Jordan derivations and antiderivations on triangular matrices

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Abstract

We define an antiderivation from an algebra $A$ into an $A$-bimodule $M$ as a linear map $\delta : A \to M$ such that $\delta(ab) = \delta(b)a + b\delta(a)$ for all $a, b \in A$. The main result states that every Jordan derivation from the algebra of all upper triangular matrices into its bimodule is the sum of a derivation and an antiderivation.

Keywords: Triangular matrix algebra; Jordan derivation; Antiderivation

1. Introduction

Let $C$ be a commutative ring with unity and let $A$ be an algebra over $C$. Recall that a $C$-linear map $\Delta$ from an algebra $A$ into an $A$-bimodule $M$ is called a Jordan derivation if

$$\Delta(ab + ba) = \Delta(a)b + a\Delta(b) + \Delta(b)a + b\Delta(a)$$

for all $a, b \in A$.

In 1957 Herstein [6] proved that every Jordan derivation from a prime ring of characteristic not 2 into itself is a derivation. This result has been extended to different
rings and algebras in various directions (see e.g. [2,5,9] and references therein); one might very roughly summarize these results by saying that proper Jordan derivations (i.e. those that are not derivations) from rings (algebras) into themselves are rather rare and very special.

More recently Johnson [7] considered a more challenging question for which (Banach) algebras \( \mathcal{A} \) there are no proper Jordan derivations from \( \mathcal{A} \) into an arbitrary (Banach) \( \mathcal{A} \)-bimodule \( \mathcal{M} \). While it has turned out that this is true for some important classes of algebras (in particular, for the algebra of all \( n \times n \) complex matrices) [7–Sections 6 and 7], there are on the other hand simple counterexamples on some algebras and their (special) bimodules. The fundamental example is given on the algebra \( \mathcal{F}_2 \) of \( 2 \times 2 \) upper triangular matrices over \( \mathbb{C} \). Let us recall it. We make \( \mathcal{C} \) an \( \mathcal{F}_2 \)-bimodule by defining \( a\lambda = a_{22} \lambda \) and \( \lambda a = \lambda a_{11} \) for all \( \lambda \in \mathbb{C}, a \in \mathcal{F}_2 \). A map \( \delta : \mathcal{F}_2 \to \mathcal{C} \) defined by \( \delta(a) = a_{12} \) is a proper Jordan derivation, which, as Johnson pointed out, in fact satisfies

\[
\delta(ab) = \delta(b)a + b\delta(a)
\]

for all \( a, b \in \mathcal{F}_2 \). In view of the analogy with antihomomorphisms (which are together with homomorphisms basic examples of Jordan homomorphisms) we shall call linear maps satisfying (1.2) antiderivations. In Section 2 we shall gather together some general observations on antiderivations and in particular provide their examples. Section 3 is devoted to the proof of the main result which we will now state. But first we fix some notation. Throughout this paper, by \( \mathcal{M}_n \), \( n \geq 2 \), we denote the algebra of all \( n \times n \) matrices over \( \mathbb{C} \), by \( \mathcal{T}_n \) its subalgebra of all upper triangular matrices, and by \( \mathcal{D}_n \) its subalgebra of all diagonal matrices. We shall assume, without further mention, that any algebra and any module considered in this paper is 2-torsionfree, i.e. if \( a \) is its nonzero element then \( 2a \) is nonzero as well.

**Theorem 1.1.** Let \( \mathcal{M} \) be a \( \mathcal{T}_n \)-bimodule and let \( \Delta : \mathcal{T}_n \to \mathcal{M} \) be a Jordan derivation. Then there exist a derivation \( d : \mathcal{T}_n \to \mathcal{M} \) and an antiderivation \( \delta : \mathcal{T}_n \to \mathcal{M} \) such that \( \Delta = d + \delta \) and \( \delta(\mathcal{D}_n) = 0 \). Moreover, \( d \) and \( \delta \) are uniquely determined.

This result hopefully suggests what kind of results one might expect for some more general algebras admitting proper Jordan derivations.

