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Carathéodory Approximate Solutions for a Class of Semilinear Stochastic Evolution Equations with Time Delays

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The paper is concerned with Carathéodory approximate solutions for a class of infinite-dimensional stochastic evolution equations with time delays. In addition to the second moment convergence, it is shown that, under suitable conditions, the approximate solution converges almost surely to the mild solution of a given stochastic evolution system. © 1998 Academic Press

1. INTRODUCTION

There is a wide literature on procedures for approximating the solution of a stochastic differential equation. We first mention the classical Picard and Euler procedures for approximating the solution of a finite-dimensional stochastic differential equation (see Ikeda and Watanabe [4]). In the finite-dimensional situation, we also mention, for instance, the Cauchy–Maruyama approximation (see G. Maruyama [10] and J. Mashane [11]) and the Carathéodory approximation (see D. R. Bell and S. E. A. Mohammed [1] and X. Mao [6]). On the other hand, the discretization of infinite-dimensional stochastic evolution equations is still a subject which has received a great deal of attention recently. For instance, the classical Euler's approximation procedure is considered for the mild solution of a class of stochastic evolution equations in P. L. Chow and J. L. Jiang [3]. For the time delay case, X. Mao [7, 8] and K. Liu [5] treat the Carathéodory approximation scheme for the strong solutions for a class of stochastic evolution equations.

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This paper is concerned with the Carathéodory successive approximation mild solution properties for a class of stochastic evolution equations in infinite dimensions with time delays. Precisely, consider a class of delay stochastic evolution equations in Hilbert space of the form

$$\begin{aligned} dX(t) = \big[AX(t) + F(X(t), X(t - \tau(t)))\big]\, dt \\ +G(X(t), X(t - \tau(t)))\, dW(t), \qquad t > 0, \\ X(t) = \psi(t), \qquad -\hat{\tau} \le t \le 0, \end{aligned}$$

where A, generally unbounded, generates a strongly continuous semigroup $S(t)$, $t \ge 0$, over a real separable Hilbert space H, and $W(t)$ is a certain Hilbert space-valued Q-Wiener process. The terms $F(x, y), G(x, y), x, y \in H$, are nonlinear and satisfy certain given Lipschitz conditions and linear growth conditions. That is, F and G are regarded as bounded perturbations. The $\psi(t)$ is a proper H-valued stochastic process on $[-\hat{\tau}, 0]$, $\hat{\tau}$ is a positive constant, and $\tau(t)$ is an appropriate non-negative time delay function defined over R^+. The Carathéodory scheme is to define the approximation solution, for each $n = 1, 2, \dots,$ via a delay equation (see the next section for the details). In fact, the proof of the convergence of the Carathéodory approximation represents an alternative to the standard procedure for establishing the existence and uniqueness of the solution to the stochastic delay differential equation.

In this paper, we shall present two results on the Carathéodory convergence of the mild approximation solution. In particular, in Section 2, a constructive way of proving the existence theorem is presented. The other approximation procedure for the mild solution will be discussed in detail elsewhere. Firstly, we get our convergence results proved in the sense of the second moment. Next, as another major result it is shown that under suitable conditions the approximate solution converges almost surely to the mild solution of the given equation.

2. THE MAIN RESULTS

The objective of this paper is to show that the Carathéodory approximation procedure is applicable to a class of delay stochastic evolution equations in Hilbert space. Assume H is a real separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let K be another real separable Hilbert space and $W(t)$, $t \ge 0$, be a K-valued Wiener process with mean zero and covariance operator Q with $\operatorname{tr} Q < \infty$ (tr denotes the trace of operator) defined by

