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# of first- and higher-order linear logic

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**Abstract**

We give a natural extension of Girard's phase semantic completeness proof of the (first order) linear logic Girard (Theoret. Comput. Sci., 1987) to a phase semantic cut-elimination proof. Then we extend this idea to a phase semantic cut-elimination proof for higher order linear logic. We also extend the phase semantics for *provability* to a phase semantics-like framework for *proofs*, by modifying the phase space of monoid domain to that of *proof-structures* (*untyped proofs*) domain, in a natural way. The resulting phase semantic-like framework for *proofs* provides various versions of proof-normalization theorem. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Phase space semantics was introduced by Girard [6] for a completeness proof (with respect to provability) of linear logic. Although it was first introduced as a tool for proving such an abstract property as completeness, it has been recently recognized that phase semantics could be used as a concrete tool to provide some concrete information; For example, when one would like to show "if A is provable, then property P(A) holds" for some concrete property P, it is very natural to try to find a suitable phase model in which the satisfiability in that model implies the property P. In particular, the key point of such a method is to set up a suitable phase semantic interpretation (with a desired property) for atoms (atomic formulas) so that the desired property is automatically expanded to the whole phase semantic interpretation (for all complex formulas), in the framework of phase model. An elegant example of using this paradigm has recently been shown by Lafont [10] in which provability of A in MALL2 (second-order

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multiplicative-additive fragment of linear logic without modality) implies acceptability of configuration $A^*$ (a certain interpretation of A) in a Minsky machine, which provides the undecidability result of MALL2. The result is extended by Lafont-Scedrov [13]. In the first part of this paper, we shall show another kind of example of the use of this paradigm to give a uniform cut-elimination proof for higher order linear logics. One of the differences between Lafont's case and ours is that the property P (acceptability of a Minsky machine) of Lafont is a property weaker than the completeness property while our P (cut-eliminability) is a property stronger than the completeness, hence our argument implies a strong form of completeness.[1]

Our paradigm to use phase semantics has an obvious and close relationship with the Tait–Girard's computability/reducibility argument (and its higher order version using the notion of candidates of reducibility [5] for normalization proofs, especially when one interprets this Tait–Girard argument as a general machinery to prove a universal property P on the domain of the proofs (as explained in [8]). It is a very fascinating question whether or not the Tait–Girard's style of normalizability proof argument can be viewed in the phase semantics framework. An affirmative answer to this question is given in the second part of this paper (Sections 5–8): we can extend our phase semantic-framework (for *provability*) to that for *proofs*, by modifying the phase space of the usual monoid domain to that of the (untyped) proof-structures domain in a natural way. Then we can view the proofs of (various versions of) normalization theorem as a natural extension of the phase semantic (strong) completeness proof.

The composition of this paper is as follows. In Section 2 below we recall the first-order phase semantics (for "provability") of Girard [6] and introduce the higher-order phase semantics. In Section 3 we demonstrate our phase semantic cut elimination proof in a simplest setting, namely a phase semantic cut elimination proof for the full first-order linear logic, by slightly refining the original phase semantic completeness proof of Girard [6]. The proof method of this section will be the central paradigm through this paper. In Section 4 we apply this method to our higher-order phase semantics so that we obtain a phase semantic cut-elimination proof (as well as a completeness proof at the same time) for higher-order linear logic. In the course of the proof we introduce the "provability"-semantics version of "candidates of reducibility" (à la Girard [5] for his strong normalization proof). In Section 5 we introduce the (untyped) proof-structures domain and phase semantics-like framework on this proof-structures domain (instead of a monoid domain for the case of the usual phase semantics for *provability* in Section 2). We introduce the linear logical operators on this domain, in the manner similar to the case of phase semantics for *provability*. The soundness theorem is proved for our phase semantics-like framework for "proofs". We use the intersection-type inference rule and union-type inference rule, rather than the additive connectives, in order to define our phase semantics-like framework for proofs. In Section 6 we apply the basic technique

[1] Recently, Lafont [12] and Okada-Terui [15] used our method introduced in this paper to show a stronger property for some restricted subsystems of linear logic. Another application of the method introduced in this paper may be seen in [9]

introduced in Section 3 to our phase semantics-like framework for *proofs* and give a proof of normalization theorems in our phase semantics-like framework. In Section 7 we consider the additive connectives (the first-order additive connectives & and $\oplus$, and the higher-order quantifiers $\forall$ and $\exists$), instead of the intersection-type and union type. Then we give the phase semantics-like framework including those additive connectives and apply our method to obtain a normalization proof for first and higher order full linear logic. In Section 8 we give a remark on some relationships between the phase semantics for *provability* and the phase semantic framework for *proofs*. In particular, we show that the *projection* of the canonical model for *proofs* is exactly the canonical model for *provability*. As a direct consequence of the relationships it follows that the theorems (Soundness Theorem, Main Lemma, Strong Completeness) for *provability* in Sections 2, 3 and 4 are immediate corollaries of the corresponding theorems for *proofs* in Section 7.

## 2. First- and higher-order phase space

In this section, we first recall the first-order phase semantics, due to Girard [6], then extend it to second-order and higher-order phase semantics. In the later sections we shall use these phase semantics for a cut-elimination proof.

Let $M$ be a commutative monoid. Let $\perp$ be a special subset of $M$, called *bottom*. We have the definition of the linear negation; For any $\alpha \subseteq M$, let $\alpha^\perp$ be $\{b : \text{for all } a \in \alpha\ a \cdot b \in \perp\} = \{b : \alpha \cdot b \subseteq \perp\}$.

Let $I = \mathbf{1} \cap J$ where $J$ is a submonoid which satisfies the weak idempotent property $\forall a \in J\ \{a\}^{\perp\perp} \subseteq \{a \cdot a\}^{\perp\perp}$ (after Y. Lafont). In particular, $J$ can be $\{a : aa = a\}$. Here $\mathbf{1} = \{b : \forall a \in \perp\ ab \in \perp\} = \perp^\perp$. (Note that we may omit the monoid operator, and write $ab$ instead of $a \cdot b$ for $a, b \in M$.)

The following is easily proved.

**Lemma 2.1.** *For any* $\alpha \subseteq M$, $\beta \subseteq M$,
1. $\alpha \subseteq \alpha^{\perp\perp}$.
2. $(\alpha^{\perp\perp})^{\perp\perp} \subseteq \alpha^{\perp\perp}$.
3. $\alpha \subseteq \beta \Rightarrow \alpha^{\perp\perp} \subseteq \beta^{\perp\perp}$.
4. $\alpha^{\perp\perp} \cdot \beta^{\perp\perp} \subseteq (\alpha \cdot \beta)^{\perp\perp}$.

$\alpha \subseteq M$ is called a *fact* iff $\alpha^{\perp\perp} = \alpha$. The set of facts is denoted by $D_M$. Then, as easily seen, for any $\alpha \subseteq M$, $\alpha^{\perp\perp}$ is the smallest fact that includes $\alpha$, and for any facts $\alpha$ and $\beta$, $\alpha \cap \beta$ is a fact. $\alpha^{\perp\perp\perp} = \alpha^\perp$, hence $\alpha^\perp$ is a fact for any $\alpha \subseteq M$. We define the phase space operators and constants as follows:
- $\mathbf{1} = \perp^\perp = \{b : \forall a \in \perp\ ab \in \perp\} (= \{1\}^{\perp\perp})$, where 1 denotes the unit element of $M$.
- $\mathbf{0} = \top^\perp$, where $\top = M$.
- $\alpha \,\&\, \beta = \alpha \cap \beta$.
- $\alpha \oplus \beta = (\alpha \cup \beta)^{\perp\perp}$.

- $\alpha \otimes \beta = (\alpha \cdot \beta)^{\perp\perp}$.
- $\alpha \parr \beta = (\alpha^{\perp} \cdot \beta^{\perp})^{\perp}$.
- $!\alpha = (I \cap \alpha)^{\perp\perp}$.
- $?\alpha = (I \cap \alpha^{\perp})^{\perp} (= (!\alpha^{\perp})^{\perp})$.

For any $D \subseteq D_M$, if $\perp \in D$ and $D$ is closed under all above operators, $(D, I, \perp)$ is called a *phase space*. In particular, $(D_M, I, \perp)$ is a phase space for any commutative monoid $M$. A phase space of the form $(D_M, I, \perp)$ is called a *standard* phase space.

On a phase space, as in [6], the interpretation $A^*$ of a formula $A$ is defined to be a fact ($\in D$) in the following way, when an assignment (a valuation) $\varphi$ of facts for (propositional) variables occurring in $A$ is given. We call this value $A^*$ the inner-value of $A$ through this paper:

$R^* = \varphi(R)$ for assignment (valuation) $\varphi : A\text{-}Form \to D$, where *A-Form* stands for the set of atomic formulas:

- $(R^{\perp})^* = (R^*)^{\perp}$ for atomic $R$.
- $(A \,\&\, B)^* = A^* \,\&\, B^*$.
- $(A \oplus B)^* = A^* \oplus B^*$.
- $(A \otimes B)^* = A^* \otimes B^*$.
- $(A \parr B)^* = A^* \parr B^*$.
- $(?A)^* = ?(A^*)$.
- $(!A)^* = !(A^*)$.

Note that $A^{\perp}$ is defined as the De Morgan dual, hence $(A^{\perp})^* = A^{*\perp}$ is easily shown. It is obvious that any inner value $A^*$ is a fact. One can easily extend these interpretations to the case of the first order quantifiers (by interpreting these as additive operators), as usual.

A formula $A$ is said to be *true* if $1 \in A^*$, where 1 is the unit element of underlying monoid $M$.

For a given phase space $(D, I, \perp)$ and a given assignment $\varphi$, $(D, I, \perp, \varphi)$ is called a *phase model*. A phase model $(D, I, \perp, \varphi)$ is called a *standard* phase model if $D = D_M$ (namely, if $(D, I, \perp)$ is a standard phase space).

Now we extend Girard's phase semantics above to a higher order phase semantics. The interpretation of the second order operators is added as follows.

For any $\xi : D \to D$,

$$\forall X.\xi(X) = \bigcap_{\alpha \in D} \xi(\alpha) \quad \exists X.\xi(X) = \left(\bigcup_{\alpha \in D} \xi(\alpha)\right)^{\perp\perp} \left(= (\forall X.\xi^{\perp}(X))^{\perp}\right).$$

By $A[X]$ we mean that $X$ is the list of free propositional variables occurring in $A$; by $A[B/X]$ or $A[B]$ we mean the formula obtained from $A[X]$ by substituting a vector $B$ of formulas; by $A^*[\alpha/X]$ or $A^*[\alpha]$ we mean the result of the inner value construction starting the vector $\alpha$ for the value (i.e., assignment) of the variable list $X$. In this paper we use *Form* for the set of (second-order) formulas.

A phase space $(D, I, \perp)$ is called a *second-order phase space* if $D$ is closed under the operators $\forall$ and $\exists$ above; namely if $\forall X.\xi(X) \in D$ and $\exists X.\xi(X) \in D$ for $\xi : D \to D$ of the form $\xi(X) \equiv A^*[X, \alpha_1/Y_1, \ldots, \alpha_n/Y_n]$ for $A \in Form$ and $\alpha_i \in D$. Therefore, the above condition can be expressed as follows:

(∗) For any (second order) formula $A$ (possibly with free second order variables $X \equiv X_1, \ldots, X_n$), if $\alpha_i \in D$ then $A^*[\alpha/X] \in D$, where $\alpha \equiv \alpha_1, \ldots, \alpha_n$.

One can easily extend our phase semantics of the second-order linear logic to higher-order linear logic of finite types.

According to the finite type structure of the syntax, we consider the functional space based on D. More precisely, for each type $\tau$ we define $D_\tau$ as follows;

$$D_{\langle 0 \rangle} = D.$$

$$D_{\langle \tau_1 \times \cdots \times \tau_n \to \tau_{n+1} \rangle} = D_{\tau_{n+1}}^{D_{\tau_1} \times \cdots \times D_{\tau_n}}$$

The interpretation of higher-order operators is added as follows.

For any $\xi : D_\tau \to D$, $\forall X.\xi(X) = \bigcap_{\alpha \in D_\tau} \xi(\alpha)$. $\exists X.\xi(X) = (\bigcup_{\alpha \in D_\tau} \xi(\alpha))^{\perp\perp}$.

A phase space is called a *higher-order phase space* if the above condition (∗) is satisfied for a higher order formula $A$.

## 3. Phase-semantic cut-elimination for first-order case

In this section, we shall demonstrate our paradigm in a simplest setting – the first-order case. Since the one-sided sequent calculus formulation is usually used for the classical linear logic, we shall follow that manner in this paper. In the same way as in [6], we have

**Theorem 3.1** (Soundness Theorem; Girard [6]). *If a formula $A$ is provable in (first-order) linear logic, then $A$ is true (namely, $1 \in A^*$) for any phase model. More generally, if a sequent $\vdash A_1, \ldots, A_n$ is provable, then $1 \in A_1^* \mathbin{⅋} \ldots \mathbin{⅋} A_n^*$, or equivalently, $A_1^{*\perp} \cdot \ldots \cdot A_n^{*\perp} \subseteq \perp$ for any phase model.*

**Theorem 3.2** (Completeness Theorem; Girard [6]). *If a formula $A$ is true for any (standard) phase model, $A$ is provable in (first-order) linear logic.*

On the other hand, the paradigm through this paper is to consider the following silghtly refined form of the above completeness theorem;

**Theorem 3.3** (Strong Completeness Theorem). *If a formula $A$ is true for any (standard) phase model, $A$ is provable in (first-order) linear logic without the cut-rule.*

The argument similar to the proof of this case will be repeatedly used in later sections.

We construct a canonical phase model $(D_M, I, \perp, \varphi)$ for which truth of a formula $A$ implies its cut-free provability. We take the commutative free monoid generated by the set of formulas and denote it as $M$. An element of $M$ is a multiset of formulas where multiple occurrences of a formula of the form $?A$ counts only once. For example, $\{A,A,A,?B,?B,C\}$ is identified by $\{A,A,A,?B,C\}$. The construction of the phase space is the same as in Girard [6] in his completeness proof, except that we use *cut-free provability* instead of provability in the definition of $[\![A]\!]$ below. For any formula $A$ (of linear logic), we define $[\![A]\!]=\{\Delta : \vdash_{\mathrm{cf}} \Delta, A\}$, where $\vdash_{\mathrm{cf}}$ means "cut-free provable". We call $[\![A]\!]$ the *outer value* of A in this paper. $\perp = [\![\perp]\!] = \{\Delta : \vdash_{\mathrm{cf}} \Delta, \perp\} = \{\Delta : \vdash_{\mathrm{cf}} \Delta\}$; the unit element 1 of $M$ is $\phi$ (the empty sequence). $I$ is defined as $\{?\Gamma : \Gamma$ is an arbitrary sequence of formulas$\}$, where $?\Gamma$ means $?A_1,\ldots,?A_n$ if $\Gamma \equiv A_1,\ldots,A_n$. Finally, the assignment $\varphi$ of the canonical model is defined as $\varphi(R)=[\![R]\!]$ for any atomic $R$.

**Lemma 3.1** (Main Lemma). *For any formula $A$, $A^* \subseteq [\![A]\!]$.*

It is easy to see that this Main Lemma directly implies the Strong Completeness; if formula $A$ is true, then $\phi \in A^*$. On the other hand $A^* \subseteq [\![A]\!]$, hence $\phi \in [\![A]\!]$, which means "*$A$ is cut-free provable*".

By combining this with the soundness theorem, we have

**Theorem 3.4** (The Cut-Elimination Theorem). *If $A$ is provable* (*with the cut rule*) *then it is provable without cut.*

**Remark.** We can actually prove $A^* = [\![A]\!]$ if we interpret $[\![A]\!]$ as $\{\Gamma : \vdash \Gamma, A$ is provable with the cut rule$\}$, which was the essential part of the original completeness proof by Girard [6]. On the other hand, we have taken a more restricted interpretation $[\![A]\!] = \{\Gamma : \vdash \Gamma, A$ is provable without the cut rule$\} = \{\Gamma : \vdash_{\mathrm{cf}} \Gamma, A\}$ and show a weaker version of the corresponding lemma (Main Lemma). As we shall see later, the form of $A^* \subseteq [\![A]\!]$ is essential when we extend this to the higher order cases. (However, after having proved the cut-elimination theorem, it can be shown that $A^* = [\![A]\!]$. Also cf. Lemma 4.2 and the remark following the lemma.

**Lemma 3.2.** *If $A^* \subseteq [\![A]\!]$, then $A \in A^{*\perp}$.*

**Proof.** Assume $A^* \subseteq [\![A]\!]$. Then $A \cdot A^* \subseteq A \cdot [\![A]\!] \subseteq [\![\perp]\!]$. Therefore $A \in A^{*\perp}$. □

**Lemma 3.3.** $[\![A]\!]^{\perp\perp} = [\![A]\!]$ *for any formula A.* (*Therefore, any outer value is a fact.*)

**Proof.** Since $[\![A]\!] \subseteq [\![A]\!]^{\perp\perp}$ is trivial, we prove $[\![A]\!]^{\perp\perp} \subseteq [\![A]\!]$. Let $\Gamma \in [\![A]\!]^{\perp\perp}$. By definition, $\forall \Delta \in [\![A]\!]^{\perp} \vdash_{\mathrm{cf}} \Gamma, \Delta$. On the other hand, $A \in [\![A]\!]^{\perp}$ is trivial. Hence $\vdash_{\mathrm{cf}} \Gamma, A$, therefore $\Gamma \in [\![A]\!]$. □

Now we prove the Main Lemma.

**Proof of Lemma 3.1** (*Main Lemma*). The proof of this is carried by induction on the complexity of a formula A.

*Case* 1: *When $A$ is atomic.* $A^* = [\![A]\!]$ by definition.

*Case* 2: *When $A$ is the form $R^\perp$ for atomic $R$.* If $\Gamma \in R^{*\perp}$, then by the definition of $R^{*\perp}$, $R^*\Gamma \subseteq [\![\perp]\!]$. On the other hand, $R^\perp \in [\![R]\!] = R^*$. Hence $R^\perp, \Gamma \in [\![\perp]\!]$, therefore $\Gamma \in [\![R^\perp]\!]$.

*Case* 3: *When $A$ is of the form $B \otimes C$.* By the induction hypothesis, $B^* \cdot C^* \subseteq [\![B]\!] \cdot [\![C]\!]$. On the other hand,

$$\frac{\vdash_{\mathrm{cf}} B, \Gamma \quad \vdash_{\mathrm{cf}} C, \Delta}{\vdash_{\mathrm{cf}} B \otimes C, \Gamma, \Delta}$$

Hence, $[\![B]\!] \cdot [\![C]\!] \subseteq [\![B \otimes C]\!]$. Therefore, $B^* \cdot C^* \subseteq [\![B \otimes C]\!]$. Since $[\![B \otimes C]\!]$ is a fact, $B^* \otimes C^* = (B^* \cdot C^*)^{\perp\perp} \subseteq [\![B \otimes C]\!]^{\perp\perp} = [\![B \otimes C]\!]$.

*Case* 4: *When $A$ is of the form $B \parr C$.* Assume that $\Gamma \in (B^{*\perp} \cdot C^{*\perp})^\perp$. Hence, $\Gamma \cdot (B^{*\perp} \cdot C^{*\perp}) \subseteq [\![\perp]\!]$. On the other hand, by the induction hypothesis, $B^* \subseteq [\![B]\!]$ and $C^* \subseteq [\![C]\!]$. Hence by Lemma 3.2, $B \in B^{*\perp}$ and $C \in C^{*\perp}$, hence $B, C \in B^{*\perp} \cdot C^{*\perp}$. Therefore, $\Gamma, B, C \in [\![\perp]\!]$. Since

$$\frac{\vdash_{\mathrm{cf}} \Gamma, B, C}{\vdash_{\mathrm{cf}} \Gamma, B \parr C}$$

it follows $\Gamma \in [\![B \parr C]\!]$.

*Case* 5: *When $A$ is of the form $B \& C$.* By the induction hypothesis, $B^* \subseteq [\![B]\!]$ and $C^* \subseteq [\![C]\!]$. Hence $B^* \& C^* = B^* \cap C^* \subseteq [\![B]\!] \cap [\![C]\!]$. On the other hand,

$$\frac{\vdash_{\mathrm{cf}} B, \Gamma \quad \vdash_{\mathrm{cf}} C, \Gamma}{\vdash_{\mathrm{cf}} B \& C, \Gamma}$$

Hence, $[\![B]\!] \cap [\![C]\!] \subseteq [\![B \& C]\!]$. Therefore, the claim holds.

*Case* 6: *When $A$ is of the form $B \oplus C$.* By the induction hypothesis, $B^* \cup C^* \subseteq [\![B]\!] \cup [\![C]\!]$. On the other hand,

$$\frac{\vdash_{\mathrm{cf}} B, \Gamma}{\vdash_{\mathrm{cf}} B \oplus C, \Gamma}, \qquad \frac{\vdash_{\mathrm{cf}} C, \Gamma}{\vdash_{\mathrm{cf}} B \oplus C, \Gamma}$$

Hence $[\![B]\!] \cup [\![C]\!] \subseteq [\![B \oplus C]\!]$. Therefore, $B^* \cup C^* \subseteq [\![B \oplus C]\!]$. Since $[\![B \oplus C]\!]$ is a fact, $B^* \oplus C^* = (B^* \cup C^*)^{\perp\perp} \subseteq [\![B \oplus C]\!]^{\perp\perp} = [\![B \oplus C]\!]$.

*Case* 7: *When $A$ is of the form $!B$.* Assume $?\Gamma \in (I \cap B^*)$. By the induction hypothesis, $B^* \subseteq [\![B]\!]$. Hence $?\Gamma \in [\![B]\!]$. Since

$$\frac{\vdash_{\mathrm{cf}} ?\Gamma, B}{\vdash_{\mathrm{cf}} ?\Gamma, !B}$$

$?\Gamma \in [\![!B]\!]$. Therefore, $(I \cap B^*) \subseteq [\![!B]\!]$. Hence $!B^* = (I \cap B^*)^{\perp\perp} \subseteq [\![!B]\!]^{\perp\perp} = [\![!B]\!]$.

*Case* 8: *When $A$ is of the form $?B$.* Assume $\Gamma \in ?B^* = (I \cap B^{*\perp})^\perp$. Hence, $\Gamma \cdot (I \cap B^{*\perp}) \subseteq [\![\perp]\!]$. On the other hand, by the induction hypothesis, $B^* \subseteq [\![B]\!]$. Hence, by Lemma 3.2, $B \in B^{*\perp}$. Since

$$\frac{\vdash_{\mathrm{cf}} B, \Delta}{\vdash_{\mathrm{cf}} ?B, \Delta}$$

$?B \in B^{*\perp}$. Hence, $?B \in I \cap B^{*\perp}$. Therefore, $\Gamma, ?B \in [\![\perp]\!]$. Hence, $\Gamma \in [\![?B]\!]$.

