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Non-BPS attractors in $5d$ and $6d$ extended supergravity

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Abstract

We connect the attractor equations of a certain class of $N = 2$, $d = 5$ supergravities with their $(1, 0)$, $d = 6$ counterparts, by relating the moduli space of non-BPS $d = 5$ black hole/black string attractors to the moduli space of extremal dyonic black string $d = 6$ non-BPS attractors. For $d = 5$ real special symmetric spaces and for $N = 4, 6, 8$ theories, we explicitly compute the flat directions of the black object potential corresponding to vanishing eigenvalues of its Hessian matrix. In the case $N = 4$, we study the relation to the $(2, 0)$, $d = 6$ theory. We finally describe the embedding of the $N = 2$, $d = 5$ magic models in $N = 8$, $d = 5$ supergravity as well as the interconnection among the corresponding charge orbits.

1. Introduction

Recently the study of the attractor equations for extremal black holes (BHs) [1–5] in four dimensions received special attention, especially in relation with new results on non-BPS, non-supersymmetric solutions [6–48].

Not much is known about non-BPS attractors in five dimensions, although general results for symmetric special geometries in BHs (and black strings) backgrounds were derived in [49]. More recently, it has been shown [36] that real special symmetric spaces have, in the non-BPS

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case, a moduli space of vacua, as it was the case for their $d = 4$ special Kähler descendants [21]. In four dimensions, massless Hessian modes for generic cubic geometries were shown to occur for the non-BPS case with non-vanishing central charge in [10,34]. Some additional insight on the correspondence among (the supersymmetry preserving features of) extremal BH attractors in four and five dimensions have been gained in [40], by relating the $d = 4$ and 5 BH potentials and the corresponding attractor equations. In particular, it was shown that the moduli space of non-BPS attractors in $d = 5$ real special symmetric geometries must be in the intersection of the moduli spaces of non-BPS $Z \neq 0$ and non-BPS $Z = 0$ attractors in the corresponding $d = 4$ special Kähler homogeneous geometries.

Aim of the present investigation is to perform concrete computations of the massless modes of the non-BPS $d = 5$ Hessian matrix, and further relate the $d = 5$ BH (or black string) potential to the $d = 6$ dyonic extremal black string potential and its BPS and non-BPS critical points, following the approach of [49] and [50]. This analysis reveals a noteworthy feature of the relation between $d = 5$ and $d = 6$. Namely, the moduli space of $d = 6$ non-BPS (with vanishing central charge[1]) dyonic string [51] attractors is a submanifold of the moduli space of $d = 5$ non-BPS attractors of symmetric real special geometries. The only exception is provided by the cubic reducible sequence of real special geometries, for which the non-BPS $d = 6$ and $d = 5$ moduli spaces actually coincide. It is worth pointing out that moduli spaces also exist, for particular non-BPS-supporting charge configurations, for all real special geometries with a $d = 6$ uplift [52]. This is the case for the homogeneous non-symmetric real special geometries studied in [53]. For $N = 2$, $d = 5$ magic supergravities, with the exception of the octonionic case, the non-BPS moduli spaces can also be obtained as suitable truncations of the moduli space of BPS sttractors of $N = 8$, $d = 5$ supergravity. In all cases, the Hessian matrix is semi-positive definite.

It is worth pointing out that in this work we consider only extremal black p-extended objects which are asymptotically flat, spherically symmetric and with an horizon geometry $AdS_{p+2} \times S^{d-p-2}$ [54]. Thus, we do not deal with, for instance, black rings and rotating BHs in $d = 5$, which however also exhibit an attractor behaviour (see, e.g., [55]).

The paper is organized as follows.

In Section 2 we recall some relevant facts about $N = 2$, $d = 6$ self-dual black string attractors and the properties of the black string effective potential in terms of the moduli space spanned by the tensor multiplets' scalars. In Section 3 we discuss the $d = 5$ effective potential in a six-dimensional language for the $d = 5$ models admitting a $d = 6$ uplift (including all homogeneous real special geometries classified in [53]), in the absence (Section 3.1) or presence (Section 3.2) of $d = 6$ vector multiplets. In Section 3.2.1 we perform an analysis of the attractors in $d = 5$, $N = 2$ magic supergravities, and comment on the moduli spaces of attractor solutions for such theories. Thence, in Sections 4.1, 4.2 and 4.3 we recall a similar analysis of the attractors respectively in $d = 5$, $N = 8$, 6 and 4 supergravities [49,56,57]. The analysis holds for all $N = 2$ symmetric spaces, as well as for homogeneous spaces by considering particular charge configurations. In Section 5 we comment on the conditions to be satisfied in order to obtain an anomaly-free $(1, 0)$, $d = 6$ supergravity by uplifting $N = 2$, $d = 5$ theories. Section 6 is devoted to final remarks and conclusions.

Appendix A discusses some group embeddings, relevant in order to elucidate the relation between the $N = 8$, $d = 5$ BPS unique orbit and the non-BPS orbits of the $N = 2$, $d = 5$ theories

[1] This means that non-BPS dyonic strings are neutral with respect to the central extension of the $(1, 0)$, $d = 6$ supersymmetry algebra.

obtained as consistent truncations of $N=8$ supergravity. Such $N=2$ theories include the magic supergravities based on the Jordan algebras $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$, $J_3^{\mathbb{R}}$ with $n_H=0,1,2$ hypermultiplets, respectively.

2. $(1,0)$, $d=6$ attractors for extremal dyonic strings

In $d=6$, $(1,0)$ and $(2,0)$ chiral supergravities[2] there are no BPS BH states, because the central extension of the corresponding $d=6$ superalgebras does not contain scalar central charges [59]. However, there are BPS (dyonic) string configurations, as allowed from the superalgebra, and extremal black string BPS attractors exist [49,50]. Such attractors preserve 4 supersymmetries, so they are the $d=6$ analogue of $d=5$ and $d=4$ $\frac{1}{2}$-BPS extremal BH attractors. Interestingly enough, extremal black string non-BPS attractors also exist in such $d=6$ theories [49], as it is the case for (extremal BH attractors) in $d=5$ and $d=4$. The next sections are partially devoted to such an issue.

Let us start by recalling the general structure of the minimal supergravity in $d=6$, the chiral $(1,0)$ theory. The field content of the minimal theory is:

- Gravitational multiplet[3]:

$$\left(V^a_\mu, \psi_{A\mu}, B^+_{\mu\nu}\right) \quad (\mu=0,1,\dots,5;\ A=1,2). \tag{2.1}$$

- Tensor multiplets:

$$\left(B^-_{\mu\nu}, \chi^A, \phi\right)^i \quad (i=1,\dots,q+1). \tag{2.2}$$

The scalar fields in the tensor multiplets sit in the coset space [60]

$$\frac{G}{H}=\frac{O(1,q+1)}{O(q+1)}. \tag{2.3}$$

They may be parametrized in terms of $q+2$ fields X^Λ, $(\Lambda=0,1,\dots,q+1)$, contstrained by the relation

$$X^\Lambda X^\Sigma \eta_{\Lambda\Sigma} \equiv X^\Lambda X_\Lambda = 1, \tag{2.4}$$

where $\eta_{\Lambda\Sigma}=\mathrm{diag}[1,-1,\dots,-1]$. The kinetic matrix for the tensors is:

$$G_{\Lambda\Sigma}=2X_\Lambda X_\Sigma-\eta_{\Lambda\Sigma}, \tag{2.5}$$

whose inverse matrix is:

$$G^{\Lambda\Sigma}=2X^\Lambda X^\Sigma-\eta^{\Lambda\Sigma}. \tag{2.6}$$

As for any $d=6$ theory, the field strengths of the antisymmetric tensors $H^\Lambda=dB^\Lambda$ have definite self-duality properties:

$$G_{\Lambda\Sigma}{}^\star H^\Sigma=\eta_{\Lambda\Sigma}H^\Sigma. \tag{2.7}$$

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[2] In the literature they are sometimes referred to as $(2,0)$ and $(4,0)$ respectively [58].

[3] Here and below the $SU(2)$ indices A up and down denote opposite chiralities.

As a consequence, there is no distinction between the associated electric and magnetic charges

$$e^\Lambda = \eta^{\Lambda\Sigma} e_\Sigma = \int_{S^3} H^\Lambda. \tag{2.8}$$

- Vector multiplets:

$$(A_\mu, \lambda_A)^\alpha \quad (\alpha = 1, \ldots, m). \tag{2.9}$$

The kinetic matrix for the vector field strengths is given in terms of a given constant matrix $C^\Lambda_{\alpha\beta}$ by [61]:

$$\mathcal{N}_{\alpha\beta} = X_\Lambda C^\Lambda_{\alpha\beta}. \tag{2.10}$$

- Hypermultiplets:

$$\big(\zeta^A, 4q\big)^\ell \quad (\ell = 1, \ldots, p). \tag{2.11}$$

The hypermultiplets do not play any role in the attractor mechanism, and will not be discussed further here.