As a corollary to Theorem 1.1 we shall easily derive

**Corollary 1.2.** Let \( 2 \leq n \leq m \). There are no proper Jordan derivations from \( \mathcal{T}_n \) into \( \mathcal{M}_m \). In particular, there are no proper Jordan derivations from \( \mathcal{T}_n \) into itself.

The last assertion may be considered as a derivation analogue of results on Jordan isomorphisms of \( \mathcal{T}_n \) which have recently appeared in the series of papers [1,4,8,10].
2. Remarks on antiderivations

Until further notice \( \mathcal{A} \) will denote an arbitrary algebra over \( \mathbb{C} \) and \( M \) will denote an arbitrary \( \mathcal{A} \)-bimodule. As usual, by \( [a, b] \) we denote the commutator \( ab - ba \) of elements \( a \) and \( b \).

Remark 2.1. Let \( \delta : \mathcal{A} \to \mathcal{M} \) be an antiderivation. Then \( \delta(a)[b, c] = [a, b]\delta(c) \) for all \( a, b, c \in \mathcal{A} \). In particular, \( \delta(a)[a, c] = 0 = [a, c]\delta(a) \) for all \( a, c \in \mathcal{A} \).

Proof. Compute \( \delta(cba) \) in two different ways:

\[
\begin{align*}
\delta((cb)a) &= \delta(a)cb + a\delta(b)c + ab\delta(c), \\
\delta(c(ba)) &= \delta(a)bc + a\delta(b)c + ba\delta(c),
\end{align*}
\]

and compare the results so obtained. \( \square \)

Antiderivations were also introduced (however, under a different name) by Herstein [6] who also noticed the above formulae and used them to prove that \( \delta \) must be 0 in the case when \( \mathcal{A} = \mathcal{M} \) is prime and noncommutative.

We shall say that an antiderivation \( \delta \) is improper if it is simultaneously a derivation; otherwise we shall say that \( \delta \) is proper. Note that \( \delta \) is improper if and only if \( \delta(ab) = \delta(b)a + b\delta(a) = \delta(ba) \) for all \( a, b \in \mathcal{A} \). That is, we have

Remark 2.2. An antiderivation \( \delta : \mathcal{A} \to \mathcal{M} \) is improper if and only if \( \delta([\mathcal{A}, \mathcal{A}]) = 0 \).

Next we present a general method of constructing antiderivations. Suppose that \( \mathcal{M} \) is such that \( [\mathcal{A}, \mathcal{A}]\mathcal{M} = \mathcal{M}[\mathcal{A}, \mathcal{A}] = 0 \). Then the \( \mathbb{C} \)-module of \( \mathcal{M} \) becomes an \( \mathcal{A} \)-bimodule if we introduce the new multiplication by \( a \cdot m = ma \) and \( m \cdot a = am \) for all \( a \in \mathcal{A}, m \in \mathcal{M} \). We set \( \mathcal{M}^{op} = (\mathcal{M}, +, \cdot) \). Given any map \( d : \mathcal{A} \to \mathcal{M} \) we denote by \( d^{op} \) a map from \( \mathcal{A} \) to \( \mathcal{M}^{op} \) defined by \( d^{op}(a) = d(a) \) for all \( a \in \mathcal{A} \).

Remark 2.3. Suppose that \([\mathcal{A}, \mathcal{A}]\mathcal{M} = \mathcal{M}[\mathcal{A}, \mathcal{A}] = 0\). If \( d : \mathcal{A} \to \mathcal{M} \) is a derivation, then \( d^{op} : \mathcal{A} \to \mathcal{M}^{op} \) is an antiderivation.

Similarly, if \([\mathcal{A}, \mathcal{A}]\mathcal{M} = \mathcal{M}[\mathcal{A}, \mathcal{A}] = 0\) then a derivation \( d : \mathcal{A} \to \mathcal{M} \) gives rise to an antiderivation from the opposite algebra \( \mathcal{A}^{op} \) of \( \mathcal{A} \) (which is derived from \( \mathcal{A} \) by reversing the multiplication in \( \mathcal{A} \)) into \( \mathcal{M} \) (which can also be considered as an \( \mathcal{A}^{op} \)-bimodule). However, since it is our aim to construct antiderivations from a fixed algebra \( \mathcal{A} \) into some \( \mathcal{A} \)-bimodule, dealing with \( d^{op} : \mathcal{A} \to \mathcal{M}^{op} \) is more appropriate for our purposes.