The cases for constants are proved in the similar way and left to the reader. □

The Main Lemma may be expressed in the following form, which will be very essential when we extend the method of this section to the higher-order case in the next section.

**Lemma 3.4** (Main Lemma, modified version). *For any formula $A$, $A^{\perp} \in A^{*} \subseteq [\![A]\!]$.*

**Proof.** The second half is the Main Lemma itself. $A^{\perp} \in A^{*}$ is proved by Lemma 3.2 with the help of the Main Lemma. □

## 4. Phase semantic higher-order cut-elimination and the "candidates of reducibility"

In this section we shall extend the semantical cut-elimination proof of the previous section to the higher-order case, using the higher-order phase space. In particular, we introduce a notion analogious to Girard's "candidates of reducibility" in his well-known higher-order normalization proof [5]. We shall discuss the correspondence between our cut-elimination proof by phase semantics and Girard's syntactical normalization proof in a later section.

For simplicity of the argument, we shall demonstrate the cut-elimination proof based on the second-order phase semantics in detail below. The higher-order case can be obtained exactly in the same way, using the higher-order phase semantics of finite types, instead of the second-order one. First we can extend the *Soundness Theorem* in the previous section in the obvious way.

**Theorem 4.1** (Soundness Theorem (second-order version)). *For any second-order phase space, if $A[X]$ is provable (in the second-order linear logic with cut), then $A[X]$ is true in any second-order phase space. Here, "$A[X]$ is true" means "$1 \in A^{*}[\alpha/X]$ for any $\alpha_i \in D$". More generally, if $\vdash A_1[X], \dots, A_n[X]$ is provable, then for any $\alpha_i \in D$,*

$$1 \in A_1^{*}[\alpha/X] \parr \dots \parr A_n^{*}[\alpha/X], \text{ or equivalently } A_1^{*\perp}[\alpha/X] \cdot \dots \cdot A_n^{*\perp}[\alpha/X] \subseteq \perp,$$

*where $X$ and $\alpha$ denote vectors whose lengths are the same.*

**Proof.** The proof is carried out by the induction on the length of proof essentially in the same way as that of the usual first-order case, except for the following second-order quantifier cases.

(1) $\forall$-rule:

$$\frac{\vdash \Gamma[X], A[X, Y]}{\vdash \Gamma[X], \forall Y\ A[X, Y]}$$

where $Y$ does not appear as a free variable in $\Gamma$.

By the induction hypothesis, for any $\alpha_i \in D$, and for any $\beta \in D$, $\Gamma^{*\perp}[\alpha] \cdot A^{*\perp}[\alpha,\beta] \subseteq \perp$. Hence $\Gamma^{*\perp}[\alpha] \subseteq A^{*\perp\perp}[\alpha,\beta] = A^{*}[\alpha,\beta]$. Since this holds for any $\beta \in D$,

$$\Gamma^{*\perp}[\alpha] \subseteq \bigcap_{\beta\in D} A^{*}[\alpha,\beta] = \forall Y\ A^{*}[\alpha, Y].$$

Hence $\Gamma^{*\perp}[\alpha] \cdot (\forall Y\ A^{*}[\alpha,Y])^{\perp} \subseteq \perp$, which means $1 \in \Gamma^{*}[\alpha] \mathbin{⅋} \forall Y\ A^{*}[\alpha,Y]$.

(2) $\exists$-rule:

$$\frac{\vdash \Gamma[X], A[B[X],X]}{\vdash \Gamma[X],\ \exists\, Y\ A[Y,X]}$$

By the induction hypothesis, for any $\alpha_i \in D$, $1 \in \Gamma^{*}[\alpha] \mathbin{⅋} A^{*}[B^{*}[\alpha],\alpha]$, namely $\Gamma^{*\perp}[\alpha] \subseteq A^{*}[B^{*}[\alpha],\alpha]$. By the condition on the second-order phase space, $B^{*}[\alpha] \in D$. Therefore,

$$\Gamma^{*\perp}[\alpha] \subseteq \bigcup_{\beta\in D} A^{*}[\beta,\alpha] \subseteq \exists\, Y\ A^{*}[Y,\alpha].$$

Hence, for any $\alpha_i \in D$, $(\Gamma^{*}[\alpha])^{\perp} \cdot (\exists Y\ A^{*}[Y,\alpha])^{\perp} \subseteq \perp$,which means $1 \in \Gamma^{*}[\alpha] \mathbin{⅋} \exists\, Y\ A^{*}[Y,\alpha]$. $\square$

Now we shall prove the Main Lemma for the second-order case. As we did for the first-order case, we shall specify one canonical phase model $(D, I, \perp, \varphi)$. The definition of the canonical phase model is exactly the same as before except for the following changes; For any formula A (of the second-order linear logic), we define $[\![A]\!] = \{\Delta : \vdash_{\mathrm{cf}} \Delta, A\}$, where $\vdash_{\mathrm{cf}}$ means "*cut-free provable* in the second-order linear logic". Recall that we consider the phase space based on the commutative monoid composed of the finite sequences of formulas, $1 = \phi$ (the empty sequence), where $\perp = [\![\perp]\!]$; $D$ is defined as $D = \bigcup_{A\in Form}\langle A\rangle$, where *Form* is the set of second-order formulas, and for any formula $A$, $\langle A\rangle = \{\alpha \in D_M : A^{\perp} \in \alpha \subseteq [\![A]\!]\}$.

The set $\langle A\rangle$ corresponds to the set of candidates of reducibility of type $A$ in Girard [5]. Then, we can prove;

**Lemma 4.1** (Main Lemma (second-order case)). *For any formulas $A[X]$ where $X \equiv X_1,\dots,X_m$ and for any $C \equiv C_1,\dots,C_m$ where $C_i \in Form$, and for any $\alpha_i \in \langle C_i\rangle$, $A[C/X]^{\perp} \in A^{*}[\alpha/X] \subseteq [\![A[C/X]]\!]$.*

**Proof.** We first prove the second half; $A^{*}[\alpha/X] \subseteq [\![A[C/X]]\!]$. The proof is carried out in the way similar to the proof of the Main Lemma in the previous section except for the following cases.

*Case* 1: *$A[X]$ is of the form $X_k$.* Then by the definition of $\langle C\rangle$, $A^{*}[\alpha/X] \subseteq [\![A[C]]\!]$ is obvious for any $\alpha_i \in \langle C_i\rangle$ and any $C_i \in Form$ since $A^{*}[\alpha/X] \subseteq [\![A[C]]\!]$ means $\alpha_k \subseteq [\![C_k]\!]$.

*Case* 2: *$A[X]$ is of the form $\forall Y\ B[X,Y]$.* We prove $\forall Y\ B^{*}[\alpha/X,Y] \subseteq [\![\forall Y B[C/X,Y]]\!]$ for any $\alpha_i \in \langle C_i\rangle$ and any $C_i \in Form$. Assume that

$$\Gamma \in \forall Y\ B^{*}[\alpha, Y] = \bigcap_{\beta\in\langle D\rangle, D\in Form} B^{*}[\alpha,\beta].$$

By the induction hypothesis, $B^*[\alpha,\beta] \subseteq [\![B[C,D]]\!]$ for any $\beta \in \langle D \rangle$ and any $D \in Form$. Hence, $\Gamma \in [\![B[C,D]]\!]$. In particular, for a variable $Y$ which does not occur in $\Gamma$, $\Gamma \in [\![B[C,Y]]\!]$. On the other hand,

$$\frac{\vdash_{\mathrm{cf}} \Gamma, B[C,Y]}{\vdash_{\mathrm{cf}} \Gamma, \forall Y\ B[C,Y]}$$

is an LL-rule. Hence, $\Gamma \in [\![\forall Y\ B[C,Y]]\!]$.

*Case* 3: *$A[X]$ is of the form $\exists Y\ B[X,Y]$*. We prove $\exists Y\ B^*[\alpha/X,Y] \subseteq [\![\exists Y\ B[C/X,Y]]\!]$ for any $\alpha_i \in \langle C_i \rangle$ and any $C_i \in Form$. Take arbitrary $C_i \in Form$ and $\alpha_i \in \langle C_i \rangle$. Assume that

$$\Gamma \in \exists Y\ B^*[\alpha/X,Y] = \left( \bigcup_{\beta \in \langle D \rangle, D \in Form} B^*[\alpha,\beta] \right)^{\perp\perp}.$$

It suffices to show that $\exists Y B[C,Y] \in (\bigcup_{\beta \in \langle D \rangle, D \in Form} B^*[\alpha,\beta])^{\perp}$, since then $\Gamma, \exists Y\ B[C,Y] \in [\![\perp]\!]$, hence $\Gamma \in [\![\exists Y\ B[C,Y]]\!]$. By the induction hypothesis, $B^*[\alpha,\beta] \subseteq [\![B[C,D]]\!]$ for any $D \in Form$ and any $\beta \in \langle D \rangle$. Therefore, for any $\Delta \in B^*[\alpha,\beta]$, $\vdash_{\mathrm{cf}} \Delta, B[C,D]$. On the other hand,

$$\frac{\vdash_{\mathrm{cf}} \Delta, B[C,D]}{\vdash_{\mathrm{cf}} \Delta,\ \exists Y\ B[C,Y]}$$

is an LL-rule. Hence $\vdash_{\mathrm{cf}} \Delta,\ \exists Y\ B[C,Y]$. Since this holds for any $D \in Form$ and any $\beta \in \langle D \rangle$,

$$\exists X\ B[C,Y] \in \left( \bigcup_{\beta \in \langle D \rangle, D \in Form} B^*[\alpha,\beta] \right)^{\perp}.$$

Therefore, the claim has been proved.

Now we prove the following sublemma, which concludes the above Main Lemma.

**Sublemma.** $A^{\perp}[C/X] \in A^*[\alpha/X]$ *for all* $\alpha \in \langle C \rangle$.

**Proof.** Since $A^{*\perp}[\alpha/X] \subseteq [\![A^{\perp}[C/X]]\!]$, $A^{\perp}[C/X] \cdot A^{*\perp}[\alpha/X] \subseteq A^{\perp}[C/X] \cdot [\![A^{\perp}[C/X]]\!] \subseteq [\![\perp]\!]$. Hence $A^{\perp}[C/X] \in A^{*\perp\perp}[\alpha/X] = A^*[\alpha/X]$. $\square$

In other words, the canonical phase space and the assignment ($\varphi(A) = [\![A]\!]$ for atomic $A$) just defined form a second-order phase model. Note that the condition "if $\alpha_i \in D$ then $A^*[\alpha/X] \in D$, for any $A$" is verified at the same time when the Main Lemma above is proved. Then with the same argument in the previous section we have the following theorems.

**Theorem 4.2** (Strong Completeness (second-order version)). *If $A[X]$ is true for any second-order phase space, $A$ is provable without the cut rule.*

**Theorem 4.3** (The Cut-Elimination Theorem (second-order version)). *If A is provable with the cut rule in the second-order linear logic, it is also provable without the cut rule.*

We can extend this proof of *Main Lemma* to more general case of higher-order linear logic using the higher-order phase space of finite types, exactly in the same way.

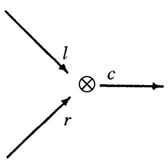
## 5. Phase semantics-like framework for "proofs"

In this section we give a phase semantics-like framework for "proofs" with the presence of the intersection types and the union types by naturally generalizing the phase semantics for *provability* of Section 2. The replacement of the additive connectives and the second order quantifiers by the intersection types and union types simplifies the phase semantic framework very much. We shall outline the phase semantics-like framework with additive connectives and second order quantifiers in Section 7.

We consider the set $P$ of *proof-structures* (or sometimes called *untyped proofs* in this paper).

A proof-structure is a graphic structure obtained by the set of links (i.e., axiom-links, $\otimes$-links, $⅋$-links, contraction-links, weakening-links, !-links and ?-links), with the help of !-boxes, in the obvious way.

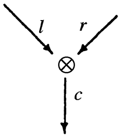
Here each inference link (except for the axiom-link, the weakening link and the cut-link) has one or two in-edges and one out-edge. The out-edge is called the conclusion edge (marked by $c$) of the link. When the link has two in-edges, the two in-edge ports are distinguished by marking the port with $l$ (left) or $r$ (right). For example, the $\otimes$-link has the form



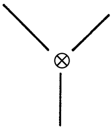

&-link and !-link are defined with the help of the box-notation (cf. [6])[2] in the natural way. A proof-structure may be unconnected and may have a directed cycle, in general. When a proof-structure has no directed cycle, hence is a finite set of trees i.e., acyclic and connected components (when ignoring the axiom-links and cut-links, whose edges are not directed), we can represent it in such a way that every conclusion edge is always drawn downward and the left-edge and the right-edge (if any) are drawn on the left hand-side and on the right hand-side, respectively; for example, the $\otimes$-link

[2] We use a weakening node, instead of a weakening box of Girard [6], following Lafont [11].

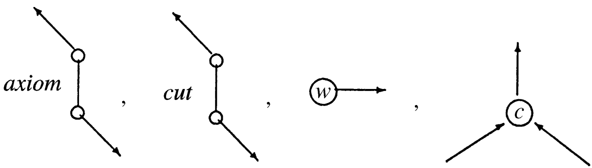
above may be drawn as



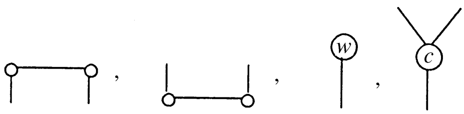

or, in short without the marks and arrows,



An axiom-link, a cut-link and a weakening-link are of the form

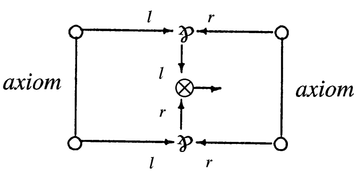



respectively, and may be drawn as

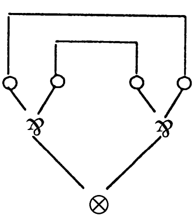



respectively.

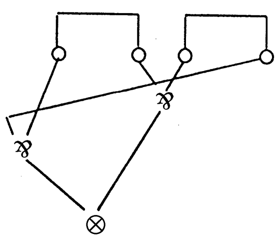
Hence, for example, a proof structure of the form




may be often represented as

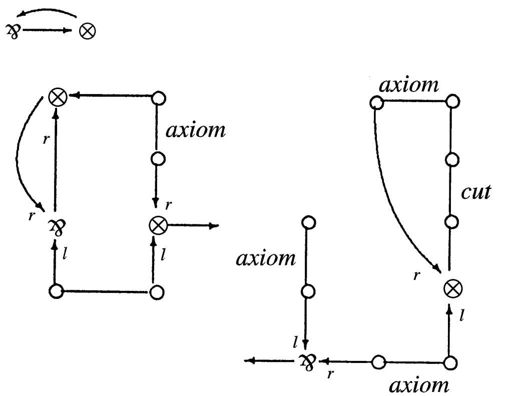


or

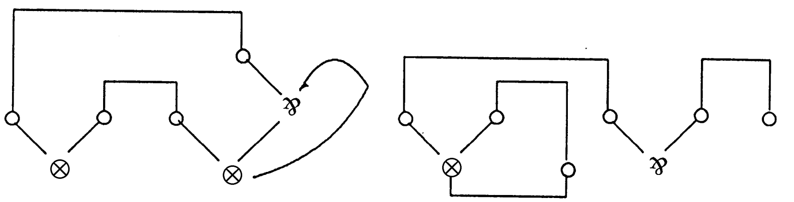


or the like.

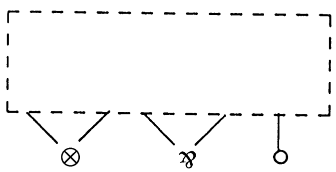
We often represent a proof-structure as if it were of a tree-form; the end nodes and some logical connective links connected to the end nodes downwards on the bottom. This can be done because a neighborhood of each end node has a tree-structure locally. For example, the following is a proof-structure which has a directed cycle



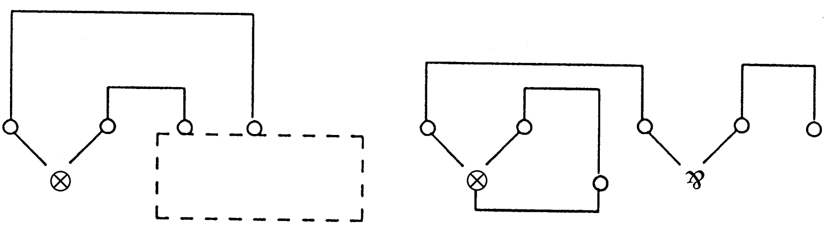

We could represent this as, for example,



or, when the upper part is not needed to represent in detail we could represent this as

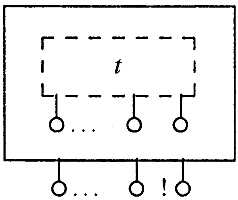


or

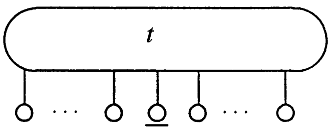


Note that our convention of drawing proof-structures does not ensure anything like typability nor sequentializability of a proof-structure (hence, in particular, our notion of proof-structure does not satisfy the Danos–Regnier's switching condition [4], nor equivalently Girard's long trip condition [6], even for the multiplicative fragment, and, hence is essentially the same as the notion of so called "Danos–Regnier graph" but extended to the modality links.)

For the !-rule we use the box notation for the proof-structures, in the usual way (as [6]). With the help of !-box, !-rule is denoted as

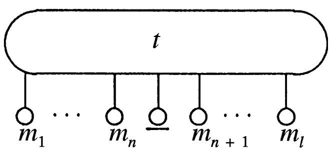



We assume that there is at most one distinguished end node for any proof-structure. The distinguished end node is denoted as




by underlining the distinguished end node. An end node which is not a distinguished node is called an environment node.

For a given set $M$, a proof-structure with labels from $M$ is a proof-structure whose environment end nodes have labels from $M$; for example, a proof-structure $t$ with labels $m_1,\dots,m_n,\ m_{n+1},\dots,m_l$ is of the form

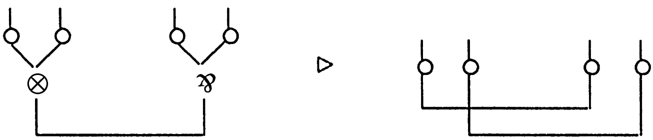



For a set $M$, $P_M$ is the set of proof-structures with labels from $M$. From now on we assume that a set $M$ is given, and that a proof-structure with labels is called just a proof-structure.

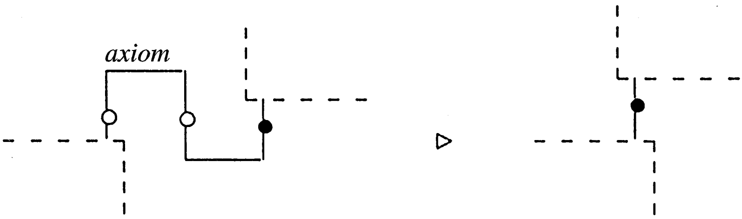
Here we identify the words "type" and "formula" in the rest of this paper. A typable proof (or a typable proof-structure) is a proof-structure in which there is a suitable assignment of a type (i.e. formula) to each axiom node and weakening node such that the resulting proof-structure becomes a (well-typed) proofnet in the usual sense, (therefore, the two nodes of each cut-link become exactly dual, the types of the three nodes involved by the contraction link are the same (and of the form $?A$ for some $A$), and the environment types of !-rule are of the form $?B_1,\dots,?B_n$ for some $B_1,\dots,B_n$, and the proof-structure can be reconstructed by the inductive formation rules of well-typed proofnet (i.e., there is a type-inference proof for the typability of the proof-structure. Hence, a typable proof means a proof-net. By a well-typed proof we mean a typable proof with a suitable type assignment.)

We assume the usual cut-elimination reduction (in short, *reduction*) and its reverse, *expansion*. Namely, a reduction $t \triangleright s$ is a reduction from $t$ to $s$ by a one-step reduction of a cut. The reductions are the following and their duals. The reverse relation $s \triangleleft t$ of a reduction relation $t \triangleright s$ is called an expansion. The reverse of a reduction rule is called an expansion rule. The labels for the environment end nodes should be preserved by reductions and by expansions, namely, we assume that the corresponding end nodes (before and after a reduction) have the same labels (from $M$). The following are the reduction rules. In the below, the left-hand side expresses the redex part of each reduction rule.