Since the vector multiplets do not contain scalar fields, the only contribution to the black string effective potential comes from the tensor multiplets, and reads [50]:

$$V^{(6)} = G^{\Lambda\Sigma} e_\Lambda e_\Sigma = 2\big(X^\Lambda e_\Lambda\big)^2 - e^\Lambda e_\Lambda \tag{2.12}$$

or equivalently, in terms of the dressed central and matter charges $Z = (X^\Lambda e_\Lambda)$ and $Z_i = P_{i\Lambda} e^\Lambda$ (where $P^{\Lambda\Sigma}$, $P^{\Lambda\Sigma} X_\Sigma = 0$, is the projector orthogonal to the central charge):

$$V^{(6)} = Z^2 + Z_i Z^i. \tag{2.13}$$

The criticality conditions for the effective black string potential (2.13) reads

$$\partial_i V^{(6)} = 0 \quad \Leftrightarrow \quad Z Z_i = 0, \quad \forall i, \tag{2.14}$$

and therefore two different extrema are allowed, the BPS one for $Z_i = 0\ \forall i$, and a non-BPS one for $Z = 0$, both yielding the following critical value of $V^{(6)}$:

$$V^{(6)}\big|_{\mathrm{extr}} = \big|e^\Lambda e_\Lambda\big|. \tag{2.15}$$

3. $N = 2$, $d = 5$ attractors with a six-dimensional interpretation

In the absence of gauging, the minimal five-dimensional theory generally admits the following field content (omitting hypermultiplets):

- Gravitational multiplet:

$$\big(V^a_\mu, \psi_{A\mu}, A_\mu\big) \quad (\mu = 0, 1, \ldots, 4;\ A = 1, 2). \tag{3.1}$$

- Vector multiplets:

$$\big(A_\mu, \chi^A, \phi\big)^a \quad (a = 1, \ldots, n). \tag{3.2}$$

The scalar fields do not necessarily belong to a coset manifold, but their σ-model is described by real-special geometry. In particular, the scalar manifold is described by the locus

$$\mathcal{V}(L) = 1 \tag{3.3}$$

where $L^I(\phi)$, $I = 0, 1, \dots, n$ are function of the scalars and $\mathcal{V}$ is the cubic polynomial:

$$\mathcal{V}(L) = \frac{1}{3!} d_{IJK} L^I L^J L^K, \tag{3.4}$$

written in terms of an appropriate totally symmetric, constant matrix d_{IJK}. Note that in order to have a $d = 6$ uplift the real special geometry must have a certain structure, as discussed in [63]. Namely

$$\mathcal{V} = z X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma + X^\Lambda C_{\Lambda\alpha\beta} X^\alpha X^\beta. \tag{3.5}$$

This is always the case for the homogeneous spaces discussed in [53], where $C_{\Lambda\alpha\beta}$ is written in terms of the γ-matrices of $SO(q+1)$ Clifford algebras.

The kinetic matrix for the vector field-strengths has the general form:

$$a_{IJ} = -\partial_I \partial_J \log \mathcal{V}|_{\mathcal{V}=1}. \tag{3.6}$$

The BH effective potential in five-dimensions is given by

$$V^{(5)} = a^{IJ} q_I q_J, \tag{3.7}$$

where $q_I = \int_{S^3} \frac{\partial \mathcal{L}}{\partial F^I}$ are the electric charges and a^{IJ} the inverse of (3.6).

3.1. No $d = 6$ vector multiplets

We are interested in finding the relation of the six-dimensional attractor behavior to the five-dimensional one. Let us first consider the simplest case of a six-dimensional supergravity theory only coupled to $q + 1$ tensor multiplets (no vector multiplets). In this case, $n = q + 1$ and the scalar content is given by the six-dimensional scalars X^Λ plus the Kaluza–Klein (KK) dilaton z. The five-dimensional scalar fields are related by the constraint (3.3), where the surface expression (3.4) takes here the simple form[4]:

$$\mathcal{V}(L) = \mathcal{V}(z, X) = \frac{1}{2} z X^\Lambda X^\Sigma \eta_{\Lambda\Sigma}. \tag{3.1.1}$$

The constraint (3.3) then becomes:

$$\frac{1}{2} X^\Lambda X_\Lambda = z^{-1}. \tag{3.1.2}$$

The components of the kinetic matrix are in this case:

$$a_{IJ} = \begin{cases} a_{zz} = z^{-2}, \\ a_{z\Lambda} = 0, \\ a_{\Lambda\Sigma} = z \tilde{G}_{\Lambda\Sigma}, \end{cases} \tag{3.1.3}$$

[4] This corresponds to the $d = 5$ symmetric real spaces of the "generic sequence" $SO(1,1) \times \frac{SO(1,q+1)}{SO(q+1)}$ [62].

where the matrix $\tilde{G}$

$$\tilde{G}_{\Lambda\Sigma}(X) = 2\frac{X_\Lambda X_\Sigma}{X^\Gamma X_\Gamma} - \eta_{\Lambda\Sigma} \tag{3.1.4}$$

is related to G in (2.5) by

$$\tilde{G}_{\Lambda\Sigma}\big|_{X^\Lambda X_\Lambda = 1} = G_{\Lambda\Sigma}. \tag{3.1.5}$$

More precisely, setting:

$$\hat{X}^\Lambda \equiv \frac{X^\Lambda}{\sqrt{X^\Lambda X_\Lambda}} \quad \big(\hat{X}^\Lambda \hat{X}_\Lambda = 1\big), \tag{3.1.6}$$

we have:

$$\tilde{G}_{\Lambda\Sigma}(X) = G_{\Lambda\Sigma}(\hat{X}). \tag{3.1.7}$$

The matrix (3.1.3) is easily inverted giving:

$$a^{IJ} = \begin{cases} a^{zz} = z^2, \\ a^{z\Lambda} = 0, \\ a^{\Lambda\Sigma} = z^{-1}\tilde{G}^{\Lambda\Sigma}. \end{cases} \tag{3.1.8}$$

Then, in this case the BH effective potential takes the form:

$$V^{(5)} = z^2 e_z^2 + z^{-1}\tilde{G}^{\Lambda\Sigma}(X) e_\Lambda e_\Sigma = z^2 e_z^2 + z^{-1} V^{(6)}(\hat{X}), \tag{3.1.9}$$

where $(e_z, e_\Lambda) \equiv q_I$ denote the electric charges and, to obtain the last expression, we made use of (3.1.7). The physical interpretation of the charges e_z and e_Λ is the following: e_z is the Kaluza–Klein charge and e_Λ are the charges of dyonic strings [51]wrapped around S^1.

The extrema of $V^{(5)}$ are found for:

$$\frac{\partial V^{(5)}}{\partial z} = 0 \quad \Rightarrow \quad 2z e_z^2 - \frac{1}{2} z^{-2} V^{(6)}(\hat{X}) = 0, \tag{3.1.10}$$

which is the stabilization equation for the KK dilaton, solved by:

$$z = \left(\frac{V^{(6)}|_{\mathrm{extr}}}{2e_z^2}\right)^{\frac{1}{3}} \tag{3.1.11}$$

and for:

$$\frac{\partial V^{(5)}}{\partial \hat{X}^\Lambda} = 0 \quad \Rightarrow \quad \frac{\partial V^{(6)}}{\partial \hat{X}^\Lambda} = 0, \tag{3.1.12}$$

which shows that in this case the attractor solutions of the five-dimensional theory are precisely the same of the parent six-dimensional theory.

The BH entropy is now given by [49]:

$$\big(S_{\mathrm{BH}}^{(5)}\big)^{4/3} = V^{(5)}\big|_{\mathrm{extr}} = 3\left(\frac{1}{2} e_z V^{(6)}\big|_{\mathrm{extr}}\right)^{\frac{2}{3}} = 3\left(\frac{1}{2} e_z e^\Lambda e_\Lambda\right)^{\frac{2}{3}}. \tag{3.1.13}$$

The solution of Eq. (3.1.12) depends on whether the $d = 6$ attractor is BPS or not. As previously mentioned, the $d = 6$ BPS attractors correspond to $Z_i = 0$ $\forall i$, whereas the non-BPS ones are given by $Z = 0$ (and $Z_i \neq 0$ for at least some i) [49,50]. Thus, all $q + 1$ $d = 6$ BPS moduli are fixed, while there are q non-BPS flat directions, spanning the $d=6$ non-BPS moduli space $\frac{SO(1,q)}{SO(q)}$ [49].

The supersymmetry-preserving features (BPS or non-BPS) of the $d=6$ attractors solutions depend on the sign of $e^{\Lambda}e_{\Lambda}$: it is BPS for $e^{\Lambda}e_{\Lambda}>0$ and non-BPS for $e^{\Lambda}e_{\Lambda}<0$. In this latter case, also the $d=5$ solution is non-BPS, because in a given frame [52] $e_z e^{\Lambda}e_{\Lambda}=e_z e_+ e_-$ (with $e_{\pm}\equiv e_1\pm e_2$), and if $e_+e_-<0$ the three charges cannot have the same sign [40]. On the other hand, if $e^{\Lambda}e_{\Lambda}>0$ one can have both BPS and non-BPS $d=5$ solutions [40].

Thus, we can conclude that for the "generic sequence" of $d=5$ symmetric real special spaces the non-BPS moduli space, predicted in [36], does indeed coincide with the above mentioned $d=6$ (tensor multiplets') non-BPS moduli space, found in [49].

3.2. Inclusion of $d=6$ vector multiplets

Let us now generalize the discussion to the case where s extra vector multiplets:

$$\left(A_{\mu},\lambda^{A},Y\right)^{\alpha},\quad \alpha=1,\ldots,s, \tag{3.2.1}$$

corresponding to the dimensional reduction of six-dimensional ones, are present [63]. The reduction may be done preserving the $SO(1,q+1)$ symmetry when the number s of $d=6$ vector multiplets coincides with the dimension of the spinor representation of $SO(1,q+1)$:

$$s=\dim\left[\operatorname{spin} SO(1,q+1)\right]. \tag{3.2.2}$$

This implies that the kinetic matrix of the $d=6$ vector fields is positive definite and no phase transitions, as discussed in [63,64], occur in this class of models.

