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On mixed variational relation problems

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ABSTRACT

In this paper we exploit the method of variational relations to establish existence of solutions to a general inclusion problem. The result is applied to variational relation problems in which several relations are simultaneously considered. Particular cases of variational inclusion and intersection of set-valued maps are also discussed.

1. Introduction

Let A, B and C be nonempty sets, $S_1 : A \rightrightarrows A$, $S_2 : A \rightrightarrows B$, $T : A \times B \rightrightarrows C$ be set-valued mappings with nonempty values and $R(a, b, c)$ be a relation linking elements $a \in A$, $b \in B$ and $c \in C$. In a general setting R is a subset of the product space $A \times B \times C$. In practice, it is often given by a system of inequalities of real functions or a system of inclusions of set-valued maps on $A \times B \times C$. The following variational relation problem was very recently introduced by Luc [1] (see also Khanh and Luc [2], Lin and Wang [3], Lin and Ansari [4], Luc et al. [5], Balaj and Lin [6] for further studies) as a model for many problems in optimization, equilibrium theory, variational inclusions or variational inequalities:

(VR) Find $\bar{a} \in A$ such that
(i) $\bar{a} \in S_1(\bar{a})$;
(ii) $R(\bar{a}, b, c)$ holds for all $b \in S_2(\bar{a})$ and $c \in T(\bar{a}, b)$.

In the paper [1] quoted above, a weaker problem that we describe below is mentioned, without any developments:

(WVR) Find $\bar{a} \in A$ such that
(i) $\bar{a} \in S_1(\bar{a})$;
(ii) $R(\bar{a}, b, c)$ holds for all $b \in S_2(\bar{a})$ and some $c \in T(\bar{a}, b)$.

Let X, Y and Z be nonempty sets, let $P_1, P_2 : X \times Y \rightrightarrows Z$, $Q_1, Q_2 : X \rightrightarrows Y$ be set-valued maps and let $r_1(x, y, z)$, $r_2(x, y, z)$ be two relations linking $x \in X$, $y \in Y$ and $z \in Z$. In this paper we consider the following mixed variational relation problem simultaneously involving r_1 and r_2:

(I) Find $\bar{x} \in X$ such that
(i) $r_1(\bar{x}, \bar{y}, z)$ holds for some $\bar{y} \in Q_1(\bar{x})$ and for all $z \in P_1(\bar{x}, \bar{y})$
(ii) $r_2(\bar{x}, y, z)$ holds for all $y \in Q_2(\bar{x})$ and $z \in P_2(\bar{x}, y)$.

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Notice that problem (I) contains relations of both types (VR) and (WVR). Nevertheless it can be seen as a particular case of problem (VR). In fact, set $A = X \times Y$, $B = Y$, $C = Z \times Z$, $S_1(x, y) = X \times Q_1(x)$, $S_2(x, y) = Q_2(x)$, $T((x, y), y') = P_1(x, y) \times P_2(x, y')$ and define R as follows: $R((x, y), y', (z_1, z_2))$ holds if and only if $r_1(x, y, z_1)$ and $r_2(x, y', z_2)$ hold. Then $(\bar{x}, \bar{y})$ is a solution of (VR) if and only if $\bar{x}$ is a solution of (I) and $\bar{y}$ satisfies (i) of (I). The general scheme of [1,2] can be applied to derive existence results as well as stability for the problem (I). However, as it was shown in [3] for many theoretical and applicative purposes it is sometimes more convenient to split the relation R into two or more parts. Some typical examples of problem (I) are given below.

Example 1. Let $G_1, G_2 : X \times Y \rightrightarrows Z$ and the two variational relations r_1 and r_2 defined as follows:

$$r_1(x, y, z) \quad \text{holds iff } z \in G_1(x, y), \text{ and}$$
$$r_2(x, y, z) \quad \text{holds iff } z \in G_2(x, y).$$

Then problem (I) becomes: Find $\bar{x} \in X$ such that

(i) $P_1(\bar{x}, \bar{y}) \subseteq G_1(\bar{x}, \bar{y})$ for some $\bar{y} \in Q_1(\bar{x})$
(ii) $P_2(\bar{x}, y) \subseteq G_2(\bar{x}, y)$ for all $y \in Q_2(\bar{x})$.

Example 2. Let V be a nonempty set and $F_1, F_2, C_1, C_2 : X \times Y \times Z \rightrightarrows V$. Consider the following variational relations:

$$r_{1,1}(x, y, z) \quad \text{holds iff } F_1(x, y, z) \subseteq C_1(x, y, z),$$
$$r_{1,2}(x, y, z) \quad \text{holds iff } F_1(x, y, z) \cap C_1(x, y, z) \neq \emptyset,$$
$$r_{2,1}(x, y, z) \quad \text{holds iff } F_2(x, y, z) \subseteq C_2(x, y, z),$$
$$r_{2,2}(x, y, z) \quad \text{holds iff } F_2(x, y, z) \cap C_2(x, y, z) \neq \emptyset.$$

Taking all possible combinations of $r_1 \in \{r_{1,1}, r_{1,2}\}$ and $r_2 \in \{r_{2,1}, r_{2,2}\}$ we obtain four systems of type (I) which may have different practical meanings but may be mathematically treated in a similar manner. Though there is a large number of papers in which each one or more of the corresponding problems I (i), I (ii), or particular forms thereof are studied (see for instance [7–20], to our best knowledge there is no paper dealing with any one of the four systems above.

The paper is structured as follows. In the next section we study an auxiliary inclusion problem by using two generalizations of KKM maps. The results of this section are then applied to establish existence criteria for Problem (I) in Section 3. Section 4 is devoted to a particular problem described in Example 2 above for all possible combinations of intersection and inclusion of set-valued maps. A weak version of Problem (I) is discussed in the final section.

2. Variational inclusions

In order to establish existence of solutions to problem (I), we consider the following auxiliary variational inclusion problem. Let X and Y be topological spaces and let $Q_1, Q_2, U : X \rightrightarrows Y$ and $W : Y \rightrightarrows X$ be set-valued maps.

(VI) Find $\bar{x} \in X$ such that
(i) $\bar{x} \in W \circ Q_1(\bar{x})$
(ii) $Q_2(\bar{x}) \subseteq U(\bar{x})$.

This problem corresponds to a variational relation problem (VR) in which $A = X$, $B = Y$, S_1 is replaced by $W \circ Q_1$, $S_2 = Q_2$, T is absent, and for $(x, y) \in X \times Y$ the relation $R(x, y)$ holds if and only if $y \in U(x)$.

In [1,5] it has been shown that under a certain closedness hypothesis on the data, (VI) has a solution if and only if it is finitely solvable, which means that for every finite subset D of Y, there is a point $x_D \in X$ such that for every $y \in D$, either $y \notin Q_2(x_D)$, or x_D is a fixed point of Q_1 and $y \in U(x_D)$ (see Proposition 3.1 [1] and Theorem 3.1 [5]). In its turn, the finite solvability of (VI) is closely related to the finite intersection property of the following map $P : Y \rightrightarrows X$:

$$P(y) = \left(X \setminus Q_2^{-1}(y)\right) \cup \left(Fix(W \circ Q_1) \cap U^{-1}(y)\right), \tag{1}$$

where $Fix(W \circ Q_1)$ denotes the set of all fixed points of the map $W \circ Q_1$ on X and $Q_2^{-1}(y)$ is the fiber of Q_2 on y, that is $Q_2^{-1}(y) = \{x \in X : y \in Q_2(x)\}$. On the other hand the finite intersection property is guaranteed by a property of the so-called KKM-maps.

