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Supersolvable Frame-matroid and Graphic-lift Lattices

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A geometric lattice is a *frame* if its matroid, possibly after enlargement, has a basis such that every atom lies under a join of at most two basis elements. Examples include all subsets of a classical root system. Using the fact that finitary frame matroids are the bias matroids of biased graphs, we characterize modular coatoms in frames of finite rank and we describe explicitly the frames that are supersolvable. We apply the characterizations to three kinds of example.

A geometric lattice is a *graphic lift* if it can be extended to contain an atom whose upper interval is graphic. We characterize modular coatoms in and supersolvability of graphic lifts of finite rank and we examine families analogous to the frame examples.

INTRODUCTION

One of the outstanding problems concerning arrangements of hyperplanes and finite matroids is to understand when and why the characteristic polynomial of the associated geometric lattice has a complete integral factorization. A sufficient condition for such a factorization, which even implies a simple combinatorial interpretation of the roots, is that the lattice be supersolvable, which means that it has a complete chain of modular flats [14, Corollary 2.3]. However, it can be hard to decide whether a particular geometric lattice is supersolvable. Here we completely settle that question for two kinds of geometric lattice which can be presented in terms of graphs.

The first kind is a geometric lattice (of finite rank) whose matroid has a basis, or can be extended to have one, such that each point lies on a line generated by a pair of basis elements. We call this a (finite-rank) *frame* matroid. The primary example is the matroid of a subset of a classical root system; more generally, of a subgeometry of a Dowling group geometry. One can think of a frame matroid as an abstraction of a *two-term arrangement of hyperplanes*: a finite set of homogeneous hyperplanes in F^n, each of whose equations involves at most two variables; thus $x_i = gx_j$ or $x_k = 0$. If $g \in \{\pm 1\}$ and $F = \mathbb{R}$ or $\mathbb{C}$, we have a subarrangement of $B_n^* = \{x_i = \pm x_j, x_k = 0 : 1 \leq i < j \leq n, 1 \leq k \leq n\}$, the real or complex arrangement dual to the root system B_n. By [18], a finitary frame matroid is the matroid of a graph with certain additional structure which I call a 'bias' (to be explained in Section 1); and conversely every such 'bias matroid' is a finitary frame. This representation theorem is what makes two-term arrangements and their abstraction to frame matroids relatively tractable, since it permits one to employ the rich theory of biased graphs to characterize modular copoints and supersolvability. Since an ordinary graph can be treated as a certain kind of biased graph, our theorem generalizes Stanley's that the lattice of contractions of a graph is supersolvable if and only if the graph is chordal.

The second kind of geometric lattice is one which contains, or can be extended to contain, an atom e_0 whose upper interval is graphic; that is, the lattice of contractions of a graph. We call this a *graphic-lift* because it is obtained from a graphic matroid by a standard lift construction specified by a bias on the graph. Graphic lifts abstract a different kind of arrangement of hyperplanes, which we call *affinographic* because the defining equations have the form $x_i - x_j = g$, whence the hyperplanes are affine flats in F^n. Some types of real affinographic arrangements have been studied in [1, 9, 15]. Again, biased graph theory helps us to treat these arrangements and all graphic lift matroids and in particular to determine all modular copoints and supersolvable lattices of this kind.

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We find that very few frame-matroid or graphic-lift lattices are supersolvable. Yet we know several families of vector sets (or dually, arrangements of hyperplanes) that correspond to frame and graphic-lift matroids whose characteristic polynomials have integral roots but which are on the whole not supersolvable: the root subsystems obtained from B_n by removing some coordinate vectors, for example, and generalizations; certain bicircular matroid lattices; lattices related to the semilattices of composed partitions studied in [9]; and some lift analogs. This shows that supersolvability does not completely explain integral factorization even for frame matroids. There are several broader properties of an arrangement of hyperplanes or a matroid that guarantee integral roots, such as freeness (see [11]), factorization [8, 16], and existence of an atom decision tree [3]. Why, then, characterize a comparatively weak property? The best reason is that it can be done—in considerable generality. There is no known characterization in comparable generality of any other integrality property, although it has been done for a few special types, notably graphic arrangements (subarrangements of $A^*_{n-1} = \{x_i = x_j : 1 \leq i < j \leq n\}$); arrangements between A^*_{n-1} and B^*_n, for which freeness and supersolvability are characterized in the beautiful theorem of [7], extended to factorizability by [2];† and the projectivizations ('cones') of arrangements between A^*_{n-1} and the Shi arrangement, for which Athanasiadis characterized freeness and supersolvability in [1]. In some of these types, most of the properties that guarantee integral roots turn out to nearly coincide. That is not so in the whole class of frame matroids, but, to extend a question raised by Bailey [2], might it be true of all subarrangements of B^*_n, i.e., of all signed graphs? This question is open.

After developing the general results we turn to three families of examples, looking for supersolvability and for nonsupersolvable cases that nonetheless have integral roots. Examples 4.1 and 4.2 include the root system D_n and Dowling's group geometries. In Example 4.3 we obtain a mild generalization of Edelman and Reiner's supersolvability theorem. In Example 4.4 we characterize the supersolvable bicircular matroids. We close with comments on algorithmics, chordality, and freeness and some open questions.

1. Biased Graphs, Matroids, Etc.

We give a quick exposition of the relevant portions of biased graph theory, from definitions through their matroids to their representations as vector sets and arrangements of hyperplanes. The source for biased graphs is [17], especially the cryptomorphic definitions of the matroids in Theorems II.2.1 (that is, Theorem 2.1 of Part II) and II.3.1 and the gain-graph and matroid invariant theory of Sections III.4 and III.5. The reader need not be acquainted with the sources in order to read this paper.

1.1. Biased graphs. A *biased graph* $\Omega = (V, E, \mathcal{B})$ is a graph $||\Omega|| = (V, E)$, not necessarily finite, together with a *linear subclass* $\mathcal{B} = \mathcal{B}(\Omega)$ of its polygons (or 'circuits', but we reserve this term for matroid circuits): a class of polygons such that, if in a theta subgraph two polygons belong to $\mathcal{B}$, so does the third. In biased graph theory we find it helpful to have four kinds of edges: links (two distinct endpoints), loops (two coincident endpoints), half edges (one endpoint), and loose edges (no endpoints; this term is due to Tutte). Neither of the latter can belong to a polygon. A subgraph or edge set is *balanced* if it contains no half edge and any polygon in it belongs to $\mathcal{B}$. It is *contrabalanced* if it contains no balanced polygon

†The present work, except Example 4.3, and that of Edelman, Reiner, and Bailey were originally done independently, including Bailey's independent discovery of Theorem 2.2 for signed graphs (with a longer proof). However, this report and [2] have been revised to take account of the connections.

or loose edge. Thus a loose edge is balanced, a half edge is not, and a loop may or may not be. For matroidal purposes a loose edge behaves like a balanced loop and a half edge like an unbalanced one, but for technical reasons it is helpful to allow all four types of edge.