The extra scalars contribute to the general relations (3.6) and (3.7) via a modification of the cubic form $\mathcal{V}$ into [53]:

$$\mathcal{V}=\frac{1}{2}zX^{\Lambda}X^{\Sigma}\eta_{\Lambda\Sigma}+\frac{1}{2}X_{\Lambda}Y^{\alpha}Y^{\beta}\Gamma^{\Lambda}_{\alpha\beta}. \tag{3.2.3}$$

The total number of five-dimensional scalars is then $q+2+s$. Of particular interest are the four magic models which are associated with the simple Jordan algebras having an irreducible norm form (displayed in Table 4 of [36]). In these cases $q=1,2,4,8$ and $s=2q$. Also the "generic sequence" $L(0,P)$ can be viewed as a particular case of Eq. (3.2.3) with $q=0$ and $s=P$.

The $d=6$ origin of the second term in Eq. (3.2.3) is the kinetic term of the $d=6$ vector fields, which reads [61,63] ($\Lambda=0,1,\ldots,q+1$, $\alpha=1,\ldots,s$, $C^{\Lambda}_{\alpha\beta}=C^{\Lambda}_{\beta\alpha}$)

$$X_{\Lambda}C^{\Lambda}_{\alpha\beta}F^{\alpha}\wedge^{*}F^{\beta}. \tag{3.2.4}$$

Thus, in the presence of $d=6$ BH charges Q^{α}, it originates an effective $d=6$ BH potential of the form

$$V^{(6)}_{\mathrm{BH}}=X_{\Lambda}C^{\Lambda}_{\alpha\beta}Q^{\alpha}Q^{\beta}. \tag{3.2.5}$$

Such a potential has run-away extrema at $d=6$ [58]. This can be seen for instance in the case $n_T=1\Leftrightarrow q=0$, where Eq. (3.2.5) reduces to ($\alpha=1,\ldots,P$, $X_0=\cosh\phi$, $X_1=\sinh\phi$)

$$V^{(6)}_{\mathrm{BH}}(\phi)=\cosh\phi C^{0}_{\alpha\beta}Q^{\alpha}Q^{\beta}+\sinh\phi C^{1}_{\alpha\beta}Q^{\alpha}Q^{\beta}=e^{\phi}Q^{\alpha}Q^{\alpha} \tag{3.2.6}$$

(in the last step we used the fact that in the $n_T = 1$ case we may set $C^0_{\alpha\beta} = C^1_{\alpha\beta} = \delta_{\alpha\beta}$ without loss of generality). Consequently

$$\frac{\partial V^{(6)}_{\mathrm{BH}}(\phi)}{\partial\phi} = 0 \quad \Leftrightarrow \quad V^{(6)}_{\mathrm{BH}}(\phi) = 0 \quad \Leftrightarrow \quad \phi = -\infty. \tag{3.2.7}$$

We then conclude that non-BPS extremal BH attractors are excluded in $(1,0)$ supergravity in six dimensions. However, we can have a 0-dimensional black object by an intersection of a $d = 6$ BH with a $d = 6$ black string. Its reduction to $d = 5$ gives a BH which carries both the string charge and the BH charge, with cubic invariant of the form [65]

$$I_3 = e_z e^\Lambda e_\Lambda + e_\Lambda C^\Lambda_{\alpha\beta} Q^\alpha Q^\beta, \tag{3.2.8}$$

and the $d = 5$ resulting BH entropy $S^{(5)}_{\mathrm{BH}} \sim \sqrt{|I_3|}$. Thus, even if the KK charge e_z vanishes, one gets a contribution from the second term of Eq. (3.2.8). This is in contrast with the case of the $d = 6$ dyonic extremal black string treated in Section 3.1, where the non-vanishing of the KK charge e_z was needed in order to get a non-vanishing entropy for the corresponding $d = 5$ BH, obtained by wrapping the $d = 6$ string on S^1.

The inclusion of extra multiplets corresponding to $d = 6$ vector multiplets entails a significative complication in the model. In particular, the moduli space of the non-BPS attractors drastically changes with respect to the case described in Section 3.1. As we shall prove below, in the magic models the number of moduli becomes equal to $s = 2q$ instead of q as it was in the absence of these extra multiplets.

Before entering into the detail of the magic models, let us argue the existence, at least for the homogeneous spaces $L(q, P)$ (and, for $q = 4m$, $L(q, P, P')$) [53], of particular non-BPS critical points where the same results of Section 3.1 may still be directly applied. Indeed, it turns out that for the four magic models the non-BPS attractor moduli spaces of dimension $2q$ always contain as a subspace precisely the coset $\frac{SO(1,q)}{SO(q)}$ (that is the moduli space of $d = 6$ non-BPS attractors for $q + 1$ strings, as discussed above). Such submanifold of the moduli space may be obtained by considering the particular critical point where $Y^\alpha = 0$. This critical point may always be reached because, as (3.2.3) and (3.6) show, the Y coordinates always appear quadratically in the effective potential (3.7). Then, for $Y^\alpha = 0$ the effective potential reduces to the one previously considered (Eq. (3.1.9)), whose non-BPS attractor solution is known to have q flat directions belonging to the coset $\frac{SO(1,q)}{SO(q)}$. This is in fact only half the total number of flat directions for these solutions. It may be understood because the non-compact stabilizer of the non-BPS orbit (that is for example $F_{4(-20)} \supset SO(1, 8)$ for $q = 8$ [66,67]), mixes the X with Y variables, so that the restriction $\{Y^\alpha\} = 0$ implies the reduction of the orbit to its subgroup $SO(1, q)$. The same considerations may be directly extended, for charge configurations where the spinorial charges are set to zero, to the series of homogeneous non-symmetric spaces $L(q, P)$ (and, for $q = 4m$, $L(q, P, P')$) [53], which always admit a non-BPS attractor point where all the spinorial moduli are zero. As before, this condition selects the submanifold $\frac{SO(1,q)}{SO(q)}$ of the non-BPS attractor moduli space, with the only difference that in this case the number q is not directly related to the number of spinorial moduli.

3.2.1. $N = 2$ magic models

For $N = 2$ supergravity, one can apply the general relations of real special geometry [49,62], so that the effective potential

$$V(\phi, q) = a^{IJ} q_I q_J \tag{3.2.1.1}$$

takes a simpler form. Indeed, for $N=2$ supergravity the vector kinetic matrix a_{IJ} is related to the metric g_{xy} of the scalar manifold via

$$a_{IJ} = h_I h_J + \frac{3}{2} h_{I,x} h_{J,y} g^{xy}, \tag{3.2.1.2}$$

$a^{IJ} = h^I h^J + \frac{3}{2} h^I_{,x} h^J_{,y} g^{xy}$ or conversely

$$g_{xy} = \frac{3}{2} h_{I,x} h_{J,y} a^{IJ}. \tag{3.2.1.3}$$

In terms of these quantities the central charge is

$$Z = q_I h^I \tag{3.2.1.4}$$

and we can write the potential as

$$V(q,\phi) = Z^2 + \frac{3}{2} g^{xy} \partial_x Z \partial_y Z, \tag{3.2.1.5}$$

where $\partial_x Z = q_I h^I_{,x} = P^a_x Z_a$ are the matter charges. The index $x = 1, \ldots, n_V$ is a world index labelling the scalar fields while a is the corresponding rigid index. P^a_x denotes the scalar vielbein. The matter charges obey the differential relations:

$$\begin{aligned} \nabla Z &= P^a Z_a, \\ \nabla Z_a &= \frac{2}{3} g_{ab} P^b Z - \sqrt{\frac{2}{3}} T_{abc} P^b g^{cd} Z_d. \end{aligned} \tag{3.2.1.6}$$

To make explicit computations of the attractor points of the potential and of the corresponding Hessian matrix, let us use the property that both T_{abc} and g_{ab}, written in rigid indices, are invariants of the group $SO(q+1)$, where $q = 1, 2, 4, 8$ for the magic models, corresponding to the symmetric spaces $L(q,1)$. The H-representation $\mathbf{R}$ of the scalar fields branch with respect to $SO(q+1)$ in the following way

$$\mathbf{R} \to \mathbf{1} + (\mathbf{q}+\mathbf{1}) + \mathbf{R_s}, \tag{3.2.1.7}$$

where $\mathbf{R_s}$ is the real Clifford module of $SO(q+1)$ of dimensions $\dim(\mathbf{R_s}) = 2, 4, 8, 16$ corresponding to the four values of q. The index a split into the indices $1, m, \alpha$, where $m = 1, \ldots, q+1$ and $\alpha = 1, \ldots, \dim(\mathbf{R_s})$. Let us write the general form for T_{abc} and g_{ab}:

$$\begin{aligned} &g_{11} = \alpha, \qquad g_{mn} = \beta \delta_{mn}, \qquad g_{\alpha\beta} = \gamma \delta_{\alpha\beta}, \\ &T_{111} = \sqrt{\frac{\alpha}{2}} g_{11}, \qquad T_{1mn} = -\sqrt{\frac{\alpha}{2}} g_{mn}, \qquad T_{1\alpha\beta} = \frac{1}{2}\sqrt{\frac{\alpha}{2}} g_{\alpha\beta}, \\ &T_{n\alpha\beta} = -\frac{1}{2}\gamma \sqrt{\frac{3}{2}\beta} \Gamma_{n\alpha\beta}, \end{aligned} \tag{3.2.1.8}$$

where Γ_n are the (symmetric, real) $SO(q+1)$ gamma matrices in the $\mathbf{R_s}$ representation. The coefficients of T_{abc} are determined in terms of the coefficients of g_{ab} by the following relation:

$$T_{a(bc} T^a_{ef)} = \frac{1}{2} g_{(bc} g_{ef)}. \tag{3.2.1.9}$$

The potential V can be written in the following useful form:

$$V = Z^2 + \frac{3}{2} g^{ab} Z_a Z_b = Z^2 + \frac{3}{2}\left(Z_1 Z^1 + Z_n Z^n + Z_\alpha Z^\alpha\right), \tag{3.2.1.10}$$

where the following short-hand notation is used: $Z^a \equiv g^{ab} Z_b$. Let us now compute the extrema of V. Using Eqs. (3.2.1.6) we find

$$\begin{aligned}\nabla V = {} & P^1\left[4ZZ_1 - \sqrt{3\alpha}\left(Z_1 Z^1 - Z_n Z^n + \frac{1}{2} Z_\alpha Z^\alpha\right)\right] \\ & + P^n\left(4ZZ_n + 2\sqrt{\frac{3}{\alpha}} Z_1 Z_n + \frac{3}{2\gamma}\sqrt{\beta}\,\Gamma_{n\alpha\beta} Z_\alpha Z_\beta\right) \\ & + P^\alpha\left(4ZZ_\alpha - \frac{3}{2}\sqrt{\frac{3}{\alpha}} Z_1 Z_\alpha + \frac{3}{\sqrt{\beta}}\Gamma_{n\alpha\beta} Z_n Z_\beta\right). \qquad (3.2.1.11)\end{aligned}$$

It is straightforward to see that the above expression has two zeroes corresponding to the two attractors:

- *BPS attractor*: $Z_n = Z_\alpha = Z_1 = 0$ and the potential at the extremum reads $V_0 = Z^2$.
- *Non-BPS attractor*: $Z_n = Z_\alpha = 0$, $Z = \frac{1}{4}\sqrt{\frac{3}{\alpha}} Z_1$ and the potential at the extremum reads $V_0 = 9Z^2$.

