Appl. Math. Lett. Vol. 2, No. 2, pp. 171–174, 1989
0893-9659/89 \$3.00 + 0.00

Cracks of Unequal Length at the Edge of An Elliptic Hole in Out of Plane Shear

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Abstract. Integral transforms are used to find mode III stress intensity factors for two unequal length cracks at the edge of an elliptic hole in an infinite elastic solid.

1. INTRODUCTION

The problem we wish to discuss is that of finding the stress intensity factors for two cracks of unequal length at the edge of an elliptic hole in an infinite elastic solid which is subject to out of plane shear.

In Cartesian coordinates (x, y) the ellipse is given by the equation

$$\frac{x^2}{c^2} + \frac{y^2}{h^2} = 1 \tag{1.1}$$

and the cracks by the relations $-b_1 < x \leq -c$, $y = 0$ and $c \leq x \leq b_2$, $y = 0$ respectively. The cracks and the hole are assumed to be traction free while the solid is subject to a uniform out of plane shear load $\sigma_{yz} = T$. We solve the problem for the case in which $c > h$. The case $h > c$ can be dealt with in a similar fashion and leads to exactly the same expressions for the stress intensity factors.

2. STATEMENT AND SOLUTION OF THE PROBLEM

If we define elliptic coordinates (ξ, η) by

$$x = R \operatorname{ch} \xi \cos \eta \quad , \quad y = R \operatorname{sh} \xi \sin \eta \tag{2.1}$$

where $\xi \geq 0$, $0 \leq \eta < 2\pi$ and $R = (c^2 - h^2)^{1/2}$, our ellipse becomes the coordinate line $\xi = \gamma = \operatorname{ch}^{-1}(c/R)$, $0 \leq \eta < 2\pi$ and our solution takes the form

$$u_z = \frac{TR}{\mu}[\operatorname{sh} \xi \sin \eta + \phi(\xi, \eta)] \tag{2.2}$$

$$\sigma_{\xi z} = \frac{T}{K}[\operatorname{ch} \xi \sin \eta + \frac{\partial \phi}{\partial \xi}] \tag{2.3}$$

$$\sigma_{\eta z} = \frac{T}{K}[\operatorname{sh} \xi \cos \eta + \frac{\partial \phi}{\partial \eta}] \tag{2.4}$$

where

$$K = (\operatorname{ch}^2 \xi - \cos^2 \eta)^{1/2} \tag{2.5}$$

and $\phi(\xi, \eta)$ is harmonic.

By symmetry we need only find a harmonic function $\phi(\xi, \eta)$ in the strip $\gamma < \xi < \infty$, $0 < \eta < \pi$ satisfying:

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1. $\phi(\xi,\eta) \to 0 \quad$ as $\xi \to \infty$.

2. $\frac{\partial\phi}{\partial\xi}(\gamma,\eta) = -\text{ch}\,\gamma \sin\eta \quad , \quad 0 < \eta < \pi.$

3. $\frac{\partial\phi}{\partial\eta}[\xi,(2-n)\pi] = (-)^{n-1}\text{sh}\,\xi \quad , \quad \gamma < \xi < \beta_n$

$\phi[\xi,(2-n)\pi] = 0 \quad , \quad \beta_n < \xi < \infty$

where $\beta_n = \text{ch}^{-1}(b_n/R), n = 1,2$.

On introducing new variables $X = \xi - \gamma$, $Y = \eta$, $B_n = \beta_n - \gamma$, $\psi(X,Y) = \phi(\xi,\eta)$ we obtain an equivalent problem:

P.D.E. $\dfrac{\partial^2\psi}{\partial X^2} + \dfrac{\partial^2\psi}{\partial Y^2} = 0 \quad , \quad 0 < X < \infty, \; 0 < Y < \pi$

B.C. 1. $\psi(X,Y) \to 0$ as $X \to \infty$.

