j'' = 0 < j'-j''$; therefore if $i'' \neq 0$ then $i' \neq 0$ and hence $|f^{!!}_\omega| = -j'/i' = -j''/i'' = |f^!_\omega|$; moreover if $i''=0 \neq j''$ then $i'=0$ and hence $|f^{!!}_\omega| = -\infty = |f^!_\omega|$. So far $\{f^!_\omega, f^{!!}_\omega\} \subset \mathbb{Z}^2_z$. In the left over case $\{f^!_\omega, f^{!!}_\omega\} \not\subset \mathbb{Z}^2_z$.

Finally assuming $\omega_1 = 0$, by (V.2) we get the following. If $\deg_w f > 0 < \omega_2$ then $j' = j'' > 0$; therefore if $i' \neq 0$ then $|f^!_\omega| = -j'/i' < -j''/i'' = |f^{!!}_\omega|$; moreover if $i'=0$ then $|f^!_\omega| = -\infty < -j''/i'' = |f^{!!}_\omega|$. If $\deg_w f < 0 > \omega_2$ then $j' = j'' > 0$; therefore if $i' \neq 0$ then $|f^{!!}_\omega| = -j'/i' < -j''/i'' = |f^!_\omega|$; moreover if $i'=0$ then $|f^{!!}_\omega| = -\infty < -j''/i'' = |f^!_\omega|$. If either $\deg_w f < 0 < \omega_2$ or $\deg_w f > 0 > \omega_2$ then we get a contradiction to $\deg_w f = j'\omega_2$. If $\deg_w f = 0$ then $j' = j'' = 0$; therefore if $i' \neq 0$ then $|f^!_\omega| = -j'/i' = -j''/i'' = |f^{!!}_\omega|$ or $|f^{!!}_\omega| = -j'/i' = -j''/i'' = |f^!_\omega|$ according as $\omega_2 > 0$ or $\omega_2 < 0$. So far $\{f^!_\omega, f^{!!}_\omega\} \subset \mathbb{Z}^2_z$. In the left over case $\{f^!_\omega, f^{!!}_\omega\} \not\subset \mathbb{Z}^2_z$. This completes the proof of (V.3).

In view of (V.3), to prove (V.4) we only have to show that in cases (i), (ii), (iii), we have $\deg_w f > 0$. This follows by noting that in case (i) we have $(i,j) \in \mathrm{Supp}(f)$ for some $(i,j) \in \mathbb{N}^2$ with $(i,j) \neq (0,0)$, in case (ii) we have $(i,0) \in \mathrm{Supp}(f)$ for some $i \in \mathbb{N}_+$, and in case (iii) we have $(0,j) \in \mathrm{Supp}(f)$ for some $j \in \mathbb{N}_+$. This completes the proof of (34).

Finally by (31), (33), and (34) we see that

(35) Fact (3) is true. □

8. Degraded weight systems

To generalize the concepts of weight systems still further we proceed thus.

Definition (8.1). Let $\theta = \theta(X,Y) \in k[X, X^{-1}, Y, Y^{-1}]$ where k is a field.

Corresponding to $\mathbb{Z}_c^2$ we introduce the *full rational numbers set* $\mathbb{Q}_c$ as the disjoint union

$$\mathbb{Q}_c = \mathbb{Q}_d \amalg \mathbb{Q} \amalg \{\pm\infty\}$$

where $\mathbb{Q}_d =$ the *set of degraded rational numbers* is a set together with a

$$\text{bijection } \mathbb{Q}_d \to \mathbb{Q} \text{ given by } w \mapsto w^\sharp = \text{ the } \textit{sharpened} \text{ version of } w.$$

We make $\mathbb{Q}_c$ into a linearly ordered set by taking the obvious linear orders on $\mathbb{Q}_d$ and $\mathbb{Q} \cup \{\pm\infty\}$, and declaring $w < \overline{w}$ for all $w \in \mathbb{Q}_d$ and $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$. Thus $\mathbb{Q}_c$ contains two copies of every rational number, the ordinary and the degraded, whereby all the degraded rationals are regarded as smaller than all the ordinary rationals (including $\pm\infty$). We get a

$$\text{bijection } \mathbb{Q}_c \to \mathbb{Z}_c^2 \text{ given by } w \mapsto w^\flat = \left(w_1^\flat, w_2^\flat\right)$$

where

$$\begin{cases} w \in \mathbb{Q}_d \Rightarrow w^\sharp = -w_2^\flat / w_1^\flat \ \text{ with } w_1^\flat < 0, \\ w \in \mathbb{Q} \Rightarrow w = -w_2^\flat / w_1^\flat \text{ with } w_1^\flat > 0, \\ w = -\infty \text{ or } \infty \Rightarrow (w_1^\flat, w_2^\flat) = (0, 1) \text{ or } (0, -1) \text{ respectively.} \end{cases}$$

Letting $\mathbb{Q}_{d+}$ be the inverse images of $\mathbb{Q}_+$ under the bijection $\mathbb{Q}_d \to \mathbb{Q}$, and letting the superscript $^\flat$ denote the image under the bijection $\mathbb{Q}_c \to \mathbb{Z}_c^2$, we clearly have

$$\mathbb{Q}^\flat = \mathbb{Z}_c^2 \cap \mathbb{Z}_x^2$$

with

$$\mathbb{Q}_+^\flat = \mathbb{Z}_c^2 \cap \mathbb{Z}_x^2 \cap \mathbb{Z}_{y-}^2 \quad \text{and} \quad \mathbb{Q}_{d+}^\flat = \mathbb{Z}_c^2 \cap \mathbb{Z}_y^2 \cap \mathbb{Z}_{x-}^2.$$

Let us observe that for any $w \in \mathbb{Q} \cup \{\pm\infty\}$, we clearly have: $\deg_w \theta = \deg_{w^\flat} \theta$, $\theta_w^+ = \theta_{w^\flat}^+$, $\theta_w^0 = \theta_{w^\flat}^0$, w-homogeneous iff it is $(w^\flat)$-homogeneous, w-automorphic pair iff it is a $(w^\flat)$-automorphic pair, having one or two (respectively at most two) points at infinity in the w-weighted sense iff in the $(w^\flat)$-sense, w-similar iff it is $(w^\flat)$-similar, $\operatorname{lag}_w(f, g) = \operatorname{lag}_{w^\flat}(f, g)$, and w-monomial iff $(w^\flat)$-monomial.

Correspondingly, for any $w \in \mathbb{Q}_d$, let us proceed to *define*: $\deg_w \theta = \deg_{w^\flat} \theta$, $\theta_w^+ = \theta_{w^\flat}^+$, $\theta_w^0 = \theta_{w^\flat}^0$, w-homogeneous iff it is $(w^\flat)$-homogeneous, w-automorphic pair iff it is a $(w^\flat)$-automorphic pair, having one or two (respectively at most two) points at infinity in the w-weighted sense iff in the $(w^\flat)$-sense, w-similar iff it is $(w^\flat)$-similar, $\operatorname{lag}_w(f, g) = \operatorname{lag}_{w^\flat}(f, g)$, and w-monomial iff $(w^\flat)$-monomial.

Symmetry (8.2). Now we come to assertions (8.3.u) and (8.4.v) which, by symmetry, mostly follow from (7.3.u) and (7.4.v) respectively.

As common notation for (8.3.u) and (8.4.v) let k be a field of characteristic 0 and let $w \in \mathbb{Q}_c$ or $w \in \mathbb{Z}_z^2$; from Remark–Definition (8.4.14) onwards, we specify the range of w more explicitly. As common notation for (8.3.u) let F, G be nonzero w-homogeneous members of

$k[X, X^{-1}, Y, Y^{-1}]$. As common notation for (8.4.v) with $v < 14$ let f, g be nonzero members of $k[X, Y]$.

Complemented Eulerian and Jacobian Lemmas with some consequences (8.3.1), (8.3.2.6) to (8.3.2.9), (8.3.3.1) to (8.3.3.4), (8.4.1) to (8.4.5), (8.4.9), and (8.4.10). Respectively same as (3.1), (3.2.6) to (3.2.9), (3.3.1) to (3.3.4), (4.1) to (4.5), (4.9), and (4.10), with $w \in \mathbb{Q}_c$ or $w \in \mathbb{Z}_z^2$. In items (8.3.2.9), (8.3.3.4), (8.4.5), (8.4.9), and (8.4.10), in case of $w \in \mathbb{Q}_c$, replace (w_1, w_2) by $(w_1^\flat, w_2^\flat)$.

Remark–Definition (8.4.14). Noting that items (8.4.11) to (8.4.13) are nonexistent, we proceed to generalize the concepts of antecedent and consequent introduced in (4.14) to $w \in \{\pm\infty\}$ and $w \in \mathbb{Q}_d$. Pictures in (9.4) to (9.6).

In (8.4.14) *we let f belong to $k[X, X^{-1}, Y, Y^{-1}]^\times$ with any commutative ring k.*

We start by extending the definitions of (4.14) *to $w \in \{\pm\infty\}$ thus.*

Note that

$$f_{-\infty}^{+} = f_{(0,1)}^{+} = \operatorname{defo}_Y f = \sum_{i' \leqslant i \leqslant i''} \alpha_i X^i Y^{j'}$$

where

$$\deg_{-\infty} f = \deg_{(0,1)} f = \deg_Y f = j' = j'' \quad \text{and} \quad \alpha_i \in k \quad \text{with } \alpha_{i'} \neq 0 \neq \alpha_{i''}$$

and

$$f_{\infty}^{+} = f_{(0,-1)}^{+} = \operatorname{info}_Y f = \sum_{i^* \leqslant i \leqslant i^{**}} \beta_i X^i Y^{j^*}$$

where

$$-\deg_{\infty} f = -\deg_{(0,-1)} f = \operatorname{ord}_Y f = j^* = j^{**} \quad \text{and} \quad \beta_i \in k \quad \text{with } \beta_{i^*} \neq 0 \neq \beta_{i^{**}}.$$

[Also note that $f_0^+ = f_{(1,0)}^+ = \operatorname{defo}_X f$ and $f_{(-1,0)}^+ = \operatorname{info}_X f$.]

Let S be the set of all (i, j) in $\operatorname{Supp}(f)$ with $j < j''$, and let T be the set of all (i, j) in $\operatorname{Supp}(f)$ with $j > j^*$.

We define the *degreewise antecedent* $(-\infty)^\dagger(f) \in \mathbb{Q}_d$ of $-\infty$ relative to f by putting

$$\big((-\infty)^\dagger(f)\big)^\sharp = \max_{(i,j)\in S} \frac{i' - i}{j' - j}$$

with the understanding that if $S = \emptyset$ then $(-\infty)^\dagger(f) = \infty$. Note that

$$(-\infty)^\dagger(f) \in \mathbb{Q}_d \cup \{\infty\}.$$

We define the *degreewise consequent* $(-\infty)^{\ddagger}(f) \in \mathbb{Q}$ of $-\infty$ relative to f by putting

$$(-\infty)^{\ddagger}(f) = \min_{(i,j)\in S} \frac{i'' - i}{j'' - j}$$

with the understanding that if $S = \emptyset$ then $(-\infty)^{\ddagger}(f) = \infty$. Note that

$$(-\infty)^{\ddagger}(f) \in \mathbb{Q} \cup \{\infty\}.$$

We define the *degreewise antecedent* $\infty^{\dagger}(f) \in \mathbb{Q}$ of ∞ relative to f by putting

$$\infty^{\dagger}(f) = \max_{(i,j)\in T} \frac{i^{**} - i}{j^{**} - j}$$

with the understanding that if $T = \emptyset$ then $\infty^{\dagger}(f) = -\infty$. Note that

$$\infty^{\dagger}(f) \in \mathbb{Q} \cup \{-\infty\}.$$

We define the *degreewise consequent* $\infty^{\ddagger}(f) \in \mathbb{Q}_d$ of ∞ relative to f by putting

$$\left(\infty^{\ddagger}(f)\right)^{\sharp} = \min_{(i,j)\in T} \frac{i^{*} - i}{j^{*} - j}$$

with the understanding that if $T = \emptyset$ then $\infty^{\ddagger}(f) = -\infty$. Note that

$$\infty^{\ddagger}(f) \in \mathbb{Q}_d \cup \{-\infty\}.$$

Now we extend the definitions of (4.14) *to* $w \in \mathbb{Q}_d$ *thus.*

First for a moment let $j^{\dagger}$ be the smallest value of j with (i, j) varying over $\mathrm{Supp}(f_w^+)$, let $i^{\dagger}$ be the unique value of i with $(i, j^{\dagger}) \in \mathrm{Supp}(f_w^+)$, and let $S^{\dagger}$ be the set of all (i, j) in $\mathrm{Supp}(f)$ with $j < j^{\dagger}$. We define the *degreewise antecedent* $w^{\dagger}(f)$ of w relative to f by putting

$$\left(w^{\dagger}(f)\right)^{\sharp} = \max_{(i,j)\in S^{\dagger}} \frac{i^{\dagger} - i}{j^{\dagger} - j}$$

with the understanding that if $S^{\dagger} = \emptyset$ then $w^{\dagger}(f) = \infty$. Note that

$$w^{\dagger}(f) \in \mathbb{Q}_d \cup \{\infty\}.$$

Next for a moment let $j^{\ddagger}$ be the largest value of j with (i, j) varying over $\mathrm{Supp}(f_w^+)$, let $i^{\ddagger}$ be the unique value of i with $(i, j^{\ddagger}) \in \mathrm{Supp}(f_w^+)$, and let $S^{\ddagger}$ be the set of all (i, j) in $\mathrm{Supp}(f)$ with $j > j^{\ddagger}$. We define the *degreewise consequent* $w^{\ddagger}(f)$ of w relative to f by putting

$$\left(w^{\ddagger}(f)\right)^{\sharp} = \min_{(i,j)\in S^{\ddagger}} \frac{i^{\ddagger} - i}{j^{\ddagger} - j}$$

with the understanding that if $S^\ddagger = \emptyset$ then $w^\ddagger(f) = -\infty$. Note that

$$w^\ddagger(f) \in \mathbb{Q}_d \cup \{-\infty\}.$$

This completes the definition of $w^\dagger(f)$ *and* $w^\ddagger(f)$ *for all* $w \in \mathbb{Q}_c$. Now for all $w \in \mathbb{Q}_c$, upon letting $\omega = w^\flat$, in the notation of Fact (3) of (7.4.14) we clearly have

$$w^\dagger(f)^\flat \in \widehat{\Omega}' \quad \text{and} \quad w^\ddagger(f)^\flat \in \widehat{\Omega}''$$

and to match up with Ω' and Ω'' we define the *full antecedent* of w relative to f and the *full consequent* of w relative to f to be the unique members $w_c^\dagger(f)$ and $w_c^\ddagger(f)$ of $\mathbb{Q}_c$ such that

$$w_c^\dagger(f)^\flat \in \Omega' \quad \text{and} \quad w_c^\ddagger(f)^\flat \in \Omega''$$

respectively. So we may tacitly use Fact (3) of (7.4.14) in dealing with the above four displayed objects.

For any $w \in \mathbb{Q}_c$ we define the *dotted* w*-degree form* $f_w^\odot$ of f and the *double dotted* w*-degree form* $f_w^{\odot\odot}$ of f by putting

$$f_w^\odot = \begin{cases} \operatorname{defo}_Y f_w^+ & \text{if } w \in \mathbb{Q}, \\ \operatorname{info}_X f_w^+ & \text{if } w = -\infty, \\ \operatorname{defo}_X f_w^+ & \text{if } w = \infty, \\ \operatorname{info}_Y f_w^+ & \text{if } w \in \mathbb{Q}_d \end{cases}$$

and

$$f_w^{\odot\odot} = \begin{cases} \operatorname{info}_Y f_w^+ & \text{if } w \in \mathbb{Q}, \\ \operatorname{defo}_X f_w^+ & \text{if } w = -\infty, \\ \operatorname{info}_X f_w^+ & \text{if } w = \infty, \\ \operatorname{defo}_Y f_w^+ & \text{if } w \in \mathbb{Q}_d. \end{cases}$$

Clearly these are monomials and hence we have unique $\kappa^\odot$ and $\kappa^{\odot\odot}$ in $k^\times$ together with $(i^\odot, j^\odot)$ and $(i^{\odot\odot}, j^{\odot\odot})$ in $\mathbb{Z}^2$ such that

$$f_w^\odot = \kappa^\odot X^{i^\odot} Y^{j^\odot} \quad \text{and} \quad f_w^{\odot\odot} = \kappa^{\odot\odot} X^{i^{\odot\odot}} Y^{j^{\odot\odot}}.$$

We put

$$f_w^? = \kappa^\odot \in k^\times \quad \text{and} \quad f_w^{??} = \kappa^{\odot\odot} \in k^\times$$

and we call these the *questioned* w*-constant* of f and the *double questioned* w*-constant* of f respectively. Likewise we put

$$f_w^! = \big((f_w^!)_1, (f_w^!)_2\big) = \big(i^\odot, j^\odot\big) \in \mathbb{Z}^2$$

and

$$f_w^{!!} = \big((f_w^{!!})_1, (f_w^{!!})_2\big) = \big(i^{\odot\odot}, j^{\odot\odot}\big) \in \mathbb{Z}^2$$

and we call these the *banged w-point* of f and the *double banged w-point* of f respectively.