$$E\langle W(t), g\rangle\langle W(s), h\rangle = (t \wedge s)\langle Qg, h\rangle, \qquad \forall g, h \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on the space K. For convenience, we introduce a subspace $K_Q \subset K$, the closure of $Q^{1/2}K$ with respect to the norm $\|\cdot\|_Q$ defined by $\|Q^{1/2}k\|_Q^2 = (Q^{1/2}k, Q^{1/2}k)_Q = \langle k, k\rangle$, $k \in K$. Let $V \subset H$ be a densely imbedding Banach subspace. Suppose that $A: V \to V^*$, the dual of V, is bounded. $F: H \times H \to H$ and $G: H \times H \to \mathscr{L}(K_Q, H)$, the space of all linear bounded operators from K_Q into H, are two measurable mappings. As an abuse of notation, we also use $\|\cdot\|$ for the norm in the linear continuous operator space $\mathscr{L}(K_Q, H)$. Let $\tau(\cdot)$ be a continuous non-negative function on R^+ and define

$$\hat{\tau} = \sup\{\tau(t) - t: t \geq 0\} < \infty.$$

Let $M^2([-\hat{\tau}, 0], H)$ denote the family of all continuous H-valued stochastic processes $\psi(t)$ defined on $[-\hat{\tau}, 0]$ such that $\psi(t)$, $-\hat{\tau} \leq t \leq 0$, are all $\mathscr{F}_0$-measurable and

$$\sup_{-\hat{\tau} \leq t \leq 0} \left\{E\|\psi(t)\|^2, -\hat{\tau} \leq t \leq 0\right\} < \infty.$$

Consider a class of delay stochastic evolution equations in Hilbert space of the form

$$\begin{aligned} dX(t) &= \left[AX(t) + F(X(t), X(t - \tau(t)))\right] dt \\ &\quad + G(X(t), X(t - \tau(t)))\, dW(t), \qquad t \geq 0, \\ X(t) &= \psi(t), \qquad -\hat{\tau} \leq t \leq 0. \end{aligned} \tag{2.1}$$

Throughout this paper we assume F, G satisfy the following conditions:

(A.1) There exists a positive constant L such that

$$\|G(x, y)\| \vee \|F(x, y)\| \leq L(1 + \|x\| + \|y\|), \qquad \forall x, y \in H \tag{2.2}$$

and

$$\begin{aligned} &\|F(x, y) - F(\tilde{x}, \tilde{y})\| \vee \|G(x, y) - G(\tilde{x}, \tilde{y})\| \\ &\quad \leq L\big(\|x - \tilde{x}\| + \|y - \tilde{y}\|\big) \end{aligned} \tag{2.3}$$

for all $x, y, \tilde{x}, \tilde{y} \in H$, and the following assumptions are also made:

(A.2) $A: V \to V^*$ is coercive such that it generates an analytic semigroup $\{S_t, t \geq 0\}$ on H.

(A.3) For arbitrarily given $T > 0$, there exist constants $\theta = \theta(T) > 0$ and $K(T) > 0$ such that for any positive integer n large enough

$$\mu\left\{t: 0 < \tau(t) < \frac{1}{n}, 0 \leq t \leq T\right\} \leq \frac{K(T)}{n^{\theta}},$$

where μ is the Lebesgue measure on R^+.

The definition of the mild solution for the infinite-dimensional stochastic delay differential equation is given as follows:

Definition 2.1. For any $T > 0$, an H-valued stochastic process $X(t), t \in [-\hat{\tau}, T]$, defined on some given probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, is a *mild solution* of (2.1) if

(1) $X(t)$ is adapted to $\mathscr{F}_t$.

(2) $X(t)$ is measurable and almost surely $\int_0^T \|X(s)\|^2\, ds < \infty$. Moreover,

$$X(t) = S_t X_0 + \int_0^t S_{t-s} F(X(s), X(s - \tau(s)))\, ds + \int_0^t S_{t-s} G(X(s), X(s - \tau(s)))\, dW(s)$$

for all $t \in [0, T]$ a.e., and

$$X(t) = \psi(t), \qquad -\hat{\tau} \leq t \leq 0.$$

On the other hand, the Carathéodory approximate solution is defined as follows: Fix $T > 0$, for arbitrary $\nu \geq 1$, $\psi(\cdot) \in M^2([-\hat{\tau}, 0], H)$ and all $n \geq 2/\hat{\tau}$, we define

