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Effectively closed sets and graphs of computable real functions

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Abstract

In this paper, we compare the computability and complexity of a continuous real function F with the computability and complexity of the graph G of the function F. A similar analysis will be carried out for functions on subspaces of the real line such as the Cantor space, the Baire space and the unit interval. In particular, we define four basic types of effectively closed sets C depending on whether (i) the set of closed intervals which with nonempty intersection with C is recursively enumerable (r.e.), (ii) the set of closed intervals with empty intersection with C is r.e., (iii) the set of open intervals which with nonempty intersection with C is r.e., and (iv) the set of open intervals with empty intersection with C is r.e. We study the relationships between these four types of effectively closed sets in general and the relationships between these four types of effectively closed sets for closed sets which are graphs of continuous functions.

Keywords: Computable analysis

1. Introduction

Computable analysis studies the effective content of theorems and constructions in analysis. In this paper, we study two of the most basic objects of computable analysis, namely continuous functions and their graphs over four natural spaces, the reals $\Re$, the unit interval $[0, 1]$, the Cantor space $\{0, 1\}^{\omega}$, and the Baire space ω^{ω}. The papers of Gregorczyk [11, 12] and Lacombe [21, 22] which initiated the study of computable analysis provide the starting point of our study since those papers provide careful definitions of computably closed sets of reals and computable real functions. More recently, Weihrauch [34–36] has provided a comprehensive foundation for computability theory on various spaces, including the space of compact sets and the space of continuous real functions.

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In this paper, we examine the complexity and computability of a continuous real function F as compared with the complexity and computability of the graph G of F. Of course, the graph of a continuous function is always a closed set and, for functions on a compact space such as the unit interval, any function with a closed graph is automatically continuous. We will give effective versions of these results, as well as counterexamples where the effective versions do not hold. Of course, it is first necessary to have a firm notion of an "effectively" closed set. Brattka and Weihrauch [2] identified three different types of effectively closed sets of Euclidean space $\Re^n$, namely, recursively enumerable (r.e.), co-recursively enumerable (co-r.e.), and recursively closed sets. Let $\{I_n\}_{n\in\omega}$ be some effective enumeration of the products of open rational intervals of $\Re^n$. Let ω denote the set of natural numbers and for each n, let $\overline{I_n}$ denote the closure of I_n. Then Brattka and Weihrauch defined a closed set K contained in $\Re^n$ to be

(a) *r.e.* if $\{n : I_n \cap K \neq \emptyset\}$ is r.e.,

(b) *co-r.e.* if $\{n : \overline{I_n} \cap K = \emptyset\}$ is r.e. and

(c) *recursive* if K is both r.e. and co-r.e.

In $\Re^n$, these notions can be characterized in several other natural ways, (see [2]). For example, a closed set K of $\Re^n$ is r.e. if and only if the distance function d_K to the set is upper semi-computable and is co-r.e. if and only if d_K is lower semi-computable. Similarly, a closed set K of $\Re^n$ is co-r.e. if and only if $\Re^n - K = \bigcup_{n\in B} I_n$ for some r.e. set $B \subseteq \omega$. These notions can easily be extended to the spaces $[0,1]$, $\{0,1\}^\omega$, and ω^ω.

This given, a number of natural questions arise. First, it is natural to ask about the relation between the computability of a continuous function F from $\Re$ to $\Re$ and computability of its graph G as a closed set. For example, we show that for any computably continuous function on either the real line or the Baire space ω^ω, the graph of F is a r.e. closed set. On the other hand, the set of closed intervals missed by the graph is r.e. for computably continuous real functions, but not necessarily r.e. for functions on ω^ω. One can also ask how the various equivalent formulations of r.e., co-r.e., and computably closed sets on the reals extend to our three other spaces. We shall show that not all of these types of results extend to our three other spaces. For example, we show that for subsets of ω^ω, the set of intervals missed by K is r.e. if and only if d_K is lower semi-computable, but these conditions are not equivalent to having the complement of K be the union of a r.e. set of intervals.

