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Oriented graph coloring ☆

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Abstract

An oriented k-coloring of an oriented graph G (that is a digraph with no cycle of length 2) is a partition of its vertex set into k subsets such that (i) no two adjacent vertices belong to the same subset and (ii) all the arcs between any two subsets have the same direction. We survey the main results that have been obtained on oriented graph colorings. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

If G is a graph or a digraph, we denote by $V(G)$ its vertex set and by $E(G)$ its edge or arc set. All the graphs we consider are simple, that is they have neither multiple edges nor loops, and connected. Let G and G' be either two graphs or two digraphs. A *homomorphism* of G to G' is a mapping $f: V(G) \to V(G')$ that preserves the edges or the arcs: $xy \in E(G)$ implies $f(x)f(y) \in E(G')$. The existence (resp. non-existence) of such a homomorphism is denoted by $G \to G'$ (resp. $G \nrightarrow G'$).

A (proper) *k-coloring* of a graph H is a partition of $V(H)$ into k subsets, called *color classes*, such that no two adjacent vertices belong to the same color class. Such a k-coloring can be equivalently regarded as a homomorphism of H to the complete graph K_k on k-vertices. Therefore, the *chromatic number* $\chi(H)$ of a graph H, defined as the smallest k such that H admits a k-coloring, corresponds to the smallest k such that $H \to K_k$ and $H \nrightarrow K_{k-1}$.

An *orientation* of a graph H is a digraph obtained from H by giving to every edge one of its two possible orientations. A digraph G is an *oriented graph* if it is an orientation of some graph H. As before, homomorphisms of oriented graphs induce the notion of oriented graph coloring as follows. An *oriented k-coloring* of an oriented graph G is a partition of $V(G)$ into k color classes such that: (i) no two adjacent vertices belong to the same color class and (ii) all the arcs linking two color classes

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have the same direction. We can then define the *oriented chromatic number* of an oriented graph G, denoted by $\mathrm{ocn}(G)$, as the smallest k such that G has an oriented k-coloring or, equivalently, as the minimum order of an oriented graph G' such that $G \to G'$. This notion can be extended to graphs as follows: the oriented chromatic number of a graph H is defined as the maximum of the oriented chromatic numbers of its orientations. Similarly, the oriented chromatic number of a family of graphs or oriented graphs is defined as the maximum of the oriented chromatic numbers of its members.

Observe that conditions (i) and (ii) above imply that any two vertices linked by a directed path of length 1 or 2 must be assigned distinct colors in any oriented coloring. We then get for instance that the oriented chromatic number of the directed cycle on five vertices is 5. This property can also be expressed by saying that homomorphisms of oriented graphs 'preserve' directed paths of length 2 (the image of such a path is still a directed path of length 2). Homomorphisms preserving other type of configurations, namely paths of length k or trees, have been considered in [18].

Homomorphisms of graphs have been extensively studied and, in particular, many papers have been devoted to the complexity of the so-called *H-coloring* decision problem: given a graph H, what is the complexity of deciding whether a graph G has a homomorphism to H or not? Hell and Nešetřil solved this question in the undirected case [12]: the H-coloring problem is polynomial if the graph H is bipartite and NP-complete otherwise. The situation seems to be more complicated in the directed case and only partial results have been obtained (see [10,13] for recent results concerning this problem).

The particular case of oriented colorings have been first considered by Courcelle [9] as a tool for encoding graph orientations by means of vertex labels. Since then, oriented colorings have attracted much attention and the aim of this paper is to survey the main results that have been obtained. The links between oriented colorings and acyclic (usual) colorings is discussed in Section 2. Section 3 deals with oriented colorings of planar graphs while Section 4 considers the case of graphs with bounded degree or bounded treewidth. The notion of universal graphs is introduced and discussed in Section 5. Finally, some other related types of colorings, namely colorings of edge-colored graphs and strong colorings, are respectively considered in Sections 6 and 7.

2. Oriented and acyclic chromatic numbers

One of the first problems that have been considered in the framework of oriented colorings was to characterize the families of graphs having bounded oriented chromatic number. It appears that these families are exactly the families having bounded acyclic chromatic number.