Note that for any $w \in \mathbb{Q}_c$, upon letting $\omega = w^\flat$, we have

$$f_\omega^{\odot} = f_w^{\odot} \quad \text{and} \quad f_\omega^{\odot\odot} = f_w^{\odot\odot}$$

and

$$f_\omega^{?} = f_w^{?} \quad \text{and} \quad f_\omega^{??} = f_w^{??}$$

and

$$f_\omega^{!} = f_w^{!} \quad \text{and} \quad f_\omega^{!!} = f_w^{!!}.$$

The following assertions (1) *to* (6) *and* (1*) *to* (6*) *enhance* assertions (1) to (6) of (4.14); details in (7.4.14)(3).

(1) If $w \in \mathbb{Q} \cup \{\infty\}$ then we have that: $w^\dagger(f) \in \mathbb{Q} \Leftrightarrow w^\dagger(f) \neq -\infty \Leftrightarrow$ there is some $(i, j) \in \operatorname{Supp}(f)$ such that $j > j^*$ for every $(i^*, j^*) \in \operatorname{Supp}(f_w^+) \Rightarrow f$ is not a $w^\dagger(f)$-monomial and for $\overline{w} = w^\dagger(f)$ we have $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \{(i^\dagger, j^\dagger)\}$ where $(i^\dagger, j^\dagger) = f_w^!$ and for every $(i, j) \in \operatorname{Supp}(f_{\overline{w}}^+) \setminus \{(i^\dagger, j^\dagger)\}$ we have $\overline{w} = \frac{i^\dagger - i}{j^\dagger - j}$.

(2) If $w \in \mathbb{Q} \cup \{-\infty\}$ then we have that: $w^\ddagger(f) \in \mathbb{Q} \Leftrightarrow w^\ddagger(f) \neq \infty \Leftrightarrow$ there is some $(i, j) \in \operatorname{Supp}(f)$ such that $j < j^*$ for every $(i^*, j^*) \in \operatorname{Supp}(f_w^+) \Rightarrow f$ is not a $w^\ddagger(f)$-monomial and for $\overline{w} = w^\ddagger(f)$ we have $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \{(i^\ddagger, j^\ddagger)\}$ where $(i^\ddagger, j^\ddagger) = f_w^{!!}$ and for every $(i, j) \in \operatorname{Supp}(f_{\overline{w}}^+) \setminus \{(i^\ddagger, j^\ddagger)\}$ we have $\overline{w} = \frac{i^\ddagger - i}{j^\ddagger - j}$.

(3) If $w \in \mathbb{Q} \cup \{\infty\}$ with $w^\dagger(f) \in \mathbb{Q}$ then we have that: $w > w^\dagger(f)$ and for all $\widehat{w} \in \{\widetilde{w} \in \mathbb{Q}_d\colon\ \widetilde{w}^\dagger(f) \in \mathbb{Q}_d$ or $\widetilde{w}^\dagger(f) > w\} \cup \{-\infty\} \cup \{w' \in \mathbb{Q}\colon\ w^\dagger(f) > w'\}$ we have $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\widehat{w}}^+) = \emptyset$ and for all $\widehat{w} \in \mathbb{Q}$ with $w > \widehat{w} > w^\dagger(f)$ we have that f is a $\widehat{w}$-monomial with $\operatorname{Supp}(f_{\widehat{w}}^+) = \{(i^\dagger, j^\dagger)\}$ where $(i^\dagger, j^\dagger)$ is as in (1).

(4) If $w \in \mathbb{Q} \cup \{-\infty\}$ with $w^\ddagger(f) \in \mathbb{Q}$ then we have that: $w < w^\ddagger(f)$ and for all $\widehat{w} \in \{\widetilde{w} \in \mathbb{Q}_d\colon\ \widetilde{w}^\ddagger(f) \in \mathbb{Q}_d$ or $\widetilde{w}^\ddagger(f) < w\} \cup \{\infty\} \cup \{w' \in \mathbb{Q}\colon\ w^\ddagger(f) < w'\}$ we have $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\widehat{w}}^+) = \emptyset$ and for all $\widehat{w} \in \mathbb{Q}$ with $w < \widehat{w} < w^\ddagger(f)$ we have that f is a $\widehat{w}$-monomial with $\operatorname{Supp}(f_{\widehat{w}}^+) = \{(i^\ddagger, j^\ddagger)\}$ where $(i^\ddagger, j^\ddagger)$ is as in (2).

(5) If $w \in \mathbb{Q} \cup \{\infty\}$ is such that $w^\dagger(f) \in \mathbb{Q}$ and f is not a w-monomial then we have $w^\dagger(f)^\ddagger(f) = w$. If $w \in \mathbb{Q} \cup \{\infty\}$ is such that f is not a $w^\dagger(f)$-monomial then we have $w_c^\dagger(f) = w^\dagger(f)$. If $w \in \mathbb{Q} \cup \{\infty\}$ is such that f is a $w^\dagger(f)$-monomial then we have $w_c^\dagger(f) = w^\dagger(f)^\dagger(f)$.

(6) If $w \in \mathbb{Q} \cup \{-\infty\}$ is such that $w^\ddagger(f) \in \mathbb{Q}$ and f is not a w-monomial then we have $w^\ddagger(f)^\dagger(f) = w$. If $w \in \mathbb{Q} \cup \{-\infty\}$ is such that f is not a $w^\ddagger(f)$-monomial then we have $w_c^\ddagger(f) = w^\ddagger(f)$. If $w \in \mathbb{Q} \cup \{-\infty\}$ is such that f is a $w^\ddagger(f)$-monomial then we have $w_c^\ddagger(f) = w^\ddagger(f)^\ddagger(f)$.

(1*) If $w \in \mathbb{Q}_d \cup \{-\infty\}$ then we have: $w^\dagger(f) \in \mathbb{Q}_d \Leftrightarrow w^\dagger(f) \neq \infty \Leftrightarrow$ there is some $(i, j) \in \operatorname{Supp}(f)$ such that $j < j^*$ for every $(i^*, j^*) \in \operatorname{Supp}(f_w^+) \Rightarrow f$ is not a $w^\dagger(f)$-monomial and for $\overline{w} = w^\dagger(f)$ we have $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) = \{(i^\dagger, j^\dagger)\}$ where $(i^\dagger, j^\dagger) = f_w^!$ and for every $(i, j) \in \mathrm{Supp}(f_{\overline{w}}^+) \setminus \{(i^\dagger, j^\dagger)\}$ we have $\overline{w}^\sharp = \frac{i^\dagger - i}{j^\dagger - j}$.

(2*) If $w \in \mathbb{Q}_d \cup \{\infty\}$ then we have: $w^\ddagger(f) \in \mathbb{Q}_d \Leftrightarrow w^\ddagger(f) \neq -\infty \Leftrightarrow$ there is some $(i, j) \in \mathrm{Supp}(f)$ such that $j > j^*$ for every $(i^*, j^*) \in \mathrm{Supp}(f_w^+) \Rightarrow f$ is not a $w^\ddagger(f)$-monomial and for $\overline{w} = w^\ddagger(f)$ we have $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) = \{(i^\ddagger, j^\ddagger)\}$ where $(i^\ddagger, j^\ddagger) = f_w^{!!}$ and for every $(i, j) \in \mathrm{Supp}(f_{\overline{w}}^+) \setminus \{(i^\ddagger, j^\ddagger)\}$ we have $\overline{w}^\sharp = \frac{i^\ddagger - i}{j^\ddagger - j}$.

(3*) If $w \in \mathbb{Q}_d$ with $w^\dagger(f) \in \mathbb{Q}_d$ then we have: $w > w^\dagger(f)$. Moreover, if we have $w \in \mathbb{Q}_d \cup \{-\infty\}$ with $w^\dagger(f) \in \mathbb{Q}_d$ then: for all $\widehat{w} \in \{\widetilde{w} \in \mathbb{Q}\colon \widetilde{w}^\dagger(f) \in \mathbb{Q}$ or $\widetilde{w}^\dagger(f) > w \neq -\infty$ or $\widetilde{w}^\dagger(f) = -\infty \neq w\} \cup \{\infty\} \cup \{w' \in \mathbb{Q}_d\colon w^\dagger(f) > w'\}$ we have $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\widehat{w}}^+) = \emptyset$ and for all $\widehat{w} \in \mathbb{Q}_d$ with $w > \widehat{w} > w^\dagger(f)$ we have that f is a $\widehat{w}$-monomial with $\mathrm{Supp}(f_{\widehat{w}}^+) = \{(i^\dagger, j^\dagger)\}$ where $(i^\dagger, j^\dagger)$ is as in (1*).

(4*) If $w \in \mathbb{Q}_d$ with $w^\ddagger(f) \in \mathbb{Q}_d$ then we have: $w < w^\ddagger(f)$. Moreover, if we have $w \in \mathbb{Q}_d \cup \{\infty\}$ with $w^\ddagger(f) \in \mathbb{Q}_d$ then: for all $\widehat{w} \in \{\widetilde{w} \in \mathbb{Q}\colon \widetilde{w}^\ddagger(f) \in \mathbb{Q}$ or $\widetilde{w}^\ddagger(f) < w \neq \infty$ or $\widetilde{w}^\ddagger(f) = \infty \neq w\} \cup \{-\infty\} \cup \{w' \in \mathbb{Q}_d\colon w^\ddagger(f) < w'\}$ we have $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\widehat{w}}^+) = \emptyset$ and for all $\widehat{w} \in \mathbb{Q}_d$ with $w < \widehat{w} < w^\ddagger(f)$ we have that f is a $\widehat{w}$-monomial with $\mathrm{Supp}(f_{\widehat{w}}^+) = \{(i^\dagger, j^\dagger)\}$ where $(i^\ddagger, j^\ddagger)$ is as in (2*).

(5*) If $w \in \mathbb{Q}_d \cup \{-\infty\}$ is such that $w^\dagger(f) \in \mathbb{Q}_d$ and f is not a w-monomial then we have $w^\dagger(f)^\ddagger(f) = w$. If $w \in \mathbb{Q}_d \cup \{-\infty\}$ is such that f is not a $w^\dagger(f)$-monomial then we have $w_c^\dagger(f) = w^\dagger(f)$. If $w \in \mathbb{Q}_d \cup \{-\infty\}$ is such that f is a $w^\dagger(f)$-monomial then we have $w_c^\dagger(f) = w^\dagger(f)^\dagger(f)$.

(6*) If $w \in \mathbb{Q}_d \cup \{\infty\}$ is such that $w^\ddagger(f) \in \mathbb{Q}_d$ and f is not a w-monomial then we have $w^\ddagger(f)^\dagger(f) = w$. If $w \in \mathbb{Q}_d \cup \{\infty\}$ is such that f is not a $w^\ddagger(f)$-monomial then we have $w_c^\ddagger(f) = w^\ddagger(f)$. If $w \in \mathbb{Q}_d \cup \{\infty\}$ is such that f is a $w^\ddagger(f)$-monomial then we have $w_c^\ddagger(f) = w^\ddagger(f)^\ddagger(f)$.

Remark–Definition (8.4.15). Since there is nothing to add to (4.15), we use this space to enhance the algebra behind the geometry of the Degreewise Newton Polygon of f to be formally discussed in Section 9. For pictures see (9.1) and (9.4) to (9.7).

In (8.4.15) *we continue letting f belong to $k[X, X^{-1}, Y, Y^{-1}]^\times$ with any commutative ring k.*

We define the *system $w(f)$ of degreewise line weights* of f, the *degreewise length $l(f)$* of f, the *system $w_d(f)$ of degraded line weights* of f, the *degraded length $l_d(f)$* of f, the *system $w_c(f)$ of full line weights* of f, and the *full length $l_c(f)$* of f, by putting

$$w(f) = \bigl\{w \in \mathbb{Q} \cup \{\pm\infty\}\colon \operatorname{card} \mathrm{Supp}\bigl(f_w^+\bigr) > 1\bigr\} \quad \text{and} \quad l(f) = \operatorname{card} w(f)$$

and

$$w_d(f) = \bigl\{w \in \mathbb{Q}_d\colon \operatorname{card} \mathrm{Supp}\bigl(f_w^+\bigr) > 1\bigr\} \quad \text{and} \quad l_d(f) = \operatorname{card} w_d(f)$$

and

$$w_c(f) = \bigl\{w \in \mathbb{Q}_c\colon \operatorname{card} \mathrm{Supp}\bigl(f_w^+\bigr) > 1\bigr\} \quad \text{and} \quad l_c(f) = \operatorname{card} w_c(f).$$

By (7.4.14)(2) we see that $l(f), l_d(f), l_c(f)$ are nonnegative integers. Let

$$w(f, 1) < w(f, 2) < \cdots < w\bigl(f, l(f)\bigr)$$

and

$$w_d(f,1) < w_d(f,2) < \cdots < w_d\big(f, l_d(f)\big)$$

and

$$w_c(f,1) < w_c(f,2) < \cdots < w_c\big(f, l_c(f)\big)$$

be the unique (strictly ascending) sequences in $w(f)$, $w_d(f)$, and $w_c(f)$ respectively. We call $w(f,i)$ (respectively $w_d(f,i)$, $w_c(f,i)$) the *ith line weight* (respectively *ith degraded line weight, ith full line weight*) of f. Note that clearly

$$l_c(f) = l_d(f) + l(f)$$

and we have

$$w_c(f,i) = \begin{cases} w_d(f,i) & \text{for } 1 \leqslant i \leqslant l_d(f), \\ w(f, i - l_d(f)) & \text{for } l_d(f) < i \leqslant l_c(f). \end{cases}$$

For any $w \in \mathbb{Q}_c$ we define the *full f-length* of w to be the unique nonnegative integer $l_c(f,w) \leqslant l_c(f)$ obtained by putting

$$l_c(f,w) = \begin{cases} 0 & \text{if } l_c(f) = 0, \\ l_c(f) & \text{if } l_c(f) \neq 0 \text{ with } w \geqslant w_c(f, l_c(f)), \\ \min\{i \in \mathbb{N}\colon\ 0 \leqslant i \leqslant l_c(f) - 1 \text{ with } w < w_c(f, i+1)\} & \\ \qquad \text{if } l_c(f) \neq 0 \text{ with } w < w_c(f, l_c(f)). \end{cases}$$

For any $w \in \mathbb{Q}_c$, by removing the subscript c or replacing it by the subscript d we get the definition of the *f-length* $l(f,w)$ of w or the *degraded f-length* $l_d(f,w)$ of w respectively.

Now essentially (cf. Section 9) the system $w(f)$ or the sequence $w(f,i)$ is the *Degreewise Newton Polygon* of f. Or letting i range over $1 \leqslant i \leqslant l(f)$, in terms of a plethora of words, $w(f,i)$, $f^{+}_{w(f,i)}$, $f^{\odot}_{w(f,i)}$, and $f^{!}_{w(f,i)}$ are the ith newton slope, ith newton line, ith newton monomial, and ith newton vertex of f respectively. Moreover, $f^{\odot\odot}_{w(f,i)}$ and $f^{!!}_{w(f,i)}$ are simply the once pushed forward incarnations of $f^{\odot}_{w(f,i)}$ and $f^{!}_{w(f,i)}$ respectively.

The above description applies to the case of a monic polynomial in Y with coefficients in $k[X]$. In the general case, the system $w_c(f)$ or the sequence $w_c(f,i)$ is the *Full Newton Polygon* of f, whose “left side” is the sequence $w_d(f,i)$ and whose “right side” is the sequence $w(f,i)$. The Degreewise Newton Polygon and the Full Newton Polygon are sometimes called the *Abhyankar Polygon* of f, since they were introduced in Abhyankar’s Purdue Lectures of 1971–1972 as the polynomial incarnation of the polygon which Newton introduced for power series.