Let us now compute the Hessian matrix:

$$\begin{aligned}\nabla^2 V = {} & \left(P^1\right)^2\left\{\frac{8\alpha}{3}\left[\left(Z - 2\frac{1}{4}\sqrt{\frac{3}{\alpha}} Z_1\right)^2 + 8\left(\frac{1}{4}\sqrt{\frac{3}{\alpha}} Z_1\right)^2\right] + 2\frac{\alpha}{\beta} Z_n^2 + \frac{\alpha}{2\gamma} Z_\alpha^2\right\} \\ & + P^1 P^n\left(8Z_1 Z_n + 16\sqrt{\frac{\alpha}{3}} Z Z_n - \frac{\sqrt{3\alpha\beta}}{\gamma}\Gamma_{n\alpha\beta} Z_\alpha Z_\beta\right) \\ & + P^1 P^\alpha\left(11 Z_1 Z_\alpha - 8\sqrt{\frac{\alpha}{3}} Z Z_\alpha + \sqrt{\frac{3\alpha}{\beta}}\Gamma_{n\alpha\beta} Z_n Z_\beta\right) \\ & + P^n P^m\left\{6 Z_n Z_m + \left[\frac{8\beta}{3}\left(Z + 2\frac{1}{4}\sqrt{\frac{3}{\alpha}} Z_1\right)^2 + \frac{3\beta}{2\gamma} Z_\alpha^2\right]\delta_{mn}\right\} \\ & + P^n P^\alpha\left(6 Z_n Z_\alpha + 8\sqrt{\beta}\,\Gamma_{n\alpha\beta} Z Z_\alpha + \sqrt{\frac{3\beta}{\alpha}}\Gamma_{n\alpha\beta} Z_\beta Z_1 + 3(\Gamma_m \Gamma_n)_{\alpha\beta} Z_m Z_\beta\right) \\ & + P^\alpha P^\beta\left[\frac{8}{3}\gamma\left(Z - \frac{1}{4}\sqrt{\frac{3}{\alpha}} Z_1\right)^2 \delta_{\alpha\beta} + \frac{3\gamma}{2\beta} Z_n^2 \delta_{\alpha\beta} + \frac{4\gamma}{\sqrt{\beta}}\Gamma_{n\alpha\beta} Z Z_n \right. \\ & \left. - \gamma\sqrt{\frac{3}{\alpha\beta}}\Gamma_{n\alpha\beta} Z_1 Z_n + \frac{3}{2}\Gamma_{n\alpha\delta}\Gamma_{n\beta\gamma} Z_\delta Z_\gamma + \frac{9}{2} Z_\alpha Z_\beta\right]. \qquad (3.2.1.12)\end{aligned}$$

At the BPS critical point it is straightforward to check that:

$$\nabla^2 V = \frac{8}{3} Z^2 g_{ab} P^a P^b. \qquad (3.2.1.13)$$

As expected, the BPS critical point is a stable attractor. At the non-BPS attractor the Hessian reads:

$$\nabla^2 V = 24 Z^2\left[g_{11}\left(P^1\right)^2 + g_{mn} P^n P^m\right]. \qquad (3.2.1.14)$$

The moduli space is therefore spanned by the scalar fields in the $\mathbf{R_s}$ representation. These can be regarded as particular coordinates of the moduli spaces of the $N = 2$, $d = 5$ non-BPS solutions of the magic models $J_3^{\mathbb{O}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$, which respectively are $\frac{F_{4(-20)}}{SO(9)}$, $\frac{USp(4,2)}{USp(4)\times USp(2)}$, $\frac{SU(2,1)}{SU(2)\times U(1)}$ and $\frac{SL(2,\mathbb{R})}{SO(2)}$ (see Table 4 of [36]). It is worth pointing out that, with the exception of $J_3^{\mathbb{O}}$, all such spaces can be obtained as consistent truncations of the $N=8$, $d=5$ BPS attractor moduli space $\frac{F_{4(4)}}{USp(6)\times USp(2)}$ (quaternionic Kähler), by performing an analysis which is the $d=5$ counterpart of the $d=4$ analysis exploited in [33]. Since for $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$ the $N=8\to N=2$ reduction preserves $n_H=1$ and $n_H=2$ hypermultiplets respectively, the following inclusions must hold:

$$J_3^{\mathbb{C}}:\quad F_{4(4)}\supset \big(SU(2,1)\big)^2 \quad\Rightarrow\quad \frac{F_{4(4)}}{USp(6)\times USp(2)} \supset \frac{SU(2,1)}{SU(2)\times U(1)}\times\frac{SU(2,1)}{SU(2)\times U(1)}, \tag{3.2.1.15}$$

$$J_3^{\mathbb{R}}:\quad F_{4(4)}\supset SL(2,\mathbb{R})\times G_{2(2)} \quad\Rightarrow\quad \frac{F_{4(4)}}{USp(6)\times USp(2)} \supset \frac{SL(2,\mathbb{R})}{SO(2)}\times\frac{G_{2(2)}}{SO(4)}. \tag{3.2.1.16}$$

The two group embeddings given by Eqs. (3.2.1.15) and (3.2.1.16) are discussed in Appendix A.

On the other hand, the truncation generating $J_3^{\mathbb{H}}$ implies

$$J_3^{\mathbb{H}}:\quad F_{4(4)}\supset USp(4,2) \quad\Rightarrow\quad \frac{F_{4(4)}}{USp(6)\times USp(2)}\supset\frac{USp(4,2)}{USp(4)\times USp(2)}. \tag{3.2.1.17}$$

In this case, the $\mathbf{42}$ of $USp(8)$ decomposes along $USp(6)\times USp(2)$ as $\mathbf{42}\to(\mathbf{14},\mathbf{1})\oplus(\mathbf{14}',\mathbf{2})$. The $\mathbf{14}$ and $\mathbf{14}'$ of $USp(6)$ further decompose with respect to $USp(4)\times USp(2)$ (maximal compact subgroup of the stabilizer $USp(4,2)$ of the non-BPS orbit) as follows:

$$\begin{aligned} \mathbf{14} &\quad\to\quad (\mathbf{1},\mathbf{1})\oplus(\mathbf{5},\mathbf{1})\oplus(\mathbf{4},\mathbf{2}),\\ \mathbf{14}' &\quad\to\quad (\mathbf{5},\mathbf{2})\oplus(\mathbf{4},\mathbf{1}). \end{aligned} \tag{3.2.1.18}$$

Thus, the decomposition of the $(\mathbf{14},\mathbf{1})$ and $(\mathbf{14}',\mathbf{2})$ of $USp(6)\times USp(2)$ with respect to $USp(4)\times USp(2)\times USp(2)$ read:

$$\begin{aligned} &\text{massive:}\quad (\mathbf{14},\mathbf{1})\quad\to\quad(\mathbf{1},\mathbf{1},\mathbf{1})\oplus(\mathbf{5},\mathbf{1},\mathbf{1})\oplus(\mathbf{4},\mathbf{2},\mathbf{1}),\\ &\text{massless:}\quad (\mathbf{14}',\mathbf{2})\quad\to\quad(\mathbf{5},\mathbf{2},\mathbf{2})\oplus(\mathbf{4},\mathbf{1},\mathbf{2}). \end{aligned} \tag{3.2.1.19}$$

Since in the non-BPS case the $N=2$ $\mathcal{R}$-symmetry is the $USp(2)\sim SU(2)$ inside $USp(6)$ (i.e., the first $USp(2)$ in the decomposition (3.2.1.19)) one obtains 8 massive and 20 massless hypermultiplets' degrees of freedom, and 6 massive and 8 massless vectors' degrees of freedom. Notice that, since in the BPS case the $N=2$ $\mathcal{R}$-symmetry is the $USp(2)\sim SU(2)$ commuting with $USp(6)$ (i.e., the second $USp(2)$ in the decomposition (3.2.1.19)), the non-BPS case differs from the BPS case only by an exchange of the $(\mathbf{4},\mathbf{2},\mathbf{1})$ representation with the $(\mathbf{4},\mathbf{1},\mathbf{2})$ one.