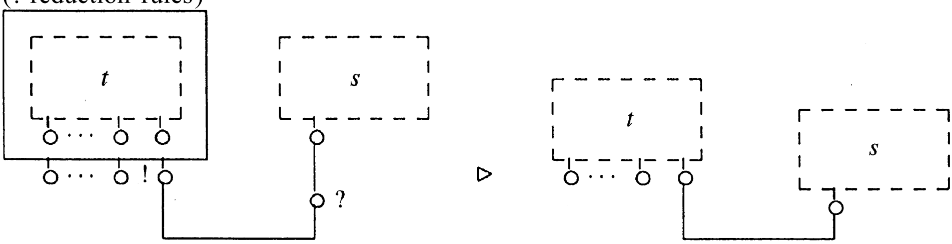
($\otimes$-reduction rule)



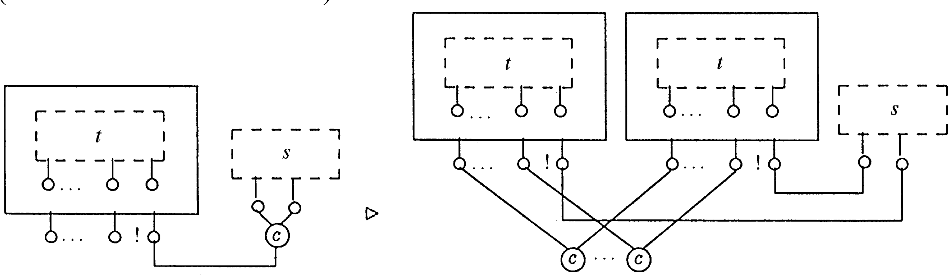
(Axiom-reduction rules)



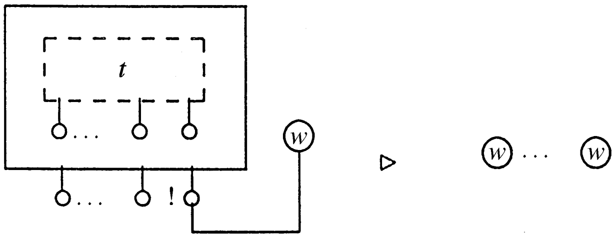

(!-reduction rules)




(?-contraction reduction rules)



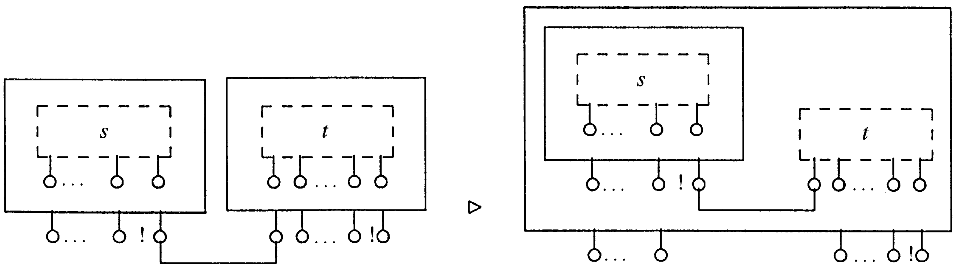
(?-weakening reduction rules)



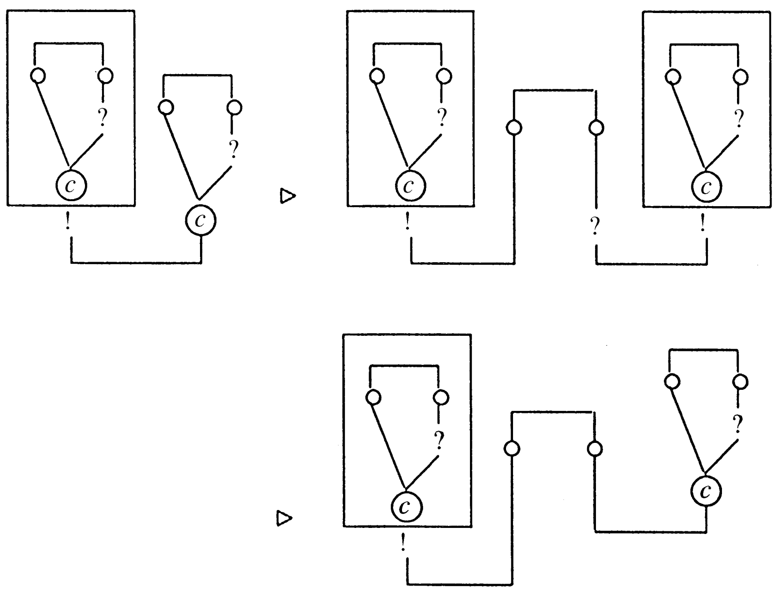

The above redex is deleted and the environment nodes of the box are replaced by the new weakening nodes.

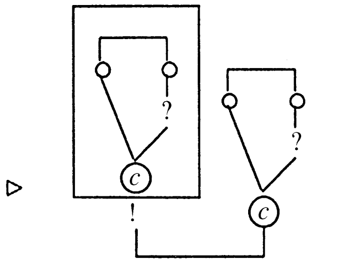
(Entering reduction rules) The entering rules were called "commutation rules" in [6].

• For !-box:




A proof-structure is not normalizable (i.e., terminating) in general; for example,





We may use the usual type-theoretic notation, namely, $x_1:A_1,\ldots,x_{k-1}:A_{k-1}$, $x_{k+1}:A_{k+1},\ldots,x_n:A_n \vdash t:A_k$ means that $t$ is typable of type $A$ (for the distinguished end node) with types $A_1,\ldots,A_{k-1}$, $A_{k+1},\ldots,A_n$ for the environment nodes $x_1,\ldots,x_{k-1}$, $x_{k+1},\ldots,x_n$. When the indication of the environment nodes is not important we abbreviate it as $\vdash t:A_1,\ldots,A_{k-1},\underline{A_k}, A_{k+1},\ldots,A_n$ or just $t:A_1,\ldots,A_{k-1},\underline{A_k},A_{k+1},\ldots,A_n$.

Here, we assume the usual type inference rules for $\otimes$, $⅋$, !, ? and the axiom (for typing the axiom-links) as well as the following natural type inference rules for $\wedge$ (intersection-type) and $\vee$ (union-type).

(the intersection type inference)

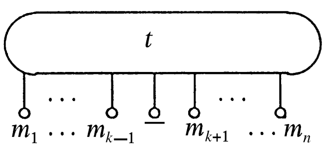
$$\frac{\vdash t:\Gamma,A \quad \vdash t:\Gamma,B}{\vdash t:\Gamma,A\wedge B}$$

(the union type inference)

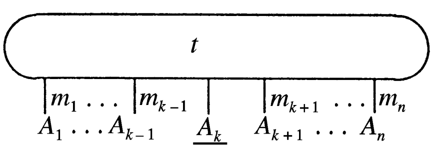
$$\frac{\vdash t:\Gamma,A}{\vdash t:\Gamma,A\vee B}\ \text{(for any } B\text{)}, \qquad \frac{\vdash t:\Gamma,A}{\vdash t:\Gamma,B\vee A}\ \text{(for any } B\text{)}.$$

Here, any end-node of $t$ may be underlined (i.e., distinguished), including the case where no end-node of $t$ is underlined.

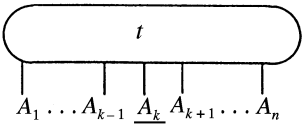
If $t$ is of the form



and is typable as $t:A_1,\ldots,A_{k-1},\underline{A_k},A_{k+1},\ldots,A_n$, we may denote it as



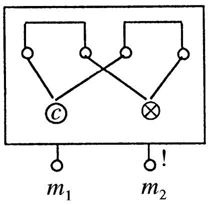
and if the labels $m_1, \dots, m_n$ are not important for the argument, we may denote it as




($A_k$ is called the type of $t$ and $A_1, \dots, A_{k-1}, A_{k+1}, \dots, A_n$ the environment types of $t$.) We may omit the underline when it is obvious in the context or not important.

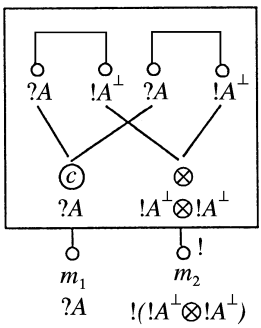
Examples of typable proof-structures.

(1)

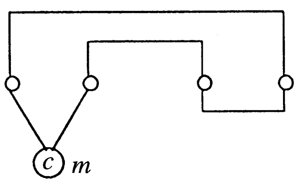
$t \equiv$ 


is typable, as $t : ?A, !(!A^{\perp} \otimes !A^{\perp})$

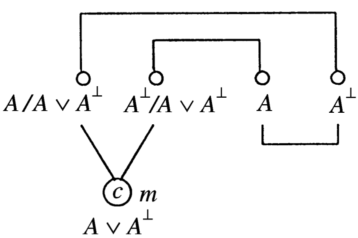
since one can make a type assignment, for example, as follows:




(2)

$t \equiv$ 

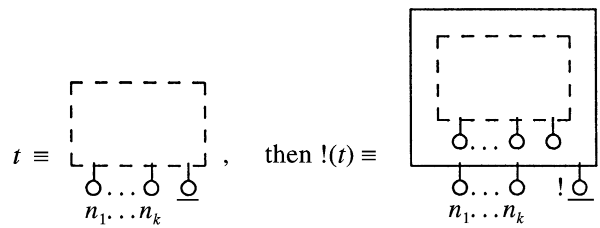

is typable as $t : A \vee A^{\perp}$ since one can make a type assignment, for example, as follows:



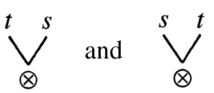
where $A/A \vee A^{\perp}$ means that one can first assign $A$ then $A \vee A^{\perp}$ by the use of the union type inference rule.

*Notation*: For two proof-structures $t$ and $s$ with a distinguished end node,  or $t \circ s$ denotes a proof-structure obtained by making a cut (link) between the two (underlined) distinguished end nodes. If $x$ denotes an environment end node of $t$ then $t[x := s]$ is a proof-structure obtained by connecting the environment node $x$ and the distinguished end node of $s$ by a cut-link. When the indicated environment end node is obvious $t[x := s]$ is also denoted as $t[s]$. When some of the environment end nodes are indicated as $x_1, \ldots, x_n$, then we use the notation $t[x_1 := s_1, \ldots, x_n := s_n]$ to denote the proof structure obtained from $t$ by connecting $x_i$ with the distinguished node of $s_i$ with a cut link. We also write $t[s_1, \ldots, s_n]$ when $x_1, \ldots, x_n$ are obvious from the context or are not important. We identify two proof-structures $t \circ s$ and $s \circ t$ in this paper. A proof-structure of the form

is abbreviated as $t \cdot s$. A proof-structure obtained from $t$ by adding the logical inference ! (with !-box) for the distinguished node is denoted by $!(t)$. Namely, if




Note that $t \circ s$ and $s \circ t$ are identified while $t \cdot s$ and $s \cdot t$ are not, namely, we distinguish two proof-structures




(more precisely, these two may be formally written as


with the indices $l$ and $r$ for the two in-edges of $\otimes$-link).

$$\alpha \cdot \beta = \{t \cdot s \mid t \in \alpha, s \in \beta\},$$
$$(\alpha)_! = \{!(t) \mid t \in \alpha\}.$$

$P_M^0$ $(\subseteq P_M)$ is the set of proof-structures without the distinguished end node. $\perp$ is an arbitrarily fixed subset of $P_M^0$. The orthogonal of $\alpha$ is defined as; $\alpha^{\perp} =_{\mathrm{def}} \{t : \forall s \in \alpha t \circ$

$s \in \perp\} = \{t : t \circ \alpha \subseteq \perp\}$. We include the empty proof-structure $\phi$ in the set $P_M$ of proof-structures, where we identify $\phi \circ t$ and $t \circ \phi$ as $t$ itself. In particular, for any $t \in \perp$, $\{t\}^\perp = \{\phi\}$. Hence, for any $\alpha \subseteq \perp$, $\alpha^\perp = \{\phi\}$ and $\alpha^{\perp\perp} = \{\phi\}^\perp = \perp$.

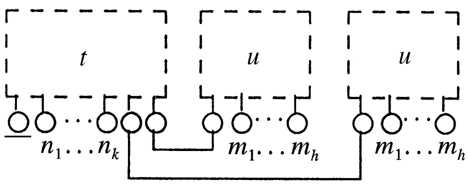
**Lemma 5.1.** *For any* $\alpha \subseteq P_M$, $\beta \subseteq P_M$,
1. $t \circ \alpha \subseteq \perp \Rightarrow t \circ \alpha^{\perp\perp} \subseteq \perp$.
2. $\alpha \circ t \subseteq \perp \Rightarrow \alpha^{\perp\perp} \circ t \subseteq \perp$.
3. $\alpha \circ \beta \subseteq \perp \Rightarrow \alpha^{\perp\perp} \circ \beta^{\perp\perp} \subseteq \perp$.

**Proof.** We only prove 1, without loss of generality. Since $t \circ \alpha \subseteq \perp$, $t \in \alpha^\perp$. On the other hand, $\forall s \in \alpha^\perp (s \circ \alpha^{\perp\perp} \subseteq \perp)$. In particular, by taking $s = t$, the claim holds. □

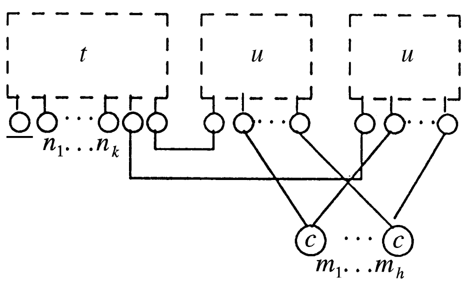
**Lemma 5.2.** *For any* $\alpha \subseteq P_M$, $\beta \subseteq P_M$,
1. $\alpha \subseteq \alpha^{\perp\perp}$.
2. $(\alpha^{\perp\perp})^{\perp\perp} \subseteq \alpha^{\perp\perp}$.
3. $\alpha \subseteq \beta \Rightarrow \alpha^{\perp\perp} \subseteq \beta^{\perp\perp}$.
4. $\alpha^{\perp\perp} \circ \beta^{\perp\perp} \subseteq (\alpha \circ \beta)^{\perp\perp}$.

$\alpha$ is a fact iff $\alpha^{\perp\perp} = \alpha$. The set of facts is denoted by $D_{P_M}$. Then, as easily seen, for any $\alpha \subseteq P_M$, $\alpha^{\perp\perp}$ is the smallest fact that includes $\alpha$. A fact $\alpha$ is called *regular* if $\alpha \neq \phi$ and $\alpha \neq P_M$. A proof-structure $t$ is called *regular* if there is a regular fact $\alpha$ such that $t \in \alpha$. The set of regular proof-structures is denoted by *Reg*. Namely, $Reg. = \bigcup\{\alpha : \alpha$ is a regular fact$\}$. The set of regular facts is denoted as $\tilde{D}_{P_M}$ $(\subseteq D_{P_M})$.

The proof structure obtained from $t[u,u]$ by making the contraction links between the corresponding environment end nodes of $u$ is denoted by $C(t[u,u])$; namely, for $t[u,u]$ of the form




$C(t[u,u])$ is of the form



A subdomain of the proof structures is a subset of the proof structures $P_M$ which is closed under the concatenation, where for any proof structures $s_1, \ldots, s_n$ and for any typable proof structure $t[x_1, \ldots, x_n]$ with exactly $n$-many environment end nodes, $t[s_1, \ldots, s_n]$ is called a concatenation of $s_1, \ldots, s_n$. Hence, $P'_M \subseteq P_M$ is a subdomain of the proof structures if for any $s_1, \ldots, s_n \in P'_M$ and for any typable proof structure $t[x_1, \ldots, x_n]$ of $n$-many environment end nodes, $t[s_1, \ldots, s_n]$ is also in $P'_M$.

Let $J \subseteq P_M$ be a subdomain of the proof-structures with a distinguished end node satisfying that for any $u \in J$ and for any proof structure $t \in P_M$ which has at least two environment end nodes, $\{C(t[u,u])\}^{\perp\perp} \subseteq \{t[u,u]\}^{\perp\perp}$.

Note that the definitions of $J$ is viewed as a natural generalization of those of $J$ in Section 2 by replacing the underlying monoid structure by the proof structure.

We assume that a subset $W \subseteq M$ is given. ($W$ is used for a closure property of $\perp$ below.)

Let $I$ be $J \lceil W = J \cap P_W$, set obtained from $J$ by restricting the labels set to $W$.

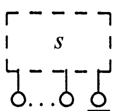
By an outer-most reduction we mean a reduction whose redex-cut link is not inside any box. The reverse of an outer-most reduction is called an outer-most expansion.

$(D_{P_M}, I, \perp)$ is a standard phase space if $\perp \subseteq P^0_M$ satisfies the following closure conditions:

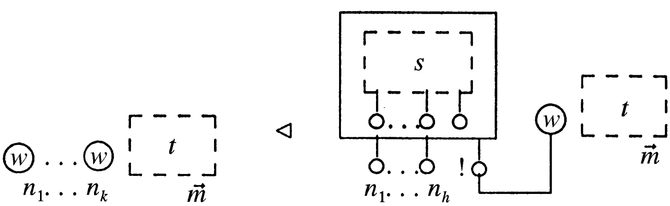
1. The closure under expansion; $s \in \perp$ if $t \in \perp$ and $t \lhd s$ by a one-step outer-most expansion by the axiom-expansion, $\otimes$-expansion, !-expansion, ?-weakening expansion, entering expansion rules, where the outer-most ?-weakening expansion is restricted to the following form.

   (?-weakening expansion rules)

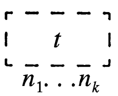
   For any regular $s$ such that




$\in I,$




2. The closure under the weakening restricted with $W(\subseteq M)$; if




$\in \perp$

and $m \in W$, then

$$\underset{n_1 \ldots n_k}{\boxed{\quad t \quad}}\ \underset{m}{w} \in \perp.$$

A standard phase space is called just a phase space in this and the next two sections except for Section 7.2.

Now we define some basic operators on the phase space domain $P$. Phase space operators for *proofs* are defined as follows.
1. $\alpha \otimes \beta = (\alpha \cdot \beta)^{\perp\perp}$.
2. $\alpha ⅋ \beta = (\alpha^{\perp} \cdot \beta^{\perp})^{\perp}$.
3. $\alpha \wedge \beta = \alpha \cap \beta$.
4. $\alpha \vee \beta = (\alpha \cup \beta)^{\perp\perp}$.
5. $!\alpha = (I \cap \alpha)_!^{\perp\perp}$.
6. $?\alpha = (I \cap \alpha^{\perp})_!^{\perp}$.

As easily seen, for fact $\alpha$ (and fact $\beta$), the resulting set is a fact. In particular, for 3 above we use the following lemma.

**Lemma 5.3.** *For any* $\alpha, \beta \subseteq P$, $(\alpha \cup \beta)^{\perp\perp} = (\alpha^{\perp} \cap \beta^{\perp})^{\perp}$. *Hence* $\alpha \vee \beta = (\alpha^{\perp} \wedge \beta^{\perp})^{\perp}$.

**Proof.** First, we show $(\alpha^{\perp} \cap \beta^{\perp})^{\perp} \subseteq (\alpha \cup \beta)^{\perp\perp}$. It suffices to show $(\alpha \cup \beta)^{\perp} \subseteq \alpha^{\perp} \cap \beta^{\perp}$. Let $t \in (\alpha \cup \beta)^{\perp}$. Hence, $t \circ (\alpha \cup \beta) \subseteq \perp$. Hence, $t \in \alpha^{\perp} \cap \beta^{\perp}$.

Second, we show $(\alpha \cup \beta)^{\perp\perp} \subseteq (\alpha^{\perp} \cap \beta^{\perp})^{\perp}$. It suffices to show $\alpha \cup \beta \subseteq (\alpha^{\perp} \cap \beta^{\perp})^{\perp}$. It is obvious since $(\alpha \cup \beta) \circ (\alpha^{\perp} \cap \beta^{\perp}) \subseteq \perp$. $\square$

**Definition 5.1.** For any formula $A$ and for any assignment (valuation) $\varphi(R) = R^* \in \tilde{D}_{P_M}$ (i.e., any assignment $\varphi$ of a regular fact) for each atomic formula $R$, the inner value $A^*$ of $A$ is defined as follows.
1. $(A^{\perp})^* = (A^*)^{\perp}$.
2. $(A \otimes B)^* = A^* \otimes B^*$.
3. $(A ⅋ B)^* = A^* ⅋ B^*$.
4. $(A \cap B)^* = A^* \wedge B^*$.
5. $(A \cup B)^* = A^* \vee B^*$.
6. $(!A)^* = !A^*$.
7. $(?A)^* = ?A^*$.

**Lemma 5.4.** *For any fact* $\alpha$ *of a phase space*, $\alpha$ *satisfies the same closure properties as those of* $\perp$.

**Proof.** We prove only the closure property under outermost expansions. The other closure properties can be proved in the same way. Assume that $t$ is obtained from $s \in \alpha$ by one-step outermost expansion. Since $s \circ \alpha^{\perp} \subseteq \perp$, by the closure under outermost expansion for $\perp$, $t \circ \alpha^{\perp} \subseteq \perp$. Hence, $t \in \alpha^{\perp\perp} = \alpha$. $\square$

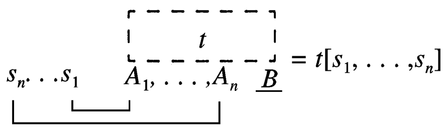
The Soundness Theorem is stated and proved (by the induction on the length of a typability proof) in the same way as that of the semantics for "provability".

The soundness theorem holds with respect to the phase models.

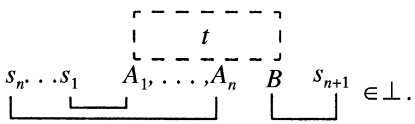
**Theorem 5.1** (Soundness). *For any set $M$ and for any phase model $(D, I, \perp, \varphi)$, for typable proof $t$ of $x_1 : A_1, \ldots, x_n : A_n \vdash t : B$,*

$$t[x_1 := A_1^{*\perp}, \ldots, x_n := A_n^{*\perp}] \subseteq B^*,$$

*namely, for any $s_1 \in A_1^{*\perp}, \ldots, s_n \in A_n^{*\perp}$, $t[s_1, \ldots, s_n] \in B^*$, where*

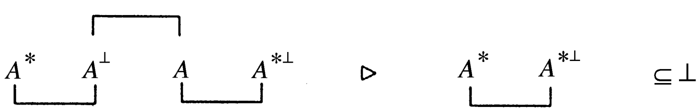


**Proof.** The above statement is equivalent to the following; for any $s_1 \in A^{*\perp}, \ldots, s_n \in A^{*\perp}$, $s_{n+1} \in B^{*\perp}$, $t[s_1, \ldots, s_n] \circ s_{n+1} \in \perp$, namely



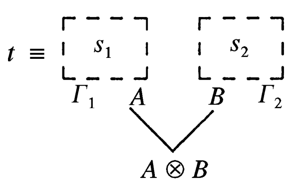
We shall use both the equivalent forms interchangably in the proof. The proof is carried out by the induction on the construction of the given "typed" proof $t$, namely the inductive structure of typability proof of $t$. Here a typable proof-structure $t$ is identified with a specifically typed proof. Below, for readability we omit the labels from $M$ in the figures.

(1) When $t$ is an axiom-link of the form $A^{\perp}$ $\quad A$ Then,



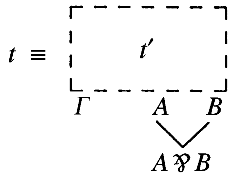
and by the closure property of $\perp$ under the axiom-expansion, the left-hand side is also in $\perp$.

(2) When the last inference of $t$ is of the form

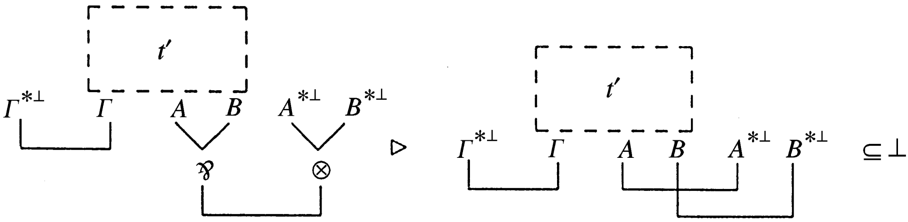


By the induction hypothesis, $s_1[\Gamma_1^{*\perp}] \subseteq A^*$, $s_2[\Gamma_2^{*\perp}] \subseteq B^*$. Hence, $s_1[\Gamma_1^{*\perp}] \cdot s_2[\Gamma_2^{*\perp}] = s_1 \cdot s_2[\Gamma_1^{*\perp}, \Gamma_2^{*\perp}] \subseteq A^* \cdot B^* \subseteq (A^* \cdot B^*)^{\perp\perp} = A^* \otimes B^*$. Therefore, $s_1 \cdot s_2[\Gamma_1^{*\perp}, \Gamma_2^{*\perp}] \circ (A^* \otimes B^*)^{\perp} \subseteq \perp$.