If \( d \) is an inner derivation (i.e. \( d(a) = am - ma \) for some \( m \in \mathcal{M} \)) then the antiderivation \( d^{op} \) is improper. So it is still not clear how to find proper antiderivations.
Remark 2.4. Let $\mathcal{S}$ be the ideal of $\mathcal{A}$ generated by $[\mathcal{A}, \mathcal{A}]$ and suppose that $\mathcal{S}^2 \neq \mathcal{S}$. Further, suppose there exists a subalgebra $\mathcal{D}$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{D} \oplus \mathcal{S}$ (as a $\mathcal{D}$-module direct sum). Then there exists an $\mathcal{A}$-bimodule $\mathcal{M}$ with a proper antiderivation from $\mathcal{A}$ into $\mathcal{M}$ and a nonzero improper antiderivation from $\mathcal{A}$ into $\mathcal{M}$.

Proof. Regard $\mathcal{S}$ as an $\mathcal{A}$-bimodule and $\mathcal{S}^2$ as its subbimodule. Let $\mathcal{M}_0$ be the quotient module $\mathcal{S}/\mathcal{S}^2$. Note that $d_0 : r + s \mapsto s + \mathcal{S}^2$, $r \in \mathcal{D}$, $s \in \mathcal{S}$, is a derivation from $\mathcal{D}$ into $\mathcal{M}_0$, and moreover, $d_0([\mathcal{A}, \mathcal{A}]) \neq 0$ since $\mathcal{S}^2 \neq \mathcal{S}$. Further, since $[\mathcal{S}, \mathcal{D}] \subseteq \mathcal{D} \cap \mathcal{S} = 0$, $[\mathcal{S}, \mathcal{S}] \subseteq \mathcal{S}^2$, and $[\mathcal{A}, \mathcal{A}] \not\subseteq \mathcal{S}^2$ it follows that $[\mathcal{D}, s_1] \not\subseteq \mathcal{S}^2$ for some $s_1 \in \mathcal{S}$. Therefore $d_1 : r + s \mapsto [r, s_1] + \mathcal{S}^2$ is a nonzero (inner) derivation from $\mathcal{D}$ into $\mathcal{M}_0$ with $d_1([\mathcal{A}, \mathcal{A}]) = 0$. Since $\mathcal{M}_0[\mathcal{A}, \mathcal{A}] = [\mathcal{A}, \mathcal{A}]$, $\mathcal{M}_0 = 0$ we may define $\mathcal{M} = \mathcal{M}_0^\oplus$. Now Remarks 2.2 and 2.3 imply that $d_1^\oplus : \mathcal{A} \to \mathcal{M}$ is a proper antiderivation and $d_1^\oplus : \mathcal{A} \to \mathcal{M}$ is an improper antiderivation. \( \square \)

The conditions of Remark 2.4 are satisfied when $\mathcal{A} = \mathcal{T}_n$, $n \geq 2$. Indeed, in this case $\mathcal{S}$ consists of all strictly upper triangular matrices and we may take $\mathcal{D}$ for $\mathcal{D}$. Thus we have

Remark 2.5. There exists an $\mathcal{T}_n$-bimodule $\mathcal{M}$ with a proper antiderivation from $\mathcal{T}_n$ into $\mathcal{M}$ and a nonzero improper antiderivation from $\mathcal{T}_n$ into $\mathcal{M}$.

We also remark that for $\mathcal{A} = \mathcal{T}_2$ an example of a proper antiderivation as constructed in Remark 2.4 essentially coincides with the one given by Johnson [7].

Next we describe the structure of any antiderivation on $\mathcal{T}_n$. Note that this result in particular shows that every antiderivation on $\mathcal{T}_n$ vanishes on $\mathcal{S}^2$ which elucidates the concept behind the construction in Remark 2.4.

By $e_{ij}$, $1 \leq i \leq j \leq n$, we denote the matrix units in $\mathcal{T}_n$.