Several special types of biased and unbiased graphs will be needed. An *ordinary graph* has only links and loops. A *link graph* has only links. A *simple graph* is a link graph that contains no digons. A *star* S_k is a simple graph having k edges (where $k \geq 1$), all incident to one vertex (the *center*). A *unicycle* is a connected ordinary graph having exactly one polygon. By $m\Gamma$ we mean a graph Γ with every edge replaced by m copies of itself. An *induced subgraph* of $\Gamma = (V, E)$ is $\Gamma{:}W = (W, E{:}W)$ where $W \subseteq V$ and $E{:}W = \{e \in E : \emptyset \neq V(e) \subseteq W\}$, $V(e)$ denoting the set of endpoints of e. For $W \subseteq V$ and $S \subseteq E$ we write $W^c = V \setminus W$, $S^c = E \setminus S$, $\Gamma \setminus W = \Gamma{:}W^c$, and $\Gamma|S = (V, S)$. W is *stable* if $E{:}W = \emptyset$. The *neighborhood* of $v \in V$ is $N(v) = \{x \in V : x$ is adjacent but not equal to $v\}$; the *complete neighborhood* is $\overline{N}(v) = N(v) \cup \{v\}$. We denote by $\langle\Gamma\rangle$ the biased graph whose underlying graph is Γ, in which every polygon is balanced.

In a biased graph Ω, we let $U(\Omega) = \{v \in V : v$ supports an unbalanced edge$\}$. Ω is *full* if $U(\Omega) = V$. We call Ω *simply biased* if it has no loose edges, balanced loops, balanced digons, or pairs of unbalanced edges at the same vertex. If $W \subseteq V$ and $S \subseteq E$, $\Omega^{(W)}$ denotes Ω with a half edge added at each vertex in $W \setminus U(\Omega)$. $\Omega{:}W$, $\Omega|S$, etc., denote subgraphs of Γ with balance of polygons the same as in Ω. Similarly, $\Gamma^{(W)}$ denotes a graph Γ with a half edge added to each vertex in W not already supporting one.

Two special unions are the disjoint union $\Omega_1 \mathbin{\dot\cup} \Omega_2$ and the one-point amalgamation $\Omega_1 \cup_p \Omega_2$, where p is a vertex of Ω_1 and Ω_2 and $\Omega_1 \cap \Omega_2 = \{p\}$. In each case the balance of a polygon is the same as in Ω_1 or Ω_2, for whichever one it is that contains the polygon.

1.2. Gain graphs. In the examples we need gain graphs, which for our purposes can be defined in the following way (simplified from [17, Section III.4]). Take a group $\mathfrak{G}$. On the vertex set $[n] = \{1, 2, \ldots, n\}$ construct a graph with edges $(i, j; g)$ for all distinct $i, j \in [n]$ and $g \in \mathfrak{G}$, but identify the edge $(i, j; g)$ with $(j, i; g^{-1})$. This is the gain graph $\mathfrak{G}K_n$. Adding a half edge to each of p vertices gives $\mathfrak{G}K_n^{(p)}$. We call g the *gain* of $(i, j; g)$ in the direction from i to j and we write $\varphi_n(i, j; g) = g$. Calling a polygon $\{(i_0, i_1; g_1), (i_1, i_2; g_2), \ldots, (i_{k-1}, i_k; g_k)\}$, where $i_0 = i_k$, *balanced* when $g_1 g_2 \cdots g_k = 1$ determines a biased graph $\langle\mathfrak{G}K_n^{(p)}\rangle$. A *gain graph* $\varphi = (V, E, \varphi)$ with gain group $\mathfrak{G}$ and gain function φ is any subgraph of $\mathfrak{G}K_n^{(n)}$, φ being the restriction to E of φ_n; $\langle\varphi\rangle$ denotes the corresponding biased graph. We call $\mathfrak{G}K_n$ the *$\mathfrak{G}$-expansion of* K_n and $\mathfrak{G}K_n^{(n)}$ the *full $\mathfrak{G}$-expansion*. When $n \leq 3$ (but not if $n \geq 4$) the construction works for any quasigroup $\mathfrak{G}$.

Switching φ by a function $\tau : V \to \mathfrak{G}$ means changing the gain of each link or loop from $\varphi(i, j; g) = g$ to $\varphi^\tau(i, j; g) = \tau(i)^{-1} g \tau(j)$. Switching preserves the balance or imbalance of polygons, hence $\langle\varphi^\tau\rangle = \langle\varphi\rangle$.

A *signed graph* Σ is a gain graph whose group is the sign group $\{+, -\}$. We write E_+ and E_- for the sets of positive and negative edges and Σ_+, Σ_- for the corresponding spanning subgraphs. $N_+(v)$ and $N_-(v)$ denote the neighborhoods of v in Σ_+ and Σ_-. We write $\pm K_n$ for the sign-group expansion of K_n. *Switching* a vertex x in Σ means reversing the sign of every link at x. Switching $X \subseteq V(E)$ means switching each vertex in X in turn. Switching does not change $\langle\Sigma\rangle$ or, consequently, any of the matroids we will define on $\langle\Sigma\rangle$.

1.3. Matroids. For matroid theory and notation we mainly follow [12]. However, we write $E(M)$ for the point set of a matroid M, S^c for $E(M) \setminus S$ if $S \subseteq E(M)$, and M/S for the contraction of M by S. The rank function is r_M or simply r. The lattice of closed sets is

Lat M. Two fundamental if elementary properties of modular elements of Lat M are that a copoint A of M is modular if and only if it is not disjoint from any line, and that M (to be precise, Lat M) is supersolvable if and only if it has a modular copoint A such that $M|A$ is supersolvable. (See for example [4, Corollary 3.4 and Proposition 3.5].)

1.4. The bias matroid. A *handcuff* is a connected graph (or its edge set) having exactly two polygons and no monovalent vertices (a half edge counting as a loop here); it is *tight* if all its edges lie in the polygons. The *bias matroid* $G(\Omega)$ is the matroid on E whose circuits are the *bias circuits*: the balanced polygons, loose edges, and contrabalanced handcuffs and thetas. For $S \subseteq E$ let $V(S) =$ the set of endpoints of edges in S and let $b(S) =$ the number of connected components of $(V(S), S)$ which are balanced. Then the rank of S in $G(\Omega)$ is $r(S) = |V(S)| - b(S)$. If Γ has no loose or half edges, then $G(\langle\Gamma\rangle) = G(\Gamma)$, the usual polygon matroid of Γ.

Bias matroids are just a graph-theoretic way of presenting finitary frame matroids. In particular, suppose $\varphi \subseteq \mathfrak{G}K_n^{(n)}$ is a gain graph whose gain group is a subgroup of F^*, the multiplicative group of a field (or skew field) F. Then $G(\varphi)$ (i.e., $G(\langle\varphi\rangle)$, but we omit the angle brackets) has a standard vector representation over F: the vector space is F^n with standard basis $b_1, \ldots, b_n$ and an edge $e = (i, j; g)$ is represented by $x(e) = b_i - gb_j$. (Thus $x(e^{-1}) = b_j - g^{-1}b_i = -g^{-1}x(e)$. Either of these vectors serves equally well in representing the matroid.) If e is a loose edge we define $x(e) = 0$; for a half edge at vertex i, $x(e) = b_i$. The linear dependence matroid of $x(\varphi) = \{x(e) : e \in E\}$ equals $G(\varphi)$ [17, Theorem IV.2.1].