Recall that a k-coloring of a graph H is *acyclic* if every subgraph of H induced by two color classes is acyclic. The *acyclic chromatic number* of H, denoted by $\chi_a(H)$, is then defined as the smallest k such that H has an acyclic k-coloring. In [19], Raspaud and Sopena proved that families of graphs with bounded acyclic chromatic number have also bounded oriented chromatic number:

Theorem 1 (Raspaud and Sopena [19]). *If a graph H has acyclic chromatic number at most k then its oriented chromatic number is at most $k \times 2^{k-1}$.*

Proof. Let a be an acyclic k-coloring of H and $\vec{H}$ be any orientation of H. We denote by $\vec{H}_{i,j}$, $1 \leqslant i < j \leqslant k$, the subgraph of $\vec{H}$ induced by vertices x with $a(x) \in \{i,j\}$ (since a is acyclic, $\vec{H}_{i,j}$ is an oriented forest). We inductively define an oriented 4-coloring $c_{i,j}$ of $\vec{H}_{i,j}$ as follows: pick any vertex x in each component of $\vec{H}_{i,j}$ and set $c_{i,j}(x) := (a(x), 0)$. Then, for every uncolored vertex z linked to some vertex y with $c_{i,j}(y) = (a(y), \alpha)$, $\alpha \in \{0,1\}$, set

- $c_{i,j}(z) := (a(z), \alpha)$ if yz (resp. zy) is an arc in $E(\vec{H}_{i,j})$ and $a(y) = i$ (resp. $a(y) = j$),
- $c_{i,j}(z) := (a(z), 1 - \alpha)$ otherwise.

This mapping $c_{i,j}$ is indeed a homomorphism of $\vec{H}_{i,j}$ to the directed 4-cycle $(i,0) \to (j,0) \to (i,1) \to (j,1) \to (i,0)$. It is then not difficult to check that the mapping c defined for every $x \in V(\vec{H})$ by

$$c(x) = (a(x), c_{1,a(x)}(x), \ldots, c_{a(x)-1,a(x)}(x), c_{a(x),a(x)+1}(x), \ldots, c_{a(x),k}(x))$$

is an oriented $(k \times 2^{k-1})$-coloring of $\vec{H}$. □

From Theorem 1 we get for instance that the families of graphs with bounded degree, bounded treewidth or bounded genus have bounded oriented chromatic number. An upper bound on the acyclic chromatic number in terms of the oriented chromatic number has been obtained in [14].

Theorem 2 (Kostochka et al. [14]). *If a graph H has oriented chromatic number at most k then its acyclic chromatic number is at most $k^{\lceil \log_2(\lceil \log_2 k \rceil + k/2) \rceil + 1}$.*

Sketch of proof. Recall that the *arboricity* of a graph H is the minimum number q such that the edges of H can be decomposed into q forests. The idea is to prove first that every graph with oriented chromatic number as most k has arboricity at most $\lceil \log_2 k + k/2 \rceil$ and then that every graph with oriented chromatic number at most k and arboricity at most q has acyclic chromatic number at most $k^{\lceil \log_2 q \rceil + 1}$. □

Theorem 1 can be generalized to the case of acyclic improper colorings, introduced by Boiron et al. in [3]. A coloring of a graph is said to be *improper* if adjacent vertices may be assigned the same color. More formally, if $P_1, P_2, \ldots, P_k$ are graph properties, a $(P_1, P_2, \ldots, P_k)$-*coloring* of a graph G is a partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that for every i, $1 \leqslant i \leqslant k$, the subgraph $G[V_i]$ induced by the color class V_i satisfies the property P_i. Such an improper coloring is *acyclic* if for every i,j, $1 \leqslant i < j \leqslant k$, the subgraph induced by all the edges linking V_i to V_j is acyclic. Then we have [3]:

Theorem 3 (Boiron et al. [3]). *If $P_1, P_2, \ldots, P_k$ are graph properties such that for every i, $1 \leqslant i \leqslant k$, the family of graphs satisfying P_i has oriented chromatic number at most α_i then every graph having an acyclic $(P_1, P_2, \ldots, P_k)$-coloring has oriented chromatic number at most $2^{k-1} \times \sum_{i=1}^{k} \alpha_i$.*

If for every i the property P_i is the property of having no edges (all the α_i's are equal to 1), acyclic $(P_1, P_2, \ldots, P_k)$-colorings are then usual acyclic colorings and we get Theorem 1. Acyclic improper colorings of planar graphs and of graphs with bounded degree have been considered in [3,4].