2. $\dfrac{\partial\psi}{\partial X}(0,Y) = -\text{ch}\,\gamma \sin Y \quad , \quad 0 < Y < \pi$

3. $\dfrac{\partial\psi}{\partial Y}[X,(2-n)\pi] = (-)^{n-1}\text{sh}(X+\gamma) \quad , \quad 0 < X < B_n$

$\psi[X,(2-n)\pi] = 0 \quad , \quad B_n < X < \infty.$

whose solution is clearly given by

$$\psi(X,Y) = \mathcal{F}_c\left[\frac{\Omega_1(\rho)\text{sh}\,\rho Y}{\rho\,\text{sh}\,\rho\pi} + \frac{\Omega_2(\rho)\,\text{sh}\,\rho(\pi - Y)}{\rho\,\text{sh}\,\rho\pi}; \rho \to X\right] + \text{ch}\,\gamma\, e^{-X}\sin Y \tag{2.6}$$

provided $\Omega_1(\rho)$ and $\Omega_2(\rho)$ satisfy the simultaneous dual integral equations

$$\begin{aligned} F_1(X) &= \mathcal{F}_c[\Omega_1(\rho)\,\text{cth}\,\rho\pi - \Omega_2(\rho)\,\text{csch}\,\rho\pi; X] = e^{\gamma}\text{ch}\,X, \quad 0 < X < B_1 \\ F_2(X) &= \mathcal{F}_c[\Omega_2(\rho)\,\text{cth}\,\rho\pi - \Omega_1(\rho)\,\text{csch}\,\rho\pi; X] = e^{\gamma}\text{ch}\,X, \quad 0 < X < B_2 \\ G_1(X) &= \mathcal{F}_c[\rho^{-1}\Omega_1(\rho); X] = 0 \;, \quad B_1 < X < \infty \\ G_2(X) &= \mathcal{F}_c[\rho^{-1}\Omega_2(\rho); X] = 0 \;, \quad B_2 < X < \infty \end{aligned} \tag{2.7}$$

involving the Fourier Cosine Transform (Sneddon [1]).
Let

$$\Omega_n(\rho) = \sqrt{\frac{2}{\pi}}\int_0^{B_n} p_n(t)\sin\rho t\,dt, \quad n = 1,2 \tag{2.8}$$

then, in terms of the Heaviside function $H(X)$,

$$G_n(X) = H(B_n - X)\int_X^{B_n} p_n(t)\,dt, \quad n = 1,2 \tag{2.9}$$

Additionally,

$$\begin{aligned} F_1(X) &= \frac{d}{dX}\mathcal{F}_s\left[\frac{\Omega_1(\rho)}{\rho}\,\text{cth}\,\rho\pi - \frac{\Omega_2(\rho)}{\rho}\,\text{csch}\,\rho\pi; X\right] \\ &= \frac{1}{\pi}\int_0^{B_1}\frac{\text{sh}\,t\,p_1(t)dt}{\text{ch}\,t - \text{ch}\,X} - \frac{1}{\pi}\int_0^{B_2}\frac{\text{sh}\,t\,p_2(t)dt}{\text{ch}\,t + \text{ch}\,X} \end{aligned} \tag{2.10}$$

where we have made use of integrals 4.116.3 and 4.121.2 on pages 516- 517 of Gradshteyn and Ryzhik [2]. A similar expression holds for $F_2(x)$ and therefore $p_1(t)$ and $p_2(t)$ must satisfy the singular integral equations

$$\frac{1}{\pi}\int_0^{B_1}\frac{\operatorname{sh}t\,p_1(t)dt}{\operatorname{ch}t-\operatorname{ch}X}-\frac{1}{\pi}\int_0^{B_2}\frac{\operatorname{sh}t\,p_2(t)dt}{\operatorname{ch}t+\operatorname{ch}X}=e^{\gamma}\operatorname{ch}X,\quad 0<X<B_1$$
$$\frac{1}{\pi}\int_0^{B_1}\frac{\operatorname{sh}t\,p_1(t)dt}{\operatorname{ch}t+\operatorname{ch}X}-\frac{1}{\pi}\int_0^{B_2}\frac{\operatorname{sh}t\,p_2(t)dt}{\operatorname{ch}t-\operatorname{ch}X}=-e^{\gamma}\operatorname{ch}X,\quad 0<X<B_2 \tag{2.11}$$