As useful subsets of $\mathbb{Q}_c$, consider the *elongated rational numbers set* $\mathbb{Q}_e$, the *augmented rational numbers set* $\mathbb{Q}_a$, the *basic rational numbers set* $\mathbb{Q}_b$, the *infinity rational numbers set* $\mathbb{Q}_\infty$, the *degraded basic rational numbers set* $\mathbb{Q}_\beta$, the *mixed rational numbers set* $\mathbb{Q}_m$, and the *full positive rational numbers set* $\mathbb{Q}_p$ obtained by putting

$$\mathbb{Q}_e = \mathbb{Q} \cup \{-\infty\} \quad \text{and} \quad \mathbb{Q}_a = \mathbb{Q}_e \setminus \mathbb{Q}_+ \quad \text{and} \quad \mathbb{Q}_b = \mathbb{Q}_+ \quad \text{and} \quad \mathbb{Q}_\infty = \{\infty\}$$

and

$$\mathbb{Q}_\beta = \mathbb{Q}_{d+} \quad \text{and} \quad \mathbb{Q}_m = \mathbb{Q}_\beta \cup \{-\infty\} \quad \text{and} \quad \mathbb{Q}_p = \mathbb{Q}_\beta \cup \mathbb{Q}_e.$$

As the corresponding subsystems of $w_c(f)$, let us define the *system* $w_e(f)$ *of elongated line weights* of f, the *elongated length* $l_e(f)$ of f, the *system* $w_a(f)$ *of augmented line weights* of f, the *augmented length* $l_a(f)$ of f, the *system* $w_b(f)$ *of basic line weights* of f, the *basic length* $l_b(f)$ of f, the *system* $w_\infty(f)$ *of infinity line weights* of f, the *infinity length* $l_\infty(f)$ of f, the *system* $w_\beta(f)$ *of degraded basic line weights* of f, the *degraded basic length* $l_\beta(f)$ of f, the *system* $w_m(f)$ *of mixed line weights* of f, the *mixed length* $l_m(f)$ of f, the *system* $w_p(f)$ *of positive line weights* of f, and the *positive length* $l_p(f)$ of f, by putting

$$w_e(f) = \{w \in \mathbb{Q}_e \colon \operatorname{card} \operatorname{Supp}(f_w^+) > 1\} \quad \text{and} \quad l_e(f) = \operatorname{card} w_e(f)$$

and ...

$$w_p(f) = \{w \in \mathbb{Q}_p \colon \operatorname{card} \operatorname{Supp}(f_w^+) > 1\} \quad \text{and} \quad l_p(f) = \operatorname{card} w_p(f).$$

Noting that $l_e(f), \dots, l_p(f)$ are nonnegative integers by (7.4.14)(2), we let

$$w_e(f,1) < w_e(f,2) < \cdots < w_e(f, l_e(f))$$

and ...

$$w_p(f,1) < w_p(f,2) < \cdots < w_p(f, l_p(f))$$

be the unique (strictly ascending) sequences in $w_e(f), \dots, w_p(f)$ respectively. These are "segmental subsequences" of the sequence $w_c(f,i)$, i.e., for a nonnegative integer l'_e with $l'_e + l_e \leqslant l_c$ we have $w_c(f,i) = w_e(f, i - l'_e)$ for $l'_e < i \leqslant l'_e + l_e$ and similarly with the subscript replaced by the subscript $a, \dots, p$. We call $w_e(f,i), \dots, w_p(f,i)$ the *ith elongated line weight* of $f, \dots,$ the *ith positive line weight* of f respectively. Note that clearly

$$l(f) = l_e(f) + l_\infty(f) \quad \text{with } l_\infty(f) \leqslant 1$$

and

$$l_e(f) = l_a(f) + l_b(f) \quad \text{with } l_p(f) = l_\beta(f) + l_e(f)$$

and we have

$$w_p(f,i) = \begin{cases} w_\beta(f,i) & \text{for } 1 \leqslant i \leqslant l_\beta(f), \\ w_e(f, i - l_\beta(f)) & \text{for } l_\beta(f) < i \leqslant l_p(f) \end{cases}$$

and so on. For any $w \in \mathbb{Q}_c$, by changing the subscript c to the subscript $e, \dots, p$ in the above definition of $w_c(f,w)$ we define of the *elongated f-length* $w_e(f,w)$ of $w, \dots,$ the *positive f-length* $w_p(f,w)$ of w respectively.

We define the *ith full vertex* $v_c(f,i)$ of f by putting

$$v_c(f,i) = \begin{cases} f^{!}_{w_c(f,i)} & \text{if } 1 \leqslant i \leqslant l_c(f), \\ f^{!!}_{w_c(f,i)} & \text{if } 1 < i = l_c(f)+1. \end{cases}$$

By dropping the subscript or replacing it by the subscript $d, \ldots, p$ we define the *ith vertex* $v(f,i)$ of $f, \ldots,$ the *ith positive vertex* $v_p(f,i)$ of f respectively.

Now the system $w_p(f)$ or the sequence $w_p(f,i)$ or the sequence $v_p(f,i)$ may be called the *Positive Newton Polygon* of f, and so on.

To illustrate the tacit use of Fact (3) of (7.4.14), we observe that

$$(1)\qquad \begin{cases} \text{if } i \text{ in } \mathbb{N}_+ \text{ with } i \leqslant l_c(f) \text{ and } w, \overline{w} \text{ in } \mathbb{Q}_c \text{ with } w < \overline{w} = w_c(f,i) \\ \quad \text{are such that in case of } i > 1 \text{ we have } w_c(f,i-1) \leqslant w, \\ \quad \text{then } w_c^{\ddagger}(f) = \overline{w} \\ \quad \text{and } \operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \{f_w^{!!}\} = \{f_{\overline{w}}^{!}\} = \{v_c(f,i)\} \end{cases}$$

and

$$(2)\qquad \begin{cases} \text{if } i \text{ in } \mathbb{N}_+ \text{ with } i \leqslant l_c(f) \text{ and } w, \overline{w} \text{ in } \mathbb{Q}_c \text{ with } w_c(f,i) = \overline{w} < w \\ \quad \text{are such that in case of } i < l_c(f) \text{ we have } w \leqslant w_c(f,i+1), \\ \quad \text{then } w_c^{\dagger}(f) = \overline{w} \\ \quad \text{and } \operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \{f_w^{!}\} = \{f_{\overline{w}}^{!!}\} = \{v_c(f,i+1)\}. \end{cases}$$

In view of (1) and (2), by (7.4.14)(3)(V) we see that

$$(3)\qquad \begin{cases} \text{if } l_p(f) > 0 \text{ and } f \in k[X,Y]^{\times} \\ \quad \text{with } f - f(0,0) \notin (Xk[X,Y]) \cup (Yk[X,Y]) \\ \quad \text{then for } 1 \leqslant i \leqslant l_p(f) \text{ we have} \\ \quad \{v_p(f,i), v_p(f,i+1)\} \subset \mathbb{Z}_z^2 \text{ with } |v_p(f,i)| < |v_p(f,i+1)|. \end{cases}$$

Sometimes it is convenient to put a different order on $\mathbb{Q}_c$ whereby ∞ moves from being the largest element to becoming the smallest element. With this new order we write $\mathbb{Q}_\gamma$ instead of $\mathbb{Q}_c$ and call it the *complete rational numbers set*. Now

$$\mathbb{Q}_\gamma = \mathbb{Q}_\epsilon \cup \mathbb{Q}_e \quad \text{where } \mathbb{Q}_\epsilon = \mathbb{Q}_\infty \cup \mathbb{Q}_d$$

and we call $\mathbb{Q}_\epsilon$ the *degraded elongated rational numbers set*. Also

$$\mathbb{Q}_\epsilon = \mathbb{Q}_\alpha \cup \mathbb{Q}_\beta \quad \text{where } \mathbb{Q}_\alpha = \mathbb{Q}_v \cup (\mathbb{Q}_d \setminus \mathbb{Q}_{d+})$$

and we call $\mathbb{Q}_\alpha$ the *degraded augmented rational numbers set*. The bijection $\mathbb{Q}_d \to \mathbb{Q}$ given by $w \mapsto w^\sharp$ now extends to an order preserving bijection $\mathbb{Q}_\epsilon \to \mathbb{Q}_e$ by taking $\infty^\sharp = -\infty$.

In an obvious manner we define the *system* $w_\gamma(f)$ *of complete line weights* of f, the *complete length* $l_\gamma(f)$ of f, the *system* $w_\epsilon(f)$ *of degraded elongated line weights* of f, the *degraded elongated length* $l_\epsilon(f)$ of f, the *system* $w_\alpha(f)$ *of degraded augmented line weights* of f, the *degraded augmented length* $l_\alpha(f)$ of f, and so on.

Now $w_\gamma(f)$ may be called the *Complete Newton Polygon* of f, and so on.

Important Remark (8.4.16). Since the proofs of (4.16)(5) and (4.16)(6) use only $w_1 \neq 0$, as further sharpenings of (3.5) we see that for all $w \in \mathbb{Q}_c \cup \mathbb{Z}_z^2$ we have facts (5) and (6) stated below. [We have *reverted* to letting f, g belong to $k[X, Y]^\times$ with a field k of characteristic 0.]

(5) If $J(f, g) = \theta$ and f is w-homogeneous then either $\deg(f) = 1$ or $f = \theta X + P(Y)$ with $P(Y) \in k[Y]$ or $f = \theta Y + Q(X)$ with $Q(X) \in k[X]$, and hence in particular (f, g) is an automorphic pair by (4.16)(2).
(6) If $J(f, g) = \theta$ and g is w-homogeneous then either $\deg(g) = 1$ or $g = \theta X + P(Y)$ with $P(Y) \in k[Y]$ or $g = \theta Y + Q(X)$ with $Q(X) \in k[X]$, and hence in particular (f, g) is an automorphic pair by (4.16)(3).

Important Lemma (8.4.17). *Assume that $w \in \mathbb{Z}_x^2 \cup \mathbb{Z}_y^2$ or $w \in \mathbb{Q}_p$ with $(w_1, w_2) = (w_1^\flat, w_2^\flat)$. Also assume that $J(f, g) = \theta$ and $\deg(f) \geqslant 2$ with $\deg(g) \geqslant 2$. Then we have the following. [We have reverted to letting f, g belong to $k[X, Y]^\times$ with a field k of characteristic* 0.]

(1) *Upon letting $\deg_w f = N$ and $\deg_w g = M$ we have $\min(N, M) \geqslant \max(w_1, w_2)$ and hence $N > 0$ and $M > 0$. Moreover f is w-similar to g, i.e., $(f_w^+)^M = \kappa (g_w^+)^N$ for some $\kappa \in k^\times$.*
(2) *Suppose that $w \in \mathbb{Q}_e$. Then $w < w^\ddagger(f) = w^\ddagger(g)$ and we have: $w^\ddagger(f) \in \mathbb{Q}_e$ iff $(f_w^{!!})_2 \neq 0$. Moreover, if $\overline{w} \in \mathbb{Q}_e$ is such that $w < \overline{w} \leqslant w^\ddagger(f)$ then f is $\overline{w}$-similar to g and, upon letting $\deg_{\overline{w}} f = \overline{N}$ with $\deg_{\overline{w}} g = \overline{M}$, we also have that: $\overline{M}, \overline{N}$ are in $\mathbb{N}_+$ with $\overline{M}/\overline{N} = M/N$ and for all $\widehat{M}, \widehat{N}$ in $\mathbb{N}_+$ with $\widehat{M}/\widehat{N} = M/N$ we obtain $(f_w^+)^{\widehat{M}} = \widehat{\kappa}(g_w^+)^{\widehat{N}}$ with $(f_{\overline{w}}^+)^{\widehat{M}} = \widehat{\kappa}(g_{\overline{w}}^+)^{\widehat{N}}$ for some $\widehat{\kappa} \in k^\times$.*
(3) *Suppose that $w \in \mathbb{Q}_m$. Then $w > w^\dagger(f) = w^\dagger(g)$ and we have: $w^\dagger(f) \in \mathbb{Q}_m$ iff $(f_w^{!})_2 \neq 0$. Moreover, if $\overline{w} \in \mathbb{Q}_m$ is such that $w > \overline{w} \geqslant w^\dagger(f)$ then f is $\overline{w}$-similar to g and, upon letting $\deg_{\overline{w}} f = \overline{N}$ with $\deg_{\overline{w}} g = \overline{M}$, we also have that: $\overline{M}, \overline{N}$ are in $\mathbb{N}_+$ with $\overline{M}/\overline{N} = M/N$ and for all $\widehat{M}, \widehat{N}$ in $\mathbb{N}_+$ with $\widehat{M}/\widehat{N} = M/N$ we obtain $(f_w^+)^{\widehat{M}} = \widehat{\kappa}(g_w^+)^{\widehat{N}}$ with $(f_{\overline{w}}^+)^{\widehat{M}} = \widehat{\kappa}(g_{\overline{w}}^+)^{\widehat{N}}$ for some $\widehat{\kappa} \in k^\times$.*

Proof. (1) follows from (4.17)(1) by "flipping" to go from $\mathbb{Z}_x^2$ to $\mathbb{Z}_y^2$; see (7.2). In view of (3.11), (8.4.14)(2), (8.4.14)(4), and (4.16)(1), (2) follows from (1); details in (8.4.18) below. In view of (3.11), (8.4.14)(1*), (8.4.14)(3*), and (4.16)(1), (3) follows from (1); details in (8.4.18) below. □

Remark (8.4.18). Since there is nothing to add to (4.18), we use this space to discuss the similarity of the Positive Newton Polygons of the members of a Jacobian pair. Again *we revert* to letting f, g belong to $k[X, Y]^\times$ with a field k of characteristic 0.

In response to the "details in (8.4.18)" in the proofs of (4.17) and (8.4.17), we only need to observe the following. Let $w, \overline{w}$ in $\mathbb{Q}_c$, and $M, N, \overline{M}, \overline{N}, \widehat{M}, \widehat{N}$ in $\mathbb{N}_+$, be as in (4.17)(2), (8.4.17)(2), (8.4.17)(3), so that we have

$$(1) \qquad \left(f_w^+\right)^M = \kappa \left(g_w^+\right)^N$$

and

$$(2) \qquad \left(f_{\overline{w}}^+\right)^{\overline{M}} = \overline{\kappa} \left(g_{\overline{w}}^+\right)^{\overline{N}}$$

with $\kappa, \bar{\kappa}$ in $k^\times$. Moreover, for suitable ω, ω^* in $\{(0,1), (0,-1), (1,0), (-1,0)\}$, upon letting

(3) $$P = \left(f_w^+\right)_\omega^+ \quad \text{and} \quad Q = \left(g_w^+\right)_\omega^+$$

we have that P and Q are nonconstant monomials with

(4) $$\left(f_{\overline{w}}^+\right)_{\omega^*}^+ = P \quad \text{and} \quad Q = \left(g_{\overline{w}}^+\right)_{\omega^*}^+ = Q.$$

By (1) and (3) we get

(5) $$P^M = \kappa Q^N.$$

By (2) and (4) we get

(6) $$P^{\overline{M}} = \bar{\kappa} Q^{\overline{N}}.$$

In view of (3.11), by (5) and (6) we see that

(7) $$\overline{M}/\overline{N} = M/N.$$

Since $\widehat{M}/\widehat{N} = M/N$, in view of (3.11), by (1) we get

(8) $$\left(f_w^+\right)^{\widehat{M}} = \widehat{\kappa}\left(g_w^+\right)^{\widehat{N}}$$

with $\widehat{\kappa} \in k^\times$. Since $\widehat{M}/\widehat{N} = M/N$, in view of (3.11), by (2) and (7) we get

(9) $$\left(f_{\overline{w}}^+\right)^{\widehat{M}} = \widehat{\kappa}^*\left(g_{\overline{w}}^+\right)^{\widehat{N}}$$

with $\widehat{\kappa}^* \in k^\times$. By (3) and (8) we see that

(10) $$P^{\widehat{M}} = \widehat{\kappa} Q^{\widehat{N}}.$$

By (4) and (9) we see that

(11) $$P^{\widehat{M}} = \widehat{\kappa}^* Q^{\widehat{N}}.$$

By (10) and (11) we see that

$$\widehat{\kappa}^* = \widehat{\kappa}$$

and hence by (9) we conclude that

(12) $$\left(f_{\overline{w}}^+\right)^{\widehat{M}} = \widehat{\kappa}\left(g_{\overline{w}}^+\right)^{\widehat{N}}.$$

This takes care of the "also have that" in (4.17)(2), (8.4.17)(2), (8.4.17)(3). The only other thing we need to note for their proof is that if f is w-similar to g then f_w^+ is a monomial iff g_w^+ is a monomial.

Now as a consequence of (8.4.17) let us prove (13) and (14) stated below.