3. Planar graphs

A celebrated result of Borodin [5] states that every planar graph has acyclic chromatic number at most five. From Theorem 1 we thus get:

Corollary 4. *Every planar graph has oriented chromatic number at most* 80.

In [20], an oriented planar graph with oriented chromatic number at least 16 has been constructed. The gap between the lower and the upper bounds for the oriented chromatic number of planar graphs is thus still large and, despite many efforts, has not been reduced up to now.

However, the upper bound can be significantly lowered when considering planar graphs with large girth [7,17] (recall that the *girth* of a graph G is the smallest size of a cycle in G). More precisely, we have the following [7]:

Theorem 5 (Borodin et al. [7]). *Every planar graph with girth at least* 14 (*resp.* 8, 6, 5) *has oriented chromatic number at most* 5 (*resp.* 7, 11, 19).

In fact, this result follows from a more general theorem. The *maximum average degree* $\mathrm{mad}(H)$ of a graph H is defined as the maximum of the average degrees $\mathrm{ad}(H') = 2|E(H')|/|V(H')|$ taken over all the subgraphs H' of H. Then we have [7]:

Theorem 6 (Borodin et al. [7]). *Every graph with maximum average degree at most* $\frac{7}{3}$ (*resp.* $\frac{11}{4}, 3, \frac{10}{3}$) *has oriented chromatic number at most* 5 (*resp.* 7, 11, 19).

If H is a planar graph with girth at least g then the number of faces in G is at most $2|E(G)|/g$. By Euler's formula we then get that

$$\frac{2|E(G)|}{|V(G)|} \leqslant \frac{2g|E(G)|}{2g + (g-2)|E(G)|}$$

and thus $\mathrm{mad}(H) < 2g/(g-2)$. Therefore, Theorem 5 directly follows from Theorem 6.

Recall that the circulant digraph $G = G(n; c_1, c_2, \ldots, c_k)$, $n > 0$, $1 \leqslant c_i < n$ for every i, is defined by $V(G) = \{0, 1, \ldots, n-1\}$ and $xy \in E(G)$ if and only if $y = x + c_i \pmod n$ for some i, $1 \leqslant i \leqslant k$. Theorem 6 has been proved by showing that the corresponding oriented graphs admit a homomorphism, respectively, to $G(7; 1, 2, 4)$, $G(11; 1, 3, 4, 5, 9)$ and $G(19; 1, 4, 5, 6, 7, 9, 11, 16, 17)$. For these three circulant tournaments, the c_i's are exactly the non-zero quadratic residues of n.

It is not difficult to construct families of graphs with maximum average degree 4 and unbounded oriented chromatic number. When the maximum average degree tends to 4, we have the following phenomenon [7]:

Theorem 7 (Borodin et al. [7]). *Every graph with maximum average degree less than* $4(1-2/(n+1))$, $n > 25$, *has oriented chromatic number at most* $(n+5) \times 2^{(n+1)/2}$. *For every* $k > 0$, *there exists a graph of arbitrarily large girth with maximum average degree less than* $4(1-1/k)$ *and oriented chromatic number at least* k.

To see that the first statement of Theorem 7 is optimal it is sufficient to consider the oriented graph B_n, $n > 2$, obtained from the undirected complete graph K_n by replacing each edge by a directed path of length two going through some new vertex. Since any two of the n vertices with degree $n-1$ are linked by a directed path of length 2, they must be assigned distinct colors in any coloring of B_n and thus $\mathrm{ocn}(B_n) \geqslant n$. Moreover, it is easy to check that the maximum average degree of B_n is

$$\mathrm{mad}(B_n) = \frac{2|E(B_n)|}{|V(B_n)|} = \frac{2n(n-1)}{n + n(n-1)/2} = 4\left(1 - \frac{2}{n+1}\right).$$

For outerplanar graphs, we have the following [20]:

Theorem 8 (Sopena [20]). *Every outerplanar graph has oriented chromatic number at most* 7.