Let $x^*(e)$ be the dual hyperplane to $x(e)$ in F^n, so $x^*(\varphi)$ is an arrangement of hyperplanes. Each hyperplane equation has the form $x_i = gx_j$ (where $i \neq j$, $g \neq 0$) or $x_i = 0$. It is these *two-term arrangements* of hyperplanes that are represented by bias matroids of gain graphs and abstracted by frame matroids and biased graphs. For example, let $\mathfrak{G} = \{+, -\}$ and $F = \mathbb{R}$ or $\mathbb{C}$. Then $x(\varphi) = B_n$, the root system, if we take $\varphi = \pm K_n^{(n)}$, or D_n if $\varphi = \pm K_n$, or an arbitrary subset of B_n if we take a suitable $\varphi \subseteq \pm K_n^{(n)}$. Hence we have, in Theorem 2.2 below with $\Omega = \langle\varphi\rangle \subseteq \langle\pm K_n^{(n)}\rangle$, a graphical characterization of the supersolvable subsets of B_n and dually of those of the hyperplane arrangement B_n^*.

Scaling an arrangement in F^n means replacing each coordinate variable x_i by a nonzero scalar multiple $\tau(i)x_i$, or equivalently, transforming F^n by an invertible diagonal matrix. Scaling a two-term arrangement $x^*(\varphi)$ is equivalent to switching φ by $1/\tau$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are homogeneous arrangements in F^n and codim $\bigcap(\mathcal{A}_1 \cup \mathcal{A}_2) =$ codim $\bigcap \mathcal{A}_1$ + codim $\bigcap \mathcal{A}_2$, we say $\mathcal{A}_1 \cup \mathcal{A}_2$ is the *direct sum* of $\mathcal{A}_1$ and $\mathcal{A}_2$.

1.5. The lift matroids. A *broken handcuff* is the union of two vertex-disjoint polygons. (It is not a handcuff. Again, a half edge counts as a loop.) The *extended lift matroid* $L_0(\Omega)$ is the matroid on $E_0 = E \cup e_0$, where e_0 is a new element called the *extra point*, whose circuits are the *lift circuits*: balanced polygons, loose edges, contrabalanced tight and broken handcuffs and thetas, and sets $C \cup e_0$ where C is an unbalanced polygon or a half edge. The *lift matroid* $L(\Omega)$ is $L_0(\Omega) \setminus e_0$. We call $S \subseteq E_0$ *balanced* if S is a balanced edge set. Letting $c(S)$ be the number of connected components of $(V(S \setminus e_0), S \setminus e_0)$, the rank function in the lift and extended lift is $r_0(S) = |V(S \setminus e_0)| - c(S) + \epsilon(S)$ where $\epsilon(S) = 0$ if S is balanced, 1 otherwise. If Γ has no loose or half edges, $L(\langle\Gamma\rangle) = G(\langle\Gamma\rangle) = G(\Gamma)$, the polygon matroid.

Suppose now that $\varphi \subseteq \mathfrak{G}K_n^{(n)}$ where $\mathfrak{G} \subseteq F^+$, the additive group of a field (or skew field). Then $L_0(\varphi)$ has a standard representation in F^{1+n} (with standard basis $b_0, b_1, \ldots, b_n$): an edge $e = (i, j; g)$ corresponds to the vector $z(e) = gb_0 - b_i + b_j$; a half edge or the extra point $e = e_0$ corresponds to $z(e) = b_0$, and a loose edge e has $z(e) = 0$. The linear dependence matroid of $z(\varphi) = \{z(e) : e \in E\}$ equals $L(\varphi)$, and that of $z_0(\varphi) = z(\varphi) \cup \{z(e_0)\}$ naturally equals $L_0(\varphi)$ [17, Theorem IV.4.1].

Dually, we regard $z^*(\varphi)$ and $z_0^*(\varphi)$ as affine hyperplane arrangements in F^n. Then $z^*(e)$ has equation $x_i - x_j = g$ if $e = (i, j; g)$, while a half edge or the extra point corresponds to the infinite hyperplane H_∞ and therefore does not appear in the affine hyperplane representation $\mathcal{A}$. However, we cannot ignore it entirely. The *projectivization* of $\mathcal{A}$ is the arrangement $\mathcal{A}_{\mathbb{P}} = \mathcal{A} \cup \{H_\infty\}$ in $\mathbb{P}^n(F)$. Note that switching the associated gain graph by τ corresponds to a translation, replacing x_i by $x_i + \tau(i)$.

An affine arrangement $\mathcal{A}_1 \cup \mathcal{A}_2$ is the *direct sum* of $\mathcal{A}_1$ and $\mathcal{A}_2$ if $r(\mathcal{A}_1) + r(\mathcal{A}_2) = r(\mathcal{A}_1 \cup \mathcal{A}_2)$, where $r(\mathcal{A}) = \max\{\operatorname{codim} S : S$ is a nonempty intersection flat of hyperplanes of $\mathcal{A}\}$. For homogeneous arrangements this specializes to the definition in Section 1.4. It entails that $\mathcal{A}_1$ and $\mathcal{A}_2$ are disjoint.

1.6. Coloring and polynomials. A 1-*coloring* of φ is a mapping $c : V \to \mathfrak{G} \cup \{\hat{0}\}$ where $\hat{0} \notin \mathfrak{G}$. It is *proper* if $c^{-1}(\hat{0})$ is stable and for all edges $(i, j; g) \in E$ with $i, j \notin c^{-1}(\hat{0})$ we have $c(j) \neq c(i)g$. If φ and $\mathfrak{G}$ are finite there is a polynomial $\chi(\lambda)$ associated with φ, called the *chromatic polynomial*, which has the property that $\chi(|\mathfrak{G}| + 1)$ is the number of proper 1-colorings of φ. Furthermore, $\lambda^{-b(E)}\chi(\lambda) = p_G(\lambda)$, the characteristic polynomial of $G(\varphi)$ [17, Theorem III.5.1]. For the *zero-free chromatic polynomial* $\chi^*(\lambda)$, $\chi^*(|\mathfrak{G}|)$ equals the number of proper 1-colorings not using the color $\hat{0}$. (If $\mathfrak{G} \subseteq F^+$, χ^* is the characteristic polynomial of the affinographic arrangement $z(\varphi)$ [17, Theorem III.5.3 and Section IV.4].) If φ is full, then $\chi^*(\lambda) = \chi(\lambda + 1) = p_G(\lambda + 1)$; and more generally,

$$p_G(\lambda + 1) = \chi_\varphi(\lambda + 1) = \sum_{\substack{X \subseteq V \\ \text{stable}}} \chi^*_{\varphi \setminus X}(\lambda) \tag{1.1}$$

[17, Theorem III.6.1]. For an arbitrary finite biased graph there are algebraic definitions of χ and χ^* which yield the same identities; see especially [17, Sections III.3 and III.6] but there is no easy way to evaluate the polynomials without coloring theory.

If Ω is a finite, connected biased graph, the characteristic polynomial of $L_0(\Omega)$ is

$$p_{L_0}(\lambda) = \lambda^{-1}(\lambda - 1)\chi^*(\lambda) = \lambda^{-1}(\lambda - 1)p_{G(\Omega^{(V)})}(\lambda + 1). \tag{1.2}$$

We see that the roots of $p_{L_0(\Omega)}(\lambda)$ are those of $p_{G(\Omega^{(V)})}(\lambda)$ decreased by one, except that the root 1 remains unchanged. If Ω is also a link graph and is unbalanced, the characteristic polynomial of $L(\Omega)$ is (from [17, Theorem III.5.2]);

$$p_L(\Omega) = p_{L_0}(\lambda) + \lambda^{-1}\chi_{||\Omega||}(\lambda). \tag{1.3}$$

2. Frames

In a biased graph Ω a vertex v is *bias simplicial* if:

(s1) for each pair of edges, e and f, from v to distinct neighbors x and y, there is an xy edge which completes a balanced triangle;

(s2) for each unbalanced digon at v, the other endpoint is in $U(\Omega)$; and

(s3) if v is in $U(\Omega)$ then every neighbor is in $U(\Omega)$.