$$X^n(t) = \psi(t), \qquad -\hat{\tau} \leq t \leq 0,$$

$$
\begin{aligned}
X^n(t) &\\
= S_t^n\psi(0) &+ \int_0^t 1_{D_n^c}(u) S_{t-u}^n F\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n(u - \tau(u))\right) du \\
&+ \int_0^t 1_{D_n^c}(u) S_{t-u}^n G\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n(u - \tau(u))\right) dW(u) \\
&+ \int_0^t 1_{D_n}(u) S_{t-u}^n F\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n\left(u - \tau(u) - \frac{1}{n^\nu}\right)\right) du \\
&+ \int_0^t 1_{D_n}(u) S_{t-u}^n G\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n\left(u - \tau(u) - \frac{1}{n^\nu}\right)\right) dW(u),
\end{aligned}
\tag{2.4}
$$

where $S_{t-s}^n = S_{t-s+1/n^\nu}$, and

$$
D_n = \left\{t : \tau(t) < \frac{1}{n^\nu}, 0 \le t \le T\right\}, \qquad D_n^c = [0, T] - D_n, 0 \le t \le T.
$$

Here 1_B denotes the indicator function on the set $B \subset R^+$. Note that each $X^n(t)$ can be determined by stepwise iterated Itô integrals over the intervals $[0, 1/n^\nu], [1/n^\nu, 2/n^\nu], \ldots,$ etc. Let $C_T(H) = C([0, T], H)$ denote the space of H-valued continuous functions on $[0, T]$ with the norm $\|u(\cdot)\|_T = \sup_{0 \le t \le T} \|u(t)\|$.

We shall show the sequence $\{X^n(t)\}$ of approximate solutions converges a.s. in the space $C_T([0, T], H)$ to the mild solution $X(t)$ of Eq. (2.1). To this end, we need the following theorem which is interesting in its own right.

THEOREM 2.1. *Under conditions* (A.1) *to* (A.3), *there exists a unique mild solution* $X(t)$ *to Eq.* (2.1). *Moreover, for any* $T > 0$, *and* $0 < \alpha \le 1/2$, *there exist positive constants* $C_1(T)$, $C_2(T)$, $C_3(\alpha, T)$, $C_4(T)$, *and* $C_5(T)$ *such that*

$$
E\left(\sup_{0 \le s \le t} \|X(s)\|^2\right) \le C_1(T) \cdot e^{C_2(T)t}, \qquad 0 \le t \le T, \tag{2.5}
$$

$$
E\|X(t) - X(s)\|^2 \le C_3(\alpha, T)(t - s)^\alpha + C_4(T)(t - s) \qquad 0 \le s \le t \le T, \quad 0 < \alpha \le 1/2 \tag{2.6}
$$

and

$$E\Big(\sup_{0\le s\le T}\|X(s)-X^n(s)\|^2\Big)$$
$$\le C_5(T)\left(\frac{1}{n^\nu}+\frac{1}{n^{\alpha\nu}}\,\mu\left\{t: 0<\tau(t)<\frac{1}{n^\nu}, 0\le t\le T\right\}\right), \quad (2.7)$$

where the $X^n(t)$ *are defined by* (2.4) *and* μ *stands for the Lebesgue measure on* R^+.

Proof.