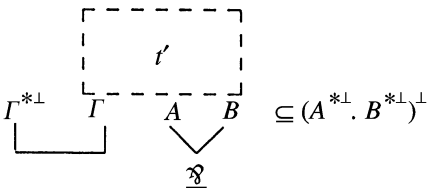
(3) When the last inference of $t$ is of the form



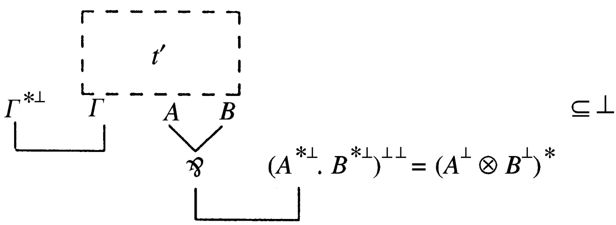

Then,




by the induction hypothesis. Hence, by the closure under outermost expansion, the left-hand side is included in $\perp$. Hence




Therefore




(4) When the last type inference rule is an $\wedge$-inference rule,

$$\frac{\vdash t : \Gamma, \underline{A} \quad \vdash t : \Gamma, \underline{B}}{\vdash t : \Gamma, \underline{A \wedge B}}$$

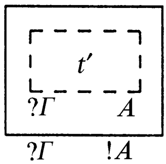
By the induction hypothesis, $t[\Gamma^{*\perp}] \in A^*, t[\Gamma^{*\perp}] \in B^*$. Hence, $t[\Gamma^{*\perp}] \in A^* \wedge B^*$.

(5) When the last type inference rule is a $\vee$-inference rule,

$$\frac{\vdash t : \Gamma, \underline{A}}{\vdash t : \Gamma, \underline{A \vee B}}$$

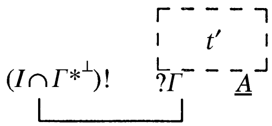
By the induction hypothesis, $t[\Gamma^{*\perp}] \in A^*$. Hence $t[\Gamma^{*\perp}] \in A^* \cup B^* \subseteq (A^* \cup B^*)^{\perp\perp}$.

(6) When the last inference of $t$ is of the form

$t \equiv$ 

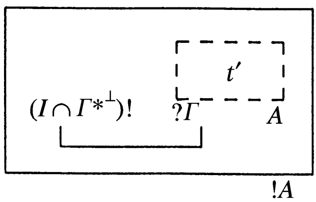
.

By the induction hypothesis,

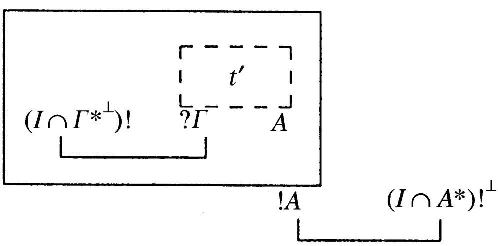


$\subseteq I \cap A^*$.

Therefore, by the closure under the concatenation of $J$, the left-hand side $\subseteq I \cap A^*$. Hence,

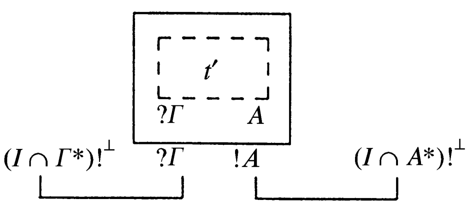


$\in (I \cap A^*)!$ .



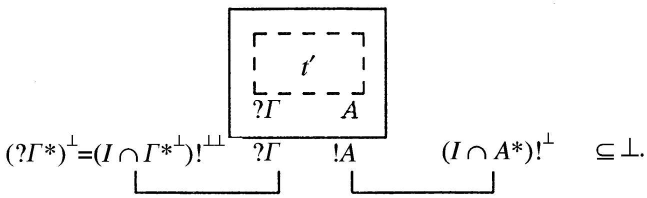
$\subseteq \perp$.

Then by the closure under outermost (entering) expansions,

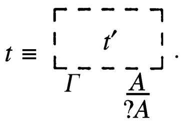


$\subseteq \perp$.

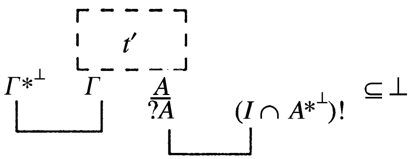
Therefore,



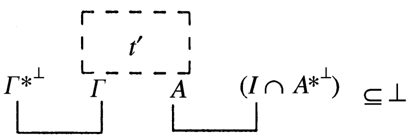
(7) When the last inference of $t$ is of the form



it suffices to show that

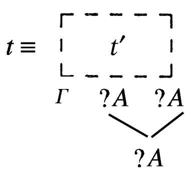


By the induction hypothesis,

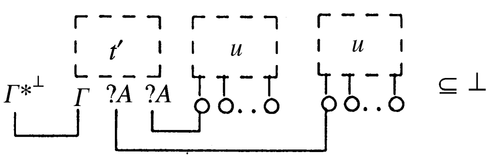


Hence, by the closure under outermost expansion for $\perp$, the claim holds.

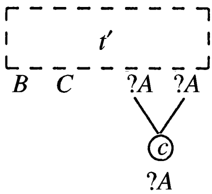
(8) When the last inference of $t$ is of the form



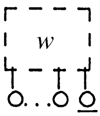
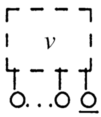
By the induction hypothesis, for any $u \in (I \cap A^{*\perp})_!$,



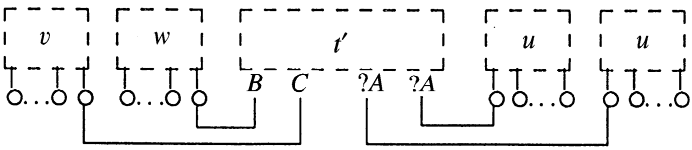
Without loss of generality we assume $t$ is of the form



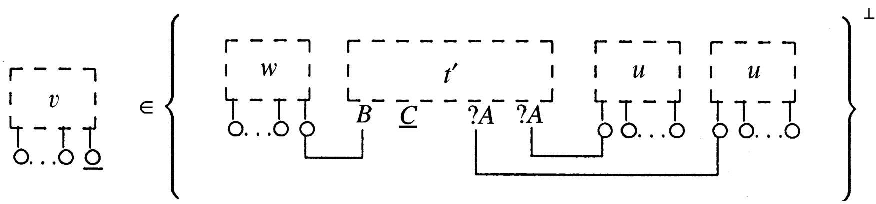

Take an arbitrary

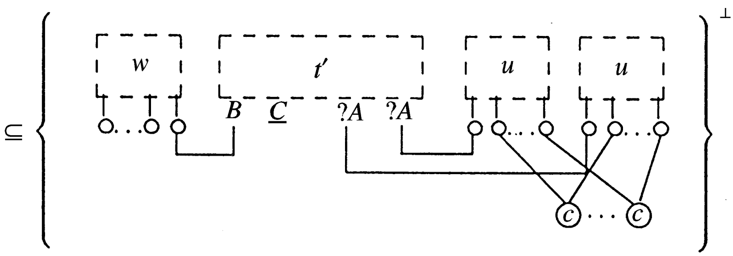
 $\in B^{*\perp}$ and  $\in C^{*\perp}$.

Then by the induction hypothesis,

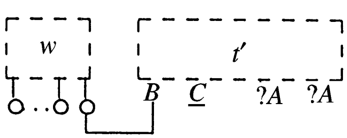
 $\in \perp$

Hence,

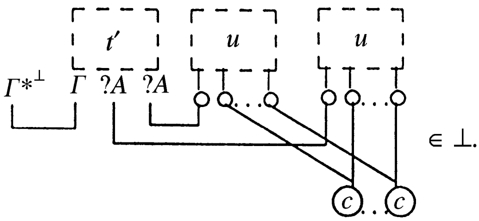




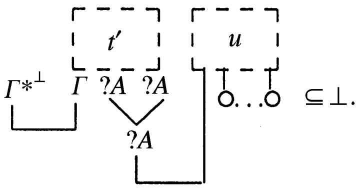
(by the definition of $I$). Here the regularity of



is obvious from the induction hypothesis. Hence

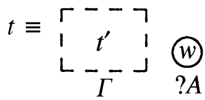


By the closure under the outermost expansion,

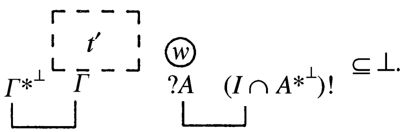


If $t$ has only one end node ⓒ (and $\Gamma$ is empty): Assume $t'[u,u] \in \perp$ for $u \in (I \cap A^{*\perp})_!$. Then by the condition of $I$, $C(t'[u,u]) \in \{C(t'[u,u])\}^{\perp\perp} \subseteq \{t'[u,u]\}^{\perp\perp} = \perp$. Hence, $C(t'[u,u]) \in \perp$. Then, by the closure under the outermost expansion, the claim holds.

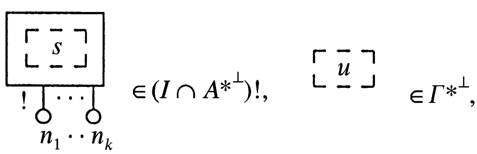
(9) When the last inference of $t$ is of the form

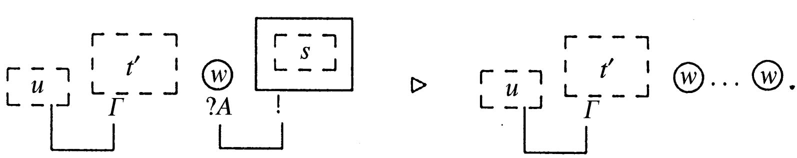


It suffices to show

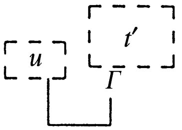


For any



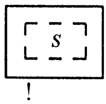


By the induction hypothesis,


$\subseteq \perp.$

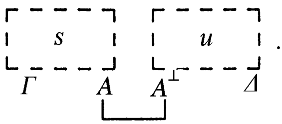
Hence, by the closure under the weakening rule and under the outermost expansions, the claim holds.

Note that since

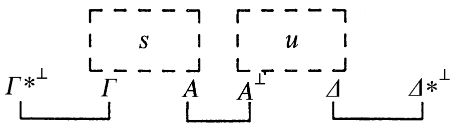

$\in (I \cap A^{*\perp})!, \quad s \in A^{*\perp},$

therefore $s$ is regular, hence the last expansion is permissible.

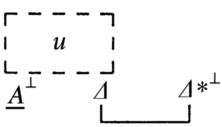
(10) When the last inference of $t$ is the cut rule of the form

$t \equiv$ 

It suffices to show


$\subseteq \perp.$

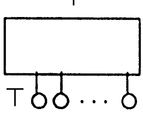
By the induction hypothesis for $u$,

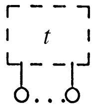

$\subseteq A^{*\perp}.$

Hence, by the induction hypothesis for $s$, the claim holds. $\square$

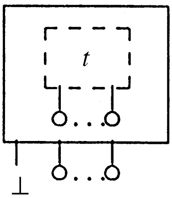
*Note on the addition of logical constants*

Now we take a slight look at constants although we do not intend to go into the details on this subject in this paper. The (untyped) proof-structures domain $P$ is extended by introducing the following.

- 1-axiom node: 
- $\top$-box:  (cf. [6]).

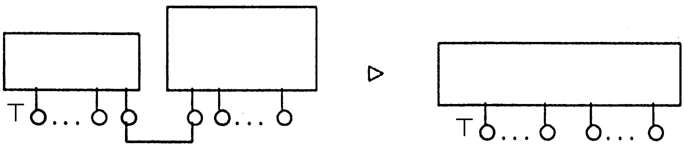
• $\perp$- box; for any proof-structure 

,

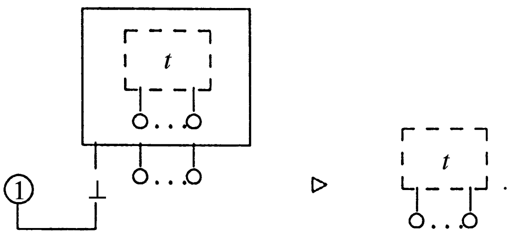



is also a proof-structure.

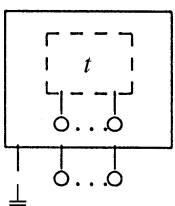
• $\top$-reduction:




• Additional axiom-reduction:




• The closure property for $\perp$ is extended by the above additional closure under axiom-reductions, by the additional closure under the additional outer-most axiom-expansions (the reverse of the above additional axiom reductions), and by the following condition; $t \in \perp$ and $t$ has no distinguished end node $\Leftrightarrow$




$\in \perp$ (where, the new node $\perp$ is distinguished).

Then, $\{1\}^{\perp} = \perp$ and $\mathbf{1} = \perp^{\perp} = \{1\}^{\perp\perp}$.

Then, the above argument for the Soundness proof goes through with these extensions.

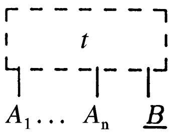
We define $\tilde{1}$ as

$$n_1,\dots,n_k \in \tilde{1} \Leftrightarrow \underset{n_1}{\textcircled{w}}\cdots\underset{n_k}{\textcircled{w}}\ \textcircled{1} \in 1=\{\textcircled{1}\}^{\perp\perp}.$$

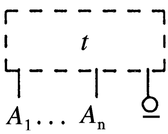
Then it is easily seen that $\tilde{1}=W$. Hence, one could define $I$ to be $J\lceil\tilde{1}=(J\cap P_{\tilde{1}})$.

## 6. Normalization proofs in the phase semantics-like framework

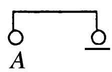
Let $M$ be the set of formulas. When a proof-structure $t$ is typable as $x_1:A_1,\dots,x_n:A_n \vdash t:B$, a proof $t'$ with labels obtained from $t$ by attaching labels $A_1,\dots,A_n$ at the environment end nodes $x_1,\dots,x_n$ is called a *well-typed proof* of type $B$, in this section. Hence, when $t$ is typable as



then a well-typed proof $t'$ of type $B$ is obtained by attaching labels $A_1,\dots,A_n$ on proof-structure $t$, of the form

$t' \equiv$ 

Hence, any well-typed proof belongs to $P_M$.

In the rest of this paper, a well-typed proof of the form  is denoted by $\{A\}$.

Let $[\![A]\!]_{SN}$ be $\{t\mid t$ is typable of type $A$ and is strongly normalizable.$\}$. Let $[\![A]\!]_N$ be $\{t\mid t$ is typable of type $A$ and is weakly normalizable.$\}$, and $[\![A]\!]_T$ be $\{t\mid t$ is typable of type $A\}$.

Now we consider phase spaces $\mathscr{M}_0\equiv(D_{P_M},I,[\![\perp]\!]_T)$, $\mathscr{M}_1\equiv(D_{P_M},I,[\![\perp]\!]_N)$ and $\mathscr{M}_2\equiv(D_{P_M},I,[\![\perp]\!]_{SN})$. When a statement holds both for $[\![A]\!]_T$ and for $[\![A]\!]_N$ we state it with the reference to $[\![A]\!]$. $I_0$ is defined to be the set $\{?\Gamma\mid\Gamma$ is a finite set of formulas$\}$. $I$ is defined to be $\{t$: $t$ is a well-typed proof and $|t|\in I_0\}$.

**Lemma 6.1.** *If* $\alpha\circ\{A\}\subseteq[\![\perp]\!]$, *then* $\alpha\subseteq[\![A]\!]$ *for* $[\![\ ]\!]=[\![\ ]\!]_T$ *or* $[\![\ ]\!]_N$ *or* $[\![\ ]\!]_{SN}$.

**Proof.** It is obvious that $\alpha$ is of type $A$. It suffices to show the following.

(When $\perp=[\![\perp]\!]_N$ or$[\![\perp]\!]_{SN}$.) By the assumption, $s\circ\{A\}$ is weakly (or strongly, resp.) normalizable for any $s\in\alpha$, hence so is $s$.

(When $\perp=[\![\perp]\!]_T$.) It is obvious. $\square$

**Lemma 6.2.** *If $\alpha$ is a fact in $\mathscr{M}_0$ or $\mathscr{M}_1$ or $\mathscr{M}_2$ and $t \in \alpha$, then if $t'$ is composed from $t$ by adding a logical inference link at some of the environment end nodes* (*with the induced new labels for the new end nodes*), *then $t' \in \alpha$.*

**Proof.** Since $t \circ \alpha^{\perp} \subseteq [\![\perp]\!]$, $t' \circ \alpha^{\perp} \subseteq [\![\perp]\!]$. Hence $t' \in \alpha^{\perp\perp}$. Since $\alpha$ is a fact, $t' \in \alpha$. $\square$

**Lemma 6.3.** $[\![A]\!] \circ \{A\} \subseteq [\![\perp]\!]$.

**Proof.** For the case $[\![A]\!] \equiv [\![A]\!]_T$ or $[\![A]\!]_N$, it is obvious. For $[\![A]\!] \equiv [\![A]\!]_{SN}$, it suffices to show that for any $t \in [\![A]\!]$, $t \circ \{A\}$ is strongly normalizable. This can be easily seen by the induction on the well-founded reduction tree of $t$; For any reduction $t \circ \{A\} \triangleright t' \circ \{A\}$, the right-hand side is strongly normalizable by the induction hypothesis. For the case that $t$ is a normal form, then $t \circ \{A\} \triangleright t$ is the only reduction. Therefore, every one step reduction of $t \circ \{A\}$ is strongly normalizable, hence so is $t \circ \{A\}$. $\square$

**Lemma 6.4.** *For any formula $A$, $[\![A]\!]$ is a fact for $[\![\ ]\!] = [\![\ ]\!]_T$ or $[\![\ ]\!]_N$ or $[\![\ ]\!]_{SN}$.*

**Proof.** By the above lemma, $[\![A]\!] \circ \{A\} \subseteq [\![\perp]\!]$. Hence, $\{A\} \in [\![A]\!]^{\perp}$. Therefore, $[\![A]\!]^{\perp\perp} \circ \{A\} \subseteq [\![\perp]\!]$. By Lemma 6.1, $[\![A]\!]^{\perp\perp} \subseteq [\![A]\!]$. Therefore, $[\![A]\!]^{\perp\perp} = [\![A]\!]$. $\square$

**Lemma 6.5.**
1. *Let $\perp = [\![\perp]\!]_T$. Then $\mathscr{M}_0 = (P, [\![\perp]\!]_T, \varphi)$ is a phase model for the assignment $\varphi(R) = [\![R]\!]_T$.*
2. *Let $\perp = [\![\perp]\!]_N$. Then $\mathscr{M}_1 = (P, [\![\perp]\!]_N, \varphi)$ is a phase model for the assignment $\varphi(R) = [\![R]\!]_N$.*

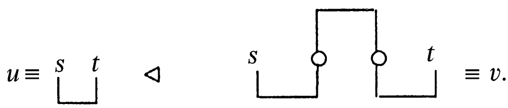
**Proof.** We first prove 2. The case 1 can be proved in the same way but the proof is much simpler.

(Closure under outer-most expansion). If $t \in [\![\perp]\!]_N$, $t$ is typable. If $t \lhd s$ by the permissible expansion rules in $\mathscr{M}_1$ then $s$ is also typable. This is because all the weakening expansions preserve the typability (due to the restriction on these expansion rules). Note that since we take $[\![\perp]\!]$ as $\perp$, any regular proof-structure is typable. The closure under the outermost expansions of $[\![\perp]\!]_N$ is obvious.
- The closure under the axiom-reduction for $[\![\perp]\!]_N$ is trivial.
- The closure under weakening rules of $[\![\perp]\!]_N$ is also obvious.
- The proof of 1 is essentially contained in the proof of 2.
- The proof for the same lemma for $\mathscr{M}_2$ is more involved. $\square$

**Lemma 6.6.** *Let $\perp = [\![\perp]\!]_{SN}$. Then $\mathscr{M}_2 = (D_{P_M}, [\![\perp]\!]_{SN}, \varphi)$ is a phase model for the assignment $\varphi(R) = [\![R]\!]_{SN}$.*

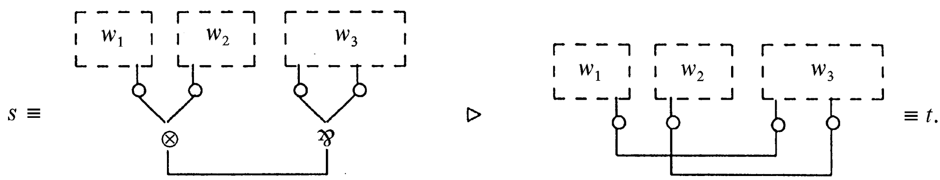
**Sketch of Proof.** All other cases are essentially the same as the proof of the previous lemma (for $\mathscr{M}_1$), we only consider the closure under the outermost expansions of $[\![\perp]\!]_{SN}$. We omit the labels from $M$ in the figures below if those are not essential for the argument.

*Case* 1: *An axiom expansion.* Assume $s \circ t \in [\![\bot]\!]_{SN}$. Consider the following axiom expansion:

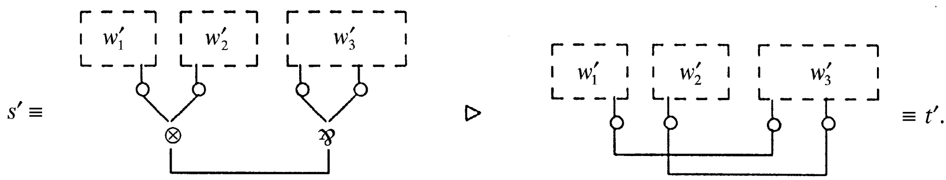


Consider an arbitrary reduction sequence starting from $v$. The above indicated axiom-reduction redex of $v$ may be duplicated finitely many times, some of which may be entered in some boxes. However, the corresponding reduction sequence starting from $u$ terminates (by the assumption) and the number of the indicated axiom-reduction redices is finite, the reduction sequence from $v$ terminates.