Remark 2.6. Let $\delta$ be an antiderivation from $\mathcal{T}_n$ to an $\mathcal{T}_n$-bimodule $\mathcal{M}$. Then there exist $d_1, \ldots, d_n, b_1, \ldots, b_{n-1} : \mathcal{T}_n \to \mathcal{M}$ such that

(i) each $d_i$ is an improper antiderivation and $d_i(e_{kl}) = 0$ whenever $e_{kl} \neq e_{ii}$,

(ii) each $b_i$ is either $0$ or is a proper antiderivation and $b_i(e_{kl}) = 0$ whenever $e_{kl} \neq e_{i,i+1}$,

(iii) $\delta = d_1 + \cdots + d_n + b_1 + \cdots + b_{n-1}$.

Proof. By Remark 2.1 we have $\delta(e_{ij})[e_{ii}, a] = 0 = [e_{ii}, a]\delta(e_{ij})$ for all $a \in \mathcal{T}_n$, which yields $\delta(e_{ij})e_{kl} = e_{kl}\delta(e_{ij}) = 0$ whenever exactly one of $k$ and $l$ equals $i$.

Therefore, since $\delta(e_{ij}) = \delta(e_{ij}e_{jj}) = \delta(e_{ii}e_{ij})$ when $i < j$, it follows that

$$\delta(e_{ij}) = e_{ij}\delta(e_{jj}) = \delta(e_{ij})e_{ii} \quad \text{whenever } i < j. \quad (2.1)$$

Set $m_i = \delta(e_{ii})$ and $n_i = \delta(e_{i,i+1})$. Considering $\delta(e_{ii}) = \delta(e_{ii}e_{ii})$ we get

$$m_i = m_i e_{ii} + e_{ii} m_i \quad \text{for all } i. \quad (2.2)$$
Further, we claim that
\[ m_i e_{kl} = e_{kl} m_i = 0 \quad \text{whenever} \quad e_{kl} \neq e_{ii}. \]  
(2.3)

As already noticed at the beginning, this is true when exactly one of \( k \) and \( l \) equals \( i \). So we assume that \( k \neq i \) and \( l \neq i \). Then we have \( \delta(e_{kl} e_{ii}) = \delta(e_{ii} e_{kl}) = 0 \).

Expanding these identities and using (2.1) we get (2.3).

Next we claim that
\[ n_i e_{ii} + 1 + e_{ii} + 1 n_i = 0 \],
(2.4)
\[ n_i e_{ij} = 0 \quad \text{and} \quad n_i e_{ij} = 0 \quad \text{for all} \quad j > i + 1, \]
(2.5)
\[ e_{i+1,i+1} + n_i = n_i \quad \text{and} \quad e_{j,i+1} + n_i = 0 \quad \text{for all} \quad j < i. \]
(2.6)

The first identity is a consequence of \( \delta(e_{ii} + 1 e_{ii} + 1) = 0 \), while the other follow, by making use of identities (2.1) and (2.3), from \( \delta(e_{ij} e_{i+1,i+1}) = \delta(e_{i+1,i+1}), \delta(e_{ij} e_{i+1,i+1}) = 0, \delta(e_{i+1,i+1} e_{i+1,i+1}) = \delta(e_{i+1,i+1}), \) and \( \delta(e_{i+1,i+1} e_{i+1,i+1}) = 0 \) respectively.

Now define \( d_i \) and \( \delta_i \) as \( \mathbb{C} \)-linear maps such that
\[ d_i(e_{ii}) = m_i \quad \text{and} \quad d_i(e_{kl}) = 0 \quad \text{whenever} \quad e_{kl} \neq e_{ii}, \]
\[ \delta_i(e_{i+1,i+1}) = n_i \quad \text{and} \quad \delta_i(e_{kl}) = 0 \quad \text{whenever} \quad e_{kl} \neq e_{i+1,i+1}. \]

To prove that each \( d_i \) is an antiderivation, we have to show that
\[ d_i(e_{kl} e_{st}) = d_i(e_{st}) e_{kl} + e_{st} d_i(e_{kl}) \]  
(2.7)
for all \( k \leq l \) and \( s \leq t \). In the case when \( e_{kl} \neq e_{ii} \) and \( e_{st} \neq e_{ii} \), it is clear that both the left-hand and the right-hand side of (2.7) are 0. Suppose that \( e_{st} = e_{ii} \). In this case (2.7) follows from (2.2) (when \( e_{kl} = e_{ii} \)) and from (2.3) (when \( e_{kl} \neq e_{ii} \)). Similarly we see that (2.7) holds true when \( e_{kl} = e_{ii} \).