Moreover, the definitions of Brattka and Weihrauch given above suggest that there are four natural notions of effectively closed sets that one can consider in each of our four spaces. In each of our spaces, there is a natural effective enumeration of the basic open sets $\{I_n : n \in \omega\}$. Again we let $\overline{I_n}$ denote the closure of I_n. Then for each of our four spaces X, we say that a closed set $K \subseteq X$ is

1. *open interval recursively enumerable* (*OIr.e.*) if $\{n : I_n \cap K \neq \emptyset\}$ is r.e.,
2. *open interval co-recursively enumerable* (*OIco-r.e.*) if $\{n : I_n \cap K = \emptyset\}$ is r.e.,
3. *closed interval recursively enumerable* (*CIr.e.*) if $\{n : \overline{I_n} \cap K \neq \emptyset\}$ is r.e., and
4. *closed interval co-recursively enumerable* (*CIco-r.e.*) if $\{n : \overline{I_n} \cap K = \emptyset\}$ is r.e.

Thus OIr.e. closed sets are just r.e. closed sets and CIco-r.e. closed sets are co-r.e. closed sets. We shall study the relationships between these four types of closed sets in each of our four spaces. Of course, for the spaces ω^ω and 2^ω, each open interval I_n is clopen so that OIr.e. = CIr.e. and OIco-r.e = CIco-r.e. However for the reals $\Re$ and the unit interval $[0,1]$, we shall show that the only implications which hold among these four types of sets is that OIco-r.e $\Rightarrow$ CIco-r.e and CIr.e. $\Rightarrow$ OIr.e.

Weihrauch has demonstrated that the two notions of open interval recursively enumerable and closed interval co-recursively enumerable are the most reasonable, since the other two notions depend on the specific basis of intervals chosen (see Theorem 5.1.14 of [36]). On the other hand, the notions of r.e. and co-r.e. closed sets are stable in that the choice of the (recursive) basis, with modest restrictions, does not affect the family of effectively closed sets so defined (see also p. 76 of [2]).

We shall also study the relationships between these four types of effectively closed sets on closed sets which are graphs of a continuous functions.

Finally, it is natural to study the same questions with regard to complexity theory. That is, the study of polynomial time computable functions on the reals was initiated by Friedman and Ko [10, 20] and the complexity theoretic study of analysis has be extensively developed, see Ko's book [18]. One can give natural complexity theoretic analogues of the notions of OIr.e., OIco-r.e., CIr.e., and CIco-r.e. closed sets by roughly replacing the occurrences of r.e. in the definitions by NP (nondeterministic polynomial time). We postpone the formal definitions of these notions until Section 5 because the notions are sensitive to the exact coding of the basic open intervals. However, one can then ask a similar set of questions about the relationships between the complexity of continuous function F and the complexity of its graph G as a closed set.

We should note that the study of effectively closed sets have a long history in computability theory. That is, our CIco-r.e. closed set are also called Π^0_1 classes in the literature of computability theory, and recursive closed sets are called *decidable* Π^0_1 classes. Just as closed sets are central to the study of computable analysis, Π^0_1 classes play a fundamental role in computability theory. For example, Π^0_1 classes have played an important role in computability theory going back to the Kleene basis theorem [17]. Many of the fundamental results about Π^0_1 classes and their members were established by Jockusch and Soare [14, 15]. For a short course on Π^0_1 classes, see [4].

Π^0_1 classes occur naturally in the application of computability to many areas of mathematics. See the recent survey of Cenzer and Remmel [7] for many examples. One important example of a CIco-r.e. closed set in Euclidean space is the set of zeroes of a computably continuous function. This leads easily to related examples of the appearance of CIco-r.e. closed sets as the set of fixed points or the set of extrema of a computably continuous function. That is, for any continuous function F, it is easy to see that the set of zeroes of F, the set of fixed points of F, and the set of points where F attains an extremum, are all closed sets. For a computably continuous function F, the corresponding closed sets are all CIco-c.e. In fact, Nerode and Huang [26] showed that any CIco-r.e. closed set of reals may be represented as the set of zeroes of a computably continuous function. Ko extended the Nerode–Huang results

[18] to show that any CIco-r.e. closed set may be represented as the set of zeroes of a polynomial time computable function. Thus CIco-r.e. closed sets also appear naturally in the theory of polynomial time computable functions on the reals. Computable aspects of dynamical systems and Julia sets have been studied by Cenzer [4], Ko [19] and by Cenzer and Remmel [7]. In particular, the Julia set of a computably continuous real function is a Π^0_1 class.