Step 1. Fix $\psi\in M^2([-\hat{\tau},0],H)$ and $T>0$. We first claim

$$E\Big(\sup_{0\le s\le t}\|X^n(s)\|^2\Big)\le C_1\cdot e^{C_2 t},\qquad 0\le t\le T. \quad (2.8)$$

Indeed, noticing that there exist constants $M\ge 1$, $\omega\in R'$ such that $\|S_t\|\le Me^{\omega t}$ and using a Burkholder type of inequality for the stochastic evolution integral (see [16]), we have

$$E\Big(\sup_{0\le s\le t}\|X^n(s)\|^2\Big)$$
$$\le C_1'(T)+C_2'(T)\int_0^t 1_{D_n^c}(u)$$
$$\times\left(1+E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2+E\|X^n(u-\tau(u))\|^2\right)du$$
$$+C_2'(T)\int_0^t 1_{D_n}(u)\left(1+E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2\right.$$
$$\left.+E\left\|X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right\|^2\right)du$$
$$\le C_1'(T)+2C_2'(T)T+C_2''(T)\int_0^t E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2du$$
$$+C_2''(T)\int_0^t 1_{D_n^c}(u)E\|X^n(u-\tau(u))\|^2du$$
$$+C_2''(T)\int_0^t 1_{D_n}(u)E\left\|X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right\|^2du$$

$$\le C_3'(T) + C_3''(T)\int_0^t E\|X^n(u)\|^2\,du$$
$$+C_4'(T)\left[\mu(D_n^c) + \int_0^t 1_{D_n^c}(u)E\left(\sup_{0\le s\le u}\|X^n(s)\|^2\right)du\right]$$
$$+C_4'(T)\left[\mu(D_n) + \int_0^t 1_{D_n}(u)E\left(\sup_{0\le s\le u}\|X^n(s)\|^2\right)du\right]$$
$$\le C_1(T) + C_2(T)\int_0^t E\left(\sup_{0\le s\le u}\|X^n(s)\|^2\right)du.$$

This implies, by the well-known Gronwall's lemma, that

$$E\left(\sup_{0\le s\le t}\|X^n(s)\|^2\right) \le C_1(T)\cdot e^{C_2(T)t}, \qquad 0\le t\le T.$$

Step 2. On the other hand, the well-known Burkholder–Davis–Gundy inequality and conditions (A.1), (A.2) imply that there exist positive constants $K_1(T), K_2(T), \ldots,$ such that if $0\le s\le t\le T$, we have

$$E\|X^n(t) - X^n(s)\|^2$$
$$\le E\|(S_t^n - S_s^n)\psi(0)\|^2 + K_1(T)\int_s^t 1_{D_n^c}(u)$$
$$\times E\left\|S_{t-u}^n F\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n(u-\tau(u))\right)\right\|^2 du$$
$$+K_1(T)\int_0^s 1_{D_n^c}(u)$$
$$\times E\left\|[(S_{t-s} - I)S_{s-u}^n]F\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n(u-\tau(u))\right)\right\|^2 du$$
$$+K_2(T)E\left\|\int_s^t 1_{D_n^c}(u)S_{t-u}^n\right.$$
$$\left.\times G\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n(u-\tau(u))\right)dW(u)\right\|^2$$
$$+K_2(T)E\left\|\int_0^s 1_{D_n^c}(u)[(S_{t-s} - I)S_{s-u}^n]\right.$$

$$\times G\left(X^n\left(u-\frac{1}{n^\nu}\right), X^n(u-\tau(u))\right) dW(u)\Big\|^2$$

$$+K_1(T)\int_s^t 1_{D_n}(u)$$

$$\times E\left\|S^n_{t-u}F\left(X^n\left(u-\frac{1}{n^\nu}\right), X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right)\right\|^2 du$$

$$+K_1(T)\int_0^s 1_{D_n}(u)E\Big\|[(S_{t-s}-I)S^n_{s-u}]$$

$$\times F\left(X^n\left(u-\frac{1}{n^\nu}\right), X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right)\Big\|^2 du$$

$$+K_2(T)E\Big\|\int_s^t 1_{D_n}(u)S^n_{t-u}$$

$$\times G\left(X^n\left(u-\frac{1}{n^\nu}\right), X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right) dW(u)\Big\|^2$$

$$+K_2(T)E\Big\|\int_0^s 1_{D_n}(u)[(S_{t-s}-I)S^n_{s-u}]$$

$$\times G\left(X^n\left(u-\frac{1}{n^\nu}\right), X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right) dW(u)\Big\|^2.$$