(13) Assume that $w \in \mathbb{Z}_x^2 \cup \mathbb{Z}_y^2$ or $w \in \mathbb{Q}_p$ with $(w_1, w_2) = (w_1^\flat, w_2^\flat)$. Also assume that $J(f,g) = \theta$ and $\deg(f) \geqslant 2$ with $\deg(g) \geqslant 2$. Then upon letting $\deg_w f = N$ and $\deg_w g = M$ we have $\min(N, M) \geqslant \max(w_1, w_2)$ and hence $N > 0$ and $M > 0$. Moreover f is w-similar to g, i.e., $(f_w^+)^M = \kappa(g_w^+)^N$ for some $\kappa \in k^\times$. Furthermore, upon letting $\widehat{M}$ and $\widehat{N}$ be any positive integers with $\widehat{M}/\widehat{N} = M/N$ we have $(f_w^+)^{\widehat{M}} = \widehat{\kappa}(g_w^+)^{\widehat{N}}$ for some $\widehat{\kappa} \in k^\times$. Finally, for any $\overline{w} \in \mathbb{Z}_x^2 \cup \mathbb{Z}_y^2$ or $\overline{w} \in \mathbb{Q}_p$, upon letting $\deg_{\overline{w}} f = \overline{N}$ with $\deg_{\overline{w}} g = \overline{M}$, we have that $\overline{M}, \overline{N}$ are positive integers with $\overline{M}/\overline{N} = M/N$ and $(f_{\overline{w}}^+)^{\widehat{M}} = \widehat{\kappa}(g_{\overline{w}}^+)^{\widehat{N}}$ with the same $\widehat{\kappa}$ as above.

Proof. Without loss of generality we may assume that w and $\overline{w}$ belong to $\mathbb{Q}_p$.

For a moment suppose that ($1^\bullet$) $l_p(f,w)=l_p(f,\overline{w})$ or ($2^\bullet$) $l_p(f,w)=l_p(f,\overline{w})+1$ with $\overline{w}=w^\dagger(f)<w$ or ($3^\bullet$) $l_p(f,w)+1=l_p(f,\overline{w})$ with $w<w^\ddagger(f)=\overline{w}$. If $w=\overline{w}$ then we have nothing to show. So also suppose that $w\neq\overline{w}$. By symmetry we may assume that $w<\overline{w}$. If $\{w,\overline{w}\}\subset\mathbb{Q}_e$ then we are done by (8.4.17)(2). If $\{w,\overline{w}\}\subset\mathbb{Q}_m$ then we are done by taking $(\overline{w},w)$ for $(w,\overline{w})$ in (8.4.17)(3). If $\{w,\overline{w}\}\not\subset\mathbb{Q}_e$ and $\{w,\overline{w}\}\not\subset\mathbb{Q}_m$ then we must have $w<-\infty<\overline{w}$ and hence we are done by taking $(-\infty,\overline{w})$ for $(w,\overline{w})$ in (8.4.17)(2) and taking $(-\infty,w)$ for $(w,\overline{w})$ in (8.4.17)(3). Thus we are done if ($1^\bullet$) or ($2^\bullet$) or ($3^\bullet$).

Now if $l_p(f,w)\neq l_p(f,\overline{w})$ then we can find $\widehat{w}\in\mathbb{Q}_p$ such that the pair satisfies ($1^\bullet$) or ($2^\bullet$) or ($3^\bullet$), and we have $0\leqslant|l_p(f,\widehat{w})-l_p(f,\overline{w})|<|l_p(f,w)-l_p(f,\overline{w})|$. So we are done by "induction" on $|l_p(f,w)-l_p(f,\overline{w})|$. $\square$

(14) Similarity of positive Newton polygons. Assume that we have $J(f,g)=\theta\!\!\!-$ and $\deg(f)\geqslant 2$ with $\deg(g)\geqslant 2$. Let $\widehat{M}$ and $\widehat{N}$ be any positive integers with $\widehat{M}/\widehat{N}=\deg(g)/\deg(f)$. Then we have the following.

(I) For any $w\in\mathbb{Z}_x^2\cup\mathbb{Z}_y^2$ or $w\in\mathbb{Q}_p$ with $(w_1,w_2)=(w_1^\flat,w_2^\flat)$, upon letting $\deg_w f=N$ and $\deg_w g=M$ we have that: $\min(N,M)\geqslant\max(w_1,w_2)$ (and hence in particular $N>0$ and $M>0$) with $M/N=\widehat{M}/\widehat{N}$, and f is w-similar to g with $(f_w^+)^{\widehat{M}}=\widehat{\kappa}(g_w^+)^{\widehat{N}}$ where $\widehat{\kappa}\in k^\times$ depends only on $f,g,\widehat{M},\widehat{N}$ and not on w.

(II) We have $l_p(f)=l_p(g)>0$ with $w_p(f)=w_p(g)$ and for $1\leqslant i\leqslant l_p(f)$ we have $w_p(f,i)=w_p(g,i)$.

(III) For $1\leqslant i\leqslant l_p(f)+1$ we have $\widehat{M}v_p(f,i)=\widehat{N}v_p(g,i)$. [Also see (7.4.14)(3)(V) and (8.4.15)(3).] For pictures see (9.7).

Proof. Clearly $-1\in\mathbb{Q}_p$ with $\deg(f)=\deg_{-1}f$ and $\deg(g)=\deg_{-1}g$; hence (I) follows from (13).

By (I), f is w-similar to g for all $w\in\mathbb{Q}_p$, and hence we get (II) except the assertion $l_p(g)>0$. If $\deg_Y g\neq\operatorname{ord}_Y g$ then $(-\infty)^\ddagger(g)\in w_p(g)$ and hence $l_p(g)>0$. If $\deg_Y g=\operatorname{ord}_Y g$ and g is not a monomial then $-\infty\in w_p(g)$ and hence $l_p(g)>0$. Since $J(f,g)=\theta\!\!\!-$ with $\deg(g)\geqslant 2$, by (4.16)(3) we see that g cannot be a monomial. This proves (II). (III) follows from (I) and (II). $\square$

To relate the above Similarity (14) with the triangularity discussed in (6.2), we make the following obvious observation.

(15) The Newton Polygon of f is a triangle if and only if either: (i) $\deg(f)=1$ or: (ii) $l_p(f)=1$ with $v_p(f,1)=(0,\nu)$ and $v_p(f,2)=(\mu,0)$ for some ν,μ in $\mathbb{N}_+$.

Remark–Definition (8.4.19). Reverting to the set-up of Section 4, let

$$\begin{cases} f \text{ and } g \text{ be nonzero members of } k[X,Y] \\ \quad\text{where } k \text{ is a field of characteristic } 0 \end{cases}$$

and for $w\in\mathbb{Q}\cup\{\pm\infty\}$ let (w_1,w_2) be the unique pair of coprime integers with $w_1\geqslant 0$ such that if $w\in\mathbb{Q}$ then $w=-w_2/w_1$ with $w_1>0$ whereas if $w=-\infty$ or ∞ then $(w_1,w_2)=(0,1)$ or $(0,-1)$ respectively. For pictures see (9.8).

In the Brief Strategy at the beginning of Section 4, we outlined an inductive procedure for constructing a sequence of polynomials $g = g_1, g_2, \dots, g_e$ such that

$$\mathrm{lag}_w(f, g_1) > \mathrm{lag}_w(f, g_2) > \cdots > \mathrm{lag}_w(f, g_e) = 0.$$

In property (XII) at the end of this subsection (8.4.19) we shall show that, under suitable conditions, as w increases the lag becomes zero earlier. In Part III of this paper we shall apply this when $g_1, \dots, g_e$ are derived from certain strict approximate roots as described in Section 3 of Part I of this paper. This will then be used for settling the sharper version of the two plus epsilon characteristic pair case which we spoke of in the Introduction of Part I.

As illustrations for Remark (3.8), and as multiplicative incarnations of the concepts of the derivative and the lag, we define the *multiplicative w-derivative* $(w)'(f)$ of f and the *multiplicative w-lag* $\lambda_w(f, g)$ of (f, g) of f by putting

$$(w)'(f) = \begin{cases} \frac{\deg_w(f)}{\deg_w(XY)} & \text{if } w \neq 1, \\ \infty & \text{otherwise} \end{cases}$$

and

$$\lambda_w(f, g) = \begin{cases} (w)'(f) + (w)'(g) - (w)'(J(f, g)) & \text{if } w \neq 1 \text{ and } J(f, g) \neq 0, \\ \infty & \text{otherwise} \end{cases}$$

and we note that

(1•) $$(w)'(f) = \infty \quad \Leftrightarrow \quad w = 1 \quad \Leftrightarrow \quad \deg_w(XY) = 0$$

and

(2•) $$(w)'(f) = 0 \quad \Leftrightarrow \quad w \neq 1 \quad \text{and} \quad \deg_w(f) = 0.$$

For any $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ we define the *compound $(w, \overline{w})$-derivative* $(w, \overline{w})'(f)$ of f by putting

$$(w, \overline{w})'(f) = \begin{cases} \frac{(w)'(f)}{(\overline{w})'(f)} & \text{if } w \neq 1 \neq \overline{w} \text{ and } \deg_{\overline{w}}(f) \neq 0, \\ \infty & \text{otherwise.} \end{cases}$$

As a consequence of (1•) and (2•) let us note that

(3•) $$(w, w)'(f) = \begin{cases} 1 & \text{if } w \neq 1 \text{ and } \deg_w f \neq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Here are 4 properties of the multiplicative derivative.

(I) If $\deg_w(XY) \neq 0$ then for all $\kappa \in k^\times$ and $c \in \mathbb{N}$ we have $(w)'(\kappa f^c) = c(w)'(f)$.

(II) For every monomial H in f_w^+ we have $(w)'(f) = (w)'(H)$.

(III) If $\deg_w(XY) \neq 0 > w - 1$ and H is any monomial above (respectively below) the 45° line then $(w)'(H)$ is a strictly decreasing (respectively increasing) function of w.

(IV) If $\deg_w(XY) \neq 0 < w - 1$ and H is any monomial above (respectively below) the 45° line then $(w)'(H)$ is a strictly decreasing (respectively increasing) function of w.

[A monomial H is an expression $H = \kappa X^i Y^j$ with $\kappa \in k^\times$ and $(i, j) \in \mathbb{N}^2$. The *monomial H is in f_w^+* means $(i, j) \in \mathrm{Supp}(f_w^+)$ and κ is the coefficient of $X^i Y^j$ in f_w^+. The monomial

H is *above* (respectively *below*) the 45° line means $i < j$ (respectively $i > j$). Property (III) says that, for fixed H, if $w < \overline{w} < 1$ in $\mathbb{Q} \cup \{-\infty\}$ are such that $\deg_w(XY) \neq 0 \neq \deg_{\overline{w}}(XY)$ then: $(\overline{w})'(H) < (w)'(H)$ or $(\overline{w})'(H) > (w)'(H)$ according as $i < j$ or $i > j$. Property (IV) says that, for fixed H, if $1 < w < \overline{w}$ in $\mathbb{Q} \cup \{\infty\}$ are such that $\deg_w(XY) \neq 0 \neq \deg_{\overline{w}}(XY)$ then: $(\overline{w})'(H) < (w)'(H)$ or $(\overline{w})'(H) > (w)'(H)$ according as $i < j$ or $i > j$. Note that by *convention*, for any $r \in \mathbb{Q}$, we have $-\infty \pm r = -\infty$ and $\infty \pm r = \infty$.]

Proof of (I) to (IV). Out of these, (I) and (II) are straightforward. To prove (III) and (IV), assume $H = \kappa X^i Y^j$ with $\kappa \in k^\times$ and $i \neq j$ in $\mathbb{N}$.

Now $w_1 + w_2 = \deg_w(XY) \neq 0$ and

$$(w)'(H) - j = \frac{iw_1 + jw_2}{w_1 + w_2} - j = \frac{(i-j)w_1}{w_1 + w_2}.$$

If $w_1 > 0$ then dividing by w_1 we get

$$(1)\qquad (w)'(H) - j = \frac{i-j}{1-w} \begin{cases} < 0 & \text{if } i < j \text{ and } w < 1, \\ > 0 & \text{if } i > j \text{ and } w < 1, \\ < 0 & \text{if } i > j \text{ and } w > 1, \\ > 0 & \text{if } i < j \text{ and } w > 1 \end{cases}$$

and reciprocating we obtain

$$(2)\qquad \frac{1}{(w)'(H) - j} = \frac{w-1}{j-i}$$

where the RHS is strictly increasing or decreasing in $w < 1$ according as $i < j$ or $i > j$, and hence the denominator of the LHS is strictly decreasing or increasing in $w < 1$ according as $i < j$ or $i > j$, and therefore the same is true after adding the constant j to the said denominator. In other words, if $w_1 > 0$ then by (2), according as $i < j$ or $i > j$, we have

$$\begin{cases} w < \overline{w} < 1 \text{ in } \mathbb{Q} \\ \quad \Rightarrow 0 > \frac{\overline{w}-1}{j-i} > \frac{w-1}{j-i} \text{ or } 0 < \frac{\overline{w}-1}{j-i} < \frac{w-1}{j-i} \\ \quad \Rightarrow (\overline{w})'(H) - j < (w)'(H) - j \text{ or } (\overline{w})'(H) - j > (w)'(H) - j \\ \quad \Rightarrow (\overline{w})'(H) < (w)'(H) \text{ or } (\overline{w})'(H) > (w)'(H). \end{cases}$$

To prove (III) it only remains to note that if $w = -\infty$ then $(w_1, w_2) = (0, 1)$ and hence $(w)'(H) = j$, whereas if $w < 1$ in $\mathbb{Q}$ then by (1) we get $(w)'(H) < j$ or $(w)'(H) > j$ according as $i < j$ or $i > j$.

In (2) the RHS is strictly increasing or decreasing in $w > 1$ according as $i < j$ or $i > j$, and hence the denominator of the LHS is strictly decreasing or increasing in $w > 1$ according as $i < j$ or $i > j$, and therefore the same is true after adding the constant j to the said denominator. In other words, if $w_1 > 0$ then by (2), according as $i < j$ or $i > j$, we have

$$\begin{cases} 1 < w < \overline{w} \text{ in } \mathbb{Q} \\ \quad \Rightarrow \frac{\overline{w}-1}{j-i} > \frac{w-1}{j-i} > 0 \text{ or } \frac{\overline{w}-1}{j-i} < \frac{w-1}{j-i} < 0 \\ \quad \Rightarrow (\overline{w})'(H) - j < (w)'(H) - j \text{ or } (\overline{w})'(H) - j > (w)'(H) - j \\ \quad \Rightarrow (\overline{w})'(H) < (w)'(H) \text{ or } (\overline{w})'(H) > (w)'(H). \end{cases}$$

To prove (IV) it only remains to note that if $w = \infty$ then $(w_1, w_2) = (0, -1)$ and hence $(w)'(H) = j$, whereas if $w > 1$ in $\mathbb{Q}$ then by (1) we get $(w)'(H) > j$ or $(w)'(H) < j$ according as $i < j$ or $i > j$. □

Writing $f \sim_w g$ *to mean* f *is* w*-similar to* g, and $f \nsim_w g$ *for its negation*, here are 6 properties of similarity and multiplicative lag.

(V) If $f \sim_w g$ and $\deg_w f \neq 0$ with $g_w^+ \notin k$ then $(\deg_w f)(\deg_w g) > 0$.

[**Note (V*).** (i) Concerning the condition $(\deg_w f)(\deg_w g) > 0$ we note that for any integers (or rational numbers) N, M we have: $NM > 0$ iff either both N and M are positive or both are negative; $NM < 0$ iff one of N and M is positive and the other is negative; $NM = 0$ iff at least one of N and M is zero.

(ii) To see the necessity of assuming $g_w^+ \notin k$, take $(w_1, w_2) = (1, -2)$ and $f = X$ with $g = 1 + Y$. Observe that now $\deg_w g = 0$ with $g_w^+ \in k$ but $g \notin k$.]

(VI) If f and g are monomials such that either $\deg_w f \neq 0$ or $\deg_w g \neq 0$, then:

$$f \sim_w g \quad \Leftrightarrow \quad f \sim_{\overline{w}} g \quad \text{for all } \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}.$$

[**Note (VI*).** This note is not related to (VI) but is meant for making a list of claims to be used in the proof of (VII). Given $\widehat{w}, \overline{w}$ in $\mathbb{Q} \cup \{\pm\infty\}$, we say $\widehat{w}$ is *between* w and $\overline{w}$ to mean that either $w > \widehat{w} > \overline{w}$ or $w < \widehat{w} < \overline{w}$. Given $\widehat{\omega}, \overline{\omega}$ in $\mathbb{Z}^2$ we say that $\widehat{\omega}$ is *proportional* to $\overline{\omega}$, and we write $\widehat{\omega} \approx \overline{\omega}$, to mean that either $\widehat{\omega} \neq (0,0) \neq \overline{\omega}$ with $|\widehat{\omega}| = |\overline{\omega}|$, or $\widehat{\omega} = (0,0)$, or $\overline{\omega} = (0,0)$; geometrically, this says that $\widehat{\omega}$ and $\overline{\omega}$ are on a line through the origin. With these definitions in hand, here is the list of *claims* (i) to (xii).