Subsets of the Baire space are investigated as so-called ω-languages in theoretical computer science. The theory of ω-languages accepted by Turing machines has been developed in a series of papers [9, 30–33]. These papers develop connections between acceptance, representability by computable or r.e. languages and classification in the arithmetical hierarchy. In particular, a Π^0_1 class may be viewed as the ω-language accepted by a deterministic Turing machine M in the sense that the infinite sequence $x(0), x(1), \ldots$ is accepted if every initial segment $x(0), \ldots, x(n)$ is accepted by M. This notion was introduced in [23].

The notion of index sets for Π^0_1 classes in ω^ω has been developed by Cenzer and Remmel [6], which builds on the work of Lempp [24] and others. The main idea is that the complexity of a problem, such as computing the measure of a closed set, may be measured by the complexity of its index set in the arithmetic hierarchy.

For example, it is shown in [8] that the index set of the computably continuous functions which have a computable zero is a Σ^0_3 complete set. This greatly strengthens the well-known fact that a computably continuous real function need not have a computable zero. It is also shown in [8] that for any computable real r, the set of indices e such that the eth Π^0_1 class in $\{0,1\}^\omega$ has measure r is a Π^0_2 complete set. This greatly strengthens the well-known fact that the measure of a Π^0_1 class need not be computable. We will prove a number of index set type results in the this paper. For example, we shall show the set of CIco-r.e. sets which are OIco-r.e. is a Σ^0_4 complete set.

The outline of this paper is as follows. Section 2 is devoted to preliminaries. In Section 3, we shall study the relationships between our four types of effectively closed sets in each of the four spaces, $\Re$, $[0,1]$, 2^ω, and ω^ω. We shall also study how the equivalent characterizations for OIr.e. and CIco-r.e. closed sets of $\Re^n$ given in [2] extend to the Cantor Space, the Baire Space and the unit interval. In Section 4, we examine the relationship of our four types of effectively closed sets for closed sets which arise as graphs of continuous functions and the relationships between the effectiveness of the graph as closed set and the computability of the function. Finally, in Section 5, we shall give some preliminary results on the relationships between OINP, OIco-NP, CINP, and CIco-NP closed sets.

2. Preliminaries

We begin with some basic definitions. Let $\omega = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. For each $n \in \omega$, let $bin(n)$ denote the binary representation of n and

$tal(n) = 1^n$ denote the tally representation of n. We then let $Bin(\omega) = \{bin(n) : n \in \omega\}$ and $Tal(\omega) = \{tal(n) : n \in \omega\}$. For any set Σ, $\Sigma^{<\omega}$ denotes the set of finite strings $(\sigma(0), \ldots, \sigma(n-1))$ of elements from Σ and Σ^{ω} denotes the set of countably infinite sequences from Σ. For any set A, we let $card(A)$ denote the cardinality of the set A.

For a string $\sigma = (\sigma(0), \sigma(1), \ldots, \sigma(n-1))$, $|\sigma|$ denotes the length n of σ. The empty string has length 0 and will be denoted by $\emptyset$. A string of n k's will be denoted k^n. For $m < |\sigma|$, $\sigma \lceil m$ is the string $(\sigma(0), \ldots, \sigma(m-1))$. We say σ is an *initial segment* of τ (written $\sigma \prec \tau$) if $\sigma = \tau \lceil m$ for some m. Given two strings σ and τ, the *concatenation* of σ and τ, denoted by $\sigma^\frown\tau$ (or sometimes $\sigma * \tau$ or just $\sigma\tau$), is defined by $\sigma^\frown\tau = (\sigma(0), \sigma(1), \ldots, \sigma(m-1), \tau(0), \tau(1), \ldots, \tau(n-1))$, where $|\sigma| = m$ and $|\tau| = n$. We write $\sigma^\frown a$ for $\sigma^\frown(a)$ and $a^\frown\sigma$ for $(a)^\frown\sigma$. For any $x \in \Sigma^{\omega}$ and any finite n, the *initial segment* $x \lceil n$ of x is $(x(0), \ldots, x(n-1))$. For a string $\sigma \in \Sigma^{<\omega}$ and any $x \in \Sigma^{\omega}$, we write $\sigma \prec x$ if $\sigma = x \lceil n$ for some n. For any $\sigma \in \Sigma^n$ and any $x \in \Sigma^{\omega}$, we have $\sigma^\frown x = (\sigma(0), \ldots, \sigma(n-1), x(0), x(1), \ldots)$.