Furthermore, an estimate in Pazy [13, p. 74] implies that for any $0<\alpha<1$,

$$E\|X^n(t)-X^n(s)\|^2$$

$$\leq E\|(S^n_t-S^n_s)\psi(0)\|^2 + K_1(T)\int_s^t 1_{D_n^c}(u)\|S^n_{t-u}\|^2$$

$$\times E\left\|F\left(X^n\left(u-\frac{1}{n^\nu}\right), X^n(u-\tau(u))\right)\right\|^2 du$$

$$+K_3(T)(t-s)^\alpha\int_0^s 1_{D_n^c}(u)\left(s-u+\frac{1}{n^\nu}\right)^{-\alpha}$$

$$\times E\left\|F\left(X^n\left(u-\frac{1}{n^\nu}\right), X^n(u-\tau(u))\right)\right\|^2 du$$

$$+K_2(T)\int_s^t 1_{D_n^c}(u)\|S^n_{t-u}\|^2$$

$$
\begin{aligned}
&\times E\left\| G\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n(u - \tau(u))\right)\right\|^2 du \\
&+K_4(T)(t-s)^\alpha \int_0^s 1_{D_n^c}(u)\left(s - u + \frac{1}{n^\nu}\right)^{-\alpha} \\
&\times E\left\| G\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n(u - \tau(u))\right)\right\|^2 du \\
&+K_1(T)\int_s^t 1_{D_n}(u)\|S^n_{t-u}\|^2 \\
&\times E\left\| F\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n\left(u - \tau(u) - \frac{1}{n^\nu}\right)\right)\right\|^2 du \\
&+K_3(T)(t-s)^\alpha \int_0^s 1_{D_n}(u)\left(s - u + \frac{1}{n^\nu}\right)^{-\alpha} \\
&\times E\left\| F\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n\left(u - \tau(u) - \frac{1}{n^\nu}\right)\right)\right\|^2 du \\
&+K_2(T)\int_s^t 1_{D_n}(u)\|S^n_{t-u}\|^2 \\
&\times E\left\| G\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n\left(u - \tau(u) - \frac{1}{n^\nu}\right)\right)\right\|^2 du \\
&+K_4(T)(t-s)^\alpha \int_0^s 1_{D_n}(u)\left(s - u + \frac{1}{n^\nu}\right)^{-\alpha} \\
&\times E\left\| G\left(X^n\left(u - \frac{1}{n^\nu}\right), X^n\left(u - \tau(u) - \frac{1}{n^\nu}\right)\right)\right\|^2 du \\
\le\ & E\|(S^n_t - S^n_s)\psi(0)\|^2 + K_5(T)\int_s^t 1_{D_n^c}(u)\|S^n_{t-u}\|^2 \\
&\times \left(1 + E\left\| X^n\left(u - \frac{1}{n^\nu}\right)\right\|^2 + E\|X^n(u-\tau(u))\|^2\right) du \\
&+K_6(T)(t-s)^\alpha \int_0^s 1_{D_n^c}(u)\left(s - u + \frac{1}{n^\nu}\right)^{-\alpha} \\
&\times \left(1 + E\left\| X^n\left(u - \frac{1}{n^\nu}\right)\right\|^2 + E\|X^n(u-\tau(u))\|^2\right) du
\end{aligned}
$$