In the classical sense a set-valued map $\Gamma : Y \rightrightarrows Y$, where Y is a convex set in a topological vector space, is called KKM if for every finite subset D of Y, its convex hull co(D) is contained in the image $\Gamma(D)$. When $\Gamma : Y \rightrightarrows X$, in which X and Y are taken from spaces with different structure, a generalized KKM property comes in force and yields also the finite intersection property of the family $\{cl(\Gamma(y)) : y \in Y\}$ (here "cl" denotes the closure). Below we exploit two generalizations of KKM maps to derive existence of solutions to (VI), the first one belongs to Park [21] (see also [22,23]) and the second one seems to be new and generalizes the concept of KKM maps by Chang and Zhang [24].

Definition 1. Let $\Gamma, W : Y \rightrightarrows X$ be set valued maps. We say that

(a) Γ is KKM with respect to W in the sense of Park (or W-KKM(a) for short) if for every finite subset D of Y, one has $W(co(D)) \subseteq \Gamma(D)$, in which case Y is assumed convex.
(b) Γ is W-KKM(b) if for every finite set $\{y_1, \dots, y_n\} \subseteq Y$, there exist $x_i \in W(y_i)$ such that for every index set $I \subseteq \{1, \dots, n\}$, one has $\mathrm{co}\{x_i : i \in I\} \subseteq \Gamma(\{y_i : i \in I\})$, in which case X is assumed convex.

We notice when X and Y coincide, KKM(a) maps with respect to the identity map are exactly KKM maps in the classical sense. In a general setting when X and Y are distinct, the finite intersection property of the family $\{cl(\Gamma(y)) : y \in Y\}$ is valid (assuming Y topological space) provided that Γ is W-KKM(a) and that W has certain additional properties. The map W that makes the family $\{\mathrm{cl}(\Gamma(y)) : y \in Y\}$ to have the finite intersection property whenever Γ is KKM(a) with respect to W is said to have the KKM property.

On the other hand the generalized KKM maps introduced by Chang and Zhang [24] correspond to the case (b) when W is the constant map $W(y) = X$ for every $y \in Y$. When X and Y coincide, every KKM map in the classical sense is W-KKM in the sense of (b) above with W being the identity map on X. It is clear that when W' is a submap of W, then every W'-KKM map is W-KKM. Hence the concept of generalized KKM maps by Chang and Zhang is the weakest. One of the remarkable properties of generalized KKM maps by Chang and Zhang is that when the intersections of $\Gamma(y)$, $y \in Y$ with finite dimensional spaces are closed, a map is generalized KKM in the sense of Chang and Zhang if and only if the family $\{\Gamma(y) : y \in Y\}$ has the finite intersection property (Theorem 3.1 [24]). For application purposes checkable sufficient conditions of W-KKM maps are more desired than the finite intersection property. Therefore, W-KKM maps with specific maps W such as listed in Proposition 4 will be of particular attention.

To proceed further we recall some continuity properties of set-valued maps. Assume that X and Y are topological spaces. A set-valued mapping $Q : X \rightrightarrows Y$ is said to be upper semicontinuous (respectively, lower semicontinuous) if for every $x \in X$ and for every open set B of Y with $Q(x) \subseteq B$ (respectively, $Q(x) \cap B \neq \emptyset$) there is a neighborhood N of x such that $Q(x') \subseteq B$ (respectively $Q(x') \cap B \neq \emptyset$) for all $x' \in N$; and it is said to be closed if its graph is a closed subset of $X \times Y$.

The following facts are known (see for instance [25]):

(i) If Q has compact values, then Q is upper semicontinuous if and only if for every net $\{x_t\}$ in X converging to $x \in X$ and for any net $\{y_t\}$ with $y_t \in Q(x_t)$ there exist $y \in Q(x)$ and a subnet $\{y_{t_\alpha}\}$ of $\{y_t\}$ converging to y.
(ii) Q is lower semicontinuous if and only if for any net $\{x_t\}$ in X converging to $x \in X$ and each $y \in Q(x)$ there exist a subnet $\{x_{t_\alpha}\}$ of $\{x_t\}$ and a net $\{y_{t_\alpha}\}$ converging to y with $y_{t_\alpha} \in Q(x_{t_\alpha})$ for all α.

Let us present some conditions for Γ to be KKM with respect to W.

Proposition 3. *The map Γ is W-KKM(a) if and only if for every $x \in X$, one has inclusion*

$$\mathrm{co}(Y \setminus \Gamma^{-1}(x)) \subseteq Y \setminus W^{-1}(x).$$

In particular, each of the following conditions is sufficient for Γ to be W-KKM(a):

(i) *The map $x \mapsto Y \setminus \Gamma^{-1}(x)$ has convex values and W is a submap of Γ, that is, $W(y) \subseteq \Gamma(y)$ for every $y \in Y$.*
(ii) *The map $x \mapsto Y \setminus W^{-1}(x)$ has convex values and W is a submap of Γ.*

Proof. Assume that Γ is W-KKM. Let $x \in X$ and $y \in \mathrm{co}(Y \setminus \Gamma^{-1}(x))$. There are $y_1, \dots, y_n \notin \Gamma^{-1}(x)$ such that $y \in \mathrm{co}\{y_1, \dots, y_n\}$. Then $x \notin \bigcup_{i=1}^n \Gamma(y_i)$ and by the hypothesis, x does not belong to $W(\mathrm{co}\{y_1, \dots, y_n\})$. In particular x does not belong to $W(y)$, and hence $y \in Y \setminus W^{-1}(x)$.

Conversely, assume $\mathrm{co}(Y \setminus \Gamma^{-1}(x)) \subseteq Y \setminus W^{-1}(x)$ for all $x \in X$. Let $y_1, \dots, y_n \in Y$ and $y \in \mathrm{co}\{y_1, \dots, y_n\}$. Let $x \in W(y)$. We have to show that x belongs to $\Gamma(\{y_1, \dots, y_n\})$. Suppose to the contrary that this is not true. Then, for each index i, $x \notin \Gamma(y_i)$ which yields $y_i \in Y \setminus \Gamma^{-1}(x)$. By the hypothesis $y \in Y \setminus W^{-1}(x)$, i.e. $x \notin W(y)$; a contradiction.

Further, under (i), for every $x \in X$ one has $W^{-1}(x) \subseteq \Gamma^{-1}(x)$, and therefore $\mathrm{co}(Y \setminus \Gamma^{-1}(x)) = Y \setminus \Gamma^{-1} \subseteq Y \setminus W^{-1}(x)$. By the first part, Γ is W-KKM. Under (ii) one has $\mathrm{co}(Y \setminus \Gamma^{-1}(x)) \subseteq \mathrm{co}(Y \setminus W^{-1}(x)) = Y \setminus W^{-1}(x)$ and yields the same conclusion. □

Note that the conclusion under (i) was already presented in [26]. Regarding the KKM property let us summarize some known sufficient conditions in the next proposition.

Proposition 4. *Let X and Y be convex sets in topological vector spaces and let $W : Y \rightrightarrows X$ be a set-valued map with nonempty values. Each of the following conditions is sufficient for W to have the KKM property:*

(i) *The closure of the image of every convex subset of Y under W is convex;*
(ii) *W has convex values and open fibers;*
(iii) *W is upper semicontinuous and has convex, compact values.*

Proof. Sufficient condition (i) is Theorem 2.2 of [23]. The other conditions are found in [21]. □

The interested readers are referred to [21–23] for more details on the KKM property.
Now we are able to establish sufficient conditions for existence of solutions to (VI).