A *tree* T over $\Sigma^{<\omega}$ is a set of finite strings from $\Sigma^{<\omega}$ which contains the empty string $\emptyset$ and which is closed under initial segments. We say that $\tau \in T$ is an *immediate successor* of a string $\sigma \in T$ if $\tau = \sigma^\frown a$ for some $a \in \Sigma$. We will assume that $\Sigma \subseteq \omega$, so that $T \subseteq \omega^{<\omega}$. Such a tree is said to be *ω-branching* since each node has potentially a countably infinite number of immediate successors. Let $\langle , \rangle : \omega \times \omega \to \omega$ be a computable $1:1$ onto pairing function. We can then inductively extend $\langle , \rangle$ to code n-tuples for $n \geqslant 3$ by defining $\langle x_1, \ldots, x_n \rangle = \langle x_1, \langle x_2, \ldots, x_n \rangle\rangle$. We shall sometimes identify T with the set $\{\langle \sigma \rangle : \sigma \in T\}$. Thus we say that T is recursive, r.e., etc., if $\{\langle \sigma \rangle : \sigma \in T\}$ is recursive, r.e., etc.

For a given function $g : \omega^{<\omega} \to \omega$, a tree $T \subseteq \omega^{<\omega}$ is said to be *g-bounded* if for every $\sigma \in \omega^{<\omega}$ and every $i \in \omega$, if $\sigma^\frown i \in T$, then $i < g(\sigma)$. Thus, for example, if $g(\sigma) = 2$ for all σ, then a g-bounded tree is simply a binary tree. T is said to be *finitely branching* if T is g-bounded for some g, that is, if each node of T has finitely many immediate successors. Observe that this is equivalent to the existence of a bounding function h such that $\sigma(i) < h(i)$ for all $\sigma \in T$ and all $i < |\sigma|$. T is said to be *recursively bounded* (*r.b.*) if it is g-bounded for some recursive function g. As above, this is equivalent to the existence of a recursive bounding function h such that $\sigma(i) < h(i)$ for all $\sigma \in T$ and all $i < |\sigma|$. If T is recursive, then this is also equivalent to the existence of a partial recursive function f such that, for any $\sigma \in T$, σ has at most $f(\sigma)$ immediate successors in T. A recursive tree T is said to be *highly recursive* if it is also recursively bounded. For any tree T, an *infinite path* through T is a sequence $(x(0), x(1), \ldots)$ such that $x \lceil n \in T$ for all n. We let $[T]$ denote the set of infinite paths through T.

A subset P of ω^{ω} is a *Π^0_1 class* if $P = [T]$ for some recursive tree $T \subseteq \omega^{<\omega}$. If the tree T is g-bounded, we will say that P is *g-bounded* and similarly for other notions of boundedness. For example, this means that the Π^0_1 class P is bounded, if $P = [T]$ for *some* recursive finitely branching tree T. It is possible that there be another tree S which is not finitely branching such that $P = [S]$ also (just let S include T together with all paths (i) of length 1). We say that P is a *strong* Π^0_2 *class* if there is a tree T recursive in $\mathbf{0}'$ such that $P=[T]$.

It is important to note here that we consider a Π^0_1 *set* to signify a subset of ω and in general a Π^0_n, Σ^0_n or Δ^0_n set is a subset of ω with the appropriate form of definability in the arithmetical hierarchy (see [13]).

A node σ of the tree $T\subseteq\omega^{<\omega}$ is said to be *extendible* if there is some $x\in[T]$ such that $\sigma\prec x$. The set of extendible nodes of T is denoted by $Ext(T)$. $Ext(T)$ may be viewed as the minimal tree S such that $[S]=[T]$. A node $\sigma\in T$ is said to be a *dead end* if $\sigma\notin Ext(T)$, that is, if σ has no infinite extension in $[T]$.