Proof. We claim that every orientation G of an outerplanar graph has a homomorphism to the circulant tournament $T_7 = G(7; 1, 2, 4)$. This tournament satisfies the following property: for every $u_1, u_2 \in V(T_7)$, $u_1 \neq u_2$, and every $\alpha_1, \alpha_2 \in \{+, -\}$, there exists some $v \in V(T_7)$ such that $vu_i \in E(T_7)$ if and only if $\alpha_i = +$, for $i = 1, 2$. We then proceed by induction on the order of G. W.l.o.g. we may assume that G is maximal outerplanar. The result is immediate if $|V(G)| \leqslant 3$. Otherwise there exists a vertex x in G with degree two. Let y_1, y_2 denote the two neighbors of x, and G' the outerplanar oriented graph obtained from G by removing x and adding the arc y_1y_2 if y_1 and y_2 are not linked in G. The above discussed property of T_7 implies that we can extend any homomorphism of G' to T_7 to a homomorphism of G to T_7. □

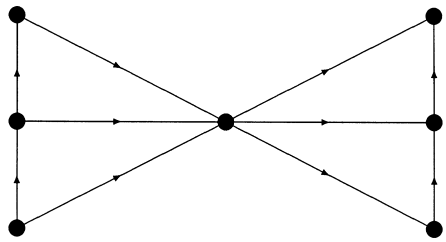

Fig. 1. An oriented outerplanar graph with oriented chromatic number 7.

This bound is tight as shown by the oriented outerplanar graph depicted in Fig. 1: this graph has 7 vertices and any two of them are linked by a directed path of length at most two; its oriented chromatic number is thus equal to 7.

4. Graphs with bounded degree or bounded treewidth

Alon, McDiarmid and Reed proved in [1] that every graph with maximum degree k has acyclic chromatic number at most $O(k^{4/3})$. From Theorem 1, we thus get that graphs with maximum degree k have oriented chromatic number at most $O(k^{4/3}) \times 2^{O(k^{4/3})} = 2^{O(k^{4/3})}$. This upper bound has been first improved in [20], where it was shown that graphs with maximum degree k have oriented chromatic number at most $(2k-1) \times 2^{2k-2}$. The best-known upper bound is the following [14]:

Theorem 9 (Kostochka et al. [14]). *Every graph with maximum degree k has oriented chromatic number at most* $2 \times k^2 \times 2^k$.

This result is not far from being optimal since it was also proved in [14] that for each $k > 1$, there exists a graph with maximum degree k and oriented chromatic number at least $2^{k/2}$. For small values of k, we know that the oriented chromatic number of every graph with maximum degree two is at most 5 (this bound is tight since the directed cycle on five vertices has oriented chromatic number 5) and that the oriented chromatic number of every graph with maximum degree three is at most 11 [21].

Recall that a *k-tree* is a graph obtained from the complete graph K_k by repeatedly inserting new vertices linked to an existing clique of size k. A graph is then said to have *treewidth at most k* if it is a subgraph of some k-tree. The 1-trees are thus the usual trees while outerplanar graphs have treewidth at most 2. From the above definition it can be easily seen that every k-tree is $(k+1)$-colorable: starting with a k-coloring of the complete graph K_k, every newly inserted vertex has exactly k neighbors and can be thus colored using a $(k+1)$th color. Moreover, this coloring is clearly acyclic since all the neighbors of a newly inserted vertex have distinct colors. Therefore, every graph with treewidth at most k has acyclic chromatic number at most $k+1$ and, by Theorem 1, oriented chromatic number at most $(k+1) \times 2^k$.

In case of graphs with treewidth at most 2 or 3, this upper bound can be improved as follows [20]:

Theorem 10 (Sopena [20]). *Every* 2*-tree has oriented chromatic number at most* 7. *Every* 3*-tree has oriented chromatic number at most* 16.

Sketch of proof. Owing to the property of the tournament $T_7 = G(7;1,2,4)$ discussed in the proof of Theorem 8, every 2-tree clearly has a homomorphism to T_7.

Let now T_{16} be the oriented graph obtained by taking two disjoint copies of T_7 (the vertices of the second copy are denote $0', 1', \ldots, 6'$) and two vertices w and w' and adding the arcs wi, $i'w$, iw', $w'i'$, for every $i = 0,1,\ldots,6$ and $j'i$, ji', for every $ij \in E(T_7)$. The oriented graph T_{16} satisfies the following property: for every $u_1,u_2,u_3 \in V(T_{16})$, $u_1 \neq u_2 \neq u_3 \neq u_1$, and every $\alpha_1,\alpha_2,\alpha_3 \in \{+,-\}$, there exists some $v \in V(T_{16})$ such that $vu_i \in E(T_{16})$ (resp. $u_iv \in E(T_{16})$) if and only if $\alpha_i = +$ (resp. $\alpha_i = -$), for $i = 1,2,3$. It is then easy to prove by induction that every 3-tree has a homomorphism to T_{16}. □

The outerplanar graph depicted in Fig. 1 shows that the bound for 2-trees is tight. A graph with treewidth 3 and oriented chromatic number 16 has been constructed in [20], showing that the second upper bound is tight too.