(i) If $f \in k$ then $\mathrm{Supp}(f) = \{(0,0)\}$ and hence $\deg_{\overline{w}} f = 0$ for all $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$. If $f_w^+ \in k$ then for all $w \leqslant \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ we have $f_{\overline{w}}^+ \in k$ with $\deg_{\overline{w}} f = 0$. If $\deg_w f = 0$ and $w = -\infty$ then $f \in k[X]$ and hence $\mathrm{Supp}(f) \subset \{(i,0)\colon i \in \mathbb{N}\}$ and so: $\deg_{\overline{w}} f = 0$ for all $\overline{w} \in \mathbb{Q} \cup \{\infty\}$ in case $f \in k$, whereas $\deg_\infty f = 0$ with $\deg_{\overline{w}} f > 0$ for all $\overline{w} \in \mathbb{Q}$ in case $f \notin k$. If $\deg_w f = 0$ and $w \neq -\infty$ with $(0,0) \notin \mathrm{Supp}(f)$ then for all $\widetilde{w} < w < \overline{w}$ in $\mathbb{Q} \cup \{\pm\infty\}$ we have $\deg_{\overline{w}} f < 0 < \deg_{\widetilde{w}} f$. If $\deg_w f = 0$ and $w \neq -\infty$ with $(0,0) \in \mathrm{Supp}(f)$ then for all $w < \overline{w}$ in $\mathbb{Q} \cup \{\infty\}$ we have $\deg_{\overline{w}} f = 0$.

(ii) If $\deg_w f < 0$ then for all $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ we have $\deg_{\overline{w}} f < 0$.

(iii) If $\deg_w f > 0$ and $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\deg_{\overline{w}} f > 0$ then for all $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}} f > 0$.

(iv) If $\deg_w f > 0$ and $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\deg_{\overline{w}} f < 0$ then there is a unique $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ for which $\deg_{\widehat{w}} f = 0$; moreover, for this $\widehat{w}$ we have $\deg_{w^*} f > 0$ for all $w^* \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\widehat{w}$, and $\deg_{w^{**}} f < 0$ for all $w^{**} \in \mathbb{Q} \cup \{\pm\infty\}$ between $\widehat{w}$ and $\overline{w}$.

(v) If $\deg_w f > 0$ and $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\deg_{\overline{w}} f = 0$ then there is a unique $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ with $w < \widehat{w} \leqslant \overline{w}$ and $\deg_{\widehat{w}} f = 0$ such that $\deg_{w^*} f > 0$ for all $w^* \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\widehat{w}$, and $\deg_{w^{**}} f = 0$ for all $w^{**} \in \mathbb{Q} \cup \{\pm\infty\}$ between $\widehat{w}$ and $\overline{w}$; moreover, for this $\widehat{w}$ we have that if $(0,0) \notin \mathrm{Supp}(f)$ then $\widehat{w} = \overline{w}$, and if $(0,0) \in \mathrm{Supp}(f)$ then f is not a $\widehat{w}$-monomial.

(vi) If $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $w \neq \overline{w}$ with $(\deg_w f)(\deg_{\overline{w}} f) > 0$ then for every $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ which is between w and $\overline{w}$ we have $(\deg_w f)(\deg_{\widehat{w}} f) > 0$.

(vii) If $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $w \neq \overline{w}$ with $(\deg_w f)(\deg_{\overline{w}} f) < 0$ then for some $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ which is between w and $\overline{w}$ we have $\deg_{\widehat{w}} f = 0$.

(viii) If $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) \neq \emptyset$ then $f_w^{!!} = f_{\overline{w}}^{!}$.

(ix) If $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \emptyset$ with $\deg_w f \leqslant 0$ then $f_w^{!!} \neq (0,0) \neq f_{\overline{w}}^!$ with $|f_w^{!!}| > |f_{\overline{w}}^!|$. If $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \emptyset$ with $\deg_w f > 0$ and either: $(')$ $\deg_{\overline{w}} f > 0$, or: $('')$ $\deg_{\overline{w}} f = 0$ with $(0,0) \notin \operatorname{Supp}(f)$, or: $(''')$ $\deg_{\overline{w}} f = 0$ with $(0,0) \in \operatorname{Supp}(f)$ but $\deg_{\widehat{w}} f \neq 0$ for all $\widehat{w} \in \mathbb{Q}$ between w and $\overline{w}$, then $f_w^{!!} \neq (0,0) \neq f_{\overline{w}}^!$ with $|f_w^{!!}| < |f_{\overline{w}}^!|$.

(x) If $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \emptyset$ then either $f_w^{!!} \neq (0,0) \neq f_{\overline{w}}^!$ with $|f_w^{!!}| > |f_{\overline{w}}^!|$, or $f_w^{!!} \neq (0,0) \neq f_{\overline{w}}^!$ with $|f_w^{!!}| < |f_{\overline{w}}^!|$, or $\deg_{\widehat{w}} f = 0$ for some $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$.

(xi) If $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \emptyset$ with $f_w^{!!} \approx f_{\overline{w}}^!$ then $\deg_{\widehat{w}} f = 0$ for some $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ which is between w and $\overline{w}$.

(xii) If $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) \neq \emptyset$ with $f \sim_w g$ and $f \sim_{\overline{w}} g$ with $(\deg_w f)(\deg_{\overline{w}} f) \neq 0$, and either $w > w^\dagger(g) > \overline{w}$ or $w < w^\ddagger(g) < \overline{w}$, then for some $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ which is between w and $\overline{w}$ we have $\deg_{\widehat{w}} g = 0$.]

(VII) Let $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ be such that $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) \neq \emptyset$. Also let $\deg_w f = N$ and $\deg_{\overline{w}} f = \overline{N}$ with $\deg_w g = M$ and $\deg_{\overline{w}} g = \overline{M}$. Then we have the following.

(VII.1) Exactly one of the following seven conditions holds:

(1) $w = \overline{w}$;
(2) $w > \overline{w} > w^\dagger(f)$;
(3) $w > \overline{w} = w^\dagger(f) > -\infty$;
(4) $w > \overline{w} = w^\dagger(f) = -\infty$;
(5) $w < \overline{w} < w^\ddagger(f)$;
(6) $w < \overline{w} = w^\ddagger(f) < \infty$;
(7) $w < \overline{w} = w^\ddagger(f) = \infty$.

(VII.2) If $f \sim_w g$ and $f \sim_{\overline{w}} g$ with $N\overline{N} \neq 0$ and for every $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ which is between w and $\overline{w}$ we have $\deg_{\widehat{w}} g \neq 0$, then corresponding to (1) to (7) of (VII.1) we respectively have:

(1*) $w = \overline{w}$;
(2*) $w > \overline{w} > w^\dagger(g)$;
(3*) $w > \overline{w} = w^\dagger(f) = w^\dagger(g) > -\infty$;
(4*) $w > \overline{w} = w^\dagger(f) = w^\dagger(g) = -\infty$;
(5*) $w < \overline{w} < w^\ddagger(g)$;
(6*) $w < \overline{w} = w^\ddagger(f) = w^\ddagger(g) < \infty$;
(7*) $w < \overline{w} = w^\ddagger(f) = w^\ddagger(g) = \infty$;

and moreover we have:

$$\operatorname{Supp}(g_w^+) \cap \operatorname{Supp}(g_{\overline{w}}^+) \neq \emptyset \quad \text{and} \quad M\overline{M} \neq 0 \quad \text{with } M/N = \overline{M}/\overline{N}$$

and

$$(w, \overline{w})'(f) = (w, \overline{w})'(g).$$

(VII.3) If $f \sim_w g$ and $f \sim_{\overline{w}} g$ with $N\overline{N} > 0$ and $w \neq 1 \neq \overline{w}$ then we have:

$$(w)'(g) \in \mathbb{Q} \quad \text{with } (\overline{w})'(g) \in \mathbb{Q}$$

and

$$(w)'(g) = (w,\overline{w})'(f)(\overline{w})'(g) \quad \text{with } (w,\overline{w})'(f) \in \mathbb{Q}^\times.$$

(VIII) Let $\overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ be such that $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) \neq \emptyset$ and $w \neq 1 \neq \overline{w}$. Also let $\deg_w f = N$ and $\deg_{\overline{w}} f = \overline{N}$ with $\deg_w g = M$ and $\deg_{\overline{w}} g = \overline{M}$. Then we have the following.

(VIII.1) Assume that $N\overline{N} \neq 0$ and $f \sim_w g$ with $f \sim_{\overline{w}} g$. Also assume that $f \sim_w J(f,g)$ with $f \sim_{\overline{w}} J(f,g)$, and for every $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}} g \neq 0 \neq \deg_{\widehat{w}} J(f,g)$. Then

$$\lambda_w(f,g) \in \mathbb{Q} \quad \text{with } \lambda_{\overline{w}}(f,g) \in \mathbb{Q}$$

and

$$\lambda_w(f,g) = (w,\overline{w})'(f)\lambda_{\overline{w}}(f,g) \quad \text{with } (w,\overline{w})'(f) \in \mathbb{Q}^\times.$$

(VIII.2) Assume that $N\overline{N} > 0$, and let $\overline{f}, \overline{g}$ in $k[X,Y]^\times$ be such that: either (i) $(\overline{f},\overline{g}) = (f,g)$, or (ii) $(\overline{f},\overline{g}) = (f_w^{\odot}, g_w^{\odot})$ with $w > \overline{w} \geqslant w^\dagger(g)$, or (iii) $(\overline{f},\overline{g}) = (f_w^{\odot\odot}, g_w^{\odot\odot})$ with $w < \overline{w} \leqslant w^\ddagger(g)$. Assume that $\overline{f} \sim_w \overline{g}$ and $\overline{f} \sim_{\overline{w}} \overline{g}$. Then we have:

$$(w)'(g) \in \mathbb{Q} \quad \text{with } (\overline{w})'(g) \in \mathbb{Q}$$

and

$$(w)'(g) = (w,\overline{w})'(f)(\overline{w})'(g) \quad \text{with } (w,\overline{w})'(f) \in \mathbb{Q}^\times.$$

Moreover, if also $f \sim_w J(f,g)$ with $f \sim_{\overline{w}} J(f,g)$, then we have

$$\lambda_w(f,g) \in \mathbb{Q} \quad \text{with } \lambda_{\overline{w}}(f,g) \in \mathbb{Q}$$

and

$$\lambda_w(f,g) = (w,\overline{w})'(f)\lambda_{\overline{w}}(f,g) \quad \text{with } (w,\overline{w})'(f) \in \mathbb{Q}^\times.$$

(IX) If $w \neq 1$ then we have:

$$\lambda_w(f,g) \neq 1 \quad \Leftrightarrow \quad \mathrm{lag}_w(f,g) \neq 0 \quad \Leftrightarrow \quad f \sim_w g$$

and if $w < 1$ then we have:

$$\lambda_w(f,g) \geqslant 1.$$

(X) Let $\overline{w} \in \mathbb{Q}$ be such that $w < \overline{w} = \min(w^\ddagger(f), w^\ddagger(g))$ and $-1 < \overline{w} < 1$. Assume that $f \sim_w J(f,g)$ and $f \sim_{\overline{w}} J(f,g)$ with $f \nsim_w g$. Also assume that $(\deg_w f)(\deg_{\overline{w}} f) > 0$ and $f_w^{\odot\odot}$ is above the 45° line. Finally let us assume that either (i) $\overline{w} < 0$ or (ii) $\overline{w} \geqslant 0 < \deg_{\overline{w}} f$. Then: $\overline{w} = 0 = w^\ddagger(f)$ and $f_{\overline{w}}^+ = \theta X^i (Y+\gamma)^j$ with $g_{\overline{w}}^+ = \theta X^{i^*}(Y+\gamma)^{j^*}$ where $\gamma \in k^\times$ and i, j, i^*, j^* are nonnegative integers such that $i - j \neq 0 \neq i + j \neq 0 \neq i^* + j^*$ with $(i^*, j^*) = (1 + ci, 1 + cj)$ and $c \in \mathbb{Q}$.

[Clearly: $(\deg_w f)(\deg_{\overline{w}} f) > 0$ and either (i) or (ii) hold iff $\deg_w f > 0 < \deg_{\overline{w}} f$. Also note that, under appropriate conditions including $\overline{w} \neq 0$, (X) essentially says: $f \sim_w J(f,g)$ & $f \sim_{\overline{w}} J(f,g) \Rightarrow f \sim_w g$.]

Proof of (V) to (X). (V) follows from (3.11) or can also be checked directly.

(V*) does not require any proof.

(VI) follows from (V).

(VI*) can be proved thus. We shall use the facts which say that: for any fixed $(i,j) \in \mathbb{N}^2$, the function $(1/w_1)\deg_w(X^iY^j) = i - jw$ is monotonically decreasing in $w \in \mathbb{Q}$ if $j \neq 0$ and constant if $j = 0$; collectively we call these the monotonicity. In addition to the basic equation $(1/w_1)\deg_w(X^iY^j) = i - jw$ for $w \in \mathbb{Q}$, we shall also use the subsidiary equations $\deg_{-\infty}(X^iY^j) = j$ and $\deg_{\infty}(X^iY^j) = -j$. For any $(i,j) \in \mathbb{N}^2$ and $w < \overline{w}$ in $\mathbb{Q} \cup \{\pm\infty\}$, by the *monotonicity* and the subsidiary equations we get (i′) to (v′):

- (i′) If $\deg_w(X^iY^j) \leqslant 0$ with $j \neq 0$ then $\deg_{\overline{w}}(X^iY^j) < 0$. If $\deg_w(X^iY^j) = 0$ with $j \neq 0$ then $\deg_{\widetilde{w}}(X^iY^j) > 0$ for all $\widetilde{w} < w$ in $\mathbb{Q} \cup \{\pm\infty\}$.
- (ii′) If $\deg_w(X^iY^j) < 0$ then $\deg_{\overline{w}}(X^iY^j) < 0$.
- (iii′) If $\deg_w(X^iY^j) > 0 < \deg_{\overline{w}}(X^iY^j)$ then for all $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}}(X^iY^j) > 0$.
- (iv′) If $\deg_w(X^iY^j) > 0 > \deg_{\overline{w}}(X^iY^j)$ then for a unique $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}}(X^iY^j) = 0$; moreover, for this $\widehat{w}$ we have $\deg_{w^*}(X^iY^j) > 0$ for all $w^* \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\widehat{w}$, and $\deg_{w^{**}}(X^iY^j) < 0$ for all $w^{**} \in \mathbb{Q} \cup \{\pm\infty\}$ between $\widehat{w}$ and $\overline{w}$.
- (v′) If $\deg_w(X^iY^j) > 0 = \deg_{\overline{w}}(X^iY^j)$ then $(i,j) \neq (0,0)$ and for all $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}} f > 0$.

The first and the third sentences of (i) are self-explanatory. Consequently, to prove (i) we may assume that $\deg_w f = 0$ and $\widetilde{w} < w < \overline{w}$ in $\mathbb{Q} \cup \{\pm\infty\}$. We want to show that: if $f_w^+ \in k$ then $f_{\overline{w}}^+ \in k$; if $(0,0) \notin \mathrm{Supp}(f)$ then $\deg_{\overline{w}} f < 0 < \deg_{\widetilde{w}} f$; and if $(0,0) \in \mathrm{Supp}(f)$ then $\deg_{\overline{w}} f = 0$. Since $w \in \mathbb{Q}$ and $\deg_w f = 0$, we can write $f = \alpha + \sum_{(i,j)\in S} \beta_{ij} X^iY^j$ with $\alpha \in k$ and $\beta_{ij} \in k^\times$ where S is a finite subset of $\mathbb{N}^2 \setminus \{(0,0)\}$ such that for all $(i,j) \in S$ we have $j \neq 0$ and $\deg_w(X^iY^j) \leqslant 0$; moreover, if $\alpha = 0$ then $\deg_w(X^iY^j) = 0$ for some $(i,j) \in S$. Now by (i′) we get $\deg_{\overline{w}}(X^iY^j) < 0$ for all $(i,j) \in S$, and if $\alpha = 0$ then $\deg_{\widetilde{w}}(X^iY^j) > 0$ for some $(i,j) \in S$. So we are done. [Similarly (ii) follows from (ii′).]

Now supposing $w < \overline{w}$ in $\mathbb{Q} \cup \{\pm\infty\}$, let us prove the weaker versions (ii*) to (v*) of (ii) to (v) obtained by assuming $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) \neq \emptyset$. By (8.4.14)(1) to (8.4.14)(6) we know that then there is a unique $(i,j) \in \mathbb{N}^2$ such that for all $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ with $w < \widehat{w} \leqslant \overline{w}$ we have $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\widehat{w}}^+) = \{(i,j)\}$ with $f_w^{!!} = (i,j) = f_{\widehat{w}}^{!}$. Hence by (ii′) to (v′) we respectively get (ii**) to (v**):

(ii**) If $\deg_w f < 0$ then $\deg_{\overline{w}} f < 0$.