Theorem 5. *Assume that X is a topological space, Y is a nonempty convex subset of a topological vector space, and that the following conditions hold:*

(i) *$Fix(W \circ Q_1)$ is a compact set;*
(ii) *Q_2 has nonempty values and open fibers, and $X \setminus Q_2^{-1}(y)$ is compact for at least one $y \in Y$;*
(iii) *$\mathrm{co}(Q_2(x)) \subseteq Q_1(x)$, for each $x \in X$;*
(iv) *U^{-1} is W-KKM(a) and its values are closed in X;*
(v) *W has the KKM property.*

Then (VI) *has solutions.*

Proof. Consider the map $P : Y \rightrightarrows X$ defined by (1). We show that P is a W-KKM(a) map. Let $\{y_1, \dots, y_n\}$ be a finite subset of Y and $x \in W(co\{y_1, \dots, y_n\})$. If $x \in Fix(W \circ Q_1)$, then since U^{-1} is W-KKM(a), one has

$$x \in Fix\,(W \circ Q_1) \cap \left(\bigcup_{i=1}^{n} U^{-1}(y_i)\right) = \bigcup_{i=1}^{n}\big(Fix(W \circ Q_1) \cap U^{-1}(y_i)\big) \subseteq \bigcup_{i=1}^{n} P(y_i).$$

If $x \in \big(X \setminus Fix(W \circ Q_1)\big) \cap Q_2^{-1}(y_i)$, for all $i \in \{1, \dots, n\}$, then $y_i \in Q_2(x)$ and by (iii), $co\{y_1, \dots, y_n\} \subseteq \mathrm{co}(Q_2(x)) \subseteq Q_1(x)$. Thus, $x \in W(co\{y_1, \dots, y_n\}) \subseteq W(Q_1(x))$; a contradiction. Hence P is a W-KKM(a) map. In view of (v), the family $\{P(y) : y \in Y\}$ has the finite intersection property. Since P has closed values and $P(y)$ is compact for at least one $y \in Y$, by (v), there exists $\bar{x} \in \bigcap_{y \in Y} P(y)$. If $\bar{x} \notin Fix(W \circ Q_1)$ it follows that $\bar{x} \in X \setminus Q_2^{-1}(y)$ for all $y \in Y$, which implies the contradiction $Q_2(\bar{x}) = \emptyset$ (see (ii)). Hence $\bar{x} \in Fix(W \circ Q_1)$. For each $y \in Q_2(\bar{x})$, i.e. $\bar{x} \notin X \setminus Q_2^{-}(y)$, since $\bar{x} \in P(y)$, we have $\bar{x} \in U^{-1}(y)$, that is $y \in U(\bar{x})$. Thus $Q_2(\bar{x}) \subseteq U(\bar{x})$. $\square$

Remark 1. The compactness of the set $Fix(W \circ Q_1)$ (condition (i) of the above theorem) is assured in each of the following situations:

(i) X is compact, Q_1 is upper semicontinuous with compact values and W is closed;
(ii) Y is compact, one of the maps W and Q_1^{-1} is closed and the other is upper semicontinuous with compact values.

Proof. (i) Let $\{x_t\}$ be a net in $Fix(W \circ Q_1)$ converging to a point x. Then, there exists a net $\{y_t\}$ in Y such that $y_t \in Q_1(x_t)$ and $x_t \in W(y_t)$, for all t. Since Q_1 is upper semicontinuous with compact values, there there exist $y \in Q_1(x)$ and a subnet $\{y_{t_\alpha}\}$ of $\{y_t\}$ converging to y. W is closed, and so $x \in W(y) \subseteq W(Q_1(x))$. Thus, $Fix(W \circ Q_1)$ is a closed subset of the compact X, hence it is compact too.

(ii) It is easy to see that the fixed point set of the map $W \circ Q_1$ coincides with the range of the map $W \cap Q_1^{-1}$. Under the given conditions the map $W \cap Q_1^{-1}$ is upper semicontinuous with compact values (see [25, p.567]). Since Y is compact, $(W \cap Q_1^{-1})(Y)$ is a compact set. $\square$

When Y is not a convex set, using the concept of W-KKM(b) maps, we may also establish existence of solutions to (VI).

Theorem 6. *Assume that X is a nonempty convex subset of a topological vector space, Y is a topological space, and that the following conditions hold:*

(i) *$Fix(W \circ Q_1)$ is compact;*
(ii) *Q_2 has nonempty values and open fibers, and $X \setminus Q_2^{-1}(y)$ is compact for at least one $y \in Y$;*
(iii) *$\mathrm{co}(W \circ Q_2(x)) \subseteq WQ_1(x)$ for every $x \in X$;*
(iv) *U^{-1} is W-KKM(b) and has closed values.*

Then (VI) *has solutions.*

Proof. We wish to show first that the map P defined by (1) is W-KKM(b). Let $\{y_1, \dots, y_n\}$ be a finite set in Y. By (iv) there are $x_i \in W(y_i)$ such that for each subset of indices $I \subseteq \{1, \dots, n\}$,

$$co\{x_i : i \in I\} \subseteq \bigcup_{i \in I} U^{-}(y_i). \tag{2}$$

Let x be a point from the convex hull of $\{x_i : i \in I\}$. We prove that (2) implies

$$x \in \bigcup_{i \in I} P(y_i). \tag{3}$$

By (1) and (2) it follows that (3) holds when $x \in Fix(W \circ Q_1)$. If $x \in \big(X \setminus Fix(W \circ Q_1)\big) \cap Q_2^{-1}(y_i)$, for all $i \in \{1, \dots, n\}$, then $y_i \in Q_2(x)$ and by (iii), $x \in co\{x_i : i \in I\} \subseteq co(W \circ Q_2(x)) \subseteq WQ_1(x)$; a contradiction. Thus P is W-KKM(b) and KKM in the sense of Chang and Zhang as well. Consequently the family $\{P(y) : y \in Y\}$ has the finite intersection property. Moreover, it follows from the hypothesis that P has closed values and at least one compact value. Hence the family $\{P(y) : y \in Y\}$ has some point $\bar{x}$ in common. Using the argument of proof of Theorem 5 we conclude that $\bar{x}$ is a solution of (VI). $\square$

We close up this section by the remark that if X is a convex subset of a locally convex space, Theorems 5 and 6 remain true if the hypothesis on open fibers of Q_2 is replaced by its lower semi-continuity. The proof is based on the method of enlargement given in Corollary 4.1 of [1]. Moreover, conditions that assure the closedness of the values $P(y)$ can be weakened to the so-called intersectional closedness recently developed in [5]. To keep the presentation as clear as possible we skip these details from our consideration.

3. Existence of solutions to simultaneous variational relations

In this section we wish to apply the existence conditions of solutions of variational inclusion problems to the model (I). To this end we need some concepts of convexity and closedness for variational relations.

Let r be a relation linking elements $x \in X, y \in Y$ and $z \in Z$. When X, Y and Z are convex sets from vector spaces, the relation r is said to be convex if whenever $r(x_i, y_i, z_i)$ holds for $x_i \in X, y_i \in Y$ and $z_i \in Z, i = 1, 2$, the relation $r(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2)$ is satisfied for all $\lambda \in [0, 1]$. In other words, r is convex if the set determining it is convex in the product space $X \times Y \times Z$.