5. Universal graphs and nice graphs

An oriented graph U is said to be *universal* for a family of graphs $\mathscr{F}$ if every orientation of every graph in $\mathscr{F}$ has a homomorphism to U. For instance, the directed cycle on three vertices is universal for the family of trees. Most of the previous results concerning upper bounds on oriented chromatic numbers have been obtained by exhibiting some special universal oriented graphs [7,17,19].

In particular, an oriented (non-planar) graph having 80 vertices which is universal for the family of planar graphs has been constructed in [19]. The existence of planar oriented graphs which are universal for families of planar graphs with high girth has been discussed in [6]. The following has been proved.

Theorem 11 (Borodin et al. [6]). *There exists no planar oriented graph which is universal for the family of planar graphs with girth at least* 4.

There exists a planar oriented graph on 6 *vertices which is universal for the family of planar graphs with girth at least* 16.

The first statement of Theorem 11 is proved by showing that every oriented graph which is universal for the family of planar graphs with girth at least 4 and inclusion minimal has minimum degree 6 and, therefore, cannot be planar. The second statement is obtained by proving that every planar graph with girth at least 16 has a homomorphism to the oriented planar circulant graph $G(6;1,2)$.

The minimum k such that there exists a planar oriented graph which is universal for the family of planar graphs with girth at least k is not known up to now. Concerning the girth of universal graphs we have [6]:

Theorem 12 (Borodin et al. [6]). *For every $k \geqslant 3$, there exists a (non-planar) oriented graph with girth at least $k+1$ which is universal for the family of planar graphs with girth at least $40k$.*

However, such a universal graph with high girth cannot be planar, as shown by the following result [11]:

Theorem 13 (Hell et al. [11]). *Every planar graph which is universal for the family of planar graphs with girth at least k, $k \geqslant 3$,* contains a triangle.

Theorem 13 follows from the fact that every such universal graph must be nice: an oriented graph G is said to be *n-nice* for some n if for every two (not necessarily distinct) vertices x and y in G, and every pattern p (given as a sequence of forward or backward arcs) of length n, there exists a directed path with pattern p in G linking x to y. An oriented graph is then *nice* if it is n-nice for some n.

More precisely, we have [17]:

Theorem 14 (Nešetřil et al. [17]). *For every $n \geqslant 3$, every n-nice oriented graph is universal for the family of planar graphs with girth at least $5n-4$.*

Proof (*Sketch*). Let G be a planar graph. W.l.o.g. we assume that G has no vertex with degree 1. Denote by V' the set of all branching vertices of G (that is vertices with degree at least 3). The graph G can thus be viewed as a subdivision of a graph G' with $V(G') = V'$. The graph G' is planar and has minimal degree at least 3. As there is a vertex in the dual of G' which has maximum degree 5 we get that some of the faces of G' have at most 5 incident edges. Now if the girth of G is at least $5n-4$ then one of the edges of G' has to be subdivided by $n-1$ points. We thus proved that if G has girth at least $5n-4$ then it contains a path of length n whose all internal vertices have degree 2.

Now let T_n be any n-nice oriented graph. Using the previous property, it is then easy to prove by induction that every planar graph with girth at least $5n-4$ has a homomorphism to T_n. $\square$

Theorem 15 (Hell et al. [11]). *Every graph which is universal for the family of planar graphs with girth at least g and minimal with respect to this property is nice.*

Characterizations of nice oriented graphs (and, more generally, of nice digraphs) have been discussed in [11].

6. Homomorphisms of edge-colored graphs

In [2], Alon and Marshall studied a new notion of the chromatic number related to homomorphisms of edge-colored graphs as introduced by Brewster [8]. An *m-edge-colored graph* is a graph whose edges are colored using the set $\{1,2,\dots,m\}$ as set of colors. Homomorphisms of edge-colored graphs are then required to preserve the edge colors.