$$
\begin{aligned}
&+K_7(T)\int_s^t 1_{D_n^c}(u)\|S_{t-u}^n\|^2\\
&\times\left(1+E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2+E\|X^n(u-\tau(u))\|^2\right)du\\
&+K_8(T)(t-s)^\alpha\int_0^s 1_{D_n^c}(u)\left(s-u+\frac{1}{n^\nu}\right)^{-\alpha}\\
&\times\left(1+E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2+E\|X^n(u-\tau(u))\|^2\right)du\\
&+K_5(T)\int_s^t 1_{D_n}(u)\|S_{t-u}^n\|^2\\
&\times\left(1+E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2+E\left\|X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right\|^2\right)du\\
&+K_6(T)(t-s)^\alpha\int_0^s 1_{D_n}(u)\left(s-u+\frac{1}{n^\nu}\right)^{-\alpha}\\
&\times\left(1+E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2+E\left\|X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right\|^2\right)du\\
&+K_7(T)\int_s^t 1_{D_n}(u)\|S_{t-u}^n\|^2\\
&\times\left(1+E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2+E\left\|X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right\|^2\right)du\\
&+K_8(T)(t-s)^\alpha\int_0^s 1_{D_n}(u)\left(s-u+\frac{1}{n^\nu}\right)^{-\alpha}\\
&\times\left(1+E\left\|X^n\left(u-\frac{1}{n^\nu}\right)\right\|^2+E\left\|X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right\|^2\right)du\\
\le\ &(t-s)^\alpha\int_0^s\left(s-u+\frac{1}{n^\nu}\right)^{-\alpha}du\\
&\cdot\left(K_9(T)+E\max_{0\le u\le T}\|X^n(u)\|^2\right)+K_{10}(T)(t-s)\\
\le\ &C_3(\alpha,T)(t-s)^\alpha+C_4(T)(t-s),\quad 0\le s\le t\le T.
\end{aligned}
\tag{2.9}
$$

Step 3. We next show that $\{X^n(t)\}$ converges to a limit in $L^2(\Omega, H)$ for each $t \in [0, T]$. To do so, letting $m > n \geq 2/\hat{\tau}$ and noticing $\|S(t)\| \leq Me^{\omega T}$ for all $t \in [0, T]$ and conditions (A.1) to (A.3), we easily see that there exist positive constants $M_1(T), M_2(T), \dots,$ such that

$$
\begin{aligned}
E\Big(\sup_{0\leq s\leq t} \|X^m(s) - X^n(s)\|^2\Big) &\leq M_1(T)\mu(D_n - D_m) + M_2(T) \\
&\quad \times \int_0^t E\left\| X^m\left(u - \frac{1}{m^\nu}\right) - X^n\left(u - \frac{1}{n^\nu}\right)\right\|^2 du \\
&\quad + M_2(T)\int_0^t 1_{D_n^c}(u) E\|X^m(u - \tau(u)) - X^n(u - \tau(u))\|^2 du \\
&\quad + M_2(T)\int_0^t 1_{D_m}(u) \\
&\qquad \times E\left\| X^m\left(u - \tau(u) - \frac{1}{m^\nu}\right) - X^n\left(u - \tau(u) - \frac{1}{n^\nu}\right)\right\|^2 du \\
&\quad + M_3(T)\left(\frac{1}{n^\nu} - \frac{1}{m^\nu}\right)^\alpha.
\end{aligned}
$$

On the other hand, we have, noting (2.9),

$$
\begin{aligned}
\int_0^t E\left\| X^m\left(u - \frac{1}{m^\nu}\right) - X^n\left(u - \frac{1}{n^\nu}\right)\right\|^2 du &\leq 2\int_0^t E\left\| X^m\left(u - \frac{1}{m^\nu}\right) - X^n\left(u - \frac{1}{m^\nu}\right)\right\|^2 du \\
&\quad + 2\int_0^t E\left\| X^n\left(u - \frac{1}{m^\nu}\right) - X^n\left(u - \frac{1}{n^\nu}\right)\right\|^2 du \\
&\leq 2\int_0^t E\|X^m(u) - X^n(u)\|^2 du \\
&\quad + 2\left(M_5(T)T\left[\frac{1}{n^\nu} - \frac{1}{m^\nu}\right] + M_6(\alpha, T)T\left[\frac{1}{n^\nu} - \frac{1}{m^\nu}\right]^\alpha\right)
\end{aligned}
$$