As stated in the introduction, we shall study effectively closed sets and effectively computable functions over four spaces, $\{0,1\}^\omega$ (the Cantor space) and ω^ω (the Baire space), the real line $\Re$ and the interval $[0,1]$. We note that the Cantor space may be represented as a closed subset of the interval $[0,1]$ in the usual manner by mapping $x\in\{0,1\}^\omega$ to the real $r_x=\sum_i 2x(i)/3^i$ and the Baire space can be represented as the space of irrational reals in $[0,1]$ under the relative topology.

We may define a distance function for each space as follows. For $\Re$ and $[0,1]$, $d(x,y)=|x-y|$ is the usual metric. Define the distance $d(x,y)$ between two elements of ω^ω or $\{0,1\}^\omega$ to be 2^{-n} if n is the least such that $x(n)\neq y(n)$ and 0 if $x=y$. Then for any closed subset K of the space X and any $x\in X$, define $d_K(x)$ to be the minimum of the set $d(x,y)$ for $y\in K$.

Each of our four spaces has a natural countable basis of basic open sets or intervals $I_0,I_1,\dots$ which we shall describe below. Thus a topology on each of our four spaces is determined by defining an open set to be a (finite or countable) union of intervals and a closed set is the complement of an open set. It is important to specify a computable enumeration of the basic open sets or intervals in each of our spaces so that our various notions of r.e. and co-r.e. closed sets can be made precise. Thus we shall specify such an enumeration $\{I_e=I^X_e\}_{e\in\omega}$ for each of our spaces X.

First consider the Baire space $X=\omega^\omega$. The topology on ω^ω is determined by a basis of intervals $\{I(\sigma):\sigma\in\omega^{<\omega}\}$ where $I(\sigma)=\{x\in\omega^\omega:\sigma\prec x\}$. Notice that each interval is also a closed set and is therefore said to be *clopen*. The finite sequences $\sigma\in\omega^{<\omega}$ may be enumerated in order $\sigma_0,\sigma_1,\dots$ by enumerating those elements of smallest weight first, where the weight of σ equals $|\sigma|+\sigma(0)+\cdots+\sigma(|\sigma|-1)$, and then by enumerating those elements of the same weight lexicographically. We then let $I_n=I(\sigma_n)$.

Next consider the Cantor space $X=\{0,1\}^\omega$. The topology on $\{0,1\}^\omega$ is determined by a basis of intervals $\{I(\sigma):\sigma\in\{0,1\}^{<\omega}\}$ where $I(\sigma)=\{x\in\{0,1\}^\omega:\sigma\prec x\}$. Notice that each interval is a clopen set. The finite sequences $\sigma\in\{0,1\}^{<\omega}$ may be enumerated as $\emptyset,(0),(1),(00),\dots$, so that in general $bin(n+1)=1^\frown\sigma_n$. Then we simply let $I_n=I(\sigma_n)$.

For the space $[0,1]$, there is a basis of open intervals (q,r) where $q<r$ are rationals, as well as the half-open intervals $[0,r)$ and $(q,1]$. Let $q_0,q_1,\dots$ effectively enumerate (without repetition) the rationals in $[0,1]$. To be explicit, let $q_0=0$, $q_1=1$ and order the rationals p/q, with p and q relatively prime, first by the sum $p+q$ and then by p.

Then we may define I_n for $n=\langle i,j\rangle$ to be (q_i,q_j) if $q_i<q_j$, to be $[0,q_i)$ if $q_j=0<q_i$, to be $(q_j,1]$ if $q_i=1>q_j$ and to be $(0,1)$ otherwise.

For the space $\mathfrak{R}$, there is a basis of rational intervals and, for convenience, we will also include infinite open intervals. Thus if $q'_0,q'_1,\ldots$ effectively enumerates the rationals $\mathbb{Q}$ (as above) plus $\{-\infty,\infty\}$, then we define I_n for $n=\langle i,j\rangle$ to be (q_i,q_j) if $q_i<q_j$ and to be $(-\infty,\infty)$ otherwise.

For any of our four spaces X and any finite $k>0$, we can define a set of basic open sets or intervals for X^k by taking sets of the form $I_{n_1}\times\cdots\times I_{n_k}$ where each I_{n_j} is a basic interval for X. These basic open sets may be enumerated via our effective pairing function by defining $I_{\langle n_1,\ldots,n_k\rangle}=I_{n_1}\times\cdots\times I_{n_k}$.