When X, Y and Z are topological spaces, r is said to be closed if the set determining it is closed in the product space $X \times Y \times Z$; and it is said to be closed in the variables x, z if for every $y \in Y$ fixed, $r(x, y, z)$ holds whenever $r(x_t, y, z_t)$ holds for all t with (x_t, z_t) converging to (x, z). The complement of r is denoted by r^c, that is $r^c(x, y, z)$ holds if and only if $r(x, y, z)$ does not hold.

The concept of KKM maps with respect to a set-valued map can be defined for relations as follows.

Definition 2. Let X be a nonempty set, Y and Z convex sets in vector spaces. Let r_1 and r_2 be two relations linking elements $x \in X, y \in Y$ and $z \in Z$. We say that r_2 is r_1-KKM in the variables y, z if for every $x \in X$ and each nonempty finite subset $A = \{(y_1, z_1), \dots, (y_n, z_n)\}$ of $Y \times Z$ the following implication holds:
$(y, z) \in coA$ and $r_1(x, y, z)$ holds $\Longrightarrow r_2(x, y_i, z_i)$ holds for some $(y_i, z_i) \in A$.

It can be seen that r_2 is r_1-KKM if and only if the map F_2 is F_1-KKM(a), where $F_1, F_2 : Y \times Z \rightrightarrows X$ are defined by

$$F_1(y, z) = \{x \in X : \ r_1(x, y, z) \text{ holds}\}$$
$$F_2(y, z) = \{x \in X : \ r_2(x, y, z) \text{ holds}\}.$$

A generalization of convexity will also be needed in our study.

Definition 3. Let Y and Z be convex sets in two vector spaces and $P_1, P_2 : Y \rightrightarrows Z$. We say that P_1 is P_2-convex if for any finite subset $\{y_1, \dots y_n\}$ of Y and $\lambda_1, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ one has

$$\sum_{i=1}^n \lambda_i P_2(y_i) \subseteq P_1\left(\sum_{i=1}^n \lambda_i y_i\right).$$

When the inclusion "$\subseteq$" is replaced by the containment "$\supseteq$", the set-valued mapping P_1 is said to be P_2-concave.

It is clear that when $P_1 = P_2$ the two concepts above reduce to the well-known convex and concave set-valued maps. Moreover, a necessary condition for P_1 to be P_2-convex is evidently that

$$P_2(y) \subseteq P_1(y) \quad \text{for all } y \in Y.$$

Here is a sufficient condition. Assume that there is a convex set-valued map $P : Y \rightrightarrows Z$ such that

$$P_2(y) \subseteq P(y) \subseteq P_1(y) \quad \text{for all } y \in Y.$$

Then P_1 is P_2-convex. Definition 3 itself is expressively for set-valued maps. It is not interesting for single-valued maps because inclusion becomes equality. But for them associated set-valued maps by epigraph do deserve attention. For instance when $Z = \mathbb{R}$ and f is a real function on Y, its associated set-valued map is defined by $F(y) = f(y) + \mathbb{R}_+$, whose graph coincides with the epigraph of f. Then the map F is F-convex in the sense of Definition 3 if and only if f is convex in the usual sense.

To formulate and prove the main results of this section we define the set-valued maps $U : X \rightrightarrows Y$ and $W : Y \rightrightarrows X$ by

$$U(y) = \{y \in Y : r_2(x, y, z) \text{ holds for all } z \in P_2(x, y)\}$$
$$W(y) = \{x \in X : r_1(x, y, z) \text{ holds for all } z \in P_1(x, y)\}.$$

In the lemma below we present sufficient conditions for the map W to have the KKM property.

Lemma 7. *Assume that X is a convex set and that for every $y \in Y$ there exists some $x \in X$ such that $r_1(x, y, z)$ holds for all $z \in P_1(x, y)$. Then each of the following conditions is sufficient for the map W to have the KKM property.*

(i) *P_1 is concave, r_1 is convex;*
(ii) *P_1 is concave in x and upper semi-continuous with compact values, r_1 is convex in x, z and open in y, z, in the sense that when $r_1(x, y, z)$ holds, there is a neighborhood V of (y, z) in $Y \times Z$ such that $r_1(x, y', z')$ holds for all $(y', z') \in V$;*
(iii) *X is assumed compact, P_1 is concave in x and lower semicontinuous, r_1 is convex in x, z and closed.*

Proof. By hypothesis, W has nonempty values. We show that under (i) the set $W(C)$ is convex whenever $C \subseteq Y$ is convex. Indeed, let $x_1, x_2 \in W(C)$ and $x = \lambda x_1 + (1-\lambda)x_2$ for some $\lambda \in [0, 1]$. Let $y_1, y_2 \in C$ such that $x_i \in W(y_i)$, $i = 1, 2$, which means that $r_1(x_i, y_i, z_i)$ holds for all $z_i \in P_1(x_i, y_i)$. Set $y = \lambda y_1 + (1-\lambda)y_2$ and consider $P_1(x, y)$. Since P_1 is concave, for every $z \in P_1(x, y)$ there are $z_i \in P_1(x_i, y_i)$, $i = 1, 2$ such that $z = \lambda z_1 + (1-\lambda)z_2$. Furthermore, as $r_1(x_i, y_i, z_i)$, $i = 1, 2$ are true, by the convexity of r_1, $r_1(x, y, z)$ holds too. Thus, $x \in W(y) \subseteq W(C)$ as requested. By Proposition 4(i), W has the KKM property.

Under (ii) the map W has convex values and open fibers. Under (iii) the map W has convex values and is closed. Since X is compact, it is upper semicontinuous. In view of Proposition 4, under these conditions the map W has the KKM property. $\square$

We are now in position to prove the main result of the present paper.

Theorem 8. *Assume that X is a topological space, Y, Z are convex sets in two topological vector spaces and that the data of the problem* (I) *satisfy the following conditions:*

(i) *P_1 is P_2-convex in the variable y and P_2 is lower semicontinuous in x;*
(ii) *Q_2 has nonempty values and open fibers, and $X \setminus Q_2^{-1}(y)$ is compact for some $y \in Y$;*
(iii) *$\mathrm{co}(Q_2(x)) \subseteq Q_1(x)$ for all $x \in X$;*
(iv) *$r_2(x, y, z)$ is r_1-KKM in the variables y, z and closed in the variables x, z;*
(v) *W has the KKM property, which can be guaranteed by any of the conditions of Lemma 7 and the range of the map $W \cap Q_1^{-1}$ is compact.*

Then the problem (I) *has solutions.*

Proof. We wish to apply Theorem 5 to obtain a solution of (VI) for W and U defined in this section. The first hypothesis of this theorem is satisfied because the range of the map $W \cap Q_1^{-1}$ is exactly the fixed point set of the map $W \circ Q_1$. To see that U^{-1} is W-KKM(a), let $y_1, \dots, y_n \in Y$ and let $x \in W(\mathrm{co}\{y_1, \dots, y_n\})$, say $x \in W(y)$ where $y = \sum_{i=1}^n \lambda_i y_i$ with $t_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$. Suppose to the contrary that $x \notin \bigcup_{i=1}^n U^{-1}(y_i)$. Then for each $i \in \{1, \dots, n\}$ there exists $z_i \in P_2(x, y_i)$ such that $r_2(x, y_i, z_i)$ does not hold. Set $z = \sum_{i=1}^n \lambda_i z_i$. By (iv), $r_1(x, y, z)$ does not hold. Moreover, since P_1 is P_2-convex in y, $z \in \sum_{i=1}^n \lambda_i P_2(x, y_i) \subseteq P_1(x, y)$. It follows that $x \notin W(y)$, a contradiction.