and

$$\int_0^t 1_{D_n^c}(u) E\| X^m(u-\tau(u)) - X^n(u-\tau(u))\|^2\, du \le \int_0^t E\Big(\sup_{0\le s\le u} \|X^m(s) - X^n(s)\|^2\Big)\, du$$

and

$$\begin{aligned}
&\int_0^t 1_{D_m}(u) E\left\| X^m\left(u-\tau(u)-\frac{1}{m^\nu}\right) - X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right\|^2 du \\
&\quad \le 2\int_0^t 1_{D_m}(u) E\left\| X^m\left(u-\tau(u)-\frac{1}{m^\nu}\right)\right. \\
&\qquad\qquad \left. - X^n\left(u-\tau(u)-\frac{1}{m^\nu}\right)\right\|^2 du \\
&\qquad + 2\int_0^t 1_{D_m}(u) E\left\| X^n\left(u-\tau(u)-\frac{1}{m^\nu}\right)\right. \\
&\qquad\qquad \left. - X^n\left(u-\tau(u)-\frac{1}{n^\nu}\right)\right\|^2 du \\
&\quad \le 2\int_0^t E\Big(\sup_{0\le s\le u}\|X^m(s)-X^n(s)\|^2\Big)\, du \\
&\qquad + 2\left(M_8(T)T\left[\frac{1}{n^\nu}-\frac{1}{m^\nu}\right] + M_7(\alpha,T)T\left[\frac{1}{n^\nu}-\frac{1}{m^\nu}\right]^\alpha\right).
\end{aligned}$$

Hence we easily obtain

$$\begin{aligned}
&E\Big(\sup_{0\le s\le t}\|X^m(s)-X^n(s)\|^2\Big) \\
&\quad \le M_1(T)\mu(D_n - D_m) \\
&\qquad + \left(M_8(T)T\left[\frac{1}{n^\nu}-\frac{1}{m^\nu}\right] + M_7(\alpha,T)T\left[\frac{1}{n^\nu}-\frac{1}{m^\nu}\right]^\alpha\right) \\
&\qquad + M_9(T)\int_0^t E\Big(\sup_{0\le s\le u}\|X^m(s)-X^n(s)\|^2\Big)\, du
\end{aligned}$$

which immediately implies that

$$E\left(\sup_{0\le s\le T}\|X^m(s)-X^n(s)\|^2\right)$$
$$\le\Bigg\{M_1(T)\mu(D_n-D_m)$$
$$+\left(M_8(T)T\left[\frac{1}{n^\nu}-\frac{1}{m^\nu}\right]+M_7(\alpha,T)T\left[\frac{1}{n^\nu}-\frac{1}{m^\nu}\right]^\alpha\right)\Bigg\}$$
$$\cdot\exp\{M_9(T)T\}. \tag{2.10}$$

Noticing $\mu(D_n-D_m)\to 0$ as $n,m\to\infty$, we immediately see that $\{X^n(t)\}$ is Cauchy in $L^2(\Omega; C([0,T],H))$. Denote the limit by $X(t)$ in $L^2(\Omega; C([0,T],H))$. A Borel−Cantelli argument easily gives that there exists a subsequence, say $\{X^{m_i}(t)\}$, which converges to $X(t)$ uniformly in $t\in[0,T]$ almost surely. Therefore $X(t)$ is an $\{\mathscr{F}\}_t$-adapted continuous H-valued process. Moreover, letting $m\to\infty$ in (2.10) we see

$$E\left(\sup_{0\le s\le T}\|X(s)-X^n(s)\|^2\right)du$$
$$\le C_5(T)\left(\frac{1}{n^\nu}+\frac{1}{n^{\alpha\nu}}+\mu\left\{t\colon 0<\tau(t)<\frac{1}{n^\nu},0\le t\le T\right\}\right).$$

Now letting $n\to\infty$ in (2.8), (2.9), we can immediately obtain our conclusion.