We call v *link simplicial* if it satisfies (s1) and *simplicial* if it is link simplicial and the set $E(v)$ of edges incident with v is balanced (that is, there is no unbalanced digon at v, and $v \notin U(\Omega)$).

In a balanced graph $\langle\Gamma\rangle$ a simplicial or link- or bias-simplicial vertex (in the biased sense) is the same as a simplicial vertex of Γ (in the ordinary sense).

We restate the conditions to apply to a gain graph φ: (s1′) whenever there are edges $e = (v, x; g)$ and $f(v, y; h)$ with $x \neq y$, there is an edge $(x, y; g^{-1}h)$; (s2′) whenever there are edges $(v, x; g)$ and $(v, x; h)$ with $g \neq h$, there is an unbalanced loop or half edge at x; (s3′) if v supports an unbalanced loop or half edge, so does every neighbor.

THEOREM 2.1. *Let Ω be a connected, simply biased graph. A subset $A \subseteq E$ is a modular copoint of $G(\Omega)$ if and only if Ω and A are of one of the following types.*

(1) Ω is a one-point amalgamation $\Omega_1 \cup_v \Omega_2$, where Ω_2 is balanced and v is a bias-simplicial vertex in Ω_1; and $A = (E_1{:}\{v\}^c) \cup E_2$.
(2) $\Omega = \langle\Gamma\rangle^{(U)}$ where Γ is connected and U is a nonempty clique in Γ; and $A = E(\Gamma)$.
(3) $\Omega = \langle\Gamma\rangle \cup (mK_2, \emptyset)^{(U)}$ where Γ is connected, $m \geq 2$, $U \subseteq V(K_2)$, and one link of the mK_2 is in Γ; and $A = E(\Gamma)$.
(4) $\Omega = \langle\Sigma\rangle$ where Σ is a connected signed link graph, E_- is a triangle parallel to a triangle of positive edges; and $A = E_+$.
(5) $\Omega = \langle\Sigma\rangle$ where Σ is a connected signed link graph, E_- is a star of one or more edges centered at a vertex v, $N_-(v)$ is a clique; and $A = E_+$.
(6) $\Omega = \langle\Sigma\rangle^{(v)}$ where Σ and A are as in (5) and furthermore $N_-(v) \subseteq N_+(v)$.

I regard the first kind of modular copoint as the normal one; the others are exceptional cases. Note that Ω_2 may equal $\{V\}$.

PROOF. We may assume $G(\Omega)$ is unbalanced. (The balanced case is known from [13, Theorem 3].)

First, we need a catalog of all types of line Λ in $G(\Omega)$. It is easy to produce one because $b(\Lambda) \leq 2$.

(a) *Contrabalanced, order* 2. $\Lambda = E{:}\{x, y\}$, where $|E{:}\{x, y\}| \geq 2$. This line is $(mK_2, \emptyset)^{(U)}$ with $m + |U| \geq 2$.
(b) *Partly balanced, order* 3. $\Lambda = M_{1\frac{1}{2}} := E(K_2 \mathbin{\dot\cup} K_1^{(v)})$; that is, a link and an unbalanced edge at a third vertex.
(c) *Balanced, order* 3 *or* 4. Here there are three kinds of Λ. (i) $\langle C_3\rangle$, a balanced triangle. (ii) A *proper angle* $A_2 :=$ two links having one common vertex and not contained in a $\langle C_3\rangle$. (iii) $M_2 := E(K_2 \mathbin{\dot\cup} K_2)$, a two-edge matching.

Next, we need the description of copoints in [17, Theorem II.2.1(h)]. There are two kinds of copoints: a maximal balanced edge set A, and an unbalanced edge set of the form $A = E{:}Y^c \cup A_Y$ where $\emptyset \subset Y \subset V$, $b(\Omega{:}Y^c) = 0$, (Y, A_Y) is connected, and A_Y is a maximal balanced edge set in $\Omega{:}Y$.

In the former case we see from lines of types M_2 and $M_{1\frac{1}{2}}$ that A^c can contain no two vertex-disjoint edges except a pair of unbalanced edges, which, due to lines of order 2, must be at vertices which are adjacent in A. If A^c contains no links, A has type (3). Otherwise A has type (4), (5), or (6). The details are routine.

Suppose now that A is the second kind of copoint. Since the set D of edges between Y and Y^c can contain no M_2, we have $|Y| = 1$ or $\Omega = \Omega_1 \cup_p \Omega_2$ where Ω_1 and $\Omega_2 \subset \Omega$, $Y \subseteq V(\Omega_1)$, $Y^c \subseteq V(\Omega_2)$, and any unbalanced edge at p is in Ω_2. It is easy to see that, if $|Y| = 1$, say $Y = \{y\}$ (so $A = E{:}\{y\}^c$), then y is bias simplicial. We show that $|Y| > 1$ is impossible.

We show first that when a point p exists, Ω_1 is balanced. Take $p \in Y$ if possible. (Note that p is not uniquely determined if D is incident with only two vertices.) Let $e \in D$. If $p \notin Y$, let $f \in D$, not parallel to e. Then $M_{1\frac{1}{2}}$ shows that no vertex of $Y \setminus V(e)$ (or $Y \setminus V(f)$, if $p \notin Y$) can support an unbalanced edge. Thus Ω_1 has no unbalanced edges. Suppose $A^c{:}Y$ contained a link g. Then eg would be an M_2 or A_2 or else $p \notin Y$ and eg would be in a $\langle C_3 \rangle$ at p, so we would have a line disjoint from A, which contradicts the hypothesis. We conclude that $E(\Omega_1) = A_Y$, whence Ω_1 is balanced.

But then $G(\Omega)$ is the direct sum of $G(\Omega_1)$ and $G(\Omega_2)$, contrary to the assumption. □

Now we can characterize supersolvable frame matroids of finite rank. A *bias-simplicial vertex ordering* (briefly, b.s.v.o.) of a biased graph Ω of finite order is a linear ordering of the vertices, say $(v_1, v_2, \dots, v_n)$, such that each v_i is bias simplicial in $\Omega{:}\{v_1, \dots, v_i\}$. A *link-simplicial vertex ordering* (l.s.v.o.) is similar. If Ω is balanced we call this a *simplicial vertex ordering* (s.v.o.); in reverse order it is a perfect vertex elimination scheme of $||\Omega||$ [10, Section 4.2]. We call Ω a *simplicial* (or *bias-* or *link-simplicial*) *extension* of Ω_0 if Ω_0 is an induced subgraph of Ω and $V(\Omega_0)^c$ can be linearly ordered, say $(w_1, \dots, w_k)$, so that each w_i is simplicial (or bias- or link-simplicial) in $\Omega{:}(V(\Omega_0) \cup \{w_1, \dots, w_i\})$. Incidentally, in a b.s.v.o any vertex $v \notin U(\Omega)$ must follow all its neighbors that are in $U(\Omega)$. Consequently, any b.s.v.o can be rearranged so that $U(\Omega)$ is an initial segment.