Similarly, using (2.4), (2.5) and (2.6) one proves that each \( \delta_i \) is an antiderivation.

Moreover, by Remark 2.2 we see that each \( d_i \) is an improper antiderivation and each \( \delta_i \) is either 0 or a proper antiderivation. Finally, using (2.1) we see that whenever \( j > i + 1 \) we have
\[ \delta(e_{ij}) = \delta(e_{i+1,i+1} e_{i+1,j}) = \delta(e_{i+1,j} + e_{i+1,j} \delta(e_{i+1,j})) = 0. \]

Accordingly, \( \delta = d_1 + \cdots + d_n + \delta_1 + \cdots + \delta_{n-1} \). □

Clearly, derivations and antiderivations, as well as their sums, are examples of Jordan derivations. One might wonder whether a Jordan derivation is always the sum of a derivation and an antiderivation. Let us show that it is not true. By \( \mathcal{C}(x,y) \) we denote the free algebra in two generators, i.e. the algebra of polynomials in the noncommuting variables \( x \) and \( y \) with coefficients from \( \mathbb{C} \).

Remark 2.7. There exists a Jordan derivation from \( \mathcal{C}(x,y) \) into some \( \mathcal{C}(x,y) \)-bimodule which is not the sum of a derivation and an antiderivation.
Proof. Let $\mathcal{O}_0(x, y)$ be the ideal of $\mathcal{O}(x, y)$ consisting of elements of constant term 0. Regard $\mathcal{O}(x, y)$ and $\mathcal{O}_0(x, y)$ as $\mathcal{O}(x, y)$-bimodules and let $\mathcal{M} = \mathcal{O}(x, y)/\mathcal{O}_0(x, y)$ be the quotient module. Define $\Delta : \mathcal{O}(x, y) \to \mathcal{M}$ by

$$\Delta(f) = cyx - cyx + \mathcal{O}_0(x, y),$$

where $c_{xy}$ (resp. $c_{yx}$) is the coefficient of $f$ at $xy$ (resp. $yx$). Note that $\Delta$ is a Jordan derivation. If $d : \mathcal{O}(x, y) \to \mathcal{M}$ is any derivation then we clearly have $d(xy) = d(x) \cdot y + x \cdot d(y) = 0$. Similarly, any antiderivation must vanish on $xy$. Since $\Delta(xy) = 1 + \mathcal{O}_0(x, y) \neq 0$, $\Delta$ is not the sum of a derivation and an antiderivation. □

This example in particular shows that primitive (and so in particular prime) algebras admit proper Jordan derivations into their bimodules. On the other hand, there are no proper Jordan derivations from noncommutative unital simple algebras into their (2-torsionfree and 3-torsionfree) bimodules [3–Corollary 1].

3. Proof of the main result

We first state some (more or less well-known) facts concerning a Jordan derivation $\Delta : \mathcal{A} \to \mathcal{M}$ where $\mathcal{A}$ is an arbitrary algebra and $\mathcal{M}$ is an $\mathcal{A}$-bimodule. Setting $a = b$ in (1.1) we get (recall that all modules are assumed to be 2-torsionfree!)

$$\Delta(a^2) = \Delta(a)a + a\Delta(a) \quad (3.1)$$

for all $a \in \mathcal{A}$. Using $2aba = a(ab + ba) + (ab + ba)a - (a^2b + ba^2)$ it follows from (1.1) and (3.1) that

$$\Delta(aba) = \Delta(a)ba + a\Delta(b)a + ab\Delta(a) \quad (3.2)$$

for all $a, b \in \mathcal{A}$. Further, we have

$$\Delta(e) = \Delta(e)e + e\Delta(e) \quad \text{and} \quad e\Delta(e)e = 0 \quad \text{when} \quad e^2 = e. \quad (3.3)$$