4. Purely five-dimensional analysis of attractors in N-extended theories

For any extended supergravity in five dimensions the BH potential can be written in terms of the dressed charges in the following form [56,57]:

$$V(\phi,q)=\frac{1}{2}Z_{AB}Z^{AB}+X^2+Z_IZ^I \tag{4.1}$$

where Z_{AB} $(A, B = 1, \ldots, N)$ are the antisymmetric, $Sp(N)$-traceless graviphoton central charges, X the trace part while Z_I $(I = 1, \ldots, n)$ denote the matter charges (which only appear for $N \leqslant 4$ theories). For all the models with a scalar sector spanning a symmetric space, the dressed charges obey some known differential relations in moduli space which allow to explicitly find the attractor condition as an extremum for the scalar potential in moduli space:

$$\frac{\partial V}{\partial \phi^i} = 0. \tag{4.2}$$

We are going to study in the following the BPS and non-BPS attractors for the various cases.

4.1. $N = 8$, $d = 5$ and $(2, 2)$, $d = 6$

The scalar manifold is the coset

$$G/H = \frac{E_{6(6)}}{Sp(8)} \tag{4.1.1}$$

and the BH potential takes the form:

$$V = \frac{1}{2} Z_{AB} Z^{AB}. \tag{4.1.2}$$

The differential relations among the 27 central charges Z_{AB} (satisfying $Z_{AB}\Omega^{AB} = 0$), are:

$$\nabla Z_{AB} = \frac{1}{2} Z^{CD} P_{ABCD}, \tag{4.1.3}$$

where the vielbein $P_{ABCD} = P_{ABCD,i} d\phi^i$ satisfies the conditions

$$P^{ABCD} = P^{[ABCD]}, \qquad P^{ABCD}\Omega_{AB} = 0. \tag{4.1.4}$$

The extremum condition is then

$$\nabla V = Z_{AB} \nabla Z^{AB} = \frac{1}{2} P^{ABCD} Z_{AB} Z_{CD} = 0. \tag{4.1.5}$$

To explicitly find the solution, it is convenient to put the central-charge matrix in normal form:

$$Z_{AB} = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & -e_1 - e_2 - e_3 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.1.6}$$

and to truncate the theory to the "charged" submanifold spanned by the vielbein components that couple to the dressed charge in normal form, that is:

$$\begin{aligned} &P_1 \equiv P_{1234} = P_{5678}, \qquad P_2 \equiv P_{1256} = P_{3478}, \\ &\text{while} \quad P_{3456} = P_{1278} = -P_1 - P_2. \end{aligned} \tag{4.1.7}$$

In this way, the covariant derivatives of the charges (4.1.3) become:

$$\begin{aligned} \nabla e_1 &= (e_1 + 2e_2 + e_3) P_1 + (e_1 + e_2 + 2e_3) P_2, \\ \nabla e_2 &= (e_1 - e_3) P_1 + (-e_1 - e_2 - 2e_3) P_2, \\ \nabla e_3 &= (-e_1 - 2e_2 - e_3) P_1 + (e_1 - e_2) P_2. \end{aligned} \tag{4.1.8}$$

Using these relations, the extremum condition of V becomes

$$\nabla V = 4\{P_1(e_1 - e_3)(e_1 + 2e_2 + e_3) + P_2(e_1 - e_2)(e_1 + e_2 + 2e_3)\} = 0. \tag{4.1.9}$$

It admits only one solution with finite area, which breaks the symmetry $Sp(8) \to Sp(2) \times Sp(6)$. Up to $Sp(6)$ rotations it is:

$$e_2 = e_3 = -\frac{1}{3}e_1, \qquad V_{\text{extr}} = \frac{4}{3}e_1^2 = \frac{4}{3}M_{\text{extr}}^2. \tag{4.1.10}$$

This is a BPS attractor, supported by the unique BPS orbit [49] $\frac{E_{6(6)}}{F_{4(4)}}$, and the maximum amount of supersymmetry preserved by the solution at the horizon is $1/4$ ($N = 8 \to N = 2$).

As mentioned above, the two vielbein-components P_1 and P_2 span the submanifold of the moduli space which couples to the proper values of the central charge. This automatically projects out, in the $N = 2$ reduced theory, the 28 scalar degrees of freedom corresponding to the hypermultiplets.

The Hessian matrix reads

$$H_{ij} \equiv \nabla_i \nabla_j V = \frac{1}{4} P_{ABLM} P^{CDLM} Z^{AB} Z_{CD}. \tag{4.1.11}$$

To have the complete spectrum of massive plus flat directions, we have to consider in (4.1.11) the complete vielbein P_{ABCD}. On the solution, where $Sp(8) \to Sp(2) \times Sp(6)$ ($A \to (\alpha, a)$, $\alpha = 1, 2$, $a = 1, \dots, 6$), the vielbein degrees of freedom decompose as

$$\begin{aligned} \mathbf{42} \quad &\to \quad (\mathbf{14}, \mathbf{1}) + (\mathbf{14}', \mathbf{2}), \\ P_{ABCD} \quad &\to \quad P_{\alpha\beta ab} + P_{\alpha abc}, \end{aligned} \tag{4.1.12}$$

where $P_{\alpha\beta ab} = \epsilon_{\alpha\beta} P_{ab}$ (satisfying $P_{ab}\Omega^{ab} = 0$) is the vielbein of the $\frac{SU^*(6)}{Sp(6)}$ $N = 2$ vector multiplet sigma model, while $P_{\alpha abc}$ (satisfying $P_{\alpha abc}\Omega^{ab} = 0$) spans the $N = 2$ hyperscalar sector. Note that, at the horizon, from (4.1.6) and (4.1.10) we find, for the central charge in normal form:

$$Z_{AB} \quad \to \quad (Z_{ab} = e\Omega_{ab};\ Z_{\alpha\beta} = -3e\epsilon_{\alpha\beta}). \tag{4.1.13}$$

The Hessian matrix (4.1.11) is then:

$$\begin{aligned} H_{ij} &= \frac{1}{4}\left(P_{abLM} Z^{ab} + P_{\alpha\beta LM} Z^{\alpha\beta}\right)\left(P^{cdLM} Z_{cd} + P^{\gamma\delta LM} Z_{\gamma\delta}\right) \\ &= 9e^2 P_{LM,i} P^{LM}_{,j}. \end{aligned} \tag{4.1.14}$$

The hyperscalar vielbein $P_{\alpha abc}$ do not appear in (4.1.14) so that the corresponding directions do not acquire a mass. The moduli space of the solution is then [36] $\frac{F_{4(4)}}{USp(6) \otimes USp(2)}$.

The $N = 8$, $d = 5$ theory has an uplift to $(2, 2)$, $d = 6$ supergravity, whose scalar manifold is $\frac{SO(5,5)}{SO(5) \times SO(5)}$ [50]. In such a theory, the unique orbit with non-vanishing area is the $\frac{1}{4}$-BPS orbit $\frac{SO(5,5)}{SO(5,4)}$ [68], specified by an $SO(5, 5)$ charge vector e_Λ with non-vanishing norm $e_\Lambda e^\Lambda \neq 0$. The corresponding moduli space of $\frac{1}{4}$-BPS attractors is $\frac{SO(5,4)}{SO(5) \times SO(4)}$, and it is indeed contained [67] in the $N = 8$, $d = 5$ $\frac{1}{8}$-BPS moduli space $\frac{F_{4(4)}}{USp(6) \times USp(2)}$, as implied by our analysis. Note that the two non-compact forms of F_4 which occur in $N = 2$ and $N = 8$, $d = 5$ supergravities precisely contain the two non-compact forms of $SO(9)$ present in the corresponding moduli spaces [67]: $F_{4(-20)} \supset SO(1, 8)$ and $F_{4(4)} \supset SO(5, 4)$.

4.2. $N=6$ ($N=2$, $J_3^{\mathbb{H}}$)

The scalar manifold is the coset

$$G/H=\frac{SU^*(6)}{USp(6)}, \tag{4.2.1}$$

the BH potential takes the form:

$$V=\frac{1}{2}Z_{AB}Z^{AB}+\frac{1}{3}X^2 \tag{4.2.2}$$

and the differential relations among the $14+1$ central-charges Z_{AB} (satisfying $Z_{AB}\Omega^{AB}=0$) and X, are:

$$\begin{aligned}&\nabla Z_{AB}=\Omega^{CD}Z_{C[A}P_{B]D}+\frac{1}{6}\Omega_{AB}Z_{CD}P^{CD}+\frac{1}{3}XP_{AB},\\&\nabla X=\frac{1}{2}Z_{AB}P^{AB},\end{aligned} \tag{4.2.3}$$

where $P_{AB}=P_{AB,i}d\phi^i$ is the Ω-traceless vielbein of G/H satisfying the conditions

$$P^{AB}=P^{[AB]},\qquad P^{AB}\Omega_{AB}=0. \tag{4.2.4}$$

To study the attractors, it is convenient to put the central-charge matrix in normal form:

$$Z_{AB}=\begin{pmatrix}e_1&0&0\\0&e_2&0\\0&0&-e_1-e_2\end{pmatrix}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \tag{4.2.5}$$

so that the BH potential takes the form

$$V=e_1^2+e_2^2+(e_1+e_2)^2+\frac{1}{3}X^2. \tag{4.2.6}$$

The vielbein components that couple to the dressed charges in normal form are:

$$\begin{aligned}&P_1\equiv P_{12},\qquad P_2\equiv P_{34},\\&\text{while}\quad P_{56}=-P_1-P_2.\end{aligned} \tag{4.2.7}$$

In this way, the covariant derivatives of the charges (4.2.3) become:

$$\begin{aligned}&\nabla e_1=\frac{1}{3}(-e_1+e_2+X)P_1+\frac{1}{3}(e_1+2e_2)P_2,\\&\nabla e_2=\frac{1}{3}(2e_1+e_2)P_1+\frac{1}{3}(e_1-e_2+X)P_2,\\&\nabla X=(2e_1+e_2)P_1+(e_1+2e_2)P_2.\end{aligned} \tag{4.2.8}$$

Using these relations, the extremum condition of V becomes

$$\begin{aligned}\nabla V&=\frac{2}{3}XZ_{AB}P^{AB}+\Omega^{CD}Z_{CA}Z^{AB}P_{BD}\\&=2\left\{P_1(2e_1+e_2)\left(e_2+\frac{2}{3}X\right)+P_2(e_1+2e_2)\left(e_1+\frac{2}{3}X\right)\right\}=0.\end{aligned} \tag{4.2.9}$$

Two inequivalent solutions with finite area are there:

1. $e_1 = e_2 = -\frac{2}{3}X$, giving for the Bekenstein–Hawking entropy $V_{\text{extr}} = 3X^2$.
   This is the $N = 6$ 1/6-BPS solution and breaks the symmetry of the theory to $Sp(4) \times Sp(2)$.
2. $e_1 = e_2 = 0$, with Bekenstein–Hawking entropy $V_{\text{extr}} = \frac{1}{3}X^2$.
   It is a non-BPS attractor of the $N = 6$ theory, and leaves all the $Sp(6)$ symmetry of the theory unbroken.