For an ordinary graph Γ Stanley [14, Proposition 2.8] proved that Lat $G(\Gamma)$ is supersolvable if and only if Γ is a chordal ('triangulated', 'rigid-circuit') graph. A good reference for chordal graphs is [10, Section 4.2]. We mention, in particular, Dirac's lemma (see [10, Lemma 4.2]) that a chordal graph is complete or has a pair of nonadjacent simplicial vertices and the consequence that a graph is chordal if and only if it has a simplicial vertex ordering (from which Stanley's theorem follows easily).

THEOREM 2.2. *Let Ω be a simply biased graph of finite order. $G(\Omega)$ is supersolvable if and only if each connected component of Ω either:*

(i) has a bias-simplicial vertex ordering; or

(ii) is a simplicial extension of one of

- *(a) $(mK_2, \emptyset)$, where $m \geq 2$, or*
- *(b) $\langle \pm K_3 \rangle$, or*
- *(c) $\langle \Sigma \rangle$ for $\Sigma = +\Gamma \cup -S_k$, where Γ is a chordal simple graph of finite order, S_k is a k-edge star whose vertex set lies in $V(\Gamma)$, and the noncentral vertices of S_k are a clique in Γ.*

Furthermore, the bias-simplicial vertex ordering (or simplicial extension) can be chosen so that any desired bias-simplicial (or simplicial) vertex is the last vertex.

An alternate description of Σ in (ii)(c) is as $(+\Gamma_0 \cup -e)/(-e)$ where Γ_0 is a chordal simple graph of finite order and e is a link whose endpoints are not adjacent in Γ_0.

If we have a gain graph $\varphi \subseteq \mathfrak{G}K_n^{(n)}$ instead of an abstract biased graph Ω, we can restate some of Theorem 2.2. Part (ii)(b) becomes: a gain subgraph of $\mathfrak{G}K_3$ obtained by switching $\{1, g\}K_3$ where g is an involution in $\mathfrak{G}$. Part (ii)(c) becomes: a gain graph with gain group $\mathfrak{G}$ obtained by switching from $1\Gamma \cup gS_k$, where $g \in \mathfrak{G}$, $g \neq 1$, gS_k denotes S_k with every edge given gain g in the orientation away from the central vertex, and Γ and S_k are as before.

PROOF. Sufficiency is clear by Theorem 2.1 and the properties of modular flats cited in Section 1.3.

For necessity we have only to clear up some technicalities. We may assume Ω is connected and unbalanced. We proceed by induction on the order. A supersolvable $G(\Omega)$ has a modular copoint A such that $G(\Omega)|A$ is supersolvable. Consequently, Ω and the modular copoint A are as described in Theorem 2.1.

Let us first take up the exceptional cases (2)–(6). Here $\Gamma = (V, A)$ is balanced, hence chordal by Stanley's theorem. By Dirac's lemma Γ is K_n or has two nonadjacent simplicial vertices. It is easy to deduce that Ω has a b.s.v.o in cases (2), (3) when $U \neq \emptyset$, and (6). In cases (3) with $U = \varphi$ and (4), if $\Omega \neq \Omega_0 := (mK_2, \varphi)$ or $\langle \pm K_3 \rangle$, respectively, then Γ has a simplicial vertex $v \notin V(\Omega_0)$. Then v is clearly simplicial in Ω, so it can be eliminated; by induction, Ω is a simplicial extension of Ω_0. As for case (5), here Ω is as in case (ii)(c).

Now consider case (1). Here $G(\Omega) = G(\Omega_1) \oplus G(\Omega_2)$; thus $G(\Omega)$ is supersolvable if and only if $G(\Omega_1)$ and $G(\Omega_2)$ are so. Ω_2, being balanced, has a simplicial vertex ordering, in which one can choose the first vertex to be v (because a chordal graph has at least two simplicial vertices; see [10, p. 82]). Ω_1 is a bias-simplicial extension of $\Omega_1{:}\{v\}^c$. Thus Ω itself is a bias-simplicial extension of $\Omega_1{:}\{v\}^c$. By successively eliminating bias-simplicial vertices we will either find a b.s.v.o or express Ω as a bias-simplicial extension of a supersolvable Ω_0 which has no bias-simplicial vertex. But then by induction Ω_0 is one of (mK_2, φ), $\langle \pm K_3 \rangle$, or $\langle +\Gamma \cup -S_k \rangle$ where Γ is chordal. It is easy to see that a bias-simplicial extension of such an Ω_0 is a simplicial extension. Therefore the theorem is proved. □

Clearly, if $G(\Omega)$ is supersolvable, then so is $G(||\Omega||)$ unless Ω is as in (ii)(c). However, we can say more. Let $||\Omega||_0$ be the graph obtained from Ω through replacing unbalanced loops and half edges by links to a new vertex v_0, then taking the underlying unbiased graph and eliminating loops and multiple edges.

COROLLARY 2.3. *If $G(\Omega)$ is supersolvable, then $||\Omega||_0$ is chordal or Ω falls under case (ii) (c) of Theorem* 2.2.

Now, the geometry of Theorem 2.2. Let $\mathcal{A}$ be a two-term arrangement of hyperplanes in F^n and U the set of coordinates i for which $x_i = 0$ is in $\mathcal{A}$. Call a coordinate i *transitive* if: (i) whenever it participates in hyperplanes $x_i = gx_j$ and $x_k = hx_i$ (where i, j, k are distinct, and we always tacitly assume $g, h \neq 0$), then $\mathcal{A}$ has a hyperplane $x_k = hgx_j$; (ii) the same if $k = j \neq i$ and $hg \neq 1$ (whence $x_k = hgx_j$ is the coordinate hyperplane $x_j = 0$); and (iii) if $\mathcal{A}$ contains $x_i = 0$ and $x_i = gx_j$, then it contains $x_j = 0$. Call i *strictly transitive* if it is transitive, $x_i = 0$ is not in $\mathcal{A}$, and for each $j \neq i$, there is at most one hyperplane equation involving both x_i and x_j. Let $\mathcal{A}_{(i)} = \{H \in \mathcal{A}$: the equation of H involves only $x_1, \ldots, x_i$, at most$\}$.

COROLLARY 2.4. *Let $\mathcal{A}$ be a two-term arrangement of hyperplanes in F^n, where F is a skew field. Then $\mathcal{A}$ is supersolvable if and only if it is a direct sum of arrangements $\mathcal{B}$ of any of the following four forms (after suitably scaling and renumbering the coordinates):*

(i) $\mathcal{B}$ is such that each i is transitive in $\mathcal{B}_{(i)}$.
(ii) Each $i \geq 3$ is strictly transitive in $\mathcal{B}_{(i)}$.
(iii) $\mathcal{B}$ consists of hyperplanes $x_i = \pm x_j$ for $1 \leq i < j \leq 3$ and additional hyperplanes (if any) so that each $i \geq 4$ is strictly transitive in $\mathcal{B}_{(i)}$.
(iv) There is an r such that every coordinate $i > r$ is strictly transitive in $\mathcal{B}_{(i)}$; $\mathcal{B}_{(r)} = \mathcal{B}_+ \cup \mathcal{B}_-$, where $\mathcal{B}_+$ is graphic and supersolvable (i.e., its hyperplanes have equations $x_i = x_j$ and each $i \geq 3$ is strictly transitive in $\mathcal{B}_{(i)}$); $\mathcal{B}_-$ consists of hyperplanes $x_m = g_0 x_i$ for fixed m, fixed $g_0 \in F^* \backslash \{1\}$, and all $i \in \{i_1, \ldots, i_k\}$ (where $k \geq 1$ and no $i_j = m$); and $\mathcal{B}_+$ contains all hyperplanes $x_{i_j} = x_{i_l}$ for $1 \leq j < l \leq k$.