Alon and Marshall proved the following:

Theorem 16 (Alon and Marshall [2]). *For every $m>0$, $k>0$, there exists an m-edge-colored graph $H_{m,k}$ on $k\times m^{k-1}$ vertices such that every m-edge-colored graph with acyclic chromatic number at most k has a homomorphism to $H_{m,k}$.*

When $n=2$, this result is similar to Theorem 1 although there is no natural relation between oriented graphs and 2-edge-colored graphs. These two results have been unified by Nešetřil and Raspaud [15], who considered the so-called colored mixed graphs. A *mixed graph* is a graph whose vertices are linked by edges or by arcs (in such a way that the underlying graph remains simple). An (n,m)*-colored* mixed graph is a mixed graph whose arcs (resp. edges) are colored using the set $\{1,2,\dots,n\}$ (resp. $\{1,2,\dots,m\}$) as set of colors. By convention, $(n,0)$-colored mixed graphs correspond to oriented graphs whose arcs are n-colored and $(0,m)$-colored mixed graphs correspond to m-edge-colored graphs. Homomorphisms of colored mixed graphs are then required to map edges to edges, arcs to arcs, and to preserve the colors. Nešetřil and Raspaud proved the following [15]:

Theorem 17 (Nešetřil and Raspaud [15]). *For every $n\geqslant 0$, $m\geqslant 0$, $k>0$, there exists an (n,m)-colored mixed graph $M_{n,m,k}$ on $k\times(2n+m)^{k-1}$ vertices such that every (n,m)-colored mixed graph with acyclic chromatic number at most k has a homomorphism to $M_{n,m,k}$.*

By, respectively, setting $n=1$, $m=0$ and $n=0$ we get Theorems 1 and 16. By adapting a construction given in [2], it can be shown that there exists (n,m)-colored mixed graphs with acyclic chromatic number at most k having no homomorphism to an (n,m)-colored mixed graph with less than $(2n+m)^{k-1}+k-1$ vertices. The upper bound given in Theorem 17 is thus in a sense best possible.

7. Strong colorings and antisymmetric flows

Nešetřil and Raspaud introduced in [16] a restriction of the notion of oriented colorings. Let J be an oriented graph whose set of vertices $V(J)=M$ is an abelian additive group with q elements. An M*-strong-oriented coloring* of an oriented graph G is a homomorphism $f:G\to J$ such that for every two arcs (not necessarily disctinct)

xy and $x'y'$ in G, we have $f(x) - f(y) \neq -(f(x') - f(y'))$. The smallest q such that G admits an M-strong-oriented coloring, $|M| = q$, is the *strong oriented chromatic number* of G and is denoted by $\mathrm{socn}(G)$. Clearly, $\mathrm{ocn}(G) \leqslant \mathrm{socn}(G)$ for every graph G.

Strong oriented chromatic numbers and acyclic chromatic numbers are related as follows [16]:

Theorem 18 (Nešetřil and Raspaud [16]). *If H is a graph with acyclic chromatic number at most k then every orientation of H has strong oriented chromatic number at most 6^k.*

In particular, every planar graph has strong oriented chromatic number at most $6^5 = 7776$.

By duality, strong oriented colorings induce the notion of antisymmetric flow. Recall that if M is an abelian additive group and G an oriented graph, an *M-flow* is a mapping $\Phi : E(G) \to M$ such that for every subset S of $V(G)$, $\sum_{e \in w^+(S)} \Phi(e) - \sum_{e \in w^-(S)} \Phi(e) = 0$, where $w^+(S)$ (resp. $w^-(S)$) stands for the set of arcs starting inside S (resp. outside S) and ending outside S (resp. inside S). An *antisymmetric M-flow* is then defined as an M-flow such that (i) no arc is mapped to 0 and (ii) no two arcs are mapped to opposite elements. By comparing the corresponding definitions, it is easy to observe that an oriented planar graph G has an antisymmetric M-flow if and only if its dual G^* has an M-strong oriented coloring.

The main result concerning antisymmetric flows is the following [16]:

Theorem 19 (Nešetřil and Raspaud [16]). *Every orientation of every 3-edge-connected graph admits an antisymmetric flow.*

Several new questions related to antisymmetric flows are discussed in [16]. In particular, it is still unknown whether the so-called *upper AF-number* of every 3-edge-connected graph H, defined as the smallest q such that every orientation $\vec{H}$ of H has an antisymmetric $M_{\vec{H}}$-flow with $|M_{\vec{H}}| \leqslant q$, is bounded or not by some constant.

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