Since the bosonic sector of this theory coincides with the one of an $N = 2$ theory based on the same coset space [49], these are also the attractor solutions of the corresponding $N = 2$ model. In the $N = 2$ version, however, the interpretation of the attractor solutions as BPS and non-BPS are interchanged.

To study the stability of the solutions, let us consider the Hessian matrix

$$\begin{aligned} H_{ij} &\equiv \nabla_i \nabla_j V \\ &= P_{AB} P_{CD} \left[Z^{AC} Z^{BD} + \frac{2}{9} X^2 \Omega^{AC} \Omega^{BD} - \frac{4}{3} X Z^{AC} \Omega^{BD} + Z^{AL} Z_{LM} \Omega^{BC} \Omega^{MD} \right] \\ &= P_{AB} P_{CD} \left(Z^{AC} - \frac{1}{3} X \Omega^{AC} \right) \left(Z^{BD} - \frac{1}{3} X \Omega^{BD} \right) \\ &\quad - P_{AB} P^{DB} \left(Z^{AC} - \frac{1}{3} X \Omega^{AC} \right) \left(Z_{CD} - \frac{1}{3} X \Omega_{CD} \right) \end{aligned} \tag{4.2.10}$$

and evaluate it on the two extrema. In the first case (BPS $N = 6$, non-BPS $N = 2$) the solution breaks the symmetry to $Sp(4) \times Sp(2)$, ($A \to (\alpha, a)$, $\alpha = 1, 2$, $a = 1, \ldots, 4$), since at the horizon we find, for the central charge in normal form:

$$Z_{AB} \quad \to \quad \left(Z_{ab} = -\frac{2}{3} X \Omega_{ab};\ Z_{\alpha\beta} = \frac{4}{3} X \epsilon_{\alpha\beta} \right), \tag{4.2.11}$$

so that

$$Z_{AB} - \frac{1}{3} X \Omega_{AB} \quad \to \quad \begin{cases} Z_{ab} - \frac{1}{3} X \Omega_{ab} = -X \Omega_{ab}, \\ Z_{\alpha\beta} - \frac{1}{3} X \epsilon_{\alpha\beta} = X \epsilon_{\alpha\beta}. \end{cases} \tag{4.2.12}$$

Corresponding to the group decomposition of the degrees of freedom:

$$\begin{aligned} &\mathbf{14} \quad \to \quad (\mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{4}, \mathbf{2}), \\ &P_{AB} \quad \to \quad (P_{ab}; P; P_{a\alpha}), \end{aligned} \tag{4.2.13}$$

the scalar vielbein decomposes as

$$P_{AB} \to \begin{cases} \epsilon_{\alpha\beta} P, \\ P_{\alpha a} \equiv -P_{a\alpha}, \\ P_{ab} - \frac{1}{2} \Omega_{ab} P, \end{cases} \tag{4.2.14}$$

where P_{ab} is the $\frac{SO(1,5)}{SO(5)}$ vielbein, satisfying $P_{ab} \Omega^{ab} = 0$.

On the solution, the Hessian matrix (4.2.10) is then:

$$H_{ij} = 2X^2 \left(P^{ab} P_{ab} + 3P^2 \right). \tag{4.2.15}$$

As expected, the directions corresponding to the scalars in the $(\mathbf{4}, \mathbf{2})$ of $\frac{Sp(4,2)}{Sp(4) \times Sp(2)}$ are flat. When the theory is interpreted as an $N = 6$ one, this is the BPS solution whose states, regarded as

$N=2$ BPS multiplets, have flat directions corresponding to the hyperscalar sector. On the other hand, in the $N=2$ interpretation this is instead the non-BPS solution, and now the flat directions correspond to degrees of freedom in the vector multiplets' moduli space.

The second solution (non-BPS $N=6$, BPS $N=2$) leaves all the $Sp(6)$ symmetry unbroken since the horizon value of the central charge matrix in normal form is now:

$$Z_{AB} \quad \to \quad 0. \tag{4.2.16}$$

Now the vielbein degrees of freedom do not decompose at all

$$\begin{aligned} &\mathbf{14} \quad \to \quad \mathbf{14}, \\ &P_{AB} \quad \to \quad P_{AB} \end{aligned} \tag{4.2.17}$$

and correspondingly all the scalar degrees of freedom become massive.

Let us end this section by writing the quantities used here in the $N=2$ formalism adopted in Section 3.2.1. In this case the rigid index a labelling the tangent space directions are replaced by the antisymmetric traceless couple $[AB]$ (recall that we use the convention that any summation over an antisymetrized couple always requires a factor $1/2$):

$$\begin{aligned} T_{A_1A_2,B_1B_2,C_1C_2} &= 2\sqrt{\frac{3}{2}}\bigg(\Omega_{A_1B_1}\Omega_{B_2C_1}\Omega_{C_2A_1} - \frac{1}{6}\Omega_{A_1A_2}\Omega_{B_1C_1}\Omega_{B_2C_2} \\ &\quad - \frac{1}{6}\Omega_{B_1B_2}\Omega_{A_1C_1}\Omega_{A_2C_2} - \frac{1}{6}\Omega_{C_1C_2}\Omega_{B_1A_1}\Omega_{B_2A_2} \\ &\quad + \frac{1}{18}\Omega_{A_1A_2}\Omega_{B_1B_2}\Omega_{C_1C_2}\bigg), \\ g_{A_1A_2,B_1B_2} &= \Omega_{B_1A_1}\Omega_{B_2A_2} - \frac{1}{6}\Omega_{A_1A_2}\Omega_{B_1B_2}, \end{aligned} \tag{4.2.18}$$

where antisymmetrization in the couples (A_1, A_2), (B_1, B_2), (C_1, C_2) is understood. As far as the central charges are concerned, we have the following correspondence:

$$Z = \frac{1}{\sqrt{3}}X, \qquad P^a Z_a = \frac{1}{2\sqrt{3}}P^{AB}Z_{AB}. \tag{4.2.19}$$

4.3. $N=4$, $d=5$ and $(2,0)$, $d=6$

The scalar manifold is the coset

$$G/H = O(1,1) \times \frac{SO(5,n)}{Sp(4)\times SO(n)}, \tag{4.3.1}$$

spanned by the vielbein $d\sigma$, P_{IAB} ($A, B = 1,\dots,4$, $I = 1,\dots,n$), where $d\sigma = \partial_i\sigma\, d\phi^i$ is the vielbein of the $O(1,1)$ factor while $P_{IAB} = P_{IAB,i}\, d\phi^i$ is the Ω-traceless vielbein of $\frac{SO(5,n)}{Sp(4)\times SO(n)}$ satisfying the conditions

$$P_{IAB} = P_{I[AB]}, \qquad P^{IAB}\Omega_{AB} = 0. \tag{4.3.2}$$

The bare electric charges are a $SO(5,n)$-singlet e_0 and a $SO(5,n)$-vector e_Λ (the weight with respect to $SO(1,1)$ is $+2$ for e_0 and -1 for e_Λ).

The BH potential reads:

$$V = \frac{1}{2}Z_{AB}Z^{AB} + 4X^2 + Z_I Z^I, \tag{4.3.3}$$

and the differential relations among the 5 central charges Z_{AB} (satisfying $Z_{AB}\Omega^{AB}=0$), the singlet X and the n matter charges Z_I are [56]:

$$\nabla Z_{AB} = Z^I P_{IAB} - Z_{AB} d\sigma, \tag{4.3.4}$$

$$\nabla X = 2X d\sigma, \tag{4.3.5}$$

$$\nabla Z_I = \frac{1}{2} Z^{AB} P_{IAB} - Z_I d\sigma, \tag{4.3.6}$$

yielding

$$\nabla V = 2P_{IAB}\left(Z^{AB} Z^I\right) + 2d\sigma\left(8X^2 - \frac{1}{2} Z_{AB} Z^{AB} - Z_I Z^I\right). \tag{4.3.7}$$

The central charge matrix may be put in normal form:

$$Z_{AB} = \begin{pmatrix} e_1 & 0 \\ 0 & -e_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.3.8}$$

so that the BH potential takes the form

$$V = 2e_1^2 + 4X^2 + Z_I Z^I \tag{4.3.9}$$

and the differential relations among the dressed charges become ($d\sigma$ and $P_I \equiv P_{I12} = -P_{I34}$ are the components of the scalar vielbein coupling to the charges in normal form):

$$\nabla e_1 = Z^I P_I - e_1\, d\sigma, \tag{4.3.10}$$

$$\nabla X = 2X\, d\sigma, \tag{4.3.11}$$

$$\nabla Z_I = 2e_1 P_I - Z_I\, d\sigma. \tag{4.3.12}$$

Then the extremization of the BH potential takes the form

$$\nabla V = 8P_I\left(e_1 Z^I\right) + 2\, d\sigma\left(8X^2 - 2e_1^2 - Z_I Z^I\right) = 0. \tag{4.3.13}$$

Two inequivalent solutions with finite area are there:

1. $Z_I = 0$; $e_1 = 2X$.
   This is the $N=4$ 1/4-BPS solution and breaks the $Sp(4)$ $\mathcal{R}$ -symmetry of the theory to $Sp(2) \times Sp(2)$, leaving the $SO(n)$ symmetry unbroken. It corresponds to an $\frac{SO(5,n)}{SO(4,n)}$ orbit of the charge vector.
2. $Z_{AB} = 0$; $Z_I Z^I = 8X^2$.
   It is a non-BPS attractor of the $N=4$ theory, corresponding to choose the vector Z_I to point in a given direction, say 1, in the space of charges: $Z_I = 2\sqrt{2}\delta_I^1$. This solution breaks the symmetry of the theory to $Sp(4) \times SO(n-1)$, and corresponds to an $\frac{SO(5,n)}{SO(5,n-1)}$ orbit of the charge vector.