(iii**) If $\deg_w f > 0 < \deg_{\overline{w}} f$ then for all $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}} f > 0$.

(iv**) If $\deg_w f > 0 > \deg_{\overline{w}} f$ then for a unique $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}} f = 0$; moreover, for this $\widehat{w}$ we have $\deg_{w^*} f > 0$ for all $w^* \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\widehat{w}$, and $\deg_{w^{**}} f < 0$ for all $w^{**} \in \mathbb{Q} \cup \{\pm\infty\}$ between $\widehat{w}$ and $\overline{w}$.

(v**) If $\deg_w f > 0 = \deg_{\overline{w}} f$ then for all $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}} f > 0$; moreover, if $(0,0) \in \mathrm{Supp}(f)$ then f is not a $\overline{w}$-monomial.

This completes the proof of the weaker versions.

Continuing to suppose $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$, we shall now deduce the original versions of (ii) to (v) from their weaker versions by using the *connectivity* of the Degreewise Newton Polygon $w(f)$ thus; this is similar to the use of connectivity of the Positive Newton Polygon $w_p(f)$ in the proof of (8.4.18)(14).

Let $n = l(f, \overline{w}) - l(f, w)$ and $\widetilde{w} = w^{\ddagger}(f)$. Then $n \in \mathbb{N}$ with $\widetilde{w} \in \mathbb{Q} \cup \{\pm\infty\}$ and by (8.4.14)(1) to (8.4.14)(6) we see that: $n = 0 \Rightarrow \overline{w} \leqslant \widetilde{w} \Rightarrow$ we are in the weaker version, and $\widetilde{w} < \overline{w} \Rightarrow 0 \leqslant l(f, \overline{w}) - l(f, \widetilde{w}) < l(f, \overline{w}) - l(f, w)$. So (ii) follows by induction on n. In case of (iii) we see that if $\widetilde{w} < \overline{w}$ then by (i) and (ii) we have $\deg_{\widetilde{w}} f > 0$ and hence (iii) also follows by induction on n. If $\widetilde{w} < \overline{w}$ with $\deg_w f > 0 = \deg_{\widetilde{w}} f = 0 > \deg_{\overline{w}} f$ then by the last inequality we have $(0,0) \notin \operatorname{Supp}(f)$ and hence by (i) we see that $\deg_{w^*} f > 0$ for all $w^* \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\widetilde{w}$, and $\deg_{w^*} f < 0$ for all $w^{**} \in \mathbb{Q} \cup \{\pm\infty\}$ between $\widetilde{w}$ and $\overline{w}$; consequently (iv) follows by induction on n. Similarly, in view of (i) to (iii), we also get (v) by induction on n. In all this the "initial case" of the induction is $\overline{w} \leqslant \widetilde{w}$ rather than $n = 0$.

This completes the proof of (i) to (v). To prove (vi) and (vii), by symmetry we may assume that $w < \overline{w}$ and then, in view of (V*)(i), we are done by (ii) to (iv). (viii) follows from (8.4.14)(1) to (8.4.14)(6).

To prove (ix) let $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ be such that $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \emptyset$. Upon letting $r = l(f, w^{\ddagger}(f))$ and $t = l(f, \overline{w}^{\dagger}(f))$, in view of (8.4.14)(1) to (8.4.14)(6) we see that $r \leqslant t \leqslant l(f)$in $\mathbb{N}_+$ and

$$w < w^{\ddagger}(f) = w(f, r) < w(f, r+1) < \cdots < w(f, t) = \overline{w}^{\dagger}(f) < \overline{w}.$$

Clearly

$$f_w^{!!} = f_{w(f,r)}^{!} \quad \text{and} \quad f_{w(f,t)}^{!!} = f_{\overline{w}}^{!}$$

and we have

$$f_{w(f,s)}^{!!} = f_{w(f,s+1)}^{!} \quad \text{for } r \leqslant s < t.$$

In a moment we shall show that:

(ix*) if $\deg_w f \leqslant 0$ then $\deg_{w(f,s)} f < 0$ for $r \leqslant s \leqslant t$,

(ix′) if $\deg_w f > 0 < \deg_{\overline{w}} f$ then $\deg_{w(f,s)} f > 0$ for $r \leqslant s \leqslant t$,

(ix″) if $\deg_w f > 0 = \deg_{\overline{w}} f$ with $(0,0) \notin \operatorname{Supp}(f)$, then $\deg_{w(f,s)} f > 0$ for $r \leqslant s \leqslant t$,

(ix‴) if $\deg_w f > 0 = \deg_{\overline{w}} f$ with $(0,0) \in \operatorname{Supp}(f)$ but $\deg_{\widehat{w}} f \neq 0$ for all $\widehat{w} \in \mathbb{Q}$ between w and $\overline{w}$, then $\deg_{w(f,s)} f > 0$ for $r \leqslant s \leqslant t$,

and by (7.4.14)(3)(V.3) this will complete the proof of (ix).

If $\deg_w f < 0$ then (ix*) follows from (ii). If $\deg_w f = 0$ with $w \neq -\infty$ and $(0,0) \notin \operatorname{Supp}(f)$ then (ix*) follows from (i). If $\deg_w f = 0$ with $w \neq -\infty$ and $(0,0) \in \operatorname{Supp}(f)$ then by (i) we get $(0,0) \in \operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+)$ which contradicts the assumption that $\operatorname{Supp}(f_w^+) \cap \operatorname{Supp}(f_{\overline{w}}^+) = \emptyset$. If $\deg_w f = 0$ with $w = -\infty$ then by (i) we get $\operatorname{Supp}(f) \subset \{(i,0)\colon i \in \mathbb{N}\}$ and hence $w^{\ddagger}(f) = \infty$ which contradicts the above displayed inequalities $w < w^{\ddagger}(f) < \overline{w}$. This proves (ix*).

(ix′) follows from (iii). (ix″) and (ix‴) follow from (v).

This completes the proof of (ix), and by (iv) it implies (x). (xi) follows from (x).

To prove (xii), by symmetry we may assume that $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) \neq \emptyset$ with $f \sim_w g$ and $f \sim_{\overline{w}} g$ with $(\deg_w f)(\deg_{\overline{w}} f) \neq 0$, and $w < w^\ddagger(g) < \overline{w}$. We want to show that then for some $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ which is between w and $\overline{w}$ we have $\deg_{\widehat{w}} g = 0$. Since $w < w^\ddagger(g) < \overline{w}$, by (8.4.14)(1) to (8.4.14)(6) we see that $\mathrm{Supp}(g_w^+) \cap \mathrm{Supp}(g_{\overline{w}}^+) = \emptyset$; in a moment we shall show that $g_w^{!!} \approx g_{\overline{w}}^{!}$ and then we will be done by taking g for f in (xi). If $g_w^+ \in k$ then clearly $g_w^{!!} = (0,0)$ and if $g_{\overline{w}}^+ \in k$ then clearly $g_{\overline{w}}^{!} = (0,0)$; in both the cases we have $g_w^{!!} \approx g_{\overline{w}}^{!}$. So we may assume that $g_w^+ \notin k$ and $g_{\overline{w}}^+ \notin k$.

Let $\deg_w f = N$ and $\deg_{\overline{w}} f = \overline{N}$ with $\deg_w g = M$ and $\deg_{\overline{w}} g = \overline{M}$. Then, since we are assuming $f \sim_w g$ and $f \sim_{\overline{w}} g$ with $N \neq 0 \neq \overline{N}$, in view of (3.11), by (V) we get $M \neq 0 \neq \overline{M}$ with $(f_w^+)^{|M|} = \theta (g_w^+)^{|N|}$ and $(f_{\overline{w}}^+)^{|\overline{M}|} = \theta (g_{\overline{w}}^+)^{|\overline{N}|}$. Therefore $f_w^{!!} \neq (0,0) \neq g_w^{!!}$ with $|M| f_w^{!!} = |N| g_w^{!!}$ and $f_{\overline{w}}^{!} \neq (0,0) \neq g_{\overline{w}}^{!}$ with $|\overline{M}| f_{\overline{w}}^{!} = |\overline{N}| g_{\overline{w}}^{!}$. Since $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) \neq \emptyset$, by (8.4.14)(1) to (8.4.14)(6) we see that $f_w^{!!} = f_{\overline{w}}^{!}$. Consequently $g_w^{!!} \approx g_{\overline{w}}^{!}$.

This completes the proof of (xii), and hence of the entire Note (VI*).

(VII.1) follows from (8.4.14)(1) to (8.4.14)(6).

(VII.2) can be proved thus. If $w = \overline{w}$ then we are done by ($3^\bullet$). So assume that $w \neq \overline{w}$. Now upon letting

$$S = \mathrm{Supp}\big(f_w^+\big) \cap \mathrm{Supp}\big(f_{\overline{w}}^+\big)$$

and

$$S' = \big\{(i,j) \in \mathrm{Supp}(f):\ j > \big(f_w^{!}\big)_2\big\}$$

and

$$S'' = \big\{(i,j) \in \mathrm{Supp}(f):\ j < \big(f_w^{!!}\big)_2\big\}$$

by (8.4.14)(1) to (8.4.14)(6) we see that (with (2) to (7) as in (VII.1))

$$\text{(A)}\qquad \begin{cases} (2) \Rightarrow S = \{f_w^{!}\} = \{f_{\overline{w}}^{!!}\}\ \&\ f_{\overline{w}}^+ \text{ is a monomial;} \\ (3) \Rightarrow S = \{f_w^{!}\} = \{f_{\overline{w}}^{!!}\}\ \&\ f_{\overline{w}}^+ \text{ is not a monomial } \&\ S' \neq \emptyset; \\ (4) \Rightarrow S' = \emptyset; \\ (5) \Rightarrow S = \{f_w^{!!}\} = \{f_{\overline{w}}^{!}\}\ \&\ f_{\overline{w}}^+ \text{ is a monomial;} \\ (6) \Rightarrow S = \{f_w^{!!}\} = \{f_{\overline{w}}^{!}\}\ \&\ f_{\overline{w}}^+ \text{ is not a monomial } \&\ S'' \neq \emptyset; \\ (7) \Rightarrow S'' = \emptyset \end{cases}$$

and upon letting

$$T = \mathrm{Supp}\big(g_w^+\big) \cap \mathrm{Supp}\big(g_{\overline{w}}^+\big)$$

and

$$T' = \big\{(i,j) \in \mathrm{Supp}(g):\ j > \big(g_w^{!}\big)_2\big\}$$

and

$$T'' = \big\{(i,j) \in \mathrm{Supp}(g):\ j < \big(g_w^{!!}\big)_2\big\}$$

by (8.4.14)(1) to (8.4.14)(6) we also see that

(B)
$$\begin{cases} w > \overline{w} > w^{\dagger}(g) \Rightarrow T = \{g^{!}_{w}\} = \{g^{!!}_{\overline{w}}\} \ \&\ g^{+}_{\overline{w}} \text{ is a monomial;} \\ w > \overline{w} = w^{\dagger}(g) > -\infty \Rightarrow T = \{g^{!}_{w}\} = \{g^{!!}_{\overline{w}}\} \\ \qquad \&\ g^{+}_{\overline{w}} \text{ is not a monomial } \&\ T' \neq \emptyset; \\ w > \overline{w} = w^{\dagger}(g) = -\infty \Rightarrow T' = \emptyset; \\ w < \overline{w} < w^{\ddagger}(g) \Rightarrow T = \{g^{!!}_{w}\} = \{g^{!}_{\overline{w}}\} \ \&\ g^{+}_{\overline{w}} \text{ is a monomial;} \\ w < \overline{w} = w^{\ddagger}(g) < \infty \Rightarrow T = \{g^{!!}_{w}\} = \{g^{!}_{\overline{w}}\} \\ \qquad \&\ g^{+}_{\overline{w}} \text{ is not a monomial } \&\ T'' \neq \emptyset; \\ w < \overline{w} = w^{\ddagger}(g) = \infty \Rightarrow T'' = \emptyset. \end{cases}$$

Clearly

(C)
$$\begin{cases} \text{if } f \sim_{w} g \text{ with } \deg_{w} f \neq 0 \neq \deg_{w} g \text{ then:} \\ \quad f \text{ is a } w\text{-monomial iff } g \text{ is a } w\text{-monomial.} \end{cases}$$

Under the hypothesis of (VII.2) we have

(D)
$$f \sim_{w} g \quad \text{with } f \sim_{\overline{w}} g$$

and in view of (VI*)(i) we see that

(E)
$$N \neq 0 \neq \overline{N} \quad \text{with } M \neq 0 \neq \overline{M}$$

and by (VI*)(xii) we see that

(F)
$$\text{either } w > \overline{w} \geqslant w^{\dagger}(g) \text{ or } w < \overline{w} \leqslant w^{\ddagger}(g).$$

Now (disregarding the hypothesis of (VII.2) but assuming $w \neq \overline{w}$), in view of (3.11), (8.4.14)(1) to (8.4.14)(6), (V), (VII.1), and (A) to (F), we see that (2) to (7) respectively imply (2*) to (7*) and we have

$$\begin{cases} (f^{+}_{w})^{+}_{\overline{w}} = (f^{+}_{\overline{w}})^{+}_{w} = \text{a monomial } H_f \\ \quad \text{with } (g^{+}_{w})^{+}_{\overline{w}} = (g^{+}_{\overline{w}})^{+}_{w} = \text{a monomial } H_g \\ \quad \text{and } (H_f)^{M} = \theta (H_g)^{N} \text{ with } (H_f)^{\overline{M}} = \theta (H_g)^{\overline{N}}, \\ \quad \text{and } T \neq \emptyset \text{ with } M/N = \overline{M}/\overline{N}. \end{cases}$$

Therefore by (I) and (II) we conclude that

$$(w, \overline{w})'(f) = (w, \overline{w})'(g)$$

where we note that if either $w = 1$ or $\overline{w} = 1$ then by definition both sides of the above equation are reduced to ∞. This completes the proof of (VII.2).

(VII.3) can be proved thus. In view of ($1^{\bullet}$) we see that $(w, \overline{w})'(f) \in \mathbb{Q}^{\times}$ and $(w)'(g) \in \mathbb{Q}$ with $(\overline{w})'(g) \in \mathbb{Q}$. Now clearly $1/(w, \overline{w})'(f) = (\overline{w}, w)'(f)$ and hence by symmetry we may assume that $w \leqslant \overline{w}$. If $M = 0$ then in view of (V) and (VI*)(i) we see that $\overline{M} = 0$ and hence $(w)'(g) = 0 = (\overline{w})'(g) \in \mathbb{Q}$ and therefore $(w)'(g) = (w, \overline{w})'(f)(\overline{w})'(g)$. If $M < 0$ then by (V) and (VI*)(ii) we see that all the four integers $N, \overline{N}, M, \overline{M}$ are negative and for every $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}} g < 0$, and therefore by (VII.2) we get $(w)'(g) = (w, \overline{w})'(f)(\overline{w})'(g)$. So now we may assume that $M > 0$. Then by (V) we get $N > 0 < \overline{N}$. In view of (8.4.14)(1) to (8.4.14)(6), the similarities $f \sim_w g$ and $f \sim_{\overline{w}} g$ tell us that

$$f_w^{\odot\odot} = f_{\overline{w}}^{\odot} \quad \text{and} \quad \big(f_w^{\odot\odot}\big)^M = \big(g_w^{\odot\odot}\big)^N \quad \text{with } \big(f_{\overline{w}}^{\odot}\big)^{\overline{M}} = \big(g_{\overline{w}}^{\odot}\big)^{\overline{N}}.$$

Since $g_w^{!!} \in \mathrm{Supp}(g)$, by the above display we get $\overline{M} > 0$. Now by (VI*)(iii) we see that for every $\widehat{w} \in \mathbb{Q} \cup \{\pm\infty\}$ between w and $\overline{w}$ we have $\deg_{\widehat{w}} g > 0$, and therefore again by (VII.2) we get $(w)'(g) = (w, \overline{w})'(f)(\overline{w})'(g)$.