*Case* 2: $\otimes$*-expansion*:

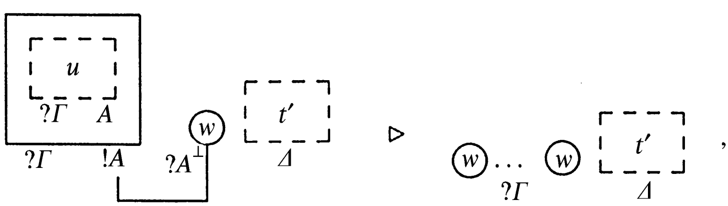


Since $t$ is strongly normalizable, so are $s_1$, $s_2$ and $s_3$. Consider an arbitrary reduction path of $s$. Then it reaches to the principal $\otimes$-reduction after finite steps of reductions;

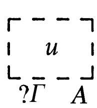
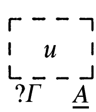


Since $t'$ is reached by finite steps of reductions from $t$, $t'$ is strongly normalizable. Hence so is $s$.

*Case* 3: ?-*weakening expansion*:



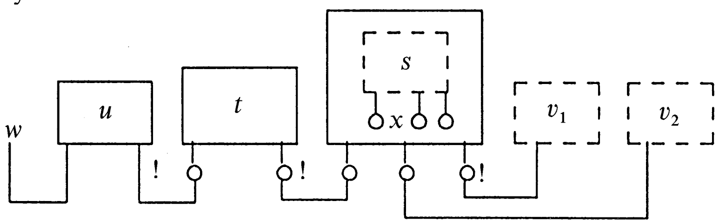
for regular

 (hence,  $\in I \cap [\![\bot]\!]_{SN}$).

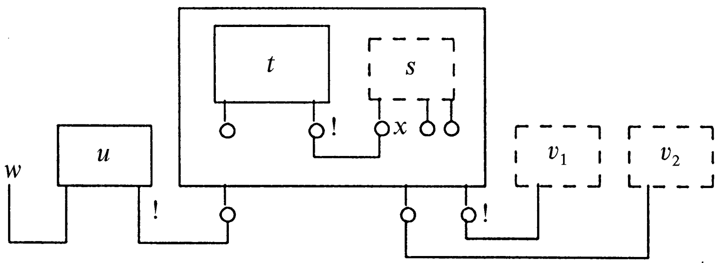
Since $t'$ and $u$ are strongly normalizable, by a finite number of steps, the left hand-side proof reaches a ?-weakening reduction. Then the resulting proof can be reached from the right hand-side proof directly.

*Case* 4: *The !-box entering expansion.* Consider the following form without loss of generality.

$q \equiv$



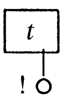

$\rhd$



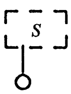

$\equiv p \in [\![\bot]\!]_{SN}$

Consider an arbitrary reduction sequence from $q$. We show that we can simulate this reduction sequence as a reduction sequence of $p$ so that the termination of the sequence for $p$ implies that for $q$.

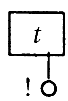
We call a box inherited from the box



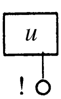

of $q$ a "$t$-box" and a position inherited from the environment end node position of




connected to the box



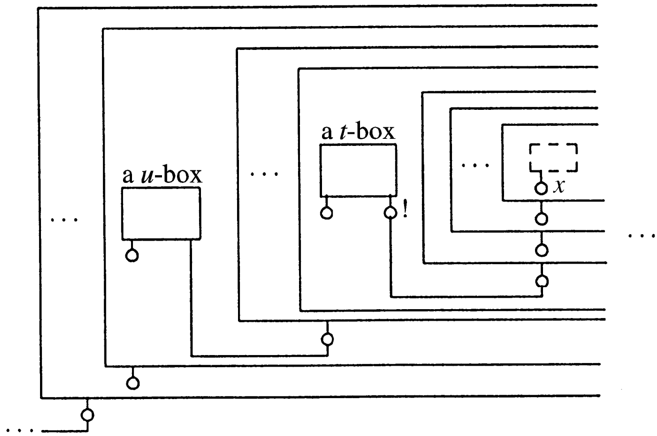

as "position $x$" (see the figure). The corresponding position in $p$ is also called "position $x$". We also call a box inherited from the box



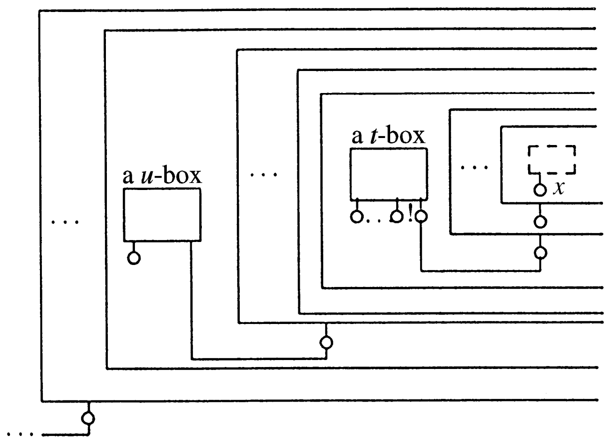

of $q$ a "$u$-box". Note that a $t$-box, a $u$-box and a position $x$ may be duplicated again and again in the reduction sequence from $q$.

The corresponding reduction sequence from $p$ can be simulated step by step by taking the same reduction except for the following cases:

(1) When a $t$-box enters into a box towards the related position $x$ of the form

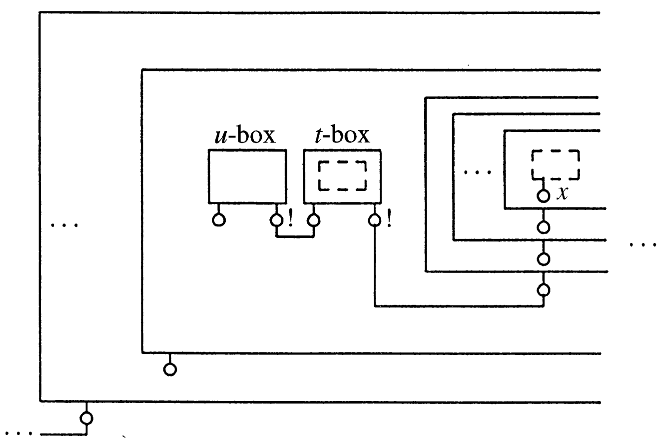



$\triangleright$

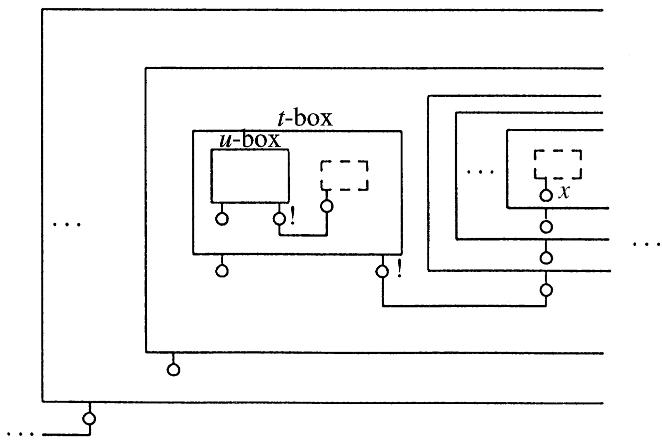



Then we do not construct any new reduction step for the corresponding reduction sequence from $p$.

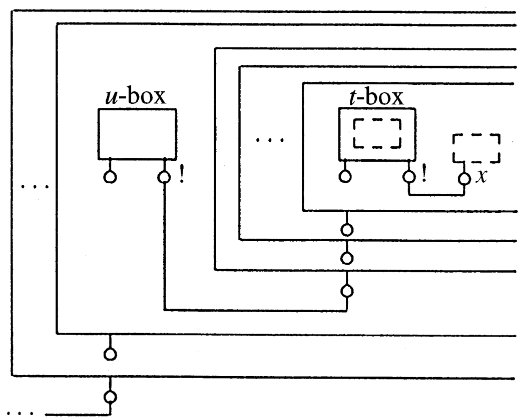
(2) When a $u$-box enters into a $t$-box of the form



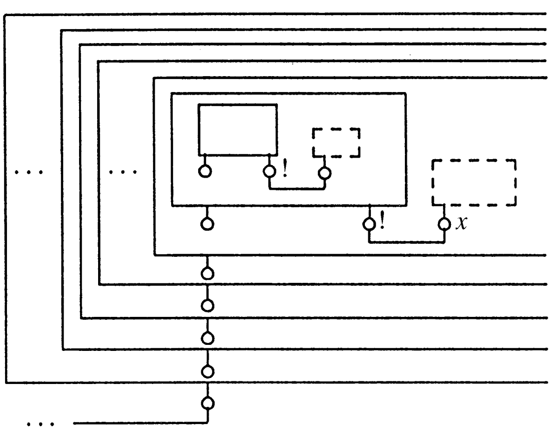
▷



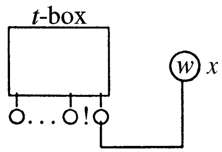

Then we repeat to apply the box-entering rule again and again until the corresponding $u$-box enters into the corresponding $t$-box, as follows:




▷ ▷ ▷

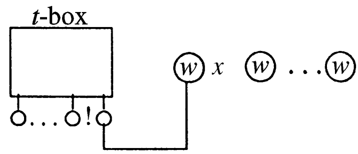


(3) When the position $x$ related to a $t$-box becomes a weakening node and it reaches the reduction with the $t$-box, of the form




due to an application of the ?-contraction reduction with the position $x$, then the corresponding ?-contraction leads to the form $w \ldots w$, which results from the environment end nodes of the $t$-box.

Then, for the reduction




we do not take any reduction on $w \ldots w$ for the corresponding reduction sequnece from $p$.

Since the above form of entering reduction of a $t$-box into another box in which an $x$ is located as well as the above form of a $t$-box contraction can be repeated only finitely many times successively and the corresponding reduction sequence from $p$ terminates (due to the fact that $p \in [\![\bot]\!]_{SN}$), the arbitrary given reduction sequence from $q$ also terminates. □

**Lemma 6.7** (Main Lemma). *In $\mathscr{M}$ (where $\mathscr{M}$ is either $\mathscr{M}_0$, $\mathscr{M}_1$ or $\mathscr{M}_2$), for any formula $A$, $A^* \subseteq [\![A]\!]$.*

The proof of the above Main Lemma is given later. From now on if a statement (and argument) holds for $[\![\ ]\!]_T$, $[\![\ ]\!]_N$ and $[\![\ ]\!]_{SN}$ at the same time we denote it as $[\![\ ]\!]$.

**Lemma 6.8.** *If $A^* \subseteq [\![A]\!]$ then $\{A\} \in A^{*\bot}$.*

**Proof.** $\{A\} \circ A^* \subseteq \{A\} \circ [\![A]\!] \subseteq [\![A]\!]$. The last inclusion means the closure under the well-typed axiom reduction, which can be easily checked. Hence, $\{A\} \in A^{*\bot}$. □

We also use the following slightly modified version of Main Lemma.

**Lemma 6.9** (Main Lemma, modified). *In $\mathscr{M}$ (where $\mathscr{M}$ is $\mathscr{M}_0$, $\mathscr{M}_1$ or $\mathscr{M}_2$), for any formula $A$, $\{A^\bot\} \in A^* \subseteq [\![A]\!]$.*

**Proof.** The second half is the above Main Lemma itself. $\{A^\bot\} \in A^*$ is proved by Lemma 6.8 with the help of the Main Lemma. □

Then with the Main Lemma and the fact that $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}_2$ are actually a phase model, we have

**Theorem 6.1** (Strong Completeness).
1. (*Completeness for proofs*). *For any proof-structure t, if* $t \in A^*$ *for any* (*standard*) *phase model, then t is a typable proof of type A.*
2. (*Strongly* (*weakly, resp.*) *Normalization*). *For any proof-structure t, if* $t \in A^*$ *for any* (*standard*) *phase model, then t is a strongly normalizable* (*weakly normalizable, resp.*) *proof of type A.*

**Proof.** Here we denote $\mathscr{M}$ for $\mathscr{M}_0$ or $\mathscr{M}_1$ or $\mathscr{M}_2$, $[\![A]\!]$ for $[\![A]\!]_T$ or $[\![A]\!]_N$ or $[\![A]\!]_{SN}$, respectively.

Since $t \in A^*$ and by the Main Lemma, $A^* \subseteq [\![A]\!]$. Hence $t \in [\![A]\!]$, which means the claims 1 and 2 above (i.e., when $[\![A]\!] = [\![A]\!]_T$, claim 1 holds; when $[\![A]\!] = [\![A]\!]_{SN}$ ($[\![A]\!]_N$, resp.), claim 2 holds). $\square$

Now we combine the Soundness Theorem and the Main Lemma to obtain a strong (weak, resp.) normalization proof, in the same manner as the former sections (Sections 3 and 4) for the provability case.

**Theorem 6.2** (Strong Normalization (weak normalization, resp.) Theorem). *Every typable proof is strongly normalizable* (*weakly normalizable, resp.*).

**Proof.** We denote $[\![A]\!]$ for $[\![A]\!]_{SN}$ (for $[\![A]\!]_N$, resp.).

By the Soundness Theorem and the Main Lemma ($\{A\} \in A^*$ for any $A$), for any proof $x_1 : A_1, \ldots, x_n : A_n \vdash t : B, t[\{A_1\}, \ldots, \{A_n\}] \in B^*$. Then by the Main Lemma, $B^* \subseteq [\![B]\!]$. Hence, $t[\{A_1\}, \ldots, \{A_n\}] \in [\![B]\!]$. Hence, by the closure under the well-typed axiom reductions of $[\![B]\!]$ (which is easily verified), $t \in [\![B]\!]$. $\square$

We now return to the Proof of Main Lemma.

**Proof of Main Lemma.** Now we show how the proof of the Main Lemma 3.1 for "provability" in Section 3 can be viewed as a proof of the Main Lemma for "provability". We explain how to interpret the notations in the original proof of Lemma 3.1. We read $\Gamma, \Delta$ as proof-structures $\in P_M$. The outer value $[\![A]\!]$ can be read as either $[\![A]\!]_{SN}$ or $[\![A]\!]_N$ or $[\![A]\!]_T$. The concatenation $\Gamma, \Delta$ (i.e., comma) should be normally understood as the cut operator $\Gamma \circ \Delta$, (but it is also sometimes understood as the product $\Gamma \cdot \Delta$. We shall remark it in such a case). The additive operators & and $\oplus$ should be understood as the intersection $\wedge$ and the union $\vee$. Then each cases (Case 1) $\sim$ (Case 8) of the proof of Lemma 3.1 can be interpreted as follows.

*Case* 1: Exactly the same.

*Case* 2: Exactly the same, except that $R^\perp \in [\![R]\!]$ should be understood as $\{R^\perp\} \in [\![R]\!]$ and $R^\perp, \Gamma \in [\![\perp]\!]$ as $\{R^\perp\} \circ \Gamma \in [\![\perp]\!]$.

*Case* 3: Exactly the same.

*Case* 4: $(B^{*\perp} \cdot C^{*\perp})$ should be understood as $(B^{*\perp} \cdot C^{*\perp})$. $B \in B^{*\perp}$ and $C \in C^{*\perp}$ should be read as $\{B\} \in B^{*\perp}$ and $\{C\} \in C^{*\perp}$. (Hence, $B, C \in B^{*\perp} \cdot C^{*\perp}$ means $\{B\} \cdot \{C\} \in B^{*\perp} \cdot C^{*\perp}$, and $\Gamma \cdot (B^{*\perp} \cdot C^{*\perp})$ means $\Gamma \circ (B^{*\perp} \cdot C^{*\perp})$.)

*Case* 5: Exactly the same, except that in order to show $[\![B]\!] \cap [\![C]\!] \subseteq [\![B \wedge C]\!]$, we use the intersection and union type inferences, instead of the &-inference:

$$\frac{\vdash_{\mathrm{cf}} t : \Gamma, \underline{B} \qquad \vdash_{\mathrm{cf}} t : \Gamma, \underline{C}}{\vdash_{\mathrm{cf}} t : \Gamma, \underline{B \vee C}}$$

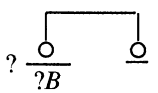
Hence, $[\![B]\!] \cap [\![C]\!] \subseteq [\![B \wedge C]\!]$. Therefore, $B^* \wedge C^* = B^* \cap C^* \subseteq [\![B \wedge C]\!]$.

*Case* 6: Exactly the same, except that in order to show $[\![B]\!] \cup [\![C]\!] \subseteq [\![B \vee C]\!]$, we use the union type inference, instead of the $\oplus$-inference:

$$\frac{\vdash_{\mathrm{cf}} t : \Gamma, \underline{A}}{\vdash_{\mathrm{cf}} t : \Gamma, \underline{A \vee B}} \qquad \frac{\vdash_{\mathrm{cf}} t : \Gamma, \underline{B}}{\vdash_{\mathrm{cf}} t : \Gamma, \underline{A \vee B}}$$

*Case* 7: Now $?\Gamma$ is an element in $(I \cap B^*) \subseteq P$. Instead of $?\Gamma \in [\![!B]\!]$, we actually have $(?\Gamma)_! \in [\![!B]\!]$ with the exactly same argument as (Case 7). Hence, $(I \cap B^*) \subseteq [\![!B]\!]$ should be read as $(I \cap B^*)_! \subseteq [\![!B]\!]$. Therefore the claim holds since $!B^*$ is now $(I \cap B^*)_!^{\perp\perp}$, rather than $(I \cap B^*)^{\perp\perp}$.

*Case* 8: $(I \cap B^{*\perp})$ should be read as $(I \cap B^{*\perp})_!$. (Hence, for example, $\Gamma \in (I \cap B^{*\perp})^{\perp}$ should be read as $\Gamma \in (I \cap B^{*\perp})_!^{\perp}$.) $?B \in B^{*\perp}$ should be read as $\{\{?B\}\} \in B^{*\perp}$, where $\{\{?B\}\}$ means



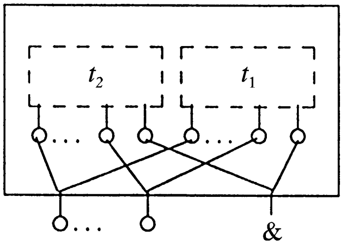

.

Then, the argument in (Case 8) implies $\Gamma, ?B \in [\![\perp]\!]$ which should be understood as $\Gamma \circ \{\{?B\}\} \in [\![\perp]\!]$. This directly implies $\Gamma \in [\![?B]\!]$ (for either $[\![\,]\!] = [\![\,]\!]_T$ or $[\![\,]\!]_N$ or $[\![\,]\!]_{SN}$). □

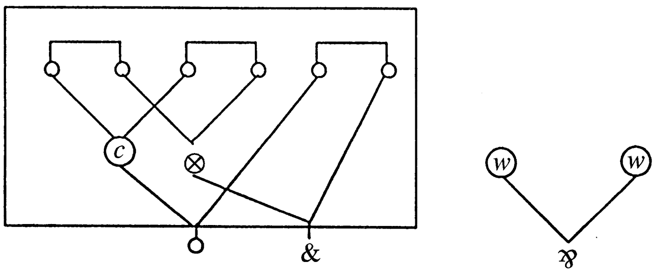
## 7. Phase semantics-like framework for "proofs" with additive connectives and higher-order quantifiers

### *7.1. Phase semantics-like framework with additive connectives*

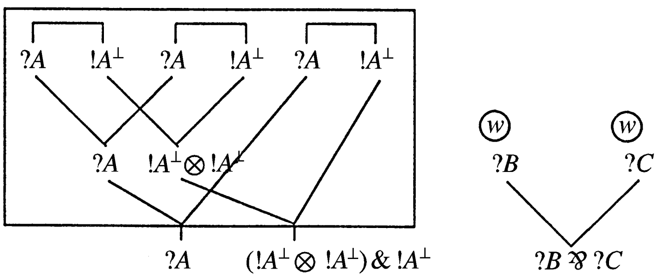
In this section we consider the additive connectives & and $\oplus$, instead of the intersection and union types. For that purpose we extend the notion of proof structure by introducing the $\oplus$-link (with one in-edge and one out-edge) and the &-box of the following form:




Typability of a proof-structure is defined in the same way as in Section 5. For example,



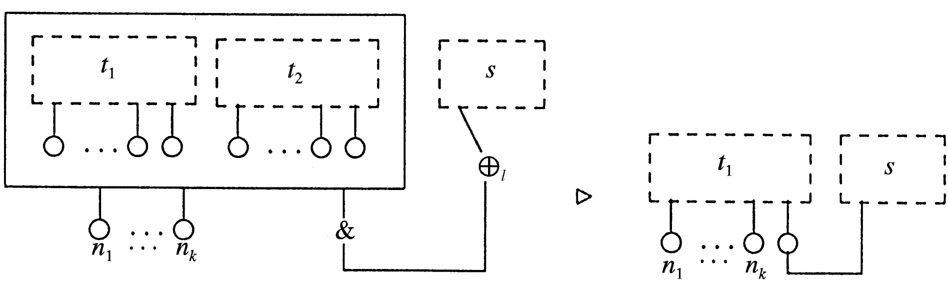

is typable as it is typed as the following well-typed proof:

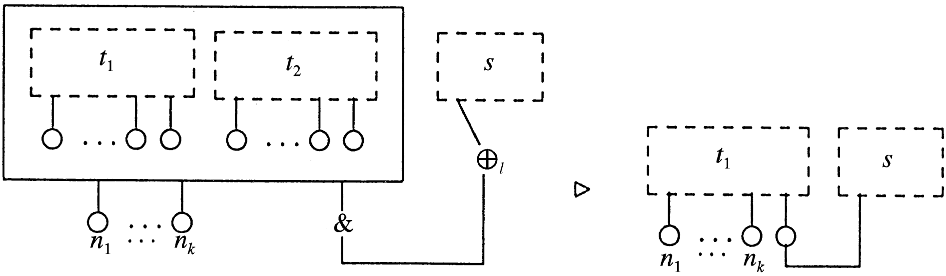



For the extended set $P$ of proof-structures and for a given set $M$ of labels, we define the set $P_M$ of proof-structures with labels from $M$, as in Section 5.

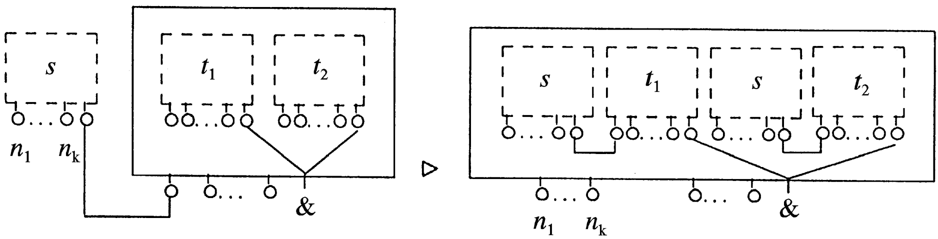
We also add the following restricted proof reduction rules for additive connectives. (Here, $n_1, \dots, n_k, m_1, \dots, m_h$ are labels from a given set $M$ of labels.)