To see that U^{-1} has closed values, for $y \in Y$, let $\{x_t\}$ be a net in $U^{-1}(y)$ converging to some $x \in X$. For each $z \in P_2(x, y)$, since $P(\cdot, y)$ is lower semicontinuous there is a subnet $\{x_{t_\alpha}\}$ of $\{x_t\}$ and a net $\{z_{t_\alpha}\}$ converging to z such that $z_{t_\alpha} \in P_2(x_{t_\alpha}, y)$. Since $r_2(x_{t_\alpha}, y, z_{t_\alpha})$ holds for all t_α, in view of (iv), $r_2(x, y, z)$ is satisfied, hence $x \in U^{-1}(y)$. Thus $U^{-1}(y)$ is closed in X.

Now, by Theorem 5 there is a solution $\bar{x}$ of (VI). By the definition of W, inclusion $\bar{x} \in W(Q_1(\bar{x}))$ shows that $r_1(\bar{x}, \bar{y}, z)$ holds for some $\bar{y} \in Q_1(\bar{x})$ and for all $z \in P_1(\bar{x}, \bar{y})$, while the inclusion $Q_2(\bar{x}) \subseteq U(\bar{x})$ shows that $r_1(\bar{x}, y, z)$ holds for all $y \in Q_2(\bar{x})$ and $z \in P_2(\bar{x}, y)$. The proof is complete. $\square$

When X is a compact space the condition that Q_2 has open fibers implies that $X \setminus Q_2^{-1}(y)$ is compact. In the same case, the range of $W \cap Q_1^{-1}$ is compact whenever P_1 is lower semicontinuous, Q_1 is upper semicontinuous with compact values and r_1 is closed.

Theorem 6 can also be applied to deduce existence of solutions to (I).

Theorem 9. *Assume that X, Y and Z are convex sets of topological vector spaces, and that the data of the problem* (I) *satisfy the following conditions:*

(i) *P_1 is P_2-convex in y and concave, P_2 is lower semicontinuous in x;*
(ii) *Q_2 has nonempty values and open fibers, and $X \setminus Q_2^{-1}(y)$ is compact for some $y \in Y$;*
(iii) *$\mathrm{co}Q_2(x) \subseteq Q_1(x)$ for all $x \in X$;*
(iv) *r_1 is convex, r_2 is r_1-KKM in the variables y, z and closed in the variables x, z;*
(v) *the range of the map $W \cap Q_1^{-1}$ is compact.*

Then the problem (I) *has solutions.*

Proof. We wish apply Theorem 6 to obtain a solution to (VI). Condition (i) of Theorem 6 and the fact that U^{-1} has closed values are proven by the same arguments as in the proof of Theorem 8. To see that U^{-1} is W-KKM(b) we suppose to the contrary that there are some $y_1, \dots, y_n \in Y$ such that for all $x_i \in W(y_i)$ one can find an index set $I \subseteq \{1, \dots, n\}$ and $x_I \in \mathrm{co}\{x_i : i \in I\}$, say $x_I = \sum_{i\in I} \lambda_i x_i$ ($\lambda_i \geq 0$ and $\sum_{i\in I} \lambda_i = 1$), with $x_I \notin U^{-1}(y_i)$ for all $i \in I$. Since $x_I \notin U^{-1}(y_i)$, for some $z_i \in P_2(x_I, y_i)$, $r_2(x_I, y_i, z_i)$ does not hold. Set $y_I = \sum_{i\in I} \lambda_i y_i$ and $z_I = \sum_{i\in I} \lambda_i z_i$. Then, since r_2 is r_1-KKM, $r_1(x_I, y_I, z_I)$ does not hold. Notice that z_I belongs to $P_1(x_I, y_I)$ because P_1 is P_2-convex in y. On the other hand, as P_1 is concave, z_I belongs to $\sum_{i\in I} \lambda_i P_1(x_i, y_i)$, say $z_I = \sum_{i\in I} \lambda_i w_i$ for some $w_i \in P_1(x_i, y_i)$, $i \in I$. Moreover, $r_1(x_i, y_i, w_i)$ being true one deduces from (iv) that $r_1(x_I, y_I, z_I)$ holds, which is a contradiction.

To see condition (iii) of Theorem 6 let $x \in X$ and $x' = \sum_{i=1}^n \lambda_i x_i$ ($\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$) with $x_i \in W(y_i)$ and $y_i \in Q_2(x)$, $i = 1, \dots, n$. Set $y = \sum_{i=1}^n \lambda_i y_i$. By (iii), y belongs to $Q_1(x)$. For every $z \in P_1(x', y)$, by the concavity of P_1, there are $z_i \in P_1(x_i, y_i)$ such that $z = \sum_{i=1}^n \lambda_i z_i$. Then for each $i \in \{1, \dots, n\}$ $r_1(x_i, y_i, z_i)$ holds which implies that $r_1(x', y, z)$ holds too. By this, x' belongs to $W \circ Q_1(x)$ as requested. Now we apply Theorem 6 to obtain a solution of (VI), which is also a solution of (I) by the same argument of the proof of Theorem 8. $\square$

4. Particular cases

Throughout this section X, Y, Z and V are nonempty convex sets in topological vector spaces and $Q_1, Q_2 : X \rightrightarrows Y$, $P_1, P_2 : X \times Y \rightrightarrows Z$, $F_1, F_2, C_1, C_2 : X \times Y \times Z \rightrightarrows V$, are set-valued mappings. As applications of the results of Section 3, we derive existence theorems of solutions for the problems considered in Example 2 in the first section of the paper. Theorem 8 is solicited in our proofs, but a similar application can be done with Theorem 9 as well. Example 1 is a particular case of Example 2, and so the results established in this section are applicable to it too. It is clear that the concept of convexity of Definition 3 can be extended to maps with several variables. For instance F_1 is F_2-convex in the variables y, z if for any nonempty finite set $\{(y_1, z_1), \ldots, (y_n, z_n)\} \subseteq Y \times Z$ and each convex combination $(y, z) = \sum_{i=1}^n \lambda_i(y_i, z_i)$ ($\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$)

$$\sum_{i=1}^n \lambda_i F_2(x, y_i, z_i) \subseteq F_1\left(x, \sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i z_i\right) \quad \text{for all } x \in X.$$

When the inclusion "$\subseteq$" is replaced by the containment "$\supseteq$", the set-valued mapping F_1 is said to be F_2-concave in the variables y, z.