In both cases the Bekenstein–Hawking entropy turns out to satisfy [56]

$$S_{\mathrm{BH}}^{(5)} = \left(V|_{\mathrm{extr}}\right)^{3/4} = \sqrt{\left|e_0 e^\Lambda e_\Lambda\right|}. \tag{4.3.14}$$

To study the stability of the solutions, let us consider the Hessian matrix

$$H_{ij} \equiv \frac{1}{4} \nabla_i \nabla_j V \tag{4.3.15}$$

$$=P^{IAB}_{,i} P_{JCD,j}\left(\frac{1}{4}Z_{AB}Z^{CD}\delta^I_J + \frac{1}{2}Z^I Z_J \delta^{CD}_{AB}\right)$$
$$- 2P^{IAB}_{,(i}\partial_{j)}\sigma Z_{AB} Z_I + \partial_i\sigma\partial_j\sigma\left(\frac{1}{2}Z_{AB}Z^{AB} + Z_I Z^I + 16X^2\right) \tag{4.3.16}$$

and evaluate it on the two extrema.

On the BPS attractor solution, the $\mathcal{R}$-symmetry $Sp(4)$ is broken to $Sp(2)\times Sp(2)$ ($A \to (\alpha\tilde{\alpha})$) and the dressed charges in normal form become

$$Z_{AB} \to 2X\begin{pmatrix}\epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon_{\tilde{\alpha}\tilde{\beta}}\end{pmatrix}, \quad Z_I \to 0. \tag{4.3.17}$$

Correspondingly, the vielbein P^{IAB} decomposes to $(P^I\epsilon^{\alpha\beta}, -P^I\epsilon^{\tilde{\alpha}\tilde{\beta}}, P^{I\alpha\tilde{\alpha}})$ where P^I and $P^{I\alpha\tilde{\alpha}}$ are the vielbein of the submanifold $\frac{SO(1,n)}{SO(n)}$ (spanning $N=2$ vector multiplets) and $\frac{SO(4,n)}{Sp(2)\times Sp(2)\times SO(n)}$ (spanning $N=2$ hypermultiplets) respectively. Since on the solution $\frac{1}{2}Z_{AB}P^{IAB} \to 4XP^I$, the Hessian matrix (4.3.15) then becomes:

$$H_{ij} = 8X^2\big(2P^I_{,i}P_{I,j} + 3\partial_i\sigma\partial_j\sigma\big), \tag{4.3.18}$$

showing that the $4n$ scalars parametrized by $P^{I\alpha\tilde{\alpha}}$, which correspond to $N=2$ hypermultiplets, have massless Hessian modes.

On the other hand, the non-BPS solution breaks the symmetry $SO(n)$ to $SO(n-1)$ ($I \to 1, k$; $k = 1,\dots,n-1$) so that the vielbein P_{IAB} decomposes into (P_{1AB}, P_{kAB}). The Hessian matrix on the solution is:

$$H_{ij} = 8X^2\left(\frac{1}{2}P^{1AB}_{,i}P_{1AB,j} + 3\partial_i\sigma\partial_j\sigma\right). \tag{4.3.19}$$

Note in particular that the $5(n-1)$ scalars corresponding to the vielbein P_{kAB}, spanning the submanifold $\frac{SO(5,n-1)}{SO(5)\times SO(n-1)}$, are flat directions.

For the $N=4$ theory it is easy to find a six-dimensional uplift in terms of the IIB, $(2,0)$ chiral $d=6$ theory coupled to n tensor multiplets [50] (at least for the anomaly-free case $n=21$) on similar lines as performed in Section 3. Indeed, similarly to the dimensional reduction of the $N=2$ theory coupled to tensor multiplets only, in the dimensional reduction of the IIB theory from six to five dimensions the scalar content is incremented only by the KK-dilaton, which provides a $O(1,1)$ factor commuting with the $\frac{SO(5,n)}{SO(5)\times SO(n)}$ coset. Moreover, the vector content in the gravitational multiplet is also incremented by one graviphoton (whose integral corresponds to the singlet charge X). Since the KK-dilaton is stabilized on the attractor solutions, then the five-dimensional attractors are in one to one correspondence with the six-dimensional ones: on the BPS attractor there are $4n$ flat directions (corresponding to the quaternionic manifold $\frac{SO(4,n)}{SO(4)\times SO(n)}$), while on the non-BPS solution there are $5(n-1)$ flat directions (spanning the coset $\frac{SO(5,n-1)}{SO(5)\times SO(n-1)}$).

5. Anomaly-free $(1,0)$, $d=6$ supergravity with neutral matter

In this section we comment on the constraints that an $N=2$, $d=5$ supergravity should satisfy in order to be uplifted to an anomaly-free $N=(1,0)$, $d=6$ theory.

It is well known that in a $(1,0)$ supergravity with neutral matter the absence of the gravitational anomaly demands a relation among the triple n_T, n_V, n_H of possible matter multiplets (tensor, vector and hyper multiplets, respectively), namely [69,70]

$$n_H - n_V + 29n_T = 273. \tag{5.1}$$

Moreover, the consistency of the gauge invariance of tensor and (Abelian) vector multiplets requires that the gauge vector current is conserved, i.e., [52,63,71–73]

$$d^* J_\alpha = \eta_{\Lambda\Sigma} C^\Lambda_{\alpha\beta} C^\Sigma_{\gamma\delta} F^\beta \wedge F^\gamma \wedge F^\delta = 0, \tag{5.2}$$

implying that ($C^\Lambda_{\alpha\beta} = C^\Lambda_{(\alpha\beta)}$)

$$\eta_{\Lambda\Sigma} C^\Lambda_{(\alpha\beta} C^\Sigma_{\gamma\delta)} = 0. \tag{5.3}$$

Such a condition holds true for all symmetric real special manifolds [53], with the exception of the sequence $L(-1, P)$, $P > 0$ (whose corresponding Kähler and quaternionic sequences are not symmetric [74]). Disregarding such a sequence, among all homogeneous real special spaces (see, e.g., the Table 2 of [53]) the symmetric spaces are $L(q,0) = L(0, P)$, $q, P \geqslant 0$ ("generic sequence", extended to consider also the $d = 5$ uplift of the so-called $d = 4$ stu model), $L(q,1)$ for $q = 1,2,4,8$ (magic supergravities over $J_3^{\mathbb{R}}$, $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$, respectively) and $L(-1,0)$ (the $d = 5$ uplift of the so-called $d = 4$ st^2 model).

The condition (5.1) for the magic models respectively gives the following allowed triples (n_T, n_V, n_H) [52]:

$$\begin{aligned} &J_3^{\mathbb{R}}: \quad (2,2,217), \qquad && J_3^{\mathbb{C}}: \quad (3,4,190), \\ &J_3^{\mathbb{H}}: \quad (5,8,136), \qquad && J_3^{\mathbb{O}}: \quad (9,16,28). \end{aligned} \tag{5.4}$$

Notice that for the $J_3^{\mathbb{O}}$-based supergravity $n_H = 28$, so its corresponding quaternionic manifold could be identified with the exceptional quaternionic Kähler coset [75] $\frac{E_{8(-24)}}{E_7 \times SU(2)}$ (which is the quaternionic reduction—or equivalently the hypermultiplets' scalar manifold—of the $d = 4$ $J_3^{\mathbb{O}}$-based supergravity [53,75]).

On the other hand, for the "generic sequence" there are two possible uplifts to $d = 6$, depending whether one starts with $L(q,0)$ or $L(0, P)$. Indeed, starting from $L(q,0)$ the condition (5.1) implies

$$n_H = 244 - 29q, \tag{5.5}$$

which demands $0 \leqslant q \leqslant 8$, whereas starting from $L(0, P)$ the same anomaly-free condition yields

$$n_H = 244 + P, \tag{5.6}$$

which always admits a solution.

The $(1,0)$, $d = 6$ theory obtained by uplifting the real special symmetric sequence $L(q,0)$ has $n_V = 0$ and $n_T = q + 1$, and thus $1 \leqslant n_T \leqslant 9$. On the other hand, the anomaly-free $(1,0)$, $d = 6$ uplift of the real special symmetric sequence $L(0, P)$ has $n_T = 1$ and n_V arbitrary, thus it may be obtained from the standard compactification of heterotic superstrings on K_3 manifolds (see, e.g., [76]).

The model $L(-1,0)$ admits an anomaly-free uplift to $d = 6$, having $n_V = n_T = 0$ and $n_H = 273$.

All other homogeneous non-symmetric real special spaces do not fulfill the condition (5.2)–(5.3) in presence of only neutral matter, so they seemingly have a $d = 6$ uplift to $(1, 0)$ supergravity which is not anomaly-free, unless they are embedded in a model where a non-trivial gauge group is present, with charged matter [77,78].