(&-reduction rules)

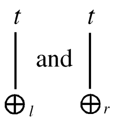





(&-box entering rule) We consider the following restricted form of the entering rule where $s$ is a separated sub-proof structure with labels.

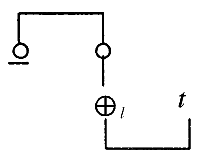


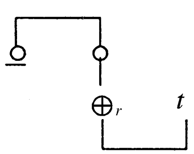

Proof-structures,




(i.e., obtained from $t$ by adding $\oplus_l$-rule and $\oplus_r$-rule at the distinguished end node of $t$, respectively) are denoted as $\oplus_l(t)$ and $\oplus_r(t)$, respectively.

For any $\alpha \subseteq P_M$ (and $\beta \subseteq P_M$),

$(\alpha)_{\&_1} = \Big\{ t \;\Big|$  $\in \alpha \Big\}$


$(\alpha)_{\&_2} = \Big\{ t \;\Big|$  $\in \alpha \Big\}$


$$(\alpha)\oplus_l = \{\oplus_l(t) \mid t \in \alpha\}$$
$$(\alpha)\oplus_r = \{\oplus_r(t) \mid t \in \alpha\}$$

$$(\alpha)_{\otimes_l} = \{\otimes_l(t) \mid t \in \alpha\}$$

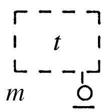
$$(\alpha)_{\otimes_r} = \{\otimes_r(t) \mid t \in \alpha\}$$

The notions of fact, regular fact, regular proof-structure and (standard) phase space $(D_{P_M}, I, \perp)$ are defined in the same way as Section 5.

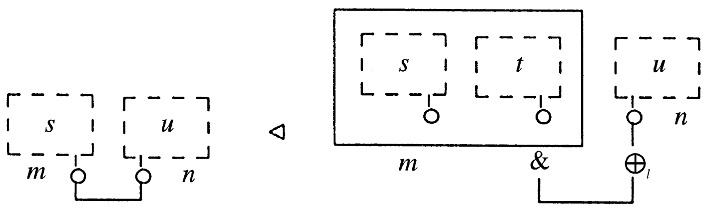
The closure conditions of $\perp$ in Section 5 are naturally extended. Here, we include an additional closure condition under axiom reductions (Condition 3) for some technical reason.

1. The closure under expansion; $s \in \perp$ if $t \in \perp$ and $t \lhd s$ by a one-step outer-most expansion by the axiom-expansion, $\otimes$-expansion, !-expansion, ?-weakening expansion, !-box entering expansion, &-expansion and &-box entering rules, where the outer-most ?-weakening expansions is restricted as before with the notion of regular proof-figure. Here, we also restrict the &-expansion rule with the notion of regular proof-figure, as follows.

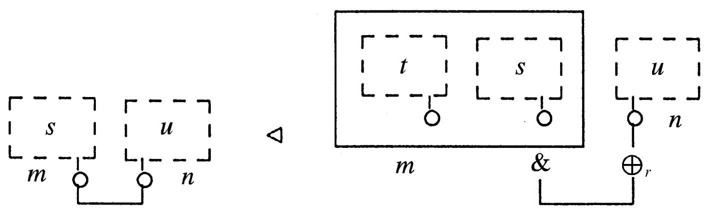
(&-expansion rules)

For any regular 

,




and




2. The closure under the weakening as before.

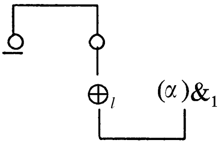
3. The closure under axiom-reduction; if $t \in \perp$ and $t \rhd s$ by an axiom-reduction, then $s \in \perp$.

Instead of operators $\wedge$ and $\vee$ in Section 5, we introduce the following additive operators.

- $\alpha \,\&\, \beta = (\alpha)_{\&_1} \cap (\beta)_{\&_2}$.
- $\alpha \oplus \beta = ((\alpha)_{\oplus_l} \cup (\beta)_{\oplus_r})^{\perp\perp}$.

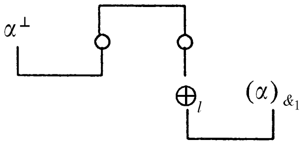
**Lemma 7.1.** *For a phase space, if $\alpha$ and $\beta$ are facts, then $\alpha \,\&\, \beta = (\alpha)_{\&_1} \cap (\beta)_{\&_2}$ is also a fact.*

**Proof.** It suffices to show that if $((\alpha)_{\&_1} \cap (\beta)_{\&_2})^{\perp} \circ u \subseteq \perp$ then $u \in (\alpha)_{\&_1} \cap (\beta)_{\&_2}$.



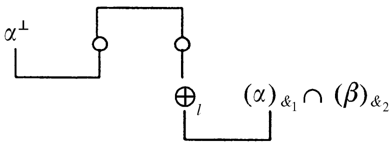

$\subseteq \alpha$ by definition.

Hence



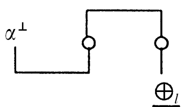

$\subseteq \perp.$

Therefore,



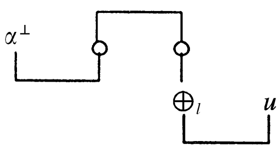

$\subseteq \perp.$

Hence,



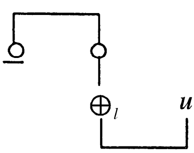

$\in ((\alpha)_{\&_1} \cap (\beta)_{\&_2})^{\perp}.$

Assume




$\subseteq \perp.$

Hence




$\in \alpha^{\perp\perp} = \alpha.$

Therefore $u \in (\alpha)_{\&_1}$. In the same way, we can show $u \in (\beta)_{\&_2}$. Therefore, $u \in (\alpha)_{\&_1} \cap (\beta)_{\&_2}$. □

**Lemma 7.2.** *For any* $\alpha, \beta \subseteq P_M$, $\alpha \oplus \beta = (\alpha^{\perp} \& \beta^{\perp})^{\perp}$.

**Proof.** Left to the reader. □

For a formula $A$, we can define the inner value $A^*$ in the same way as in Section 5.

**Lemma 7.3.** *For any fact* $\alpha$ *of a phase space,* $\alpha$ *satisfies the same closure properties as those of* $\perp$.

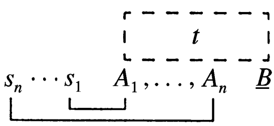
**Proof.** The same as the proof of Lemma 5.4 in Section 5. □

The Soundness Theorem is stated as before.

**Theorem 7.1** (Soundness). *For any set* $M$ *and for any phase model* $(D_{P_M}, \perp, \varphi)$, *for typable proof* $t$ *of* $x_1 : A_1, \dots, x_n : A_n \vdash t : B$,

$$t[x_1 := A_1^{*\perp}, \dots, x_n := A_n^{*\perp}] \subseteq B^*,$$

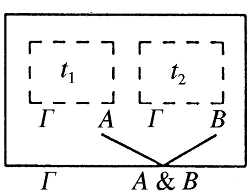
*namely, for any* $s_1 \in A_1^{*\perp}, \dots, s_n \in A_n^{*\perp}$, $t[s_1, \dots, s_n] \in B^*$, *where*



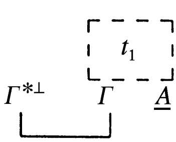
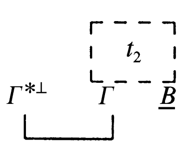
$= t[s_1, \dots s_n]$.

**Proof.** The proof is the same as in Section 5, except that we need the following additive cases, instead of the intersection-type and union-type cases.

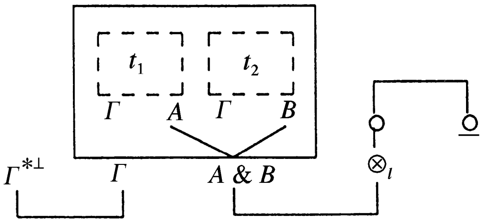
(4) When the last inference of $t$ is of the form

$t \equiv$  .

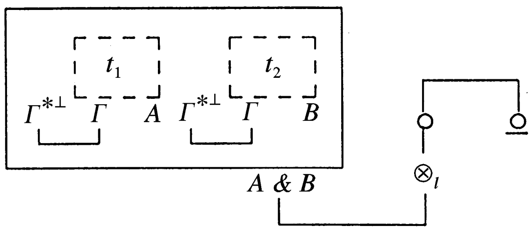
By the induction hypothesis,

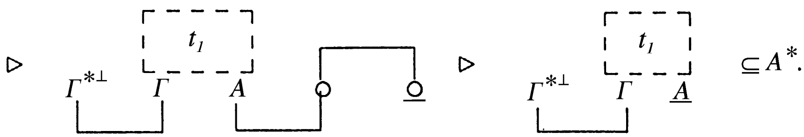
 $\subseteq A^*$ and  $\subseteq B^*$.

On the other hand,

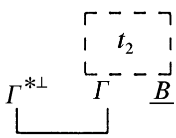
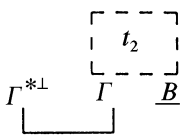


entering reductions
$\triangleright \cdots \triangleright$

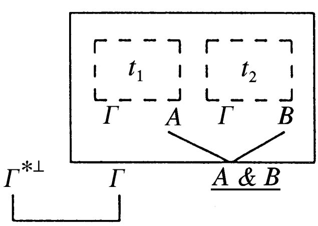




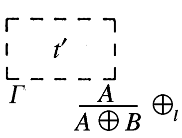
Therefore, by closure under outermost expansions for $A^*$, the left-hand side is included in $A^*$. Here, the outermost &-expansion above is allowed since by the induction hypothesis,

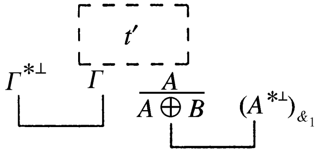
 $\subseteq B^*$ holds, hence  are regular.

The dual case also holds in the same way. Therefore,

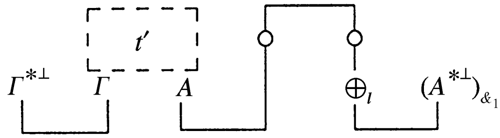
 $\subseteq (A^*)_{\&_1} \cap \subseteq (B^*)_{\&_2} = A^* \,\&\, B^*$

(5) When the last inference of $t$ is of the form

$t \equiv$  .

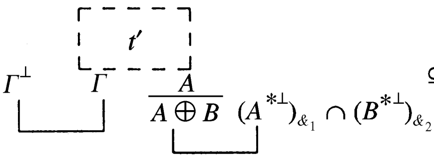


$\lhd$
(axiom-reduction)



$\subseteq \perp$

by the induction hypothesis and the definition of $(\alpha)_{\&_1}$. Hence, by the closure under the axiom reduction, the left-hand side is includend in $\perp$. Therefore,

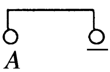


$\subseteq \perp$.

The same holds for $\oplus_r$. $\square$

Let $M$ be the set of formulas. The notion of well-typed proof is introduced in the same way as in Section 6.

As is Section 6, a well-typed proof of the form



is denoted by $\{A\}$.

The canonical models $\mathscr{M}_1$, $\mathscr{M}_2$ and $\mathscr{M}_3$ are constructed in the same way as in Section 6.

Let $[\![A]\!]_{SN}$ be $\{t \mid t$ is typable of type $A$ and is strongly normalizable.$\}$, Let $[\![A]\!]_N$ be $\{t \mid t$ is typable of type $A$ and is weakly normalizable.$\}$, and $[\![A]\!]_T$ be $\{t \mid t$ is typable of type $A\}$.

Now we consider phase spaces $\mathscr{M}_0 \equiv (D_{P_M}, I, [\![\perp]\!]_T)$, $\mathscr{M}_1 \equiv (D_{P_M}, I, [\![\perp]\!]_N)$ and $\mathscr{M}_2 \equiv (D_{P_M}, I, [\![\perp]\!]_{SN})$. When a statement holds both for $[\![A]\!]_T$, $[\![A]\!]_N$ and $[\![A]\!]_{SN}$, we state it with the reference to $[\![A]\!]$. $I_0$ is defined to be the set $\{?\Gamma \mid \Gamma$ is a finite set of formulas$\}$. $I$ is defined to be $\{t : t$ is a well-typed proof and $|t| \in I_0\}$.

**Lemma 7.4.** *If* $\alpha \circ \{A\} \subseteq [\![\perp]\!]$, *then* $\alpha \subseteq [\![A]\!]$ *for* $[\![\ ]\!] = [\![\ ]\!]_T$ *or* $[\![\ ]\!]_N$ *or* $[\![\ ]\!]_{SN}$.

**Proof.** The same as the proof of Lemma 6.1. $\square$

**Lemma 7.5.** *If $\alpha$ is a fact in $\mathcal{M}_0$ or $\mathcal{M}_1$ or $\mathcal{M}_2$ and $t \in \alpha$, then if $t'$ is composed from $t$ by adding a logical inference link (except for &-link) at some of the environment end nodes (with the induced new labels for the new end nodes), then $t' \in \alpha$.*

**Proof.** The same as the proof of Lemma 6.2. $\square$

**Lemma 7.6.** $[\![A]\!] \circ \{A\} \subseteq [\![\perp]\!]$.

**Proof.** The same as the proof of Lemma 6.3. $\square$

**Lemma 7.7.** *For any formula $A$, $[\![A]\!]$ is a fact for $[\![\ ]\!] = [\![\ ]\!]_T$ or $[\![\ ]\!]_N$ or $[\![\ ]\!]_{SN}$.*

**Proof.** The same as the proof of Lemma 6.4. $\square$

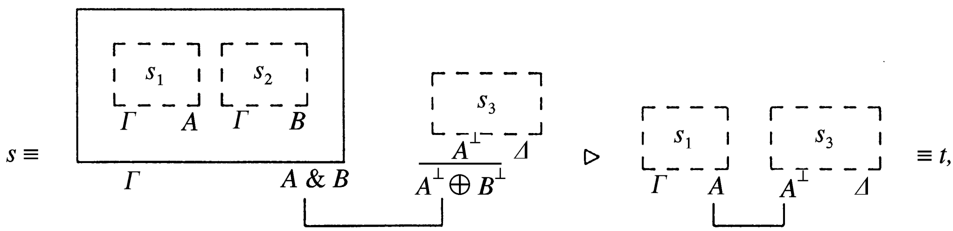
**Lemma 7.8.** *If $A^* \subseteq [\![A]\!]$ then $\{A\} \in A^{*\perp}$.*

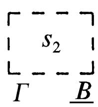
**Proof.** The same as the proof of Lemma 6.8. $\square$

The closure properties of $\perp$ in the canonical models can be shown in the same way as in Section 6 (as the proof of Lemmas 6.5 and 6.6), where we need the following modification for the proof of Lemma 6.6. Note that the typability is preserved under the &-box entering expansions due to the restricted form of the rule.

**Sketch of Modified Proof of Lemma 6.6.** First, we add the following case for the closure under &-expansion.

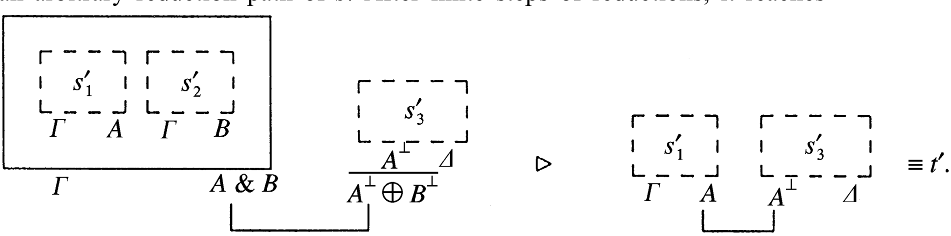
*Case* 5: *&-expansion*:




where  is regular.



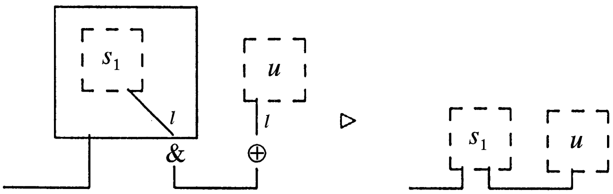
Since $t$ is strongly normalizable, so are $s_1$ and $s_3$. By the condition of regularity, $s_2$ is a sub-proof of a proof in $[\![\perp]\!]_{SN}$. Hence, $s_2$ is also strongly normalizable. Consider an arbitrary reduction path of $s$. After finite steps of reductions, it reaches



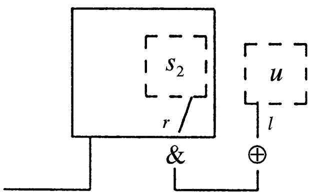
Since $t'$ is reached from $t$, so $t'$ is strongly normalizable, hence so is $s$.

In the proof of Case 4 of Lemma 6.6, we first need to reduce the current situation (with &-boxes) into the situation without &-boxes so that all argument of the original proof of Case 4 of Lemma 6.6 is applicable to the current case. For that purpose, we consider the following observation.

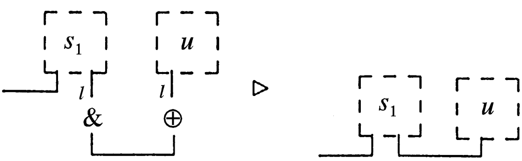
First we note that the strong normalizability of a (typable) proof $t$ is equivalent to the strong normalizability of all slices of $t$, where a slice of $t$ is obtained from $t$ by erasing one of the two sub-proofs inside each &-box (cf. [6]). Here, the &-reductions are defined as



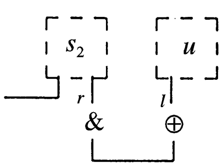

and its dual. On the other hand, the following is not a redex:




The strong normalizability of a slice $t_1$ is equivalent to the strong normalizability of $t_2$ which is obtained from the slice $t_1$ by erasing all &-boxes of $t_1$. Now the &-reductions are naturally defined as




and its dual. On the other hand, the following is not a redex:




The last equivalence relation is obtained by the fact that the &-box entering rules can be applicable at most finite many times at each reduction step of $t_2$.

Hence, from now on we assume that $q$ and $p$ are slices without &-boxes, without loss of generality. This means that we can consider !-boxes only. Then the original argument of Case 4 follows.

*Case* 6: *&-box entering expansion*: This case is essentially the same as Case 4 above, but the proof is simpler. □

Main Lemma, Strong Completeness and Cut-Elimination Theorem are proved as in Section 6, where the proof of the Main Lemma needs a slight modification due to the presence of the additive connectives.

**Lemma 7.9** (Main Lemma). *In $\mathscr{M}$ (where $\mathscr{M}$ is either $\mathscr{M}_0$, $\mathscr{M}_1$ or $\mathscr{M}_2$), for any formula $A$, $A^* \subseteq [\![A]\!]$.*

The proof of the above Main Lemma is given later. From now on if a statement (and argument) holds for $[\![\,]\!]_T$, $[\![\,]\!]_N$ and $[\![\,]\!]_{SN}$ at the same time we denote it as $[\![\,]\!]$.

We also use the following slightly modified version of Main Lemma, which can be proved by Lemma 7.7, in the same way as the proof of Lemma 6.8.

**Lemma 7.10** (Main Lemma, modified). *In $\mathscr{M}$ (where $\mathscr{M}$ is $\mathscr{M}_0$, $\mathscr{M}_1$ or $\mathscr{M}_2$), for any formula $A$, $\{A^\perp\} \in A^* \subseteq [\![A]\!]$.*

Then with the Main Lemma and the fact that $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}_2$ are actually a phase model, we have the strong completeness and the normalization theorem, in the same way as in Section 6.

**Theorem 7.2** (Strong Completeness).

1. (*Completeness for proofs*). *For any proof-structure $t$, if $t \in A^*$ for any* (*standard*) *phase model, then $t$ is a typable proof of type $A$.*
2. (*Strongly* (*weakly, resp.*) *normalization*). *For any proof-structure $t$, if $t \in A^*$ for any* (*standard*) *phase model, then $t$ is a strongly normalizable* (*weakly normalizable, resp.*) *proof of type $A$.*

**Proof.** Here we denote $\mathscr{M}$ for $\mathscr{M}_0$ or $\mathscr{M}_1$ or $\mathscr{M}_2$, $[\![A]\!]$ for $[\![A]\!]_T$ or $[\![A]\!]_N$ or $[\![A]\!]_{SN}$, respectively.

Since $t \in A^*$ and by the Main Lemma, $A^* \subseteq [\![A]\!]$. Hence $t \in [\![A]\!]$, which means the claims 1 and 2 above (i.e., when $[\![A]\!] = [\![A]\!]_T$, claim 1 holds; when $[\![A]\!] = [\![A]\!]_{SN}$ ($[\![A]\!]_N$, resp.), claim 2 holds). □

**Theorem 7.3** (Strong normalization (weak normalization, resp.) Theorem). *Every typable proof is strongly normalizable* (*weakly normalizable, resp.*).

**Proof.** We denote $[\![A]\!]$ for $[\![A]\!]_{SN}$ (for $[\![A]\!]_N$, resp.).

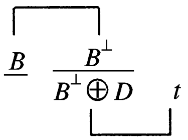
By the Soundness Theorem and the Main Lemma ($\{A\} \in A^*$ for any $A$), for any proof $x_1 : A_1, \ldots, x_n : A_n \vdash t : B$, $t[\{A_1\}, \ldots, \{A_n\}] \in B^*$. Then by the Main Lemma, $B^* \subseteq [\![B]\!]$.

Hence, $t[\{A_1\},\ldots,\{A_n\}]\in[\![B]\!]$. Hence, by the closure under the well-typed axiom reductions of $[\![B]\!]$ (which is easily verified), $t\in[\![B]\!]$. $\square$

We now return to the Proof of Main Lemma.

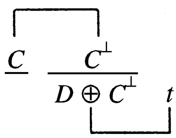
**Proof of Main Lemma.** The proof is the same as that of Lemma 6.6 (Main Lemma) of Section 6, by reinterpreting the proof of Lemma 3.1 (Main Lemma) of Section 3, except for the following Case 5′ and Case 6′, instead of Case 5 and Case 6 of the proof of Lemma 6.6. We recall that $[\![A]\!]$ means $[\![A]\!]_T$ or $[\![A]\!]_N$ or $[\![A]\!]_{SN}$ depending on the context.