This given, we can formally define our various notions of effectively closed sets for each of spaces X.

Definition 2.1. Let K be a closed set in the space X where X is either ω^ω, $\{0,1\}^\omega$, $\mathfrak{R}$, or $[0,1]$.

(i) K is *open interval recursively enumerable* (*OIr.e.*) if $\{w : I_w\cap K\neq\emptyset\}$ is recursively enumerable.
(ii) K is *open interval co-recursively enumerable* (*OIco-r.e.*) if $\{w : I_w\cap K=\emptyset\}$ is recursively enumerable.
(iii) K is *closed interval recursively enumerable* (*CIr.e.*) if $\{w : \overline{I_w}\cap K\neq\emptyset\}$ is recursively enumerable.
(iv) K is *closed interval co-recursively enumerable* (*CIco-r.e.*) if $\{w : \overline{I_w}\cap K=\emptyset\}$ is recursively enumerable.
(v) K is *recursive* if K is both OIr.e. and CIco-r.e.
(vi) K is *open interval decidable* if $\{w : I_w\cap K\neq\emptyset\}$ is recursive.
(vii) K is *closed interval decidable* if $\{w : \overline{I_w}\cap K\neq\emptyset\}$ is recursive.
(viii) K is *decidable* if K is both open interval decidable and closed interval decidable.

The most natural notions are (i) and (iv). The open interval r.e. closed sets are called r.e. closed sets and the closed interval co-r.e. sets are called co-r.e. closed sets in [2], where a recursive closed set is defined to be one satisfying both conditions, which agrees with our definition of recursive closed set. For the real line and the Cantor space, the CIco-r.e. sets are the usual Π^0_1 classes studied in computability theory [4, 7]. For the Cantor space and the Baire space, $\overline{I_w}=I_w$, so that (i) is equivalent to (iii), (ii) is equivalent to (iv), and (vi)–(viii) are all equivalent.

A uniform approach to the notion of a continuous function and a computably continuous function may be given via the concept of representing functions. We say that $f:\omega\to\omega$ is a *representing function* for a function $F : X\to Y$ if the following conditions hold:

(i) For each a and b, if $I^X_a\subset I^X_b$, then $I^Y_{f(a)}\subset I^Y_{f(b)}$.
(ii) For each $x\in X$ and for any decreasing sequence $\{I^X_{e_k}\}_{k\in\omega}$ of intervals with $\bigcap_k I^X_{e_k}=\{x\}$, $\bigcap_k I^Y_{f(e_k)}=\{F(x)\}$.

In the case where F has a representing function f, we may view the input element $x \in X$ as being given to F as list of the intervals to which x belongs. It is easy to see that a function $F : X \to Y$ is continuous if and only if F has a representing function f. We then define a function $F : X \to Y$ to be *computably continuous* if and only if F has a computable representing function f.

A function F is said to be *partial* computable if it has a partial computable representing function f which satisfies (i) and (ii) whenever f is defined and such that, whenever $I^X_a \subset I^X_b$ and $f(a)$ is defined, then $f(b)$ is defined. Then the domain $dom(F)$ is the set of x such that there is a decreasing sequence of intervals with $\bigcap_k I^X_{e_k} = \{x\}$ and $f(e_k)$ defined for all k. The partial computable function F is said to be *strongly* computable if $dom(g)$ is computable.

Next, we discuss the effective versions of rational and real numbers. The set $\mathscr{Q}$ of rational numbers is countable and may clearly be viewed, via a numbering as given above, as a recursive set equipped with a recursive ordering and recursive operations of addition, subtraction, multiplication and division. A real number x has a (lower) Dedekind cut $L(x) = \{q \in \mathscr{Q} : q < x\}$ and also an upper Dedekind cut $U(x) = \{q : x < q\}$. A real x is said to be *lower semi-computable* if and only if $L(x)$ is r.e. and is said to be *upper semi-computable* if and only if $U(x)$ is r.e.