3. Graphic Lifts

First in our treatment of graphic lifts we ought to verify that a finitary graphic lift lattice, or matroid rather, really is the lift or extended lift of a graphic matroid. In other words we should prove that for a finitary matroid M to have a nonloop point e_0 such that M/e_0 is graphic, it is necessary and sufficient that $M = L_0(\Omega)$ for some biased graph Ω. This is implicit in [6, Section 6] as amplified in [17, Section II.3 near Theorem II.3.1]. Here is how Ω is constructed: if $M/e_0 = G(\Gamma)$, then $||\Omega|| = \Gamma$; and a polygon is unbalanced when its closure in M contains e_0.

To simplify the results we shall often assume that Ω is connected. We can do so because, if it is not and if one identifies just enough vertices to make it connected, the lift and extended lift matroids are not changed.

Now we state the main theorems. We leave the proofs to the reader since they are along the same lines as those for frame matroids. We may without loss of generality take Ω to be a link graph, for an unbalanced edge is parallel to e_0, whence $\operatorname{Lat} L(\Omega) = \operatorname{Lat} L_0(\Omega)$ if Ω has any such edge.

Recall from Section 2 that a vertex v is *link simplicial* if any two nonparallel links at v are contained in a balanced triangle. If $\Omega_1 \subseteq \Omega$ and $v \in V(\Omega_1)$, then $E_1(v)$ is the set of edges of Ω_1 incident with v.

Theorem 3.1. *Let Ω be a simply biased link graph.*

(A) A subset $A \subseteq E_0$ is a modular copoint in $L_0(\Omega)$ if and only if:

(1) Ω has a block Ω_1 which has a link-simplicial vertex v, and $A = E_0 \setminus E_1(v)$; or
(2) Ω is balanced and $A = E$.

(B) A subset $A \subseteq E$ is a modular copoint of $L(\Omega)$ if and only if:

(1) Ω has a block Ω_1 which has a simplicial vertex v, and $A = E \setminus E_1(v)$; or
(2) Ω and A are as in Theorem 2.1 (3, 4, *or* 5).

Theorem 3.2. *Let Ω be a simply biased, connected link graph of finite order.*

(A) $L_0(\Omega)$ is supersolvable if and only if Ω has a link-simplicial vertex ordering.
(B) $L(\Omega)$ is supersolvable if and only if:

(i) Ω is balanced and $||\Omega||$ is chordal; or
(ii) Ω is as in Theorem 2.2 (ii).

Furthermore, the link-simplicial vertex ordering (or simplicial extension) can be chosen so that any desired bias-simplicial (or simplicial) vertex is last.

Corollary 3.3. *For a simply biased link graph Ω of finite order, $L_0(\Omega)$ is supersolvable precisely when $G(\Omega^{(V)})$ is supersolvable.*

This and the remark on p_{L_0} in Section 1.6 show that, for a biased graph Ω, the properties of supersolvability, and of positive integrality of characteristic roots, of $L_0(\Omega)$ parallel those of $G(\Omega^{(V)})$.

For the geometric interpretation of Theorem 3.2 consider an affinographic arrangement $\mathcal{A}$ in F^n and its projectivization $\mathcal{A}_{\mathbb{P}}$ in $\mathbb{P}^n(F) = F^n \cup H_\infty$. Call a coordinate i in F^n *affinely transitive in* $\mathcal{A}$ if, whenever $\mathcal{A}$ has hyperplanes $x_i - x_j = g$ and $x_k - x_i = h$, it has a hyperplane $x_k - x_j = g + h$. Call i *strictly affinely transitive* if, in addition, for each $j \neq i$ there is at most one hyperplane of the form $x_i - x_j = g$.

COROLLARY 3.4. *Let $\mathcal{A}$ be an affinographic arrangement of hyperplanes in F^n, where F is a skew field.*

(A) *$\mathcal{A}_{\mathbb{P}}$ is supersolvable if and only if, after suitable translation and renumbering of coordinates, each coordinate i is affinely transitive in $\mathcal{A}_{(i)}$.*

(B) *$\mathcal{A}_{\mathbb{P}} \setminus \{H_\infty\}$ is supersolvable, as an arrangement in $\mathbb{P}^n(F)$, if and only if $\mathcal{A}$ is a direct sum of (affine) arrangements $\mathcal{B}$ that have any of the following forms, after translation and renumbering coordinates:*

(i) *Every coordinate $i \geq 3$ in $\mathcal{B}$ is strictly affinely transitive in $\mathcal{B}_{(i)}$.*

(ii) *Every coordinate $i \geq 4$ is strictly affinely transitive in $\mathcal{B}_{(i)}$, char $F = 2$, and $\mathcal{B}$ contains $x_j - x_k = 0, 1$ for $1 \leq j < k \leq 3$.*

(iii) *There are a fixed $g_0 \in F^*$, a set $\{i_1, \ldots, i_k, m\}$ of coordinates with $k \geq 1$, and a coordinate $r \geq i_1, \ldots, i_k, m$, such that every coordinate $i > r$ is strictly affinely transitive in $\mathcal{B}_{(i)}$,*

$$\mathcal{B}_0 = \{x_m = x_{i_j} + g_0 : 1 \leq j \leq k\} \subseteq \mathcal{B},$$

$$\{x_{i_j} = x_{i_l} : 1 \leq j < l \leq k\} \subseteq \mathcal{B}_{(r)} \setminus \mathcal{B}_0 \subseteq \{x_i = x_j : 1 \leq i < j \leq r\},$$

and every coordinate $i \leq r$ is strictly affinely transitive in $\mathcal{B}_{(i)} \setminus \mathcal{B}_0$.

4. EXAMPLES

4.1. Group expansions and biased expansions. Take a simple graph Γ with vertex set $[n]$ and a subset $U \subseteq V(\Gamma)$. In the definition of $\mathfrak{G}K_n$, include only those edges $(i, j; g)$ for which $ij \in E(\Gamma)$. Then one has the *partially filled $\mathfrak{G}$-expansion* $\mathfrak{G}\Gamma^{(U)}$ of Γ. If $\mathfrak{G} = \mathbb{Z}_\gamma$, this gain graph corresponds to the complex hyperplane arrangement

$$\{x_i = \omega^k x_j,\ x_l = 0 : ij \in E(\Gamma),\ 0 \leq k < \gamma,\ l \in U\},$$

where ω is a primitive γ^{th} root of unity. (Of course, if $\gamma = 2$, then $\omega = -1$ and we can regard this as a real arrangement.)