6. Conclusion

There are three theories with eight supercharges which admit black hole/black string attractors, namely $N = 2$ supergravity in $d = 4, 5, 6$ dimensions. For symmetric special geometries, the entropy is respectively given by the quartic, cubic and quadratic invariant of the corresponding U-duality group in the three diverse dimensions. In this paper we extend previous work [40] on the investigation of the BPS and non-BPS attractor equations of such theories, by relating them as well as the corresponding moduli spaces of (non-BPS) critical points.

Furthermore, we related the moduli space of the $N = 8$, $d = 5$ BPS unique orbit to the moduli space of $N = 2$, $d = 5$ non-BPS orbit for all magic supergravities, as well as for the "generic sequence" of real special symmetric spaces. This latter is directly related to the $d = 6$ tensor multiplets' non-BPS moduli space, which describes a neutral dyonic superstring in $d = 6$.

We also considered $N = 4$, $d = 5$ supergravity, and related its $\frac{1}{4}$-BPS and non-BPS attractors to the ones of $(2, 0)$, $d = 6$ theory. Also in this case the moduli space of non-BPS attractors is spanned by the $d = 6$ non-BPS flat directions, studied in [50].

We stress that our analysis is purely classical and it does not deal with quantum corrections to the entropy, so it should apply only to the so-called "large" black objects. We leave the study of the quantum regime to future work.

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Appendix A. Relevant embeddings

Let us first fix the notations to be used in the this appendix. If α is a root of a complex Lie algebra $\mathfrak{g}$, the normalizations of the corresponding non-compact Cartan generator H_α and of the shift generators $E_{\pm\alpha}$ will be defined as follows [79]:

$$H_\alpha = \frac{2}{(\alpha \cdot \alpha)} \alpha^i H_i, \qquad (H_i, H_j) = \delta_{ij},$$
$$E_{-\alpha} = (E_\alpha)^\dagger, \qquad (E_\alpha, E_{-\alpha}) = \frac{2}{(\alpha \cdot \alpha)}, \tag{A.1}$$

where $(\cdot,\cdot)$ is the Killing form. The above normalizations imply the following commutation relations

$$[H_\alpha, E_\beta] = \langle\beta,\alpha\rangle E_\beta, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha, \qquad \langle\beta,\alpha\rangle = \frac{2}{(\alpha\cdot\alpha)}\beta\cdot\alpha. \tag{A.2}$$

$J_3^{\mathbb{C}}$, $d=5$: *The* $SU(2,1)^2 \subset \mathrm{F}_{4(4)}$ *embedding*

The simple roots of the $\mathfrak{sl}(3,\mathbb{C})^2$ subalgebra of $\mathfrak{f}_4$ over $\mathbb{C}$ are defined in terms of the simple roots of the latter α_k ($k=1,\ldots,4$, α_1,α_2 being long roots) as follows

$$a_1 = \alpha_4, \qquad a_2 = \alpha_3, \qquad b_1 = \alpha_1, \qquad b_2 = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4. \tag{A.3}$$

The real form $\mathfrak{f}_{4(4)}$ contains an $\mathfrak{sl}(2,\mathbb{R})^4$ subalgebra defined by the following mutually orthogonal roots:

$$a_2, \qquad b_2, \qquad c = \alpha_1+\alpha_2+\alpha_3, \qquad d = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4. \tag{A.4}$$

We can define the roots of $\mathfrak{f}_{4(4)}$ using a Cartan subalgebra $\mathfrak{h}_0$ generated by two non-compact H_{a_2}, H_{b_2} and two compact $\mathrm{i}H_c, \mathrm{i}H_d$ generators, the latter corresponding to the $\mathfrak{so}(2)$ generators inside $\mathfrak{sl}(2,\mathbb{R})_c \oplus \mathfrak{sl}(2,\mathbb{R})_d$. In terms of the generators of $\mathfrak{h}_0$, we can choose a basis of Cartan generators for $\mathfrak{sl}(3,\mathbb{C})^2$ to consist of H_{a_2}, H_{b_2} as well as of

$$H_{a_1} = -\frac{1}{2}(H_{a_2} + \mathrm{i}H_c - 2\mathrm{i}H_d), \qquad H_{b_1} = -\frac{1}{2}(H_{a_2} - \mathrm{i}H_c - \mathrm{i}H_d). \tag{A.5}$$

These generators define the Cartan subalgebra of an $\mathfrak{su}(2,1)^2$ subalgebra of $\mathfrak{f}_{4(4)}$. Indeed one can verify that the $\mathfrak{sl}(3,\mathbb{C})^2$ root system defined by the simultaneous eigenvalues of the $\mathfrak{h}_0$ generators, is stable with respect to the conjugation σ relative to $\mathfrak{f}_{4(4)}$, namely that

$$a_2^\sigma = a_2; \qquad a_1^\sigma = -(a_1+a_2); \qquad b_2^\sigma = b_2; \qquad b_1^\sigma = -(b_1+b_2). \tag{A.6}$$

The $\mathfrak{su}(2,1)^2$ generators are thus defined by σ-invariant combinations of the $\mathfrak{sl}(3,\mathbb{C})^2$ shift generators. The fact that this construction defines an $\mathfrak{su}(2,1)^2$ subalgebra of $\mathfrak{f}_{4(4)}$ and not an $\mathfrak{sl}(3,\mathbb{R})^2$ algebra is proven by the existence in each factor of a compact Cartan subalgebra, defined by the generators $\{E_{a_2} - E_{-a_2}, \mathrm{i}(H_c - 2H_d)\}$ for the first factor and $\{E_{b_2} - E_{-b_2}, \mathrm{i}(H_c + H_d)\}$ for the second.

$J_3^{\mathbb{R}}$, $d=5$: *The* $SL(2,\mathbb{R}) \times G_{2(2)} \subset \mathrm{F}_{4(4)}$ *embedding*

Denoting by a the $\mathfrak{sl}(2,\mathbb{R})$ root and by b_1, b_2 the simple roots of $\mathfrak{g}_{2(2)}$, the $SL(2,\mathbb{R}) \times G_{2(2)}$ generators can be written in terms of the $\mathrm{F}_{4(4)}$ generators as follows:

$$\begin{aligned}
&H_{b_1} = H_{\alpha_1+\alpha_2} + H_{\alpha_4}; \qquad H_{b_2} = H_{\alpha_2+2\alpha_3} = H_{\alpha_2} + H_{\alpha_3}; \qquad E_{b_1} = E_{\alpha_1+\alpha_2} + E_{\alpha_4},\\
&E_{b_2} = E_{\alpha_2+2\alpha_3}; \qquad E_{b_1+b_2} = -E_{\alpha_1+2\alpha_2+2\alpha_3} + E_{\alpha_2+2\alpha_3+\alpha_4},\\
&E_{2b_1+b_2} = -E_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} + E_{\alpha_2+2\alpha_3+2\alpha_4}; \qquad E_{3b_1+b_2} = -E_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4},\\
&E_{3b_1+2b_2} = E_{\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4},\\
&H_a = 2(H_{\alpha_3+\alpha_4} + H_{\alpha_1+\alpha_2+\alpha_3}); \qquad E_a = \sqrt{2}(E_{\alpha_3+\alpha_4} + E_{\alpha_1+\alpha_2+\alpha_3}).
\end{aligned} \tag{A.7}$$

Matrix representation of $\mathfrak{f}_4$ *generators*

For the sake of completeness, let us give below an explicit realization of the generators H_{α_i}, E_{α_i} and $\mathfrak{f}_4$, in the fundamental representation.

$\mathfrak{f}_4$ *generators*:

$$\begin{aligned}
H_{\alpha_1} &= \mathrm{diag}(-1,1,0,0,0,0,0,0,0,-1,0,0,0,-1,1,-1,-1,1,1,1,0,-1,1,0,0,0,0),\\
H_{\alpha_2} &= \mathrm{diag}(0,0,1,-1,0,0,0,1,1,-1,-1,0,0,0,0,1,1,-1,-1,0,0,0,1,-1,0,0),\\
H_{\alpha_3} &= \mathrm{diag}(0,1,-1,1,-1,1,0,-1,0,1,2,-1,0,0,1,-2,-1,0,1,0,-1,\\
&\qquad 1,-1,1,-1,0),\\
H_{\alpha_4} &= \mathrm{diag}(1,-1,0,0,1,0,-1,1,-1,1,-1,2,0,0,-2,1,-1,1,-1,1,0,\\
&\qquad -1,0,0,1,-1),\\
E_{\alpha_1} &= I_{4,6}+I_{5,8}+I_{7,9}+I_{18,20}+I_{19,22}+I_{21,23},\\
E_{\alpha_2} &= I_{3,4}+I_{8,10}+I_{9,11}+I_{16,18}+I_{17,19}+I_{23,24},\\
E_{\alpha_3} &= I_{2,3}+I_{4,5}+I_{6,8}+I_{10,12}+c_1 I_{11,13}+c_2 I_{11,14}+c_1 I_{13,16}+c_2 I_{14,16}\\
&\quad + I_{15,17}+I_{19,21}+I_{22,23}+I_{24,25},\\
E_{\alpha_4} &= I_{1,2}-I_{5,7}-I_{8,9}-I_{10,11}+c_2 I_{12,13}+c_1 I_{12,14}+c_2 I_{13,15}+c_1 I_{14,15}-I_{16,17}\\
&\quad - I_{18,19}-I_{20,22}+I_{25,26},
\end{aligned} \tag{A.8}$$

where $c_1 = (1+\sqrt{3})/2$, $c_2 = (1-\sqrt{3})/2$ and $(I_{I,J})_{KL} = \delta_{IK}\delta_{JL}$. The Killing form is $(M_1, M_2) = \frac{1}{6}\,\mathrm{Tr}(M_1 M_2)$.

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