(VIII.1) can be proved thus. By ($1^\bullet$), ($2^\bullet$), and (VII.2) we see that

$$(1')\qquad \begin{cases} (w)'(f) \in \mathbb{Q}^\times \text{ with } (\overline{w})'(f) \in \mathbb{Q}^\times \\ \quad \text{and } (w)'(J(f,g)) \in \mathbb{Q} \text{ with } (\overline{w})'(J(f,g)) \in \mathbb{Q} \\ \quad \text{and } (w)'(g) \in \mathbb{Q} \text{ with } (\overline{w})'(g) \in \mathbb{Q} \end{cases}$$

and

$$(2')\qquad \begin{cases} \text{upon letting } \mu = (w, \overline{w})'(f) \\ \quad \text{we have } \mu \in \mathbb{Q}^\times \text{ with } (w)'(f) = (\overline{w})'(f)\mu \\ \quad \text{and } (w)'(J(f,g)) = (\overline{w})'(J(f,g))\mu \\ \quad \text{and } (w)'(g) = (\overline{w})'(g)\mu. \end{cases}$$

Now, in view of ($1'$) and ($2'$), by the definition of the multiplicative lag we get

$$\lambda_w(f,g) \in \mathbb{Q} \quad \text{and} \quad \lambda_{\overline{w}}(f,g) \in \mathbb{Q} \quad \text{with } \lambda_w(f,g) = \lambda_{\overline{w}}(f,g)\mu.$$

(VIII.2) can be proved thus. By ($1^\bullet$), ($2^\bullet$), and (VII.3) we see that

$$(1'')\qquad \begin{cases} (w)'(f) \in \mathbb{Q}^\times \text{ with } (\overline{w})'(f) \in \mathbb{Q}^\times \\ \quad \text{and upon letting } \mu = (w, \overline{w})'(f) \\ \quad \text{we have } \mu \in \mathbb{Q}^\times \text{ with } (w)'(f) = (\overline{w})'(f)\mu \end{cases}$$

and

$$(2'')\qquad \begin{cases} \text{if } f \sim_w J(f,g) \text{ with } f \sim_{\overline{w}} J(f,g) \\ \quad \text{then we have } (w)'(J(f,g)) \in \mathbb{Q} \text{ with } (\overline{w})'(J(f,g)) \in \mathbb{Q} \\ \quad \text{and } (w)'(J(f,g)) = (\overline{w})'(J(f,g))\mu. \end{cases}$$

By (8.4.14)(1) to (8.4.14)(6) we see that

$$\begin{cases} \text{in cases (ii) and (iii),} \\ \quad \text{the monomial } \overline{f} \text{ is in } f_w^+ \text{ as well as in } f_{\overline{w}}^+, \\ \quad \text{and the monomial } \overline{g} \text{ is in } g_w^+ \text{ as well as in } g_{\overline{w}}^+, \end{cases}$$

and hence in view of (II) we see that (it being trivial in case (i)) in all the three cases we have

$$(3'') \qquad \begin{cases} (w)'(f) = (w)'(\overline{f}) \text{ with } (\overline{w})'(f) = (\overline{w})'(\overline{f}), \\ \quad \text{and } (w)'(g) = (w)'(\overline{g}) \text{ with } (\overline{w})'(g) = (\overline{w})'(\overline{g}) \end{cases}$$

In view of $(3'')$, by $(1^\bullet)$, $(2^\bullet)$, and (VII.3) we see that

$$(4'') \qquad \begin{cases} (w)'(f) \in \mathbb{Q}^\times \text{ with } (\overline{w})'(f) \in \mathbb{Q}^\times \\ \quad \text{and } (w)'(g) \in \mathbb{Q} \text{ with } (\overline{w})'(g) \in \mathbb{Q} \end{cases}$$

and

$$(5'') \qquad \begin{cases} \text{recalling that } \mu = (w, \overline{w})'(f) \\ \quad \text{we have } \mu \in \mathbb{Q}^\times \\ \quad \text{with } (w)'(f) = (\overline{w})'(f)\mu \\ \quad \text{and } (w)'(g) = (\overline{w})'(g)\mu. \end{cases}$$

Now, in view of $(1'')$ to $(5'')$, by the definition of the multiplicative lag we see that if $f \sim_w J(f,g)$ with $f \sim_{\overline{w}} J(f,g)$ then

$$\lambda_w(f,g) \in \mathbb{Q} \quad \text{and} \quad \lambda_{\overline{w}}(f,g) \in \mathbb{Q} \quad \text{with } \lambda_w(f,g) = \lambda_{\overline{w}}(f,g)\mu.$$

(IX) follows from (7.4.1).

(X) can be proved thus. Disregarding the last two sentences which start with "Finally" but assuming that $\deg_{\overline{w}} f > 0$ and $f_w^{\odot\odot} \sim_w g_w^{\odot\odot}$, by (VI) and case (iii) of (VIII.2) we see that

$$\lambda_w(f,g) = (w, \overline{w})'(f)\lambda_{\overline{w}}(f,g).$$

But by (IX) we have $\lambda_w(f,g) = 1 \leqslant \lambda_{\overline{w}}(f,g)$ and by (III) we have $(w, \overline{w})'(f) > 1$, which contradicts the above displayed equation. Therefore $f_w^{\odot\odot} \nsim_w g_w^{\odot\odot}$. By (8.4.14)(1) to (8.4.14)(6) we have $f_w^{\odot\odot} = f_{\overline{w}}^{\odot}$ with $g_w^{\odot\odot} = g_{\overline{w}}^{\odot}$, and hence by (VI) we get $f_{\overline{w}}^{\odot} \nsim_{\overline{w}} g_{\overline{w}}^{\odot}$. By (8.4.14)(2) we also see that either f is not a $\overline{w}$-monomial or g is not a $\overline{w}$-monomial. Therefore, in view of (3.3.5*), we are done by (4.6); in greater detail, because of $-1 < \overline{w} < 1$, we must have $0 \leqslant \overline{w}$ or $0 > \overline{w} > -1$, i.e., we are respectively in case (4) or (3) of (4.6) and, because f or g is non-$\overline{w}$-monomial, in both the cases we get $\gamma \neq 0$, and moreover:

(A) If $0 \leqslant \overline{w}$ then, because of $f_{\overline{w}}^{\odot} \nsim_{\overline{w}} g_{\overline{w}}^{\odot}$ & (3.3.5*), we have: $\gamma \neq 0 \Rightarrow \overline{w} = 0$.

(B) If $0 > \overline{w} > -1$ then we have: $\gamma \neq 0 \Rightarrow -1/\overline{w}$ is an integer $\geqslant 2 \Rightarrow f_{\overline{w}}^{\odot} \sim_{\overline{w}} g_{\overline{w}}^{\odot}$ (because both are constant multiples of powers of Y) $\Rightarrow$ contradiction. □

Note (XI). Given any $w, \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$, we say that $\overline{w}$ is *f-intersecting* to w to mean that $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) \neq \emptyset$, and we say that $\overline{w}$ is *f-contiguous* to w to mean that one of the 7 conditions (1) to (7) of (VII) holds as observed in the proof of (VII). In view of (8.4.14)(1) to (8.4.14)(6) we have: f-intersecting iff f-contiguous. If $\overline{w} \neq w$ is f-contiguous to w then we may further qualify it by saying that $\overline{w}$ is *forward f-contiguous* to w or *backwards f-contiguous* to w according as $w < \overline{w}$ or $w > \overline{w}$.

Given any $w, \overline{w} \in \mathbb{Q}_c$, we say that $\overline{w}$ is *full f-intersecting* to w to mean that $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) \neq \emptyset$ (so this is the same without the adjective full), and we say that $\overline{w}$ is *full f-*

contiguous to w to mean that one of the following 5 conditions holds: (1) $w = \overline{w}$; (2) $w > \overline{w} > w_c^{\dagger}(f)$; (3) $w > \overline{w} = w_c^{\dagger}(f)$; (4) $w < \overline{w} < w_c^{\ddagger}(f)$; (5) $w < \overline{w} = w_c^{\ddagger}(f)$. In view of (7.4.14)(3) we have: full f-intersecting iff full f-contiguous. Now *forward full contiguous* and *backwards full contiguous* have obvious meanings.

The versatile example $f = 1 + XY$ can be used to avoid many pitfalls. For instance in (VI*)(ix) we could have been tempted to condense the three case ($'$), ($''$), ($'''$) into one case saying that: if $w < \overline{w} \in \mathbb{Q} \cup \{\pm\infty\}$ is such that $\mathrm{Supp}(f_w^+) \cap \mathrm{Supp}(f_{\overline{w}}^+) = \emptyset$ with $\deg_w f > 0 \leqslant \deg_{\overline{w}} f$ then $f_w^{!!} \neq (0,0) \neq f_{\overline{w}}^{!}$ with $|f_w^{!!}| < |f_{\overline{w}}^{!}|$. This would be wrong because for the above f with $-1/2 = w < \overline{w} = \infty$ we have $f_{\overline{w}}^{!} = (0,0)$. Note that now $\overline{w}$ *is not forward f-contiguous* to w.

(XII) Sequence property of similarity. Let us consider a sequence $g_1, g_2, \ldots, g_e$ in $k[X,Y]^{\times}$ with integer $e > 1$. Assume that $f \sim_w g_j$ for $1 \leqslant j < e$ with $f \nsim_w g_e$. Also assume that $f \sim_w J(f, g_j)$ for $1 \leqslant j \leqslant e$. Finally assume that $f_w^{\odot\odot}$ is above the 45° line and $\deg_w f > 0$ with $w < w^{\ddagger}(f) \in \mathbb{Q}$. For every $\overline{w} \in \mathbb{Q}$ with $w < \overline{w} \leqslant w^{\ddagger}(f)$, consider the largest positive integer $\epsilon(\overline{w}) \leqslant e$ such that $f \sim_{\overline{w}} g_j$ for $1 \leqslant j < \epsilon(\overline{w})$. Then we have the following.

(1) Let $\overline{w} \in \mathbb{Q}$ be such that $w < \overline{w} = w^{\ddagger}(g_e) < w^{\ddagger}(f)$. Assume that $-1 < \overline{w} < 1$ with $\deg_{\overline{w}} f > 0$ and $f \sim_{\overline{w}} J(f, g_j)$ for $1 \leqslant j \leqslant \epsilon(\overline{w})$. Then $\epsilon(\overline{w}) < e$ and $f \nsim_{\overline{w}} g_{\epsilon(\overline{w})}$.
(2) Let $\overline{w} \in \mathbb{Q}$ be such that $w < \overline{w} = w^{\ddagger}(f)$. Assume that for every $\widehat{w} \in \mathbb{Q}$ with $w < \widehat{w} \leqslant \overline{w}$ we have: $-1 < \widehat{w} < 1$ with $\deg_{\widehat{w}} f > 0$ and $f \sim_{\widehat{w}} J(f, g_j)$ for $1 \leqslant j \leqslant \epsilon(\widehat{w})$. Then either we have:
 (i) $\epsilon(\overline{w}) < e$ and $f \nsim_{\overline{w}} g_{\epsilon(\overline{w})}$,
 or we have:
 (ii) $\overline{w} = 0$ and $f_{\overline{w}}^{+} = \theta X^i (Y + \gamma)^j$ with $(g_e)_{\overline{w}}^{+} = \theta X^{i^*} (Y + \gamma)^{j^*}$ where $\gamma \in k^{\times}$ and i, j, i^*, j^* are nonnegative integers such that $i - j \neq 0 \neq i + j \neq 0 \neq i^* + j^*$ with $(i^*, j^*) = (1 + ci, 1 + cj)$ and $c \in \mathbb{Q}$.

Proof of (XII). In (1), assuming $\epsilon(\overline{w}) = e$, we get a contradiction by taking $g = g_e$ in (X). Now consider (2) and let $\widetilde{w} = w^{\ddagger}(g_e)$. If $\widetilde{w} \geqslant \overline{w}$ then we are done by taking $g = g_e$ in (X). [**Note:** this is the only occurrence of (ii).] If $\widetilde{w} < \overline{w}$ then by (1) we have $\epsilon(\widetilde{w}) < e$ and $f \nsim_{\widetilde{w}} g_{\epsilon(\widetilde{w})}$. If $\epsilon(\widetilde{w}) = 1$ then we are done by taking $(w, g) = (\widetilde{w}, g_1)$ in (X). Otherwise apply induction on e with $(\widetilde{w}, \epsilon(\widetilde{w}))$ replacing (w, e). □

9. More geometry or degreewise Newton polygon

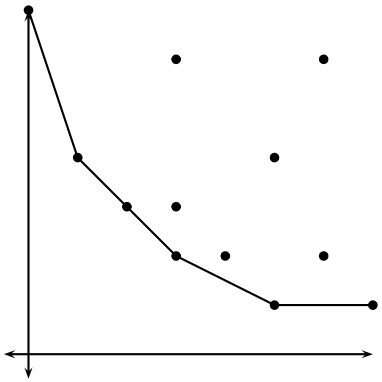
To continue with the geometric vein initiated in the Introduction, let us talk about the Newton Polygon.

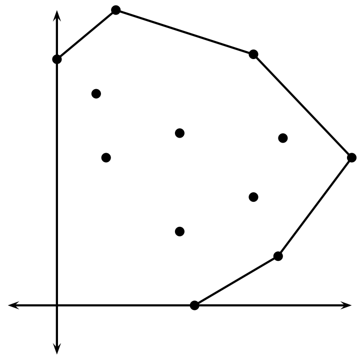
(9.1). We start with the bivariate power series $f(X,Y) = \sum a_{ij} X^i Y^j$ with coefficients a_{ij} in a field k. By applying the Weierstrass Preparation Theorem (cf. p. 85 of [Ab6]), we can arrange f to be a polynomial in Y whose coefficients are power series in X. Considering the support

$$\mathrm{Supp}(f) = \{(x,y) = (i,j) \mid a_{ij} \neq 0\}$$

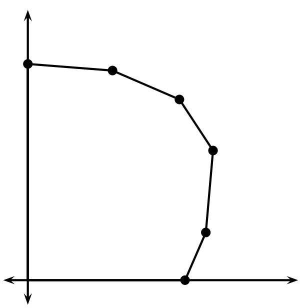
we plot these points in the xy-plane. Then taking the convex hull, we get the ordinary (orderwise) Newton Polygon; an example of it is the following diagram on the left. Since we have power series in X, there could be infinitely many dots above the polygon. See pp. 373–396 vol. II of [Chr] and pp. 98–106 of [Wal].

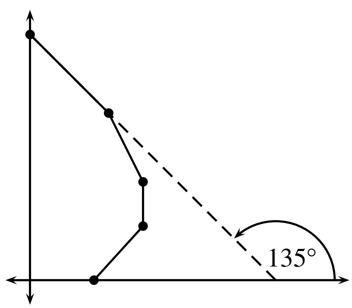
Now assume that f is a nonzero polynomial in X and Y. Then the support is a nonempty finite set. So the convex hull will look something like the following diagram on the right, and we call it the Degreewise Newton Polygon or DNP.





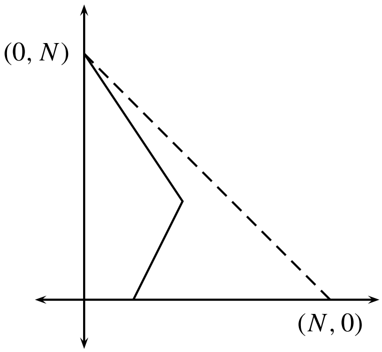
If f is monic in Y then its DNP will look like the following diagram on the left. If f is regular in Y, i.e., if its Y-degree equals its total degree, then its DNP will look like the following diagram on the right where the exhibited angle could be smaller than 135°. See (8.4.15).

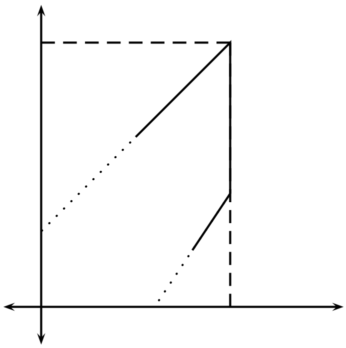




(9.2). For the concepts of 1 or 2 points at infinity see (1.2), (3.4), (4.6), and the preamble to Section 2. Here on the left is a picture for 1 point at infinity when f is regular in Y.

Moreover, on the right is a picture when f has 2 points at infinity, but f is neither monic nor regular in Y. Starting with f which is regular in Y and has two points at infinity by an appropriate linear transformation, we can arrange our polynomial to be such, as in the following shortsqueezed vertical rectangle on the right.