Corollary 10. *Suppose that:*

(i) $\{x \in X : \exists y \in Q_1(x)$ *such that* $F_1(x, y, z) \subseteq C_1(x, y, z)$ *for all* $z \in P_1(x, y)\}$ *is compact;*
(ii) Q_2 *has nonempty values and open fibers and* $X \setminus Q_2^{-1}(y)$ *is compact for at least one* $y \in Y$*;*
(iii) $\mathrm{co}Q_2(x) \subseteq Q_1(x)$ *for all* $x \in X$*;*
(iv) *for each* $y \in Y$ *there exists* $x \in X$ *such that* $F_1(x, y, z) \subseteq C_1(x, y, z)$ *for all* $z \in P_1(x, y)$*;*
(v) P_1 *is concave and* P_2*-convex in* y*;*
(vi) F_1 *is concave and* F_2*-convex in the variables* y, z*;*
(vii) C_1 *is convex and* C_1^c *is* C_2^c*-convex in the variables* y, z (C_i^c *being the map from* $X \times Y \times Z$ *into* V *defined by* $C_i^c(x, y, z) = V \setminus C_i(x, y, z)$)*;*
(viii) *for each* $y \in Y$, $P_2(\cdot, y)$ *and* $F_2(\cdot, y, \cdot)$ *are lower semicontinuous and* $C_2(\cdot, y, \cdot)$ *is closed.*

Then there exists $\bar{x} \in X$ *satisfying*

$$\begin{cases} (1) \quad F_1(\bar{x}, \bar{y}, \bar{z}) \subseteq C_1(\bar{x}, \bar{y}, \bar{z}) \quad \text{for some } \bar{y} \in Q_1(\bar{x}) \text{ and for all } \bar{z} \in P_1(\bar{x}, \bar{y}) \\ (2) \quad F_2(\bar{x}, y, z) \subseteq C_2(\bar{x}, y, z) \quad \text{for all } y \in Q_2(\bar{x}) \text{ and } z \in P_2(\bar{x}, y). \end{cases}$$

Proof. Apply Theorem 8 when the relations r_1 and r_2 are defined as follows:

$$r_1(x, y, z) \quad \text{holds iff } F_1(x, y, z) \subseteq C_1(x, y, z), \text{ and}$$
$$r_2(x, y, z) \quad \text{holds iff } F_2(x, y, z) \subseteq C_2(x, y, z).$$

We show that r_2 is r_1-KKM in the variables y, z. If not, there exist $x \in X, A = \{(y_1, z_1), \ldots, (y_n, z_n)\} \subseteq Y \times Z$ and a convex combination $(y, z) = \sum_{i=1}^n \lambda_i(y_i, z_i)$ ($\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$) such that $r_1(x, y, z)$ holds and $r_2(x, y_i, z_i)$ does not hold for all $(y_i, z_i) \in A$. This means that $F_1(x, y, z) \subseteq C_1(x, y, z)$ and for each $i \in \{1, \ldots, n\}$ there exists $v_i \in F_2(x, y_i, z_i) \cap C_2^c(x, y_i, z_i)$. By (vi) and (vii) we infer that $\sum_{i=1}^n \lambda_i v_i \in \left(\sum_{i=1}^n \lambda_i F_2(x, y_i, z_i)\right) \cap \left(\sum_{i=1}^n \lambda_i C_2^c(x, y_i, z_i)\right) \subseteq F_1(x, y, z) \cap C_1^c(x, y, z)$; a contradiction.

We prove that the relation r_2 is closed in the variables x, z. Let $y \in Y$ and $\{(x_t, z_t)\}$ a net in $X \times Z$ converging to (x, z), such that $F_2(x_t, y, z_t) \subseteq C_2(x_t, y, z_t)$ for all t. If $v \in F_2(x, y, z)$, since $F_2(\cdot, y, \cdot)$ is l.s.c., there exists a subnet $\{(x_{t_\alpha}, z_{t_\alpha})\}$ of $\{(x_t, z_t)\}$ and a net $\{v_{t_\alpha}\}$ converging to v such that $v_{t_\alpha} \in F_2(x_{t_\alpha}, y, z_{t_\alpha})$. Then $v_{t_\alpha} \in C_2(x_{t_\alpha}, y, z_{t_\alpha})$ and, since $C_2(\cdot, y, \cdot)$ is closed, $v \in C(x, y, z)$. Thus $F_2(x, y, z) \subseteq C_2(x, y, z)$, hence r_2 is closed in the variables x, z.

If $r_1(x_i, y_i, z_i)$ holds, that is $F_1(x_i y_i, z_i) \subseteq C_1(x_i y_i, z_i)$ for $i = 1, 2$, since F_1 is concave and C_1 is convex, for any $\lambda \in [0, 1]$ we have

$$\begin{aligned} &F_1(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2) \\ &\quad \subseteq \lambda F_1(x_1, y_1, z_1) + (1-\lambda)F_1(x_2, y_2, z_2) \subseteq \lambda C_1(x_1, y_1, z_1) + (1-\lambda)C_1(x_2, y_2, z_2) \\ &\quad \subseteq C_1(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2). \end{aligned}$$

Thus, $r_1(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2)$ is satisfied. Hence r_1 is convex and by Lemma 7(i), the set-valued mapping W in Theorem 8 has the KKM property. Therefore, all the requirements of Theorem 8 are fulfilled and the desired conclusion follows from this theorem. $\square$

Example 11. Let $X = Y = Z = [0, 3)$, $V = \mathbb{R}$,

$$Q_1(x) = Q_2(x) = \begin{cases} [0, x+2) & \text{if } x \in [0, 1), \\ (x-1, 3) & \text{if } x \in [1, 3), \end{cases}$$

$$F_1(x, y, z) = \left(-\infty, \left(2x + \frac{y-z}{3}\right)^2 + 1\right], \qquad F_2(x, y, z) = \left(-\infty, 4x + \frac{2(y-z)}{3}\right],$$

$$C_1(x, y, z) = \left(-\infty, 3 - \left(2x + \frac{y-z}{3}\right)^2\right], \qquad C_2(x, y, z) = (-\infty, 3],$$

$$P_1(x, y) = [0, y], \qquad P_2(x, y) = [0, min\{x, y\}].$$

Simple calculations show that

$$Q_2^{-1}(y) = \begin{cases} (0, y+1) & \text{if } y \in [0, 2) \\ (y-2, 3) & \text{if } y \in [2, 3), \end{cases}$$

hence $Q_2^{-1}(y)$ is open (in X), for each $y \in [0, 3)$. Notice also that $\{x \in [0, 3) : \exists y \in Q_1(x)$ such that $F_1(x, y, z) \subseteq C_1(x, y, z)$ for all $z \in P_1(x, y)\} = [0, \frac{1}{2}]$.

One can readily see that all requirements of Corollary 10 are satisfied. By direct checking one can see that any $\bar{x} \in [0, \frac{5}{14}]$ satisfy the conclusion of Corollary 10.

Corollary 12. *Suppose that:*

(i) $\{x \in X : \exists y \in Q_1(x)$ *such that* $F_1(x, y, z) \cap C_1(x, y, z) \neq \emptyset$ *for all* $z \in P_1(x, y)\}$ *is compact;*
(ii) Q_2 *has nonempty values and open fibers, and* $X \setminus Q_2^-(y)$ *is compact for at least one* $y \in Y$*;*
(iii) $\mathrm{co}Q_2(x) \subseteq Q_1(x)$ *for all* $x \in X$*;*
(iv) *for each* $y \in Y$ *there exists* $x \in X$ *such that* $F_1(x, y, z) \cap C_1(x, y, z) \neq \emptyset$ *for all* $z \in P_1(x, y)$*;*
(v) P_1 *is concave and* P_2*-convex in* y*;*
(vi) F_1 *is convex and* F_2*-concave in the variables* y, z*;*
(vii) C_1 *is convex and* C_1^c *is* C_2^c*-convex in the variables* y, z*;*
(viii) *for each* $y \in Y$, $P_2(\cdot, y)$ *is lower semicontinuous and one of the set-valued mappings* $F_2(\cdot, y, \cdot)$ *and* $C_2(\cdot, y, \cdot)$ *is upper semicontinuous with compact values and the other is closed.*

Then there exists $\bar{x} \in X$ *satisfying*

$$\begin{cases} (1) \quad F_1(\bar{x}, \bar{y}, \bar{z}) \cap C_1(\bar{x}, \bar{y}, \bar{z}) \neq \emptyset \quad \text{for some } \bar{y} \in Q_1(\bar{x}) \text{ and for all } z \in P_1(\bar{x}, \bar{y}); \text{ and} \\ (2) \quad F_2(\bar{x}, y, z) \cap C_1(\bar{x}, y, z) \neq \emptyset \quad \text{for all } y \in Q_2(\bar{x}) \text{ and } z \in P_2(\bar{x}, y). \end{cases}$$