*Case* 5′: *When* $A\equiv B\,\&\,C$. By the induction hypothesis, $B^*\subseteq[\![B]\!]$ and $C^*\subseteq[\![C]\!]$. Hence, for any $t\in(B^*)_{\&_1}$,



$\in[\![B]\!]$ for some $D$.

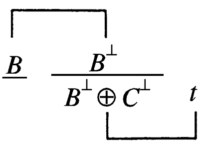
Therefore, $t$ is well-typed of type $B\oplus D$ for some $D$. In the same way, for any $t\in(C^*)_{\&_2}$,



$\in[\![C]\!]$ for some $D$.

Therefore, $t$ is well-typed of type $D\oplus C^\perp$ for some $D$. Hence, any $t\in(B^*)_{\&_1}\cap(C^*)_{\&_2}$ is well-typed of type $B\,\&\,C$. Therefore, $B^*\,\&\,C^*\subseteq[\![B\,\&\,C]\!]_T$.

On the other hand, since



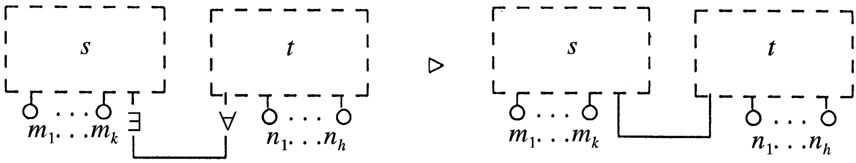
is strongly normalizable (weakly normalizable, respectively) for any $t\in(B^*)_{\&_1}\cap(C^*)_{\&_2}$ from the induction hypothesis, hence by Lemma 6.1, $t$ is also strongly normalizable (weakly normalizable respectively). Therefore, $t\in[\![B\,\&\,C]\!]_{SN}$ ($t\in[\![B\,\&\,C]\!]_N$, respectively).

*Case* 6′: *When* $A\equiv B\oplus C$. Since $(B\oplus C)^*\circ(A^\perp\,\&\,B^\perp)^*\subseteq[\![\perp]\!]$, $(B\oplus C)^*\circ\{A^\perp\,\&\,B^\perp\}\subseteq[\![\perp]\!]$. Then by Lemma 6.1, $(B\oplus C)^*\subseteq[\![B\oplus C]\!]$. $\square$

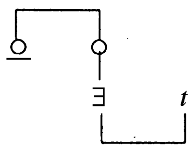
## 7.2. *Phase semantics-like framework with higher-order quantifiers*

In this subsection we extend the first-order phase semantics for proofs to the second-order one.

For the second-order proof-structures $P$, we add the $\exists$-link and the $\forall$-link. We add the following reduction rules of proof-structures with labels for the quantifier case.
$\forall$-reduction-rule:




For any $\alpha \subseteq P_M$, we define $(\alpha)_\forall$ as

$(\alpha)_\forall = \Bigg\{ t \;\Bigg|$  $\in \alpha \Bigg\}$.



$\exists(s)$ is the proof-structure obtained from $s$ by adding the $\exists$-inference link on the distinguished end node of $s$. $(\alpha)_\exists = \{\exists(s) \mid s \in \alpha\}$.

For a set $D \subseteq D_{P_M}$ (i.e., a subset $D$ of the facts $D_{P_M}$), we define new operators as follows.

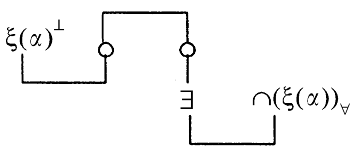
For any $\xi : D \to D$,

$$\forall X\xi(X) =_{\mathrm{def}} \bigcap_{\alpha \in D} (\xi(\alpha))_\forall,$$

$$\exists X\xi(X) =_{\mathrm{def}} (\forall X\xi^\perp(X))^\perp = \left(\bigcap_{\alpha \in D} (\xi(\alpha)^\perp)_\forall\right)^\perp = \left(\bigcup_{\alpha \in D} (\xi(\alpha))_\exists\right)^{\perp\perp}$$

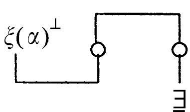
**Lemma 7.11.** *For any $\xi : D \to D$, $\forall X\xi(X) = \bigcap_{\alpha \in D} \xi(\alpha)$ is a fact, namely $\in D_{P_M}$.*

**Proof.** It suffices to show that for any $u$, if $(\forall X\xi(X))^\perp \circ u \subseteq \perp$ then $u \in \forall X\xi(X)$. It is obvious that



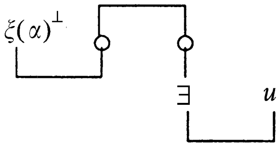

$\subseteq \perp$.

Hence,



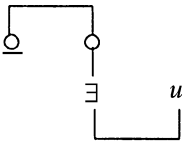

$\subseteq (\forall X\xi(X))^\perp$.

Assume




$\subseteq \perp.$

Hence,




$\in \xi(\alpha)^{\perp\perp} = \xi(\alpha).$

Therefore, $u \in (\xi(\alpha))_\forall$. This holds for any $\alpha \in D$. Hence, $u \in \bigcap_{\alpha \in D} \xi(\alpha) = \forall X \xi(X)$. □

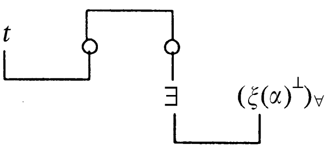
Note that obviously $\exists X \xi(X)$ is a fact because of the form of its definition above. The last equality in the definition of $\exists X \xi(X)$ is easily seen as follows: To show

$$\left( \bigcup_{\alpha \in D} (\xi(\alpha))_\exists \right)^{\perp\perp} \subseteq \left( \bigcap_{\alpha \in D} (\xi(\alpha)^\perp)_\forall \right)^\perp,$$

it suffices to show

$$\bigcup_{\alpha \in D} (\xi(\alpha))_\exists \subseteq \left( \bigcap_{\alpha \in D} (\xi(\alpha)^\perp)_\forall \right)^\perp.$$

Take an arbitrary $\exists(t) \in (\xi(\alpha))_\exists$ (namely, $t \in \xi(\alpha)$) for an arbitrary $\alpha \in D$. By the definition of $(\xi(\alpha)^\perp)_\forall$,




$\subseteq \perp.$

Hence, by the closure under the axiom reduction of $\perp$, $\exists(t) \circ (\xi(\alpha)^\perp)_\forall \subseteq \perp$. Hence,

$$\exists(t) \in \left( \bigcap_{\alpha \in D} (\xi(\alpha)^\perp)_\forall \right)^\perp.$$

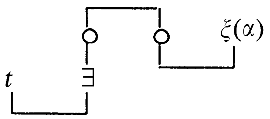
On the other hand, to show the reverse direction it suffices to show

$$\left( \bigcup_{\alpha \in D} (\xi(\alpha))_\exists \right)^\perp \subseteq \bigcap_{\alpha \in D} (\xi(\alpha)^\perp)_\forall.$$

Take an arbitrary

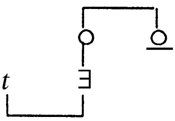
$$t \in \left( \bigcup_{\alpha \in D} (\xi(\alpha))_{\exists} \right)^{\perp}.$$

Hence, for any $\alpha \in D$, $t \circ (\xi(\alpha))_{\exists} \subseteq \perp$. By the closure under an outermost axiom expansion of $\perp$,




$\subseteq \perp$.

Hence,




$\in \xi(\alpha)^{\perp}$.

Therefore, $t \in (\xi(\alpha)^{\perp})_{\forall}$. This holds for any $\alpha \in D$. Hence, $t \in \bigcap_{\alpha \in D} (\xi(\alpha)^{\perp})_{\forall}$.

As in Section 2, we call $(D, I, \perp)$ a second-order phase space if the following conditions are satisfied.

1. $D \subseteq D_{P_M}$ satisfies the condition $(*)$ of the second-order phase spaces in Section 2, namely,

   $(*)$ For any (second-order) formula A (possibly with free second-order variables $X \equiv X_1, \ldots, X_n$), if $\alpha_i \in D$ then $A^*[\alpha/X] \in D$, where $\alpha \equiv \alpha_1, \ldots, \alpha_n$.

2. $\perp$ satisfies the three closure conditions in the previous subsection, where the closure condition under expansions should be extended with the $\forall$-expansion rule (i.e., the reverse of the $\forall$-reduction rule above).
3. $I = J \upharpoonright W$ satisfies the condition in Section 6.

The Soundness Theorem can be proved in the same way as in the former sections.

**Theorem 7.4** (Soundness Theorem; second-order case). *For any phase model and for any typable proof t such that* $x_1 : A_1[X], \ldots, x_n : A_n[X] \vdash t : B[X]$, *for any* $\alpha \in D$,

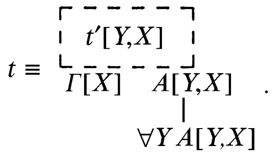
$$t[x_1 := A_1^{\perp *}[\alpha], \ldots, x_n := A_n^{\perp *}[\alpha]] \subseteq B^*[\alpha],$$

*or equivalently*,

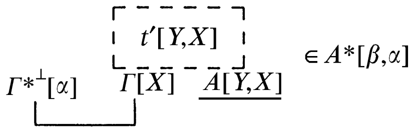
$$t[x_1 := A_1^{\perp *}[\alpha], \ldots, x_n := A_n^{\perp *}[\alpha]] \circ B^{\perp *}[\alpha] \subseteq \perp.$$

**Proof.** Since the other cases are essentially the same as the proof of Soundness Theorem in the previous two sections, we consider only the following cases. Note that the soundness proofs in Section 6 and in Section 7.1 are performed only for the standard phase models, where a phase model $(D, I, \perp, \varphi)$ is called a standard phase model if $D \equiv D_{P_M}$. However, with the closure condition $(*)$ for $D$ $(\subseteq D_{P_M})$, the former argument of the soundness proofs for the first-order case works without any change with a general (i.e., non-standard) phase model.

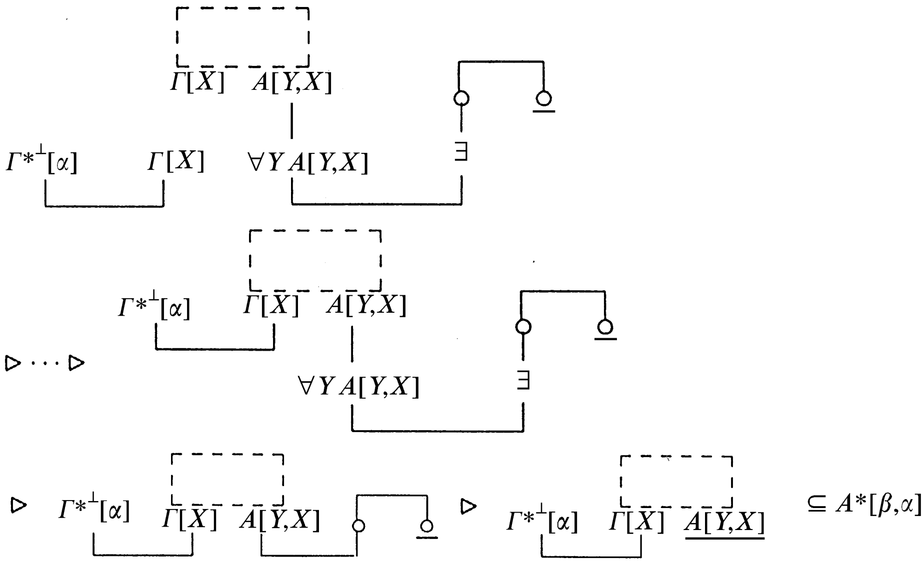
(1) When the last inference of $t$ for the typability proof is of the form



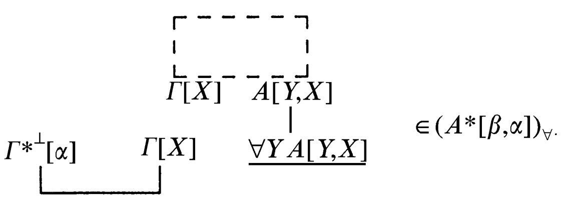
By the induction hypothesis, for any fixed $\alpha \in C$,



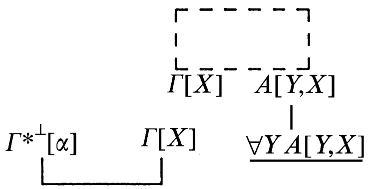
for any $\beta \in D$.



by the induction hypothesis. Hence the left-hand side is also in $A^*[\beta, \alpha]$. Hence,

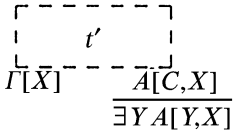


Here, this holds for any choice of $\beta \in D$,

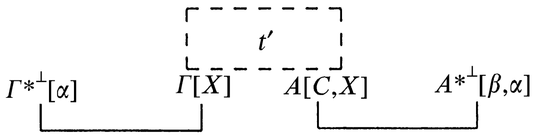


$$\in \bigcap_{\beta \in D} (A^*[\beta,\alpha])_\forall = \forall X A^*[X,\alpha].$$

(2) When the last inference of $t$ is of the form

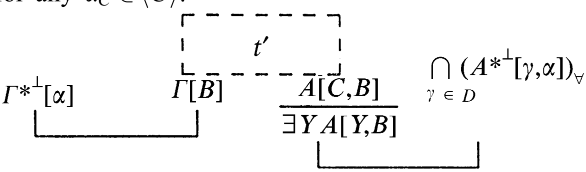
$t \equiv$  .

We take an arbitrary $\alpha \in D$. Then by the induction hypothesis,



$\subseteq \bot$,

for any $\alpha_C \in \langle C \rangle$.



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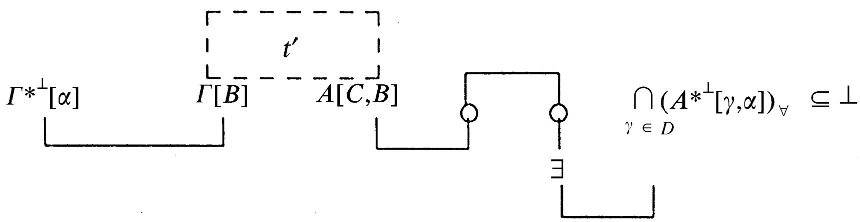


by the induction hypothesis and the definition of $(A^{*\bot}[\alpha_C, \alpha_B])_\forall$. Hence, the left-hand side is also included in $\bot$. $\square$

The canonical models $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}_2$ for the strong completeness for provability in Section 4 can be naturally extended to the scond order case, where the notion of candidates $\alpha \in \langle A \rangle$ is defined as follows:

$\alpha_B \in \langle B \rangle$ iff $\{B^\bot\} \in \alpha_B \subseteq [\![B]\!]$, where $\alpha_B$ is a fact.

This notion of candidate corresponds to the notion of candidate of reducibility in the sense of Girard [3]. Then, as in Section 4, $D$ ($\subseteq D_{P_M}$) is defined as $\bigcup_{A\in Form}\langle A\rangle$, where *Form* is the set of second-order formulas.

Using this definition of the candidates the Main Lemma can be relativized with the candidates, in the same way as in the section for "provability".

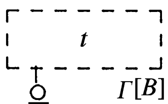
The Main Lemma states

**Lemma 7.12** (Main Lemma; second-order case). *In $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}_2$, for any (second-order) formulas $A, B \equiv B_1,\dots,B_n$, and for any candidates $\alpha_{B_i} \in \langle B_i\rangle$, $\{A^\perp[X:=B]\} \in A^*[X:=\alpha_B] \subseteq [\![A[X:=B]]\!]$.*

**Proof.** Without loss of generality we prove this for $\mathscr{M}_2$ (the strong normalizability model). The proof for $\mathscr{M}_0$ and for $\mathscr{M}_1$ is simpler, as explained at the proof of Main Lemma for the first-order case in the previous subsection. Since other cases are essentially the same as the first-order cases, we consider only the following cases.

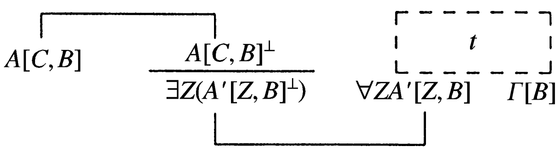
(1) When $A[X] \equiv X_j$. By the definition of $\langle B_j\rangle$, $\alpha_{B_j} \subseteq [\![B_j]\!]$.

(2) We prove $\forall Y\ A^*[Y,\alpha_B] \subseteq [\![\forall Y A[Y,B]]\!]$ for any $\alpha_B \in \langle B\rangle$, $B \in Form$. By the induction hypothesis, for any $\alpha_C \in \langle C\rangle$ and $\alpha_B \in \langle B\rangle$, $A^*[\alpha_C,\alpha_B] \subseteq [\![A[C,B]]\!]$. For



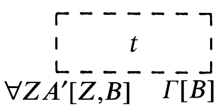
$$\in \forall Y A^*[Y,\alpha_B] = \bigcap_{\substack{\alpha_D\in\langle D\rangle\\ D\in Form}} (A^*[\alpha_D,\alpha_B])_\forall,\ \text{for any } C \text{ and } \alpha_C \in \langle C\rangle,$$

there exists $A'$ such that

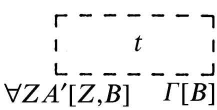


$$\in A^*[\alpha_C,\alpha_B]$$

is typable. Note that $C$ does not appear in



for some $C$ since

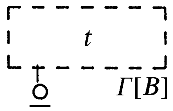


does not depend on the choice of $C$. Hence, at the $\exists$-rule

$$\frac{A[C,B]^\perp}{\exists Z(A'[Z,B]^\perp)}$$

$Z$ should be substituted on a subformula $E$ (of $A$) which contains all the occurrences of the indicated $C$'s. Hence, $A'[Z,B]$ is more general than $A[Y,B]$ in the sense that there is a substitution $F$ which satisfies $A'[Z:=F[Y],B]$ =

$A[Y,B]$. Therefore, by substituting $F[Y]$ on $Z$ in $t:\underline{\forall ZA'[Z,B]},\Gamma[B]$, we have $t:\underline{\forall Y\ A'[F[Y],B]},\Gamma[B] \equiv \underline{\forall Y\ A[Y,B]},\Gamma[B]$. Hence,




is typable of type $\forall YA[Y,B]$, namely $t:\underline{\forall YA[Y,B]},\Gamma[B]$. The strong normalizability is obvious since $t$ is a subproof of an element of $A^*[\alpha_C,\alpha_B] \subseteq [\![A[C,B]]\!]$.

(3) We prove $\exists YA^*[Y,\alpha_B] \subseteq [\![\exists YA[Y,B]]\!]$ for any $\alpha_B \in \langle B\rangle, B \in Form$. By the induction hypothesis, $A^*[\alpha_C,\alpha_B] \subseteq [\![A[C,B]]\!]$ for any $\alpha_C \in \langle C\rangle,\ C \in Form$. Hence,

$$(A^*[\alpha_C,\alpha_B])_\exists \subseteq [\![A[C,B]]\!]_\exists \subseteq [\![\exists Y\ A[Y,B]]\!],$$

for any $\alpha_C \in \langle C\rangle,\ C \in Form$. Hence,

$$\bigcup_{\substack{\alpha_C \in \langle C\rangle \\ C \in Form}} (A^*[\alpha_C,\alpha_B])_\exists \subseteq [\![\exists YA[Y,B]]\!].$$

Since $[\![\exists YA[Y,B]]\!]$ is a fact,

$$\exists Y\ A^*[Y,\alpha_B] = \left( \bigcup_{\substack{\alpha_C \in \langle C\rangle \\ C \in Form}} (A^*[\alpha_C,\alpha_B])_\exists \right)^{\perp\perp} \subseteq [\![\exists Y\ A[Y,B]]\!].$$

Now, in the same way as Lemma 6.8, for any $\alpha \in \langle C\rangle$ and $C \in Form$, if $A^*[\alpha_C] \subseteq [\![A[C]]\!]$ then $\{A[C]\} \in A^{*\perp}[\alpha_C]$. Therefore, the claim of the Main Lemma holds. $\square$

It is also proved that each canonical models, $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}_2$ are actually second-order phase models, in the same way as in the first-order case with the help of the above Main Lemma. Then these three second-order canonical models (with the help of the Soundness) provide the proof of following statement, respectively.

**Theorem 7.5** (Strong Completeness for the second-order case).
1. (*Completeness for proofs*). *For any* (*second-order*) *proof-structure* $t$, *if* $t \in A^*$ *for any second-order phase model*, *then* $t$ *is a* (*typable*) *proof of type* $A$.
2. (*Strong*(*weak*, *resp.*)*-normalization*). *For any* (*second-order*) *proof-structure* $t$, *if* $t \in A^*$ *for any phase model*, *then* $t$ *is a strongly weakly*, *resp.*) *normalizable proof of type* $A$.

**Theorem 7.6** (Strong Normalization for the second-order case). *Any typable proof is strongly normalizable* (*weakly normalizable*, *respectively*).

**Proof.** The claim follows from the Soundness Theorem and the case 2 of the above Strong Completeness Theorem using $\mathscr{M}_2$ (using $\mathscr{M}_1$, respectively). $\square$

## 8. Some relationships between the phase semantics for "provability" and the phase semantics for "proofs"

In this section, we give some relationships between the two phase semantics, the phase semantics for "provability" and that for "proofs", developed in the former Sections, so that the Soundness Theorem and the Main Lemma (for the Strong Completeness Theorem) for "provability" (in Sections 2–4) can be viewed as direct corollaries of the corresponding Theorem and Lemma for "proofs" (in Section 7).

Recall that for any proof-structure $t$ with labels (from $M$), projection $|t|$ is the multiset of labels of $t$ (where elements of $I_0 \subseteq M$ are counted as set-elements, instead of multiset-elements (cf. Section 3). For $\alpha \subseteq P_M$, $|\alpha| = \{|t| : t \in \alpha\}$. For any $X \subseteq Power(P_M)$, $|X| = \{|\alpha| : \alpha \in X\}$. For a commutative monoid $M$, a projection $|\mathscr{M}|$, of a phase model $\mathscr{M} = (D, I, \perp, \varphi)$ for "proofs" is $|\mathscr{M}| = (|D|, |I|, |\perp|, |\varphi|)$, where $|\varphi| : A\text{-}Form \to |D|$ is defined as $|\varphi|(R) = |\varphi(R)|$. Recall that *A-Form* stands for the atomic formulas and that $D_{\mathscr{M}}$ stands for the set of facts in model $\mathscr{M}$.