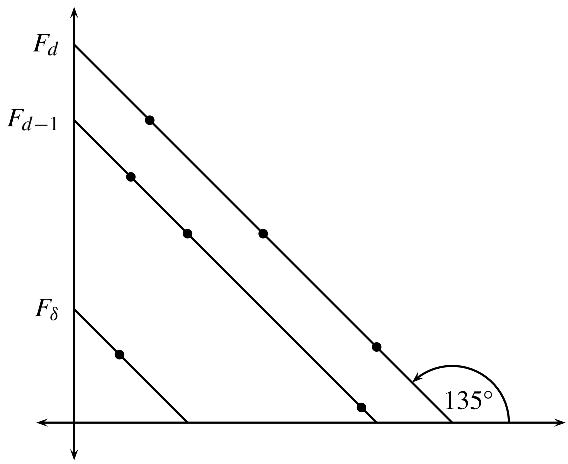




(9.3). Let us look at this Degreewise Newton Polygon more thoroughly. By collecting terms of like degree, we can decompose f into its homogeneous components, i.e.,

$$f = F_d + F_{d-1} + \cdots + F_\delta$$

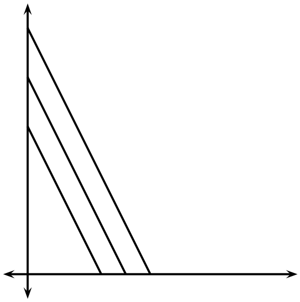
where F_d is nonzero, and either F_ℓ is homogeneous of degree ℓ or F_ℓ is identically zero. Plotting $\mathrm{Supp}(f)$ in the xy-plane, we can exhibit this decomposition by a diagram:

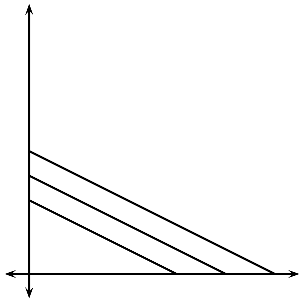


Note that the lines corresponding to homogeneous components of f will exhaust all of the points of $\mathrm{Supp}(f)$. Also note that some of the lines may not contain any points of $\mathrm{Supp}(f)$. The equation of these lines is $x + y = \ell$.

As a side note, we would like to address the angles that these newton lines make with the X-axis. You will notice that in some of the diagrams, the angles are sharper or more shallow than the 135° line. In our preparation for writing this paper, we have referred to these as 120° and 150° lines, simply to indicate that some lines are sharper or more shallow than 135°. The reason we have not labeled the diagrams with these angles is simply because such an angle would give us an irrational value of w, which we wish to avoid. In fact, we cannot find a "nice" angle that would give us a rational w. As such, we leave the extrapolation of such ideas up to the reader.

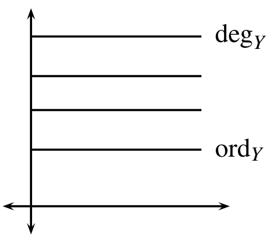
(9.4). Let us now give weights to the variables, where x has weight w_1 and y has weight w_2; see Sections 2, 7, and 8. We can look at pictures when $w_1 > w_2$ as in the following diagram on the left. Alternatively we can look at the picture when $w_1 < w_2$ as in the following diagram on the right.

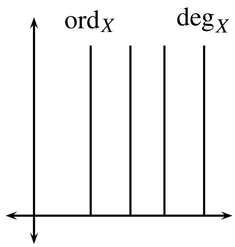




Note that in both of these diagrams, $w_1 > 0$. Then taking the generic equation of these lines as $xw_1 + yw_2 = c$ for some constant c, we can divide by w_1 to get $x - (-w_2/w_1)y = \gamma$ for some constant γ. Now by using the definition that $w = -w_2/w_1$, we have the equation $x = wy + \gamma$. Then we can clearly see that w represents what we call the y-slope. In fact, w is a slope, and we use the term y-slope to distinguish it from our usual definition of slope, and also to indicate that we are taking slopes with respect to the positive y-axis.

Now we look at two new diagrams. Our first diagram is similar to the diagram found in (9.3). However, instead of taking homogeneous components, we take the Y-degree components. The reader may consider this the same as taking homogeneous components after making the substitution $X = c$ for suitable constant c. Our second diagram is taking the X-degree components.

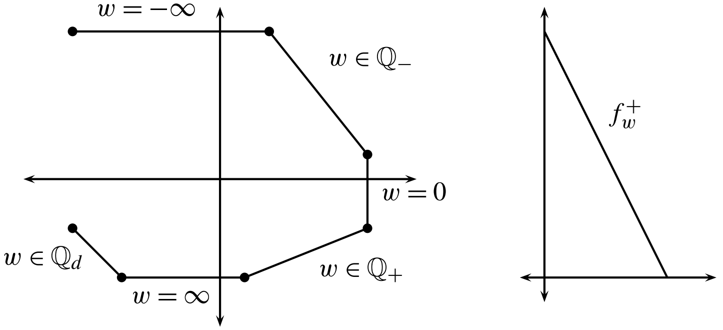




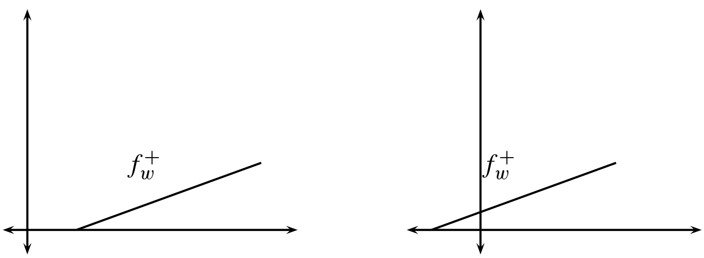
In the first diagram, we notice that the top-most such line gives the Y-degree of f, and the bottom-most line gives the Y-order of f. We have similar notions for the X-degree diagram. As noted above, for the Y-degree diagram, we can think of substituting $X = c$. However, this is equivalent to saying $w_1 = 0$ and $w_2 = 1$. Then $\deg_Y = \deg_{(0,1)} = \deg_{-\infty}$, where in the last expression, we used $\deg_w = \deg_{(w_1,w_2)}$. This makes sense when we remember that $(w_1, w_2) = (0, 1)$, so $w = -w_2/w_1 = -\infty$. With the X-degree diagram, we can similarly say that $\deg_X = \deg_{(1,0)} = \deg_0$.

These diagrams become important when we consider (4.16)(1) which implies that if f and g are two polynomials of degree $\geqslant 2$ with $J(f, g) = \theta$, then neither X nor Y can divide f. In terms of the diagrams, this says that on the Y-degree diagram, the $\operatorname{ord}_Y(f)$ line is the x-axis, and on the X-degree diagram, the $\operatorname{ord}_X(f)$ line is the y-axis.

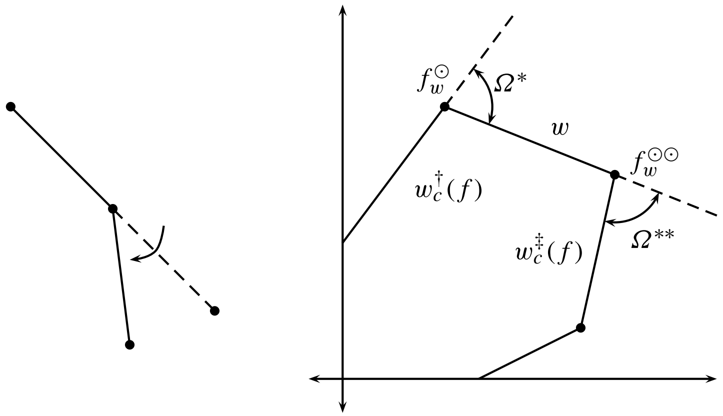
(9.5). The following picture on the right depicts the w-degree form f_w^+ of f when $-1 < w < 0$; it is similar to the first figures in (9.2) and (9.4). The following picture on the left depicts the Full Newton Polygon as described in (8.4.15); here $\mathbb{Q}_+$, $\mathbb{Q}_-$, and $\mathbb{Q}_d$ respectively denote the positive, negative, and degraded rationals; moreover, we let f belong to $k[X, X^{-1}, Y, Y^{-1}]^\times$; see (8.1).



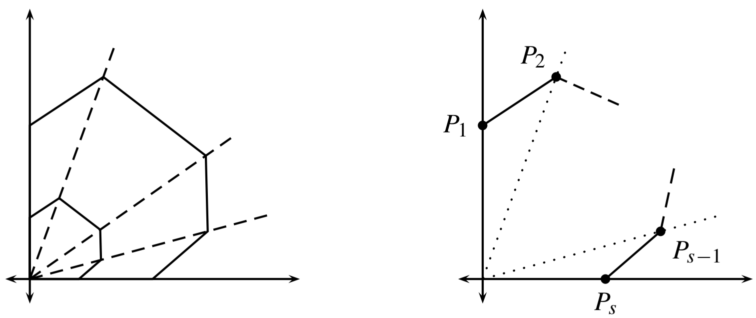
The following diagrams depict f_w^+ when $w \in \mathbb{Q}_+$. Lemma (4.17) says that if f and g are polynomials of degree $\geqslant 2$ with $J(f,g) = \theta$ then $\deg_w f > 0$ and hence the following diagram on the right cannot occur.



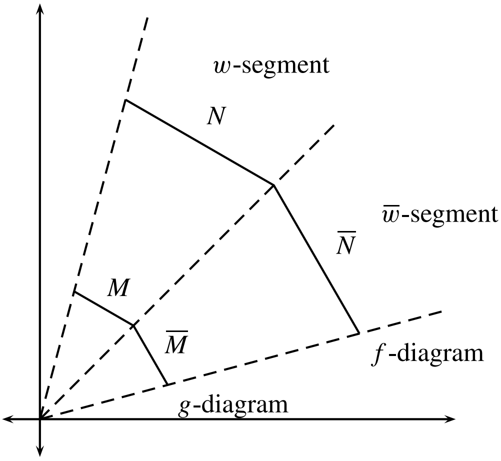
(9.6). Let us now discuss some of the terminology found in this paper. We remind the reader that at some points we talk about the antecedent and the consequent, as well as several other points of notation. Rather than rehashing them here, let us simply display the necessary terminology diagrammatically. The left hand figure below depicts the procedure for finding the consequent $w^\ddagger(f)$, i.e., the next newton line, and one can draw a similar picture for finding the antecedent $w^\dagger(f)$, i.e., the previous newton line. The right hand figure below depicts all sorts of other objects described in (4.14), (7.4.14) and (8.4.14). In particular $f_w^\odot$ and $f_w^{\odot\odot}$ are the top and bottom points of the newton line of f belonging to the weight w. Also Ω^* and Ω^{**} are respectively the segments of all weights w^* and w^{**} such that $w_c^\dagger(f) < w^* < w < w^{**} < w_c^\ddagger(f)$; these are point weights rather than line weights of f.



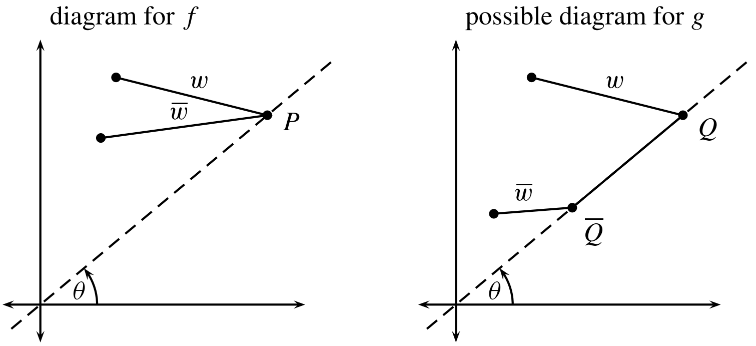
(9.7). Now referring to (8.4.18)(14), in the following diagram on the left we depict the similarity of the Newton Polygons of a Jacobian pair (f, g). Moreover referring to (7.4.14)(3)(V), (8.4.15)(3), and (8.4.18)(14), in the following diagram on the right we depict the clockwise behavior of the vertices $P_i = v_p(f, i)$ for $1 \leqslant i \leqslant s = l_p(f) + 1$.



Notice that in similar polygons, we require that each line be parallel to its mate in the other diagram. Now let us be more specific so we can analyze when similar newton polygons occur. Let us take a weight w, and let $\deg_w(f) = N$ with $\deg_w(g) = M$. Now take $\overline{w}$ to be the consequent of w relative to f as well as g, and let $\deg_{\overline{w}}(f) = \overline{N}$ with $\deg_{\overline{w}}(g) = \overline{M}$. Then in (4.17) we see the claim that $N/M = \overline{N}/\overline{M}$ which results in the local similarity of the newton polygons as in the following diagram.

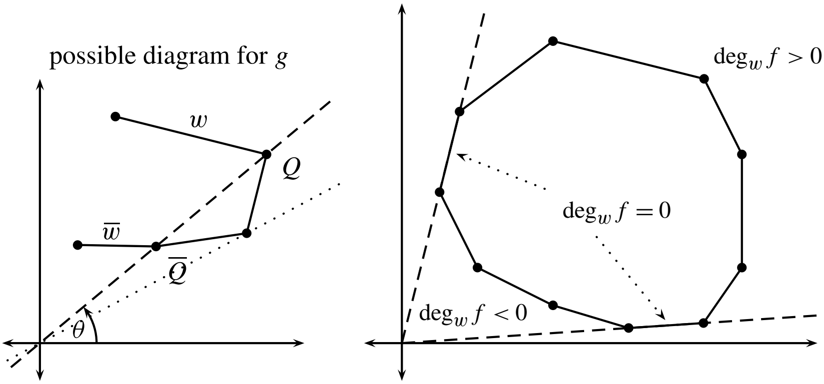


(9.8). Continuing with the discussion of (9.7), and referring to items in (8.4.19), let us point out some reasons why we need to be careful about similarity. In the following two diagrams we have $f \sim_w g$ with $f \sim_{\overline{w}} g$, and moreover the weights w and $\overline{w}$ are f-contiguous but not g-contiguous. The hypotheses of (VII.2) and (VII.3) are designed to avoid this.

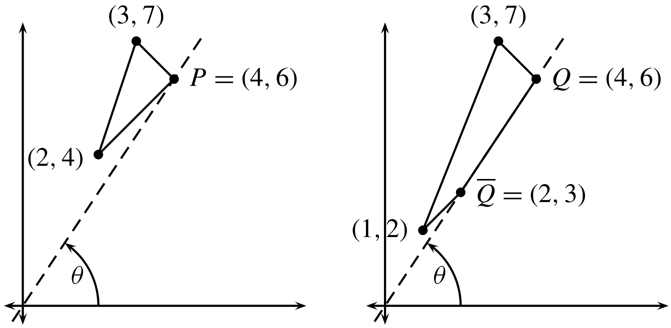


Above is a simple case of (VI*)(xii) and in it we have $f \sim_{\widehat{w}} g$ for all $w \leqslant \widehat{w} \leqslant \overline{w}$. The left diagram below is a more complicated case of (VI*)(xii) and is a refinement of its predecessor. As an explanation, we analogize this to the mean value theorem or Rolle's Theorem. Basically, given a Newton Polygon, during the transition from positive w-degrees to negative w-degrees, we must pass through a zero w-degree.

The large polygon in the right diagram below illustrates the general fact that we have $\deg_w f > 0$ for certain contiguous weights w and, provided $(0,0) \notin \mathrm{Supp}(f)$, we have $\deg_w f < 0$ for another contiguous range of weights w, with $\deg_w f = 0$ for exactly two weights w.



The final pair of polygons below illustrate a case when $f \sim_w g$ for all $w \in \mathbb{Q}_c$; see (8.1). This could be an example where $J(f, g) = 0$ but the full polygons are not similar as objects in the Euclidean plane.



Thus we see that no matter how much similarity is assumed, contiguity is not guaranteed. Notice that such discontiguities are found along the degree zero part. If we assume that both $\deg_w f$ and $\deg_{\overline{w}} f$ are positive, then we can guarantee contiguity. Note that there are at most two such discontiguities, as we can see in the large polygon above.

10. Problems about trivariate Jacobian Conjecture

Let f, g, h be nonzero polynomials in X, Y, Z with coefficients in a field k of characteristic zero. Recall that (f, g, h) is a Jacobian triple means $J(f, g, h) = \theta$, and (f, g, h) is an automorphic triple means $k[X, Y, Z] = k[f, g, h]$. The trivariate Jacobian Problem conjectures that every Jacobian triple is an automorphic triple. Here are two simple test cases of this, both of which are *good thesis problems*.

(10.1). Generalize the *important divisibility fact* (4.16)(1) by showing that if $f = X + X^2Y$ then (f, g, h) is not a Jacobian triple for any g and h in $k[X, Y, Z]^\times$.

(10.2). Show that if (f, g, h) is a Jacobian triple then all *the curves at infinity on the surface* $f = 0$ *are rational*, i.e., every nonunit irreducible factor of the degree form of f represents a rational curve in the projective plane. In particular show that if $f = X^n + Y^n + Z^n$ with $n \geqslant 3$ then (f, g, h) is not a Jacobian triple for any g and h in $k[X, Y, Z]^\times$. The hint is to use the results of my (= Abhyankar's) 1956 paper "On the valuations centered in a local domain" on pp. 321–348 of volume 78 of the American Journal of Mathematics.

Apropos the trivariate Jacobian Problem, and in continuation of the proposal made in the Introduction of giving an algorithmic rendering of the plane convexity hull, at a still further opportunity, we propose to present a generalization of convex hull construction to three and higher dimensions, which should be quite useful for the applied areas of Operations Research as well as the theoretical needs of the multivariate Jacobian Problem.

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