Indeed, the first identity is a special case of (3.1) while the second one follows by multiplying the first one from the left and from the right by $e$. Assume further that $a \in \mathcal{A}$ is such that $ea = ae = 0$. In particular, $eae = 0$ and $ae + ea = 0$, and so (3.2) and (1.1) imply that $e\Delta(a)e = 0$ and $\Delta(a)e + a\Delta(e) + \Delta(e)a + e\Delta(a) = 0$. Multiplying the latter identity from the right by $e$ we get $\Delta(a)e + a\Delta(e)e = 0$. However, since $\Delta(e)e = \Delta(e) - e\Delta(e)$ by (3.3) it follows that

$$\Delta(a)e + a\Delta(e)e = 0 = \Delta(e)a + e\Delta(a) \quad \text{when} \quad e^2 = e \quad \text{and} \quad ea = ae = 0. \quad (3.4)$$

Now we can start the proof of Theorem 1.1. So let $\Delta$ be a Jordan derivation from $\mathcal{F}_n$ into an $\mathcal{F}_n$-bimodule $\mathcal{M}$. We first record some consequences of the above formulae. By (3.3) we have

$$\Delta(e_{ii}) = \Delta(e_{ii})e_{ii} + e_{ii}\Delta(e_{ii}) \quad \text{and} \quad e_{ii}\Delta(e_{ii})e_{ij} = 0 \quad (3.5)$$
for all \( i \) and \( k \leq i \leq j \). From (1.1) and \( \Delta(e_{ij}) = \Delta(e_{ii}e_{ij} + e_{ij}e_{ii}), \) \( i < j \), we get
\[
\Delta(e_{ij}) = \Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij}) + \Delta(e_{ij})e_{ii} + e_{ij}\Delta(e_{ii}) \tag{3.6}
\]
whenever \( i < j \). Finally, (3.4) gives
\[
\Delta(e_{kj})e_{ii} + e_{kj}\Delta(e_{ii}) = \Delta(e_{ii})e_{kj} + e_{ii}\Delta(e_{kj}) \tag{3.7}
\]
whenever \( k, j \neq i \).

Now define a \( C \)-linear map \( d : T_n \to M \) by
\[
d(e_{ij}) = \frac{1}{\Delta_1(e_{ij})}e_{ij} + \frac{1}{\Delta_1(e_{ii})}e_{ii}, \quad 1 \leq i \leq j \leq n. \tag{3.8}
\]
According to (3.5) we have \( d(e_{ii}) = \Delta(e_{ii}) \) for every \( i \).

**Lemma 3.1.** \( d \) is a derivation.

**Proof.** It is enough to check that
\[
d(e_{ij}e_{kl}) = d(e_{ij})e_{kl} + e_{ij}d(e_{kl}) \tag{3.9}
\]
for all \( i, j, k, l \). We consider two cases.

**Case 1:** \( j \neq k \). Our goal is to show that \( d(e_{ij})e_{kl} + e_{ij}d(e_{kl}) = 0 \). By (3.8) we have
\[
d(e_{ij})e_{kl} + e_{ij}d(e_{kl}) = e_{ii}/\Delta_1(e_{ij})e_{kl} + e_{ii}/\Delta_1(e_{jj})e_{kl} + e_{ij}/\Delta_1(e_{jl}).
\]
In the case \( i \neq k \) (3.7) implies that
\[
d(e_{ij})e_{kl} + e_{ij}d(e_{kl}) = e_{ii}(\Delta(e_{ij})e_{kk} + e_{ii}\Delta(e_{kk}))e_{kl} = 0.
\]
On the other hand, if \( i = k \) then (3.6) gives us
\[
d(e_{ij})e_{il} + e_{ij}d(e_{il}) = (e_{ii}\Delta(e_{ij}) + e_{ij}\Delta(e_{ii}))e_{il}
\]
\[= (\Delta(e_{ij}) - \Delta(e_{ii})e_{ij} - \Delta(e_{ij})e_{ii})e_{il} = 0.
\]
Thus, (3.9) holds true in the first case.