COROLLARY 4.1. *Assuming $|\mathfrak{G}| \geq 2$ and Γ is connected, $G(\mathfrak{G}\Gamma^{(U)})$ is supersolvable if and only if Γ is chordal and U^c is a stable set of simplicial vertices in Γ, or $\mathfrak{G}\Gamma^{(U)} = \pm K_3$, or $|V(\Gamma)| \leq 2$. Furthermore, $G(\mathfrak{G}\Gamma^{(U)})$ has a modular coatom if and only if Γ has a simplicial vertex v with $N(v) \subseteq U$, or $\mathfrak{G}\Gamma^{(U)} = \pm K_3$, or $|V(\Gamma)| \leq 2$.*

The characteristic polynomial of the bias matroid of a finite $\mathfrak{G}$-expansion, provided $\gamma = |\mathfrak{G}| \geq 2$ or $U \neq \emptyset$, is

$$p_G(\lambda) = \sum_{\substack{X \subseteq V(\Gamma) \\ \text{stable}}} \gamma^{n-|X|} \chi_{\Gamma \setminus X}\left(\frac{\lambda - 1}{\gamma}\right), \tag{4.1}$$

χ_Γ being the chromatic polynomial. (See [17, Examples III.3.6 and III.4.6].) In particular, if $U = V(\Gamma)$ we have

$$p_G(\lambda) = \gamma^n \chi_\Gamma\left(\frac{\lambda - 1}{\gamma}\right), \tag{4.2}$$

so in this case $p_G(\lambda)$ has (positive) integral roots if and only if χ_Γ does. Thus if Γ is nonchordal with integral characteristic roots, then $G(\mathfrak{G}\Gamma^{(V)})$ is nonsupersolvable with integral roots.

Suppose that Γ is chordal (i.e., that $G(\Gamma)$ is supersolvable); then $p_G(\lambda)$ has positive integral roots if $U = V(\Gamma)$. More generally, let U^c be a set of simplicial vertices of Γ. It is easy to see that U^c is the disjoint union of cliques $W_1, W_2, \dots, W_q$ where every element of W_i has the same complete neighborhood, therefore the same degree d_i. Thus the characteristic polynomial of $\operatorname{Lat} G(\mathfrak{G}\Gamma^{(U)})$ is given by

$$p_G(\lambda) = \gamma^{n-q} \chi_{\Gamma \setminus Q}\left(\frac{\lambda - 1}{\gamma}\right) \prod_{i=1}^{q} (\lambda - \gamma d_i + |W_i| - 1) \tag{4.3}$$

where Q is any set consisting of one vertex from each W_i. Letting the roots of χ_Γ be $d_1, \dots, d_q, d_{q+1}, \dots, d_n$, we conclude that $p_G(\lambda)$ has roots

$$\gamma d_1 + 1 - |W_1|, \dots, \gamma d_q + 1 - |W_q|, \gamma d_{q+1} + 1, \dots, \gamma d_n + 1.$$

These are positive integers, of course. But $G(\mathfrak{G}\Gamma^{(U)})$ is supersolvable only when all $|W_i| = 1$.

Factorizability and freeness. One can show that Γ and its full $\mathfrak{G}$-expansion parallel each other's behavior in regard to properties like factorizability of the matroid or, for finite cyclic $\mathfrak{G}$, freeness of a representing hyperplane arrangement. And how about a nonfull expansion $\varphi = \mathfrak{G}\Gamma^{(U)}$? I can show by calculating characteristic polynomials that, for $G(\varphi)$ to be free or have a factorization in the sense of [8, 16], Γ must be chordal and U^c consist only of simplicial vertices. Then $G(\varphi)$ is supersolvable, hence is free and has a factorization, if U^c is stable. (What happens when U^c is not stable I have not determined.) Thus in some sense the supersolvable examples of this type are fundamental, while the nonsupersolvable ones are derived by very selective deletions. Whether this observation may generalize to any other kinds of matroids is not known.

COROLLARY 4.2. *Assume Γ is a simple graph of finite order and $|\mathfrak{G}| \geq 2$.*

(A) $L_0(\mathfrak{G}\Gamma)$ is supersolvable $\iff$ Γ is chordal. It has a modular coatom $\iff$ Γ has a simplicial vertex.

(B) Assuming Γ has no isolated vertices, $L(\mathfrak{G}\Gamma)$ is supersolvable, and indeed has a modular coatom, only when $|V(\Gamma)| \leq 2$ or $\mathfrak{G}\Gamma = \pm K_3$.

The supersolvable lift examples are trivial in a sense: they satisfy $L(\mathfrak{G}\Gamma) = G(\mathfrak{G}\Gamma)$, so they are subsumed under Corollary 4.1.

Are there any examples with integral roots that are not supersolvable? By Section 1.6, the positivity and integrality of the roots of $L_0(\mathfrak{G}\Gamma)$ are identical to those of $G(\mathfrak{G}\Gamma^{(V)})$.

Setting $\gamma = |\mathfrak{G}|$ and assuming γ and $n \geq 2$, the lift characteristic polynomial for connected Γ, from [17, Example III.6.6], is

$$p_L(\lambda) = \lambda^{-1} \left\{ (\lambda - 1)\gamma^{n-1} \chi_\Gamma\left(\frac{\lambda}{\gamma}\right) + \chi_\Gamma(\lambda) \right\}. \tag{4.4}$$

Although it seems that an integral factorization of $p_L(\lambda)$ could exist, if at all, only in the rarest circumstances, I see no way to decide this except in very special cases.

4.2. Near-Dowling and Dowling lift lattices. The most interesting group expansions are those in which $\Gamma = K_n$. The *near-Dowling lattice* $Q^\dagger_{n,p}(\mathfrak{G})$ of a group $\mathfrak{G}$ (or quasigroup, if $n \leq 3$) is $\operatorname{Lat} G(\mathfrak{G}K_n^{(p)})$; it generalizes the *Dowling lattice* $Q^\dagger_{n,n}(\mathfrak{G})$ introduced in [5]. If

$|\mathfrak{G}| = 2$, $Q^{\dagger}_{n,p}(\mathfrak{G})$ is the lattice of subspaces generated by the root system B_n with $n - p$ standard basis vectors omitted, thus in particular by D_n if $p = 0$. If $\mathfrak{G} = \mathbb{Z}_\gamma$, $Q^{\dagger}_{n,p}(\mathfrak{G})$ is a complex analog.

The characteristic polynomial of a finite near-Dowling lattice follows from (4.1), since $\chi_{K_l}(y) = y(y-1)\cdots(y-l+1)$. Evidently the roots are positive integers. Can this fact be explained by supersolvability? Mostly not, since by Corollary 4.1, if $\gamma \geq 2$, then $Q^{\dagger}_{n,p}(\mathfrak{G})$ is supersolvable if and only if $p \geq n-1$ or $(n, p, \gamma) = (3, 0, 2)$. (The supersolvability of the Dowling lattices was proved by Dowling. Nonsupersolvability for various particular values of n, p, and γ seems to have been noticed several times.) Notwithstanding this, there are explanations of the integrality of the roots of the near-Dowling lattices: algebraically, if $|\mathfrak{G}| = 2$ the corresponding arrangements are free [7]; combinatorially, the lattice has an atom decision tree if $|\mathfrak{G}| = 2$ [3], while one can regard the gain-graph coloring method of [17, Example II.4.7] as a graphical reason. On the other hand, Bailey [2] showed that if $|\mathfrak{G}| = 2$ the matroid does not have a factorization in the sense of [8, 16].

The *(extended) Dowling lift lattice* $Q^{\S\S}_n(\mathfrak{G})$ (or, $Q^{\S}_n(\mathfrak{G})$) of $\mathfrak{G}$ is $\operatorname{Lat} L(\mathfrak{G}K_n)$ (or, $\operatorname{Lat} L_0$ $(\mathfrak{G}K_n)$). $Q^{\S\S}_n(\mathfrak{G})$ equals the near-Dowling lattice $Q^{\dagger}_{n,0}(\mathfrak{G})$ if $n \leq 3$, and $Q^{\S}_1(\mathfrak{G}) = Q^{\dagger}_{1,1}(\mathfrak{G})$; otherwise the lift, extended lift, and near-Dowling lattices are all different. From Corollary 4.2 we see that all $Q^{\S}_n(\mathfrak{G})$ are supersolvable. On the other hand, $Q^{\S\S}_n(\mathfrak{G})$ is supersolvable only when $n \leq 2$ or $\gamma = 1$ or $(n, \gamma) = (2, 3)$. The roots of $Q^{\S\S}_n(\mathfrak{G})$ are positive integers when $n = 3$ because the lattice equals $Q^{\dagger}_{3,0}(\mathfrak{G})$. For larger n it seems impossible that they could all be integers, but I cannot prove there are no exceptions.