with subsidiary conditions

$$p_1(0)=p_2(0)=0 \tag{2.12}$$

Let $\tau=\operatorname{ch}t$, $\sigma=\operatorname{ch}X$, $\delta=\operatorname{ch}B_1$, $\epsilon=\operatorname{ch}B_2$, $p_1(t)=q_1(-\tau)$ and $p_2(t)=q_2(\tau)$. Define

$$q(\tau)=\begin{Bmatrix} q_1(\tau) & , & -\delta<\tau<-1 \\ q_2(\tau) & , & 1<\tau<\epsilon \end{Bmatrix} \tag{2.13}$$

and let L denote the set $(-\delta,-1)\cup(1,\epsilon)$ then (2.11) yields

$$\frac{1}{\pi}\int_{\mathrm{L}}\frac{q(\tau)}{\tau-\sigma}d\tau=e^{\gamma}\sigma,\quad \sigma\epsilon\mathrm{L} \tag{2.14}$$

and (2.12)

$$q(-1)=q(1)=0 \tag{2.15}$$

It is now readily shown ([3], [4], [5]) that

$$q(\tau)=-\frac{e^{\gamma}\operatorname{sgn}(\tau)}{\pi\Delta(\tau)}\int_{\mathrm{L}}\frac{\Delta(\sigma)}{\operatorname{sgn}(\sigma)}\frac{\sigma}{\sigma-\tau}d\sigma \tag{2.16}$$

where

$$\Delta(\tau)=\sqrt{\frac{(\tau+\delta)(\epsilon-\tau)}{(\tau^2-1)}}\ ,\quad \tau\epsilon\mathrm{L}, \tag{2.17}$$

and hence that

$$q(\tau)=\frac{1}{2}e^{\gamma}\operatorname{sgn}(\tau)(\delta-\epsilon+2\tau)\sqrt{\frac{\tau^2-1}{(\tau+\delta)(\epsilon-\tau)}}. \tag{2.18}$$

Therefore

$$p_1(t)=-\frac{e^{\gamma}(\operatorname{ch}B_1-\operatorname{ch}B_2-2\operatorname{ch}t)\operatorname{sh}t}{2\sqrt{(\operatorname{ch}B_1-\operatorname{ch}t)(\operatorname{ch}B_2+\operatorname{ch}t)}} \tag{2.19}$$

and

$$p_2(t)=\frac{e^{\gamma}(\operatorname{ch}B_1-\operatorname{ch}B_2-2\operatorname{ch}t)\operatorname{sh}t}{2\sqrt{(\operatorname{ch}B_1+\operatorname{ch}t)(\operatorname{ch}B_2-\operatorname{ch}t)}} \tag{2.20}$$

The Stress Intensity Factors

The stress intensity factors at the tips $(-b_1,0)$ and $(b_2,0)$ are defined respectively by

$$k_3(b_1)=\lim_{x\to -b_1^+}\mu[2(b_1+x)]^{1/2}\frac{\partial u_z}{\partial x}(x,0) \tag{3.1}$$

and

$$k_3(b_2)=\lim_{x\to b_2^-}\mu[2(b_2-x)]^{1/2}\frac{\partial u_z}{\partial x}(x,0)\,. \tag{3.2}$$

It follows that

$$\frac{k_3(b_n)}{T\sqrt{b_n}} = \lim_{\xi \to \beta_n -} \sqrt{\frac{2(\operatorname{ch}\beta_n - \operatorname{ch}\xi)}{\operatorname{ch}\beta_n}} \cdot \frac{p_n(\xi - \gamma)}{\operatorname{sh}\xi}, \quad n = 1, 2 \tag{3.3}$$

and hence, by virtue of (2.19), (2.20), that

$$\frac{k_3(b_n)}{T\sqrt{b_n}} = \frac{c+h}{2}\left\{s_1 + \frac{1}{s_1} + s_2 + \frac{1}{s_2}\right\}^{1/2}\left\{\frac{s_n^2 - 1}{b_n[(c+h)s_n^2 - c + h]}\right\}^{1/2} \tag{3.4}$$

where

$$s_n = \frac{b_n + \sqrt{b_n^2 - c^2 + h^2}}{c+h}, \quad n = 1, 2 \tag{3.5}$$

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