**Case 2:** \( j = k \). Now we have to show that \( d(e_{ii}) = d(e_{ij})e_{jl} + e_{ij}d(e_{jl}) \). Assume first that \( i < j < l \). Using (3.8) and (3.5) we get
\[
d(e_{ij})e_{jl} + e_{ij}d(e_{jl}) = \Delta(e_{ii})e_{jl} + e_{ii}\Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jj})e_{jl} + e_{ij}\Delta(e_{jl})
\]
\[= d(e_{ii}) - e_{ii}\Delta(e_{ij}) + e_{ii}\Delta(e_{ij})e_{jl} + e_{ij}\Delta(e_{jl}).
\]
By (3.7) we have \( \Delta(e_{ii})e_{jl} + e_{ii}\Delta(e_{jl}) = 0 \) which implies \( e_{ii}\Delta(e_{jl})e_{ij} = 0 \). Using this together with \( \Delta(e_{ii}) = \Delta(e_{ij}e_{jl} + e_{ij}e_{ij}) \) we obtain
\[
d(e_{ij})e_{jl} + e_{ij}d(e_{jl}) = d(e_{ii}) - e_{ii}(\Delta(e_{ij}) - \Delta(e_{ij})e_{jl} - e_{ij}\Delta(e_{jl}))
\]
\[= d(e_{ii}) - e_{ii}(e_{jl}\Delta(e_{ij}) + \Delta(e_{jl})e_{ij}) = d(e_{ii}).
\]
Next we assume that \( i = j < l \). Using \( d(e_{ii}) = \Delta(e_{ii}) \) and \( e_{ii}\Delta(e_{ii})e_{ii} = 0 \) (cf. (3.5)) we get
Now let $i < j = l$. We have

$$
d(e_{ij})e_{jj} + e_{ij}d(e_{jj}) = (\Delta(e_{ij})e_{ij} + e_{ij}\Delta(e_{ij}))e_{jj} + e_{ij}\Delta(e_{jj})
= \Delta(e_{ij})e_{ij} + e_{ij}\Delta(e_{ij})e_{jj} + e_{ij}\Delta(e_{jj})
= d(e_{ij}) - e_{ij}\Delta(e_{ij}) + e_{ij}\Delta(e_{jj})e_{ij} + e_{ij}\Delta(e_{jj}).
$$

By (3.7) we have $\Delta(e_{ij})e_{jj} + e_{ij}\Delta(e_{jj}) = 0$ which yields $e_{ij}\Delta(e_{jj})e_{ij} = 0$. Using this and $\Delta(e_{ij}) = \Delta(e_{ij}e_{jj} + e_{jj}e_{ij})$ we get

$$
d(e_{ij})e_{jj} + e_{ij}d(e_{jj}) = d(e_{ij}) - e_{ij}\Delta(e_{ij}) - e_{ij}\Delta(e_{ij})e_{jj} + e_{ij}\Delta(e_{jj})e_{ij} = d(e_{ij}).
$$

Finally, if $i = j = l$ then (3.9) follows from (3.5) and $d(e_{ii}) = \Delta(e_{ii})$. Therefore (3.9) holds true in every case. \[ \square \]

Now set by $\delta = \Delta - d$. Clearly $\delta$ is a Jordan derivation satisfying $\delta(D_n) = 0$. Moreover, we have

**Lemma 3.2.** $\delta$ is an antiderivation.

**Proof.** By (3.6) and (3.8) we have $\delta(e_{ij}) = \Delta(e_{ij})e_{ii} + e_{ij}\Delta(e_{ii})$ whenever $i < j$.

Since $\Delta = d + \delta$, $d$ is a derivation and $\delta(D_n) = 0$, it follows that

$$
\delta(e_{ij}) = d(e_{ij})e_{ii} + \delta(e_{ij})e_{ii} + e_{ij}d(e_{ii}) + e_{ij}\delta(e_{ii})
= d(e_{ij})e_{ii} + \delta(e_{ij})e_{ii} = \delta(e_{ij})e_{ii}.
$$