**Proposition 8.1.** (I) *For any standard phase model* $\mathscr{M} = (D_{\mathscr{M}}, I, \perp, \varphi)$ *for "provability", there is a standard phase model* $\mathscr{M}' = (D_{P_M}, I', \perp', \varphi')$ *for "proofs" such that*
(1) $\mathscr{M}$ *is a projection of* $\mathscr{M}'$, *i.e.*, $\mathscr{M} = |\mathscr{M}'|$.
(2) *The projection preserves the linear logical operators; i.e., for any facts* $\alpha, \beta$ *in* $\mathscr{M}'$,

$$|\alpha^{\perp}| = |\alpha|^{|\perp|}$$
$$|\alpha \& \beta| = |\alpha| \& |\beta|$$
$$|\alpha \oplus \beta| = |\alpha| \oplus |\beta|$$
$$|\alpha \otimes \beta| = |\alpha| \otimes |\beta|$$
$$|\alpha \mathbin{⅋} \beta| = |\alpha| \mathbin{⅋} |\beta|$$
$$|!\alpha| = !|\alpha|$$
$$|?\alpha| = ?|\alpha|.$$

*In particular, for any formula* $A$, $A^*_{\mathscr{M}} = |A^*_{\mathscr{M}'}|$.

(II) *For any second-order phase model* $\mathscr{M} = (D^{\mathscr{M}}, I^{\mathscr{M}}, \perp^{\mathscr{M}}, \varphi^{\mathscr{M}})$ *for "provability", where* $D^{\mathscr{M}} \subseteq D_M$, *there is a phase model* $\mathscr{M}' = (D^{\mathscr{M}'}, I^{\mathscr{M}'}, \perp^{\mathscr{M}'}, \varphi^{\mathscr{M}'})$ *for "proofs" such that the above* (1) *and* (2) *holds, where* (2) *is extended to*

$$|\forall X \xi(X)| = \forall X |\xi|(X),$$
$$|\exists X \xi(X)| = \exists X |\xi|(X).$$

*for any* $\xi : D' \to D'$. *Here,* $|\xi|$ *is defined as* $|\xi| : D \to D$ *and* $|\xi|(|\alpha|) = |\xi(\alpha)|$.

**Proof.** We define the inverse ($|\ |^{-1}$) of projection as follows: For any $\alpha \subseteq M$, $|\alpha|^{-1} = \{t \in P_M \mid |t| \in \alpha\}$. Then, we define as follows: $\perp^{\mathscr{M}'} =_{\text{def}} |\perp^{\mathscr{M}}|^{-1}$. $W^{M'} =_{\text{def}} |I^{\mathscr{M}}|^{-1}$ and

$I^{M'} =_{\mathrm{def}} P_{W^{M'}}$. $\varphi^{\mathcal{M}'} =_{\mathrm{def}} |\varphi^{\mathcal{M}}|^{-1}$, where $|\varphi^{\mathcal{M}}|^{-1}(R) = |\varphi^{\mathcal{M}}(R)|^{-1} = |R^*|^{-1}$. Define $\mathcal{M}' = (D_{P_M}, \perp^{\mathcal{M}'}, \varphi^{\mathcal{M}'}) = (D_{P_M}, |\perp^{\mathcal{M}}|^{-1}, |\varphi^{\mathcal{M}}|^{-1})$. Then, obviously $t \cdot s \in \perp^{\mathcal{M}'} \Leftrightarrow |t|, |s| \in \perp^{\mathcal{M}}$. Hence, for any $\alpha \subseteq P_M$,

$$|\alpha^{\perp^{\mathcal{M}'}}| = |\alpha|^{|\perp^{\mathcal{M}'}|}(=|\alpha|^{\perp^{\mathcal{M}}}) \quad \text{and} \quad D_M = |D_{P_M}|.$$

From the above, $\alpha$ is a fact in $\mathcal{M}'$ iff $|\alpha|$ is a fact in $\mathcal{M}$. It is obvious that

$$|\alpha \mathbin{\&}^{\mathcal{M}'} \beta| = |\alpha| \mathbin{\&}^{\mathcal{M}} |\beta|, \qquad |\alpha \oplus^{\mathcal{M}'} \beta| = |\alpha| \oplus^{\mathcal{M}} |\beta|,$$
$$|\alpha \otimes^{\mathcal{M}'} \beta| = |\alpha| \otimes^{\mathcal{M}} |\beta|, \qquad |\alpha \mathbin{⅋}^{\mathcal{M}'} \beta| = |\alpha| \mathbin{⅋}^{\mathcal{M}} |\beta|.$$

Because of the weak idempotent property of $J^{\mathcal{M}}$ and the property of 1, we have

$$\forall n \in I^{\mathcal{M}} \forall l \in M(lnn \in \perp \Rightarrow ln \in \perp),$$
$$\forall n \in I^{\mathcal{M}} \forall l \in M(l \in \perp \Rightarrow ln \in \perp).$$

Hence, $I^{\mathcal{M}'} = |I^{\mathcal{M}}|^{-1}$ satisfies the two closure conditions (for $J^{\mathcal{M}'}$ and $W^{\mathcal{M}'}$).

Since $I^{\mathcal{M}} = |I^{\mathcal{M}'}|$ by the definition, $|!\alpha| = !|\alpha|$ and $|?\alpha| = ?|\alpha|$ also hold.

The above relation can easily be generalized to the higher-order case by observing that the closure condition for $D^{\mathcal{M}}$ $(\subseteq D_M)$ can be lift up to that for $D^{\mathcal{M}'}$ $(\subseteq D_{P_M})$ by $|\,|^{-1}$.

For any (second order) domain $D^{\mathcal{M}}(\subseteq D_{\mathcal{M}})$, $D^{\mathcal{M}'} = |D^{\mathcal{M}}|^{-1}$,

$$|\forall X A^*| = \bigcap_{|\alpha| \in D^{\mathcal{M}}} |A^*[|\alpha|]| = \bigcap_{\alpha \in D^{\mathcal{M}'}} |A^*|(\alpha) = \forall X |A^*[X]|.$$

In the same way,

$$|\exists X A^*| = \left( \bigcap_{\alpha \in D^{\mathcal{M}'}} |A^*[|\alpha|]| \right)^{|\perp^{\mathcal{M}'}||\perp^{\mathcal{M}'}|} = \exists X |A^*[X]|.$$

Hence, for any inner value $A^*$ of $\mathcal{M}'$, $|A^*|$ is the inner value of $\mathcal{M}$.

In particular, the following closure condition holds: For any $\alpha_i \in D^{\mathcal{M}'}$, and for any (second-order) formula $A[X]$, where $X \equiv X_1, \ldots, X_n$ is the list of the second-order variable occurring in $A$, $A^*[\alpha] \in D^{\mathcal{M}'}$. $\square$

The reverse relation holds for the canonical models.

**Theorem 8.1.** (I) *Let $\mathcal{M}$ be the canonical model for the provability* (*defined in Section* 3, *and in Section* 4 *for the higher-order case*), *and $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$ be the canonical models for proofs* (*defined in Section* 6.1, *and in Section* 6.2 *for the higher-order cases*). *Then,*

(1) $|\mathcal{M}_0| = |\mathcal{M}_1| = |\mathcal{M}_2| = \mathcal{M}$, *namely, the canonical model for "provability" is the projection of the canonical models for "proofs".*

(2) *The projection preserves the linear logical operators*; *i.e.*, *for any facts* $\alpha$ *and* $\beta$ *in* $\mathscr{M}_0$ (*or in* $\mathscr{M}_1$),

$$|\alpha^{\perp}| = |\alpha|^{|\perp|}$$

$$|\alpha \& \beta| = |\alpha| \& |\beta|$$

$$|\alpha \oplus \beta| = |\alpha| \oplus |\beta|$$

$$|\alpha \otimes \beta| = |\alpha| \otimes |\beta|$$

$$|\alpha \mathbin{⅋} \beta| = |\alpha| \mathbin{⅋} |\beta|$$

$$|!\alpha| = !|\alpha|$$

$$|?\alpha| = ?|\alpha|$$

*In particular*, *for any formula* $A$, $A^*_{\mathscr{M}} = |A^*_{\mathscr{M}_0}| = |A^*_{\mathscr{M}_1}| = |A^*_{\mathscr{M}_2}|$.

(II) *The above statement also holds for the second-order canonical models. In particular*, (2) *is extended by*; *for any* $\xi : D \to D$,

$$|\forall X \xi(X)| = \forall X |\xi|(X),$$

$$|\exists X \xi(X)| = \exists X |\xi|(X),$$

*where* $|\xi| : |D| \to |D|$ *is defined by* $|\xi|(|\alpha|) = |\xi(\alpha)|$ *for* $\alpha \in D$. *In particular*, *for any second-order closed formula* $A$, $A^*_{\mathscr{M}} = |A^*_{\mathscr{M}_0}| = |A^*_{\mathscr{M}_1}| = |A^*_{\mathscr{M}_2}|$.

**Proof.** (I) By the strong normalization theorem (which followed from the strong completeness proof using $\mathscr{M}_2$), $\mathscr{M}_2$ turned out to be the same structure as $\mathscr{M}_0$ and $\mathscr{M}_1$ (since $[\![A]\!]_T = [\![A]\!]_N = [\![A]\!]_{SN}$ for any $A$, in particular $[\![\perp]\!]_T = [\![\perp]\!]_N = [\![\perp]\!]_{SN}$ and $[\![R]\!]_T = [\![R]\!]_N = [\![R]\!]_{SN}$ for any atomic $R$). Hence, $|\mathscr{M}_0| = |\mathscr{M}_1| = |\mathscr{M}_2|$, $D_{\mathscr{M}_0} = D_{\mathscr{M}_1} = D_{\mathscr{M}_2}$ and $A^*_{\mathscr{M}_0} = A^*_{\mathscr{M}_1} = A^*_{\mathscr{M}_2}$ for any closed formula $A$.

Now we show $|\mathscr{M}_0| = \mathscr{M}$ and (2) above. First note that by definition $\perp_{\mathscr{M}} = |\perp_{\mathscr{M}_0}|$ and $\varphi_{\mathscr{M}} = |\varphi_{\mathscr{M}_0}|$.

**Lemma 8.1.** *If* $\alpha$ *is a regular fact in the canonical model* $\mathscr{M}_0$, *there is a typable element* $t \in \alpha$ *of some type* $A$ *such that for any* $s \in \alpha$ *if* $s$ *is typable of type* $B$ *then* $A$ *is more general than* (*or equal to*) $B$ (*i.e.*, $B \equiv A\sigma$ *for some substitution* $\sigma$).

**Proof.** Note that if $\alpha$ is a regular fact, then $\alpha \circ \alpha^{\perp} \subseteq [\![\perp]\!]_T$, hence every element of $\alpha$ is typable. Take an arbitrary element $u$ of $\alpha^{\perp}$. Take a most general $C$ such that $u$ is typable of type $C$. Since $\alpha \circ u$ is typable, one can take $A$ as $C^{\perp}$. $\square$

**Lemma 8.2.** *For any regular fact* $\alpha$ *in* $\mathscr{M}_0$, $|\alpha|^{|\perp_{\mathscr{M}_0}|} = |\alpha^{\perp_{\mathscr{M}_0}}|$.

**Proof.** First we show $|\alpha|^{|\perp|} \subseteq |\alpha^{\perp}|$. Assume $\Gamma \in |\alpha|^{|\perp|}$. Hence, $\forall s \in \alpha \exists t \in \perp = [\![\perp]\!](|t| = |s|\Gamma)$. On the other hand, by taking $A$ for a fact $\alpha$ in the above Lemma, $s : |s|, \underline{A}$ (i.e., $s$ is typable of type $A$ with environment types $|s|$) for any $s \in \alpha$. Hence, $\{A^{\perp}\} \in \alpha^{\perp\perp} = \alpha$. Take $s$ as $\{A^{\perp}\}$. Then $\exists t \in [\![\perp]\!]_T |t| = A^{\perp}\Gamma$. By taking the end node $A^{\perp}$ of this $t$ as the distinguished node (by erasing the label $A^{\perp}$) we get $t_1$. Note that $t_1 \in \alpha^{\perp}$ since $t_1$ is typable of type $A^{\perp}$, hence $\alpha \circ t_1 \subseteq [\![\perp]\!]_T$. Hence $\Gamma = |t_1| \in |\alpha^{\perp}|$.

On the other hand, if $\Gamma \in |\alpha^{\perp}|$, then there exists $t \in \alpha^{\perp}$ such that $|t| = \Gamma$. Hence, $\forall s \in \alpha (s \cdot t \in \perp)$. Hence, $\forall \Sigma \in |\alpha| \Sigma |t| = \Sigma\Gamma \in |\perp|$. Therefore, $\Gamma \in |\alpha|^{|\perp|}$. $\square$

Note that this relation also holds for non-regular fact $\alpha$, since trivially $|\alpha|^{|\perp_{\mathscr{M}_0}|} = |\alpha^{\perp_{\mathscr{M}_0}}| = M$ for $\alpha = \phi$ and $|\alpha|^{|\perp_{\mathscr{M}_0}|} = |\alpha^{\perp_{\mathscr{M}_0}}| = \phi$ for $\alpha = P_M^0$.

**Lemma 8.3.** $I_{\mathscr{M}} = |I_{\mathscr{M}_0}| = |I_{\mathscr{M}_1}| = |I_{\mathscr{M}_2}|$.

**Proof.** This is obvious by the definitions of $I_{\mathscr{M}}$, $I_{\mathscr{M}_0}$, $I_{\mathscr{M}_1}$ and $I_{\mathscr{M}_2}$. $\square$

Now we return to the proof of Theorem. By the above lemma, $\alpha^{\perp_{\mathscr{M}}} = |\alpha^{\perp_{\mathscr{M}_0}}|$. $\alpha \& \beta = |\alpha| \cap |\beta| = |\alpha \& \beta|$. $\alpha \oplus \beta = (|\alpha| \cup |\beta|)^{|\perp||\perp|} = |\alpha \oplus \beta|^{|\perp||\perp|} = |(\alpha)_{\oplus_l} \cup (\beta)_{\oplus_r}|^{|\perp||\perp|} = |((\alpha)_{\oplus_l} \cup (\beta)_{\oplus_r})^{\perp\perp}| = |\alpha \oplus \beta|$. $\alpha \otimes \beta = |\alpha \otimes \beta|$, $\alpha \mathbin{⅋} \beta = |\alpha \mathbin{⅋} \beta|$, $!\alpha = |!\alpha|$, $?\alpha = |?\alpha|$ are proved in the similar way.

(II) The theorem can be generalized to the higher-order case in the obvious way. In fact, assume $\beta \in \langle A \rangle_{\mathscr{M}}$, i.e. $\beta$ is a fact such that $A^{\perp} \in \beta \subseteq [\![A]\!]_{\mathscr{M}}$. We claim that there is a fact $\alpha$ such that $\{A^{\perp}\} \in \alpha \subseteq [\![A]\!]_T$ and that $\beta = |\alpha|$.

Take $\gamma$ to be $\{t \in [\![A]\!]_T \,|\, |t| \in \beta\}$. Then, $\{A^{\perp}\} \in \gamma \subseteq [\![A]\!]_T$ and $\beta = |\gamma|$. Now take $\alpha = \gamma^{\perp\perp}$, then $\alpha$ is a fact and $\{A^{\perp}\} \in \alpha \subseteq [\![A]\!]_T$. Now it suffices to show $|\alpha| = \beta$.

**Lemma 8.4.** *If* $\alpha \subseteq [\![A]\!]_T (= [\![A]\!]_{SN})$, *then* $|\alpha|^{|\perp|} = |\alpha^{\perp}|$, (*where* $\alpha$ *is not necessarily a fact*).

**Proof.** By the assumption, for any $s \in \alpha$, $s : |s|, \underline{A}$. Using this, the same proof for Lemma 8.2 above implies $|\alpha|^{|\perp|} = |\alpha^{\perp}|$. $\square$

Since $\alpha \subseteq [\![A]\!]_T$, $\gamma \subseteq [\![A]\!]_T$. Hence, by the above Lemma, $|\gamma^{\perp}| = |\gamma|^{|\perp|}$. Therefore, $|\alpha| = |\gamma^{\perp\perp}| = |\gamma^{\perp}|^{|\perp|} = |\gamma|^{|\perp\perp|} = \beta^{|\perp||\perp|} = \beta$.

Hence, $\langle\ \rangle_{\mathscr{M}} = |\langle\ \rangle_{\mathscr{M}_0}| (= |\langle\ \rangle_{\mathscr{M}_1}| = |\langle\ \rangle_{\mathscr{M}_2}|)$. Therefore $\mathscr{M} = |\mathscr{M}_0| = |\mathscr{M}_1|$. Now we extend (2) to the higher-order case:

$$
\begin{aligned}
\forall X |\xi|(X) &= \bigcap_{\substack{\alpha \in \langle B \rangle_{\mathscr{M}} \\ B \in Form}} |\xi|(\alpha) \\
&= \bigcap_{\substack{\beta \in \langle B \rangle_{\mathscr{M}_0} \\ B \in Form}} |\xi|(|\beta|) \\
&= \bigcap_{\substack{\beta \in \langle B \rangle_{\mathscr{M}_0} \\ B \in Form}} |\xi(\beta)| \quad \text{(by the definition of} |\xi|)
\end{aligned}
$$

$$= \left| \bigcap_{\substack{\beta \in \langle B \rangle_{\mathscr{M}_0} \\ B \in Form}} \xi(\beta) \right|$$
$$= |\forall X \xi(X)|.$$

In the similar way (with the help of the fact $|\alpha|^{|\perp|} = |\alpha^{\perp}|$), we can prove $\exists X A^{*_{\mathscr{M}}} = |\exists X A^{*_{\mathscr{M}_0}}|$. $\square$

The relationships between the semantics for "provability" and the semantics for "proofs" established in this section show that the theorems on the semantics for "provability" given in Sections 2–4 are direct corollaries of the corresponding theorems on semantics for "proofs" given in this section, as follows.

- An alternative proof of the Soundness Theorem for "provability". Take an arbitrary model $\mathscr{M} = (D, I, \perp, \varphi)$ for "provability". Consider the inverse model $\mathscr{M}'$ for "proofs" given in the Proposition 8.1 above. By the Soundness Theorem for "proofs", for any (typed) proof $t[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ indicates the list of all end nodes, of type $t[x_1, \ldots, x_n] : A_1, \ldots, A_n$, $t[A_1^{*\perp}, \ldots, A_n^{*\perp}] \subseteq \perp$. Hence,

$$\begin{aligned} |A_1^*|^{\perp_{\mathscr{M}}} \cdot \ldots \cdot |A_n^*|^{\perp_{\mathscr{M}}} &= |A_1^{*\perp_{\mathscr{M}'}}| \cdot \ldots \cdot |A_n^{*\perp_{\mathscr{M}'}}| \\ &= |t[A_1^{*\perp_{\mathscr{M}'}}, \ldots, A_n^{*\perp_{\mathscr{M}'}}]| \\ &\subseteq |\perp_{\mathscr{M}'}| = \perp_{\mathscr{M}}. \end{aligned}$$

This means $A_1^{*\perp} \cdot \ldots \cdot A_n^{*\perp} \subseteq \perp$ on the model $\mathscr{M}$ for provability, which means $1 \in A_1^* \mathbin{⅋} \cdots \mathbin{⅋} A_n^*$ (namely, $\vdash A_1, \ldots, A_n$ is true in $\mathscr{M}$).

This proof can easily be extended to the higher-order case. For an arbitrary second-order model $\mathscr{M} = (D, I, \perp, \varphi)$ for "provability", consider the inverse second-order model $\mathscr{M}'$ for "proofs" given in Proposition 8.1. By the Soundness Theorem for "proofs", for any (typed) proof $t[x_1, \ldots, x_n] : A_1[Y], \ldots, A_n[Y]$ where $Y \equiv Y_1, \ldots, Y_m$ is a vector of the second-order free variables, and for any $\alpha_i \in \langle B_i \rangle$, $t[A_1^{*\perp}[\alpha], \ldots, A_n^{*\perp}[\alpha]] \subseteq \perp$. Hence, for any $\alpha_i \in \langle B_i \rangle_{\mathscr{M}'}$,

$$\begin{aligned} (|A_1^*|[|\alpha|])^{\perp_{\mathscr{M}}} \cdot \ldots \cdot (|A_n^*|[|\alpha|])^{\perp_{\mathscr{M}}} &= |A_1^*[\alpha]|^{\perp_{\mathscr{M}'}} \cdot \ldots \cdot |A_n^*[\alpha]|^{\perp_{\mathscr{M}'}} \\ &= |t[A_1^{*\perp}[\alpha], \ldots, A_n^{*\perp}[\alpha]]| \\ &\subseteq |\perp_{\mathscr{M}'}| = \perp_{\mathscr{M}}. \end{aligned}$$

Since for any $\alpha \in \langle B_i \rangle_{\mathscr{M}}$ on $\mathscr{M}$, $\alpha = ||\alpha|^{-1}|$ and $|\alpha|^{-1} \in \langle B_i \rangle_{\mathscr{M}'}$, it follows that for any $\alpha_i \in \langle B_i \rangle_{\mathscr{M}}$, $(A_1^*[\alpha])^{\perp} \cdot \ldots \cdot (A_n^*[\alpha])^{\perp} \subseteq \perp$ on $\mathscr{M}$.

- An alternative proof of the Main Lemma for "provability". By the Main Lemma for "proofs", $\{A^{\perp}\} \in A^* \subseteq [\![A]\!]$ on the canonical model ($\mathscr{M}_0$ or $\mathscr{M}_1$ or $\mathscr{M}_2$) for "proofs". Hence, $A^{\perp} = |\{A^{\perp}\}| \in |A^*| \subseteq |[\![A]\!]|$, which means $A^{\perp} \in A^* \subseteq [\![A]\!]$ on the canonical model $\mathscr{M}$ for "provability" by Theorem 8.1.

The above proof can easily be extended to the higher-order case. By the Main Lemma for "proofs", for any second-order formula $A[X]$, where $X \equiv X_1, \ldots, X_n$ is the list of second-order variables, and for any $\alpha_i \in \langle B_i \rangle$, $\{(A[B])^\perp\} \in A^*[\alpha] \subseteq [\![A[B]]\!]$ on the canonical model ($\mathscr{M}_0$ or $\mathscr{M}_1$ or $\mathscr{M}_2$) for "proofs". Hence, for any $\alpha_i \in \langle B_i \rangle$

$$(A[B])^\perp = |\{(A[B])^\perp\}| \in |A^*|[|\alpha|] = |A^*[\alpha]| \subseteq |[\![A[B]]\!]|.$$

By Theorem 8.1, this means that on the canonical model $\mathscr{M}$ for "provability", for any $\alpha_i \in \langle B_i \rangle_{\mathscr{M}}$, $(A[B])^\perp \in A^*[\alpha] \subseteq [\![A[B]]\!]$.

The strong completeness (for "provability") is the direct consequence from this Main Lemma.

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