4.3. *An extension of Edelman and Reiner's theorem.* Edelman and Reiner [7] showed that the two-term hyperplane representation $x^*(\Sigma^{(U)})$ of a signed graph $\Sigma^{(U)}$, where Σ has the form $+K_n \cup -\Delta$, is supersolvable if and only if Δ is a threshold graph and U satisfies certain conditions. They did so as a byproduct of their characterization of free arrangements of this type. We can derive their supersolvability criterion and a simple generalization directly from Theorem 2.2.

First we need some definitions. The (decreasing) *degree partial order* on the vertices of a graph Δ is defined by $v \prec w$ if $\deg v > \deg w$. A *degree order* on $V(\Delta)$ is any linear extension of the degree partial order. A graph Δ is a *threshold graph* if it is obtained from the empty graph by adding one vertex at a time, each new vertex being adjacent to all or none of the previous vertices. (For threshold graphs see [10, Chapter 10] or [20].)

COROLLARY 4.3. *Let $\mathfrak{G}$ be a group, $\mathfrak{H}$ a proper subgroup, Δ a spanning subgraph of K_n, and $U \subseteq V(K_n)$. Define $\varphi = \mathfrak{G}\Delta \cup \mathfrak{H}K_n$. Then $G(\varphi^{(U)})$ is supersolvable if and only if one of the following holds:*

(i) Δ is a threshold graph, U is an ideal in the degree partial order, and U^c is stable in Δ if $|\mathfrak{H}| = 1$, $|U^c| \leq 1$ if $|\mathfrak{H}| > 1$.
(ii) $|\mathfrak{H}| = 1$, $U = \emptyset$, and $|E(\Delta)| = 1$.
(iii) $|\mathfrak{H}| = 1$, $|\mathfrak{G}| = 2$, $U = \emptyset$, and $E(\Delta)$ is either a triangle or a star.

One could apply Corollary 4.2 to a finite cyclic group $\mathfrak{G} \subseteq \mathbb{C}^*$. Then the hyperplane representation $x^*(\varphi^{(U)})$ is a mild complex generalization of the arrangements treated by Edelman and Reiner.

COROLLARY 4.4. *With notation as in Corollary* 4.3, *$L_0(\varphi)$ is supersolvable if and only if Δ is a threshold graph. $L(\varphi)$ is supersolvable if and only if φ is as in Corollary* 4.3 *(ii)–(iii).*

4.4. Bicircular matroids. These are the bias matroids $G(\Gamma, \emptyset)$ of contrabalanced graphs $(\Gamma, \emptyset)$. (Loose edges are excluded here. We also exclude loops, since they behave like half edges. Multiple edges are, as usual, allowed.) Since $(\Gamma, \emptyset)$ can be embedded in $\langle \mathbb{Z}_\gamma K_n^{(n)} \rangle$ for any $\gamma \geq 2^{|E(\Gamma)|}$, we obtain a representation of $G(\Gamma, \emptyset)$ by complex hyperplanes of the forms $x_i = \omega^k x_j$ and $x_l = 0$, similar to those mentioned in Example 4.1.

A *multitree* is a link graph T whose simplification (that is, the graph resulting from replacing parallel sets by simple edges) is a tree. A *partially filled* multitree is $T^{(U)}$ where $U \subseteq V(T)$. A multitree T can be described by a tree $\overline{T}$ and a positive multiplicity function $\mu : E(\overline{T}) \to \mathbb{Z}$. $T^{(U)}$ can be described by the pair $(\overline{T}^{(U)}, \mu)$. A *leaf* is a vertex having just one neighbor.

COROLLARY 4.5. *Let Γ be a graph of finite order.*

(a) $G(\Gamma, \emptyset)$ is supersolvable if and only if each component of Γ is either a partially filled multitree $T^{(U)}$ such that $T{:}U$ is connected and every multiple edge is incident with a vertex in U, or a multitree with just one multiple edge, or a link graph containing exactly one polygon.

(b) $G(\Gamma, \emptyset)$ has a modular coatom if and only if Γ has a leaf whose degree is one or whose neighbor is filled, or $\Gamma = mK_2$ with $m \geq 2$, or Γ is a unicycle.

The matroids of partially filled multitrees are very simple. Geometrically, they consist of lines l_{ij}, corresponding to the edges joining adjacent pairs of vertices, in general position in n-space except for meeting at points P_i (called 'joints' by Kahn and Kung [19]) corresponding to vertices. A joint belongs to the matroid if and only if the corresponding vertex is in U. The total number of points, other than joints, on each l_{ij} equals the multiplicity of edges joining its two vertices. (These matroids are obviously not new: for instance, those for which $\overline{T}$ is a path are the origami geometries of [19, section 8]. Our interest in them is as examples of our biased-graphic theorems.)

Let $T^{(U)}$ be a partially filled multitree of the kind in Corollary 4.5(a). Let $B = E(\overline{T}){:}U$. One can show by coloring in suitable gain groups that $G(T^{(U)})$ has characteristic polynomial

$$p_G(\lambda) = \prod_{e \in B} (\lambda - 1 - \mu(e)) \prod_{e \notin B} (\lambda - \mu(e)), \tag{4.5}$$

assuming $T^{(U)}$ is not balanced. In fact one can calculate the characteristic polynomial of any $G(T^{(U)})$. It turns out that there do exist limited but still substantial numbers of these matroids that are not supersolvable but whose characteristic roots are positive integers. The details are complicated and mysterious, so I omit them.

We conclude with the lift analogs of bicircular matroids. A *multi-isthmus* in a connected graph is an edge or a set of parallel edges whose removal disconnects the graph.

COROLLARY 4.6. *Let Γ be a connected link graph of finite order.*

(A) $L_0(\Gamma, \emptyset)$ is supersolvable $\iff$ Γ is a multitree. It has a modular copoint $\iff$ Γ has a multi-isthmus.

(B) $L(\Gamma, \emptyset)$ is supersolvable $\iff$ Γ is a multitree with at most one multiple edge. It has a modular copoint $\iff$ Γ has an isthmus, or Γ is a multitree with one multiple edge, or Γ is a unicycle.

5. COMMENTS AND QUESTIONS

5.1. Algorithmics. Theorems 2.2 and 3.2 yield a reasonably fast algorithm for deciding whether a given biased or gain graph has supersolvable bias, lift, or extended lift matroid; and thereby whether a given two-term or graphic-lift arrangement of hyperplanes is supersolvable.

5.2. Generalized chordality. Theorem 2.2 raises the tantalizing question of generalizing criteria for chordality in graphs. For a graph Γ, supersolvability of $G(\Gamma)$ is equivalent to each of the following: existence of a simplicial vertex ordering, chordality, and Dirac's condition that minimal vertex joins be cliques (see [10, Theorem 4.1]). The first property is generalized (with necessary exceptions) to biased graphs by Theorem 2.2 but I do not know how to generalize the others.

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