Accordingly

$$
\delta(e_{ij}) = \delta(e_{ij})e_{jj} + e_{jj}\delta(e_{ij}) = \delta(e_{ij})e_{jj} + e_{jj}\delta(e_{ij}) + \delta(e_{jj})e_{ij} + e_{jj}\delta(e_{ij})
= \delta(e_{ij})e_{ii}e_{jj} + e_{jj}\delta(e_{ij}) = e_{jj}\delta(e_{ij}).
$$

We proved that

$$
\delta(e_{ij}) = \delta(e_{ij})e_{ii} \quad \text{and} \quad \delta(e_{ij}) = e_{jj}\delta(e_{ij}) \quad (3.10)
$$
whenever $i < j$.

Our goal is to prove that

$$
\delta(e_{ij}e_{kl}) = \delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}) \quad (3.11)
$$
for all $i, j, k, l$. Again we consider two cases.

**Case 1:** $j \neq k$. We have to show that $\delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}) = 0$. If $i = k$ and $j = l$ then this is true since $\delta$ is a Jordan derivation. From (3.10) we see that
\[ \delta(ekl)ejj + ekjekl \delta(eij) = \delta(ekl)ekk eij + ekjekl \delta(eij), \quad (3.12) \]

which proves the desired identity in the case when \( i \neq k \) and \( j \neq l \). Next, assume \( i = k \) and \( j \neq l \). From (3.12), (3.10) and the fact that \( \delta \) is a Jordan derivation we infer that

\[ \delta(ekl)ekj + ekjekl \delta(eij) = \delta(ekl)ekj = \delta(eij ekj + ekjekl)ejj = \delta(eij ekj)ejj. \]

But this expression is always 0. This is obvious when \( l \neq k \), while in the case when \( l = k \) this follows from (3.10). In the case \( i \neq k \) and \( j = l \) we proceed similarly as above. Now we have

\[ \delta(ekl)ekj + ekjekl \delta(eij) = \delta(ekl)ekj = \delta(ekl ekj + ekjekl)ejj = \delta(ekl ekj)ejj. \]

Case 2: \( j = k \). The case when \( i = j = l \) is trivial since \( \delta \) vanishes on \( D_n \). In cases \( i < j = l \) or \( i = j < l \) the antiderivation rule holds on account of (3.10). Namely,

\[ \delta(eij) = \delta(ejj)ejj + ejj \delta(eij), \quad \delta(eil) = \delta(eji ejl + ejl eij) = \delta(eij) ejl + ejl \delta(eij). \]

Finally, let \( i < j < l \). Then

\[ \delta(ejl)ejj + ejl \delta(eij) = \delta(ejl)ejj ejj + ejl ejj \delta(eij) = 0, \]

while

\[ \delta(eil) = \delta(eij ejl + ejl eij) = \delta(eij ejl + ejl \delta(eij) + \delta(ejl)ejj + ejl \delta(eij)) = \delta(eij) ejl ejl + ejl ejl \delta(eij) + \delta(ejl) ejj ejj + ejl ejl \delta(eij) = 0. \]

Therefore (3.11) holds true in the second case as well. \( \square \)

It remains to prove that \( \delta \) and \( d \) are unique with respect to \( \delta(D_n) = 0 \). It suffices to show that 0 is the only improper antiderivation vanishing on \( D_n \). But this clearly follows from Remark 2.2. The proof of Theorem 1.1 is thus complete.

**Proof of Corollary 1.2.** It is enough to prove that there does not exist a nonzero antiderivation from \( T_n \) into \( M_m \) vanishing on \( D_n \). So let \( \delta : T_n \to M_m \) be such an antiderivation. Note that \( \delta \) satisfies the condition (3.10). Given any \( i < j \) we thus have \( \delta(eij) = ejj \delta(eij) eii \) which shows that \( \delta(eij) = cejj \) for some \( c \in C \). However, from \( \delta(eij)ejj + ejj \delta(eij) = \delta(ejj^2) = 0 \) then follows that \( \delta(eij) = 0 \). Hence \( \delta = 0 \) and so the proof is complete. \( \square \)

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