Proof. Take the variational relations r_1 and r_2 defined as follows:

$$r_1(x, y, z) \quad \text{holds iff } F_1(x, y, z) \cap C_1(x, y, z) \neq \emptyset \quad \text{and}$$
$$r_2(x, y, z) \quad \text{holds iff } F_2(x, y, z) \cap C_2(x, y, z) \neq \emptyset.$$

We claim that r_2 is r_1-*KKM* in the variables y, z. If not, there exist $x \in X$, $A = \{(y_1, z_1), \ldots, (y_n, z_n)\} \subseteq Y \times Z$ and a convex combination $(y, z) = \sum_{i=1}^n \lambda_i(y_i, z_i)$ ($\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$) such that $r_1(x, y, z)$ holds and $r_2(x, y_i, z_i)$ does not hold for all $(y_i, z_i) \in A$. This means that $F_1(x, y, z) \cap C_1(x, y, z) \neq \emptyset$ and $F_2(x, y_i, z_i) \subseteq C_2^c(x, y_i, z_i)$, $i = 1, \ldots, n$. By (vi) and (vii) we have

$$F_1(x, y, z) \subseteq \sum_{i=1}^n \lambda_i F_2(x, y_i, z_i) \subseteq \sum_{i=1}^n \lambda_i C_2^c(x, y_i, z_i) \subseteq C_1^c(x, y, z),$$

which contradicts $F_1(x, y, z) \cap C_1(x, y, z) \neq \emptyset$.

We show that the relation r_2 is closed in the variables x, z. Let $y \in Y$ and $\{(x_t, z_t)\}$ a net in $X \times Z$ converging to (x, z), such that for each t there exists $v_t \in F_2(x_t, y, z_t) \cap C_2(x_t, y, z_t)$. Suppose that $F_2(\cdot, y, \cdot)$ is upper semcontinuous with compact values and $C_2(\cdot, y, \cdot)$ is closed. Then there exists $v \in F_2(x, y, z)$ and a subnet $\{v_{t_\alpha}\}$ of $\{v_t\}$ converging to v. Since $v_{t_\alpha} \in C_2(x_{t_\alpha}, y, z_{t_\alpha})$ and $C_2(\cdot, y, \cdot)$ is closed, it follows that $v \in C_2(x, y, z)$. Thus $v \in F_2(x, y, z) \cap C_2(x, y, z)$, hence r is closed in the variables x, z.

If $r_1(x_i, y_i, z_i)$ holds, that is there exists $v_i \in F_1(x_i, y_i, z_i) \cap C_1(x_i, y_i, z_i)$ for $i = 1, 2$, then for any $\lambda \in [0, 1]$ we have

$$\begin{aligned} \lambda v_1 + (1-\lambda)v_2 &\in \big(\lambda F_1(x_1, y_1, z_1) + (1-\lambda)F_1(x_2, y_2, z_2)\big) \cap \big(\lambda C_1(x_1, y_1, z_1) + (1-\lambda)C_1(x_2, y_2, z_2)\big) \\ &\subseteq F_1(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2) \cap C_1(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 \\ &\quad + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2). \end{aligned}$$

Hence r_1 is convex and, according to Lemma 7(i), the set-valued mapping W in Theorem 8 has the *KKM* property. It remains to apply Theorem 8 to obtain the conclusion. □

Corollary 13. *Suppose that:*

(i) $\{x \in X : \exists y \in Q_1(x)$ *such that* $F_1(x, y, z) \subseteq C_1(x, y, z)$ *for all* $z \in P_1(x, y)\}$ *is compact;*
(ii) Q_2 *has nonempty values and open fibers, and* $X \setminus Q_2^{-1}(y)$ *is compact for at least one* $y \in Y$*;*
(iii) $\text{co}Q_2(x) \subseteq Q_1(x)$ *for all* $x \in X$*;*
(iv) *for each* $y \in Y$ *there exists* $x \in X$ *such that* $F_1(x, y, z) \subseteq C_1(x, y, z)$ *for all* $z \in P_1(x, y)$*;*
(v) P_1 *is concave and* P_2*-convex in* y*;*
(vi) F_1 *is concave,* F_2^c *is convex and* $F_1(x, y, z) \cap F_2(x, y, z) \neq \emptyset$ *for all* $(x, y, z) \in X \times Y \times Z$*;*
(vii) C_1 *is* C_2*-concave in the variables* y, z*;*
(viii) *for each* $y \in Y$*,* $P_2(\cdot, y)$ *is lower semicontinuous and one of the set-valued mappings* $F_2(\cdot, y, \cdot)$ *and* $C_2(\cdot, y, \cdot)$ *is upper semicontinuous with compact values and the other is closed.*

Then there exists $\bar{x} \in X$ *satisfying*

$$\begin{cases} (1) \quad F_1(\bar{x}, \bar{y}, \bar{z}) \subseteq C_1(\bar{x}, \bar{y}, \bar{z}) \quad \text{for some } \bar{y} \in Q_1(\bar{x}) \text{ and for all } \bar{z} \in P_1(\bar{x}, \bar{y}); \text{ and} \\ (2) \quad F_2(\bar{x}, y, z) \cap C_2(\bar{x}, y, z) \neq \emptyset \quad \text{for all } y \in Q_2(\bar{x}), \text{ and } z \in P_2(\bar{x}, y). \end{cases}$$

Proof. Apply Theorem 8 when the relations r_1 and r_2 are given by

$$r_1(x, y, z) \quad \text{holds iff } F_1(x, y, z) \subseteq C_1(x, y, z) \quad \text{and}$$
$$r_2(x, y, z) \quad \text{holds iff } F_2(x, y, z) \cap C_2(x, y, z) \neq \emptyset.$$

We claim that r_2 is r_1-KKM(a) in the variables y, z. If not, there exist $x \in X$, $A = \{(y_1, z_1), \ldots, (y_n, z_n)\} \subseteq Y \times Z$ and a convex combination $(y, z) = \sum_{i=1}^{n} \lambda_i (y_i, z_i)$ ($\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$) such that $r_1(x, y, z)$ holds and $r_2(x, y_i, z_i)$ does not hold for all $(y_i, z_i) \in A$. This means that $F_1(x, y, z) \subseteq C_1(x, y, z)$ and $C_2(x, y_i, z_i) \subseteq F_2^c(x, y_i, z_i)$, $i = 1, \ldots, n$. Taking into account (vi) and (vii) we obtain:

$$F_1(x, y, z) \subseteq C_1(x, y, z) \subseteq \sum_{i=1}^{n} \lambda_i C_2(x, y_i, z_i) \subseteq \sum_{i=1}^{n} \lambda_i F_2^c(x, y_i, z_i) \subseteq F_2^c(x, y, z).$$

It follows that $F_1(x, y, z) \cap F_2(x, y, z) = \emptyset$, which contradicts the last part of (vi). From the proof of Corollary 12 it follows that the relation r is closed in the variables x, z and from the proof of Corollary 10 we get that the set-valued mapping W in Theorem 8 has the KKM property. From Theorem 8 we get the conclusion. $\square$

Condition (i) in each of Corollaries Corollaries 10, 12 and 13 becomes superfluous when the convex set X is compact, the maps P_1, Q_1 satisfy some conditions of continuity and the relation r_1 is closed. In its turn, the closedness of r_1 is assured when F_1 and C_1 are closed.