Step 4. We next extend $X(t)$ to $[-\hat\tau,T]$ by defining $X(t)=\psi(t)$ on $[-\hat\tau,T]$. We see that, to conclude the remainder of the proof, it suffices to show that $X(t)$ is the unique mild solution of Eq. (2.1) on $[-\hat\tau,T]$. Indeed, for $0\le t\le T$,

$$E\left\|X(t)-X^n\left(t-\frac{1}{n^\nu}\right)\right\|^2$$
$$\le 2E\|X(t)-X^n(t)\|^2+2E\left\|X^n(t)-X^n\left(t-\frac{1}{n^\nu}\right)\right\|^2$$
$$\le 2C_5(T)\left(\frac{1}{n^\nu}+\frac{1}{n^{\alpha\nu}}+\mu\left\{t\colon 0<\tau(t)<\frac{1}{n^\nu},0\le t\le T\right\}\right)$$
$$+\frac{2C_5'(T)}{n^\nu}+\frac{2C_5''(T)}{n^{\alpha\nu}}$$
$$\to 0,\qquad n\to\infty.$$

We can easily see that

$$\lim_{n\to\infty} E\left\| 1_{D_n^c}(t)F\left(X^n\left(t-\frac{1}{n^\nu}\right), X^n(t-\tau(t))\right) - 1_{D^c}(t)F(X(t), X(t-\tau(t)))\right\|^2 = 0$$

and

$$\lim_{n\to\infty} E\left\| 1_{D_n^c}(t)G\left(X^n\left(t-\frac{1}{n^\nu}\right), X^n(t-\tau(t))\right) - 1_{D^c}(t)G(X(t), X(t-\tau(t)))\right\|^2 = 0$$

for all $0 < t < T$, where $D^c = \{t\colon \tau(t) > 0, t \in [0,T]\}$. Therefore we also have

$$\lim_{n\to\infty} E\left\| 1_{D_n}(t)F\left(X^n\left(t-\frac{1}{n^\nu}\right), X^n\left(t-\tau(t)-\frac{1}{n^\nu}\right)\right) - 1_D(t)F(X(t), X(t-\tau(t)))\right\|^2 = 0$$

$$\lim_{n\to\infty} E\left\| 1_{D_n}(t)G\left(X^n\left(t-\frac{1}{n^\nu}\right), X^n\left(t-\tau(t)-\frac{1}{n^\nu}\right)\right) - 1_D(t)G(X(t), X(t-\tau(t)))\right\|^2 = 0$$

for all $0 < t < T$, where $D = \{t\colon \tau(t) = 0, t \in [0,T]\}$. Hence, we can let $n \to \infty$ in (2.4) to obtain

$$X(t) = S_t\psi(0) + \int_0^t S_{t-u}F(X(u), X(u-\tau(u)))\,du + \int_0^t S_{t-u}G(X(u), X(u-\tau(u)))\,dW(u)$$

on $0 < t < T$. That is, $X(t)$ is a mild solution of Eq. (2.1) over $[-\hat{\tau}, T]$.

The uniqueness of Eq. (2.1) can be obtained similarly by a Gronwall lemma argument. The proof is complete.

THEOREM 2.2. *For arbitrarily given* $T > 0$, *assume the assumptions* (A.1) *to* (A.3) *hold with* $\theta > 1$ *in* (A.3). *Let* $\nu > 2$ *in the Carathéodory approximate solution* (2.4). *Then the sequence* $\{X^n(t)\}$ *of approximate mild solutions converge in* $C_T(H)$ *almost surely to the solution* $X(t)$ *of Eq.* (2.1). *That is,*

$$\lim_{h \to \infty} \sup_{0 \le t \le T} \|X(t) - X^n(t)\| = 0 \quad a.s.$$

Proof. When $\nu > 2$, $\theta > 1$, by virtue of (2.7) and a Borel-Cantelli Lemma argument, we can easily prove our results.

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