5. A weak mixed variational relation problem

Another kind of mixed variational relation problem which is weaker than (I) is also worthwhile to be studied. It is formulated below.

(II) Find $\bar{x} \in X$ such that
  (i) $r_1(\bar{x}, \bar{y}, \bar{z})$ holds for some $\bar{y} \in Q_1(\bar{x})$ and some $\bar{z} \in P_1(\bar{x}, \bar{y})$
  (ii) $r_2(\bar{x}, y, z)$ holds for all $y \in Q_2(\bar{x})$, and some $z \in P_2(\bar{x}, y)$.

To establish existence conditions for this problem we apply the same technique of Problem (I). Namely we shall apply Theorems 5 and 6 to the modified maps $U : X \rightrightarrows Y$ and $W : Y \rightrightarrows X$ defined by

$$U(y) = \{y \in Y : r_2(x, y, z) \quad \text{holds for some } z \in P_2(x, y)\}$$
$$W(y) = \{x \in X : r_1(x, y, z) \quad \text{holds for some } z \in P_1(x, y)\}.$$

Theorem 14. *Assume that* X *is a topological space,* Y, Z *are convex sets and that the data of the problem* (II) *satisfy the following conditions:*

(i) P_1 *is* P_2*-concave in the variable* y *and* P_2 *is compact-valued, upper semicontinuous in the variable* x*;*
(ii) Q_2 *has nonempty values and open fibers, and* $X \setminus Q_2^{-1}(y)$ *compact for some* $y \in Y$*;*
(iii) $\text{co}Q_2(x) \subseteq Q_1(x)$ *for all* $x \in X$*;*
(iv) $r_2(x, y, z)$ *is* r_1*-KKM in the variables* y, z *and closed in the variables* x, z*;*
(v) W *has the KKM property and the range of the map* $W \cap Q_1^{-1}$ *is compact.*

Then the problem (II) *has solutions.*

Proof. The desired conclusion follows from Theorem 5 as soon as we prove that condition (iv) of that theorem is fulfilled. For $y \in Y$, let $\{x_t\}$ be a net in $U^{-1}(y)$ converging to $x \in X$. Then, for each t there exists $z_t \in P_2(x_t, y)$ such that $r_2(x_t, y, z_t)$

holds. Since $P_2(\cdot, y)$ is $u..c$ with compact values, there exists $z \in P_2(x, y)$ and a subnet $\{z_{t_\alpha}\}$ of $\{z_t\}$ such that $z_{t_\alpha} \longrightarrow z$. Since the relation r_2 is closed in x, z, it follows that $r_2(x, y, z)$ holds, hence $x \in U^{-1}(y)$. Thus $U^{-1}(y)$ is closed in X. To see that U^{-1} is W-KKM(a) let $\{y_1, \ldots, y_n\}$ be a finite subset of Y and $x \in W(co\{y_1, \ldots, y_n\})$. Then $x \in W(y)$, for some $y = \sum_{i=1}^n \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Suppose that $x \notin \bigcup_{i=1}^n U^{-1}(y_i)$. Then, there exists $z \in P_1(x, y)$ such that $r_1(x, y, z)$ holds and for each $i \in \{1, \ldots, n\}$, $r_2(x, y_i, z_i)$ does not hold for all $z_i \in P_2(x, y_i)$. In view of (i), $z \in P_1(x, \sum_{i=1}^n \lambda_i y_i) \subseteq \sum_{i=1}^n \lambda_i P_2(x, y_i)$, hence there exist $z_i \in P_2(x, y_i)$, $i = 1, \ldots, n$, such that $z = \sum_{i=1}^n \lambda_i z_i$. Since $r_2(x, y, z)$ is r_1-KKM in the variables y, z, it follows that $r_1(x, y, z)$ does not hold, a contradiction. $\square$

We deduce next an existence result for the solutions of problem (II) when the relations r_1 and r_2 are defined as in Example 1 in the first section of the paper.

Corollary 15. *Let $G_1, G_2 : X \times Y \rightrightarrows Z$. Suppose that:*

(i) *$\{x \in X : \exists y \in Q_1(x)$ such that $P_1(x, y) \cap G_1(x, y) \neq \emptyset\}$ is compact;*
(ii) *Q_2 has nonempty values and open fibers, and $X \setminus Q_2^-(y)$ is compact for at least one $y \in Y$;*
(iii) *$\mathrm{co}Q_2(x) \subseteq Q_1(x)$ for all $x \in X$;*
(iv) *for each $y \in Y$ there exists $x \in X$ such that $P_1(x, y) \cap G_1(x, y) \neq \emptyset$;*
(v) *P_1 is convex and P_2-concave in y;*
(vi) *G_1 is convex and G_1^c is G_2^c convex in y;*
(vii) *for each $y \in Y$, $P_2(\cdot, y)$ is u.s.c. with compact values and $G_2(\cdot, y)$ is closed.*

Then there exists $\bar{x} \in X$ satisfying

$$\begin{cases} (1) \quad P_1(\bar{x}, \bar{y}) \cap G_1(\bar{x}, \bar{y}) \neq \emptyset \quad \text{for some } \bar{y} \in Q_1(\bar{x}) \\ (2) \quad P_2(\bar{x}, y) \cap G_2(\bar{x}, y) \neq \emptyset \quad \text{for all } y \in Q_2(\bar{x}). \end{cases}$$

Proof. It is a direct application of Theorem 14 when r_1 and r_2 are defined as follows:

$$r_1(x, y, z) \quad \text{holds iff } z \in G_1(x, y), \text{ and}$$
$$r_2(x, y, z) \quad \text{holds iff } z \in G_2(x, y). \quad \square$$

Here is a numerical example to illustrate the above corollary.

Example 16. Let $X = Y = (0, 4]$, $Z = \mathbb{R}$,

$$Q_1(x) = Q_2(x) = \begin{cases} (x, x+2) & \text{if } x \in (0, 2], \\ [2, 4) & \text{if } x = 2, \\ [2, 4] & \text{if } x \in (2, 4), \\ (2, 4] & \text{if } x = 4, \end{cases}$$

$$P_1(x, y) = \left[e^{y-x}, +\infty\right), \qquad P_2(x, y) = [0, +\infty),$$

$$G_1(x, y) = (-\infty, y - x + 1], \qquad G_2(x, y) = \left(-\infty, y - x + \frac{4}{3}\right].$$

Direct calculation gives

$$Q_2^{-1}(y) = \begin{cases} (0, y) & \text{if } y \in (0, 2) \\ (0, 4) & \text{if } y = 2, \\ (y - 2, 4) & \text{if } y \in (2, 4), \\ (2, 4] & \text{if } y = 4, \end{cases}$$

hence $Q_2^-(y)$ is open (in X), for each $y \in (0, 4]$. Note that $P_1(x, y) \cap G_1(x, y) \neq \emptyset$ iff $x = y$ and, consequently, $\{x \in X : \exists y \in Q_1(x) \text{ such that } P_1(x, y) \cap G_1(x, y) \neq \emptyset\} = [2, 4]$.

One can readily verify that all requirements of Corollary 15 are satisfied. By direct checking one can see that any $\bar{x} \in [2, \frac{10}{3}]$ satisfy the conclusion of Corollary 15.

Corollaries 10, 12 and 13 can be extended to the weak version of the problem given in Example 2. The method is the same as for Example 1 and we leave it to the interested readers.

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