These notions are related to the notions of effectively closed sets as follows. The real x is lower semi-computable if and only if the closed set $(-\infty, x]$ is OIr.e. and if and only if $[x, \infty)$ is CIco-r.e. Similarly x is upper semi-computable if and only if $(-\infty, x]$ is CIco-r.e. and if and only if $[x, \infty)$ is OIr.e. We observe that if x is lower semi-computable, then we can obtain x as the limit of a computable, increasing sequence $\{q_n\}$ of rationals by letting q_n be the largest rational which has been enumerated into $L(x)$ after stage n. Conversely, if x is the limit of a computable increasing sequence q_n, then we have $q \in L(x)$ if and only if there is some n such that $q < q_n$, so that $L(x)$ is r.e. Similarly, x is upper semi-computable if and only if it is the limit of a computable, decreasing sequence of rationals.

More generally, a real function $F : \mathfrak{R}^n \to \mathfrak{R}$ is said to be lower (upper) semi-computable if there is a uniformly computable, increasing (decreasing) sequence $\{F_i\}_{i \in \omega}$ of real functions whose limit is F, that is, for all $i < j$ and $x \in X$, $F_i(x) < F_j(x)$ ($F_i(x) > F_j(x)$) and $\lim_i F_i(x) = F(x)$.

Lemma 2.2. *Let $F : \mathfrak{R}^n \to \mathfrak{R}$. Then*
(a) *F is lower semi-computable if and only if $\{(q, r) \in \mathbb{Q}^{n+1} : q < F(r)\}$ is recursively enumerable.*
(b) *F is lower semi-computable if and only if $\{(q, r) \in \mathbb{Q}^{n+1} : F(r) < q\}$ is recursively enumerable.*

Proof. (a) Suppose first that $\{F_i\}_{i \in \omega}$ is a uniformly computable increasing sequence of functions with limit F in the sense that there exists a uniformly computable sequence of functions $f_i : \omega \to \omega$ such that f_i is a representing function for F_i for all $i \in \omega$. Then for any rationals q and r, $q < F(r)$ if and only if there is an i such that $q < F_i(r)$ which, in turn, is if and only if there exists n and $c<d<r$ such that $I_w=(q-1/n,q+1/n)$ $I_{f_i(w)}=(c,d)$. This shows that $\{(q,r)\in\mathcal{Q}^n\times\mathcal{Q}:q<F(r)\}$ is recursively enumerable.

Conversely, suppose that $L=\{(q,r)\in\mathbb{Q}^n\times\mathcal{Q}:q<F(r)\}$ is recursively enumerable. Then let L_i be the set of (q,r) of L enumerated after i steps. Let $r<_i x$ denote that the rational r is known to be $<x$ after checking the first i bits of information about x. Now let $F_i(x)$ be the largest q such that for some rational $r<_i x$, $(q,r)\in L_i$. It is clear that F_i is uniformly computable with increasing limit F.

The proof of (b) is similar. □

3. Effectively closed sets

In this section, we shall completely analyze the relationships between our four basic types of effectively closed sets in each of our four spaces.

We begin by recalling some alternative characterizations of OIr.e. and CIco-r.e. sets in $\mathfrak{R}$ given by Brattka and Weihrauch [2]. We shall then consider how these characterizations extend to our other three spaces, $\{0,1\}^\omega$, ω^ω and $[0,1]$ since these characterizations will be useful in our later proofs.

Theorem 3.1 (Brattka and Weihrauch [2]). *Let $K\subset\mathfrak{R}$ be a closed set.*

(1) *The following statements are equivalent*:

   (a) *K is OIr.e.*

   (b) *$\{w:K\cap I_w\neq\emptyset\}$ is recursively enumerable.*

   (c) *d_K is upper semi-computable.*

   (d) *$K=\emptyset$ or range(f) is dense in K for some computable $f:\omega\to\mathfrak{R}$.*

(2) *The following statements are equivalent*:

   (a) *K is CIco-r.e.*

   (b) *$\{w:K\cap\overline{I_w}=\emptyset\}$ is recursively enumerable.*

   (c) *d_K is lower semi-computable.*

   (d) *$K=F^{-1}\{0\}$ for some computably continuous function $F:\mathfrak{R}\to\mathfrak{R}$.*

   (e) *$K=dom(G)$ for some computable $f:\mathfrak{R}\to\omega$.*

   (f) *$\mathfrak{R}-K=\bigcup_{w\in B}I_w$ for some recursively enumerable $B\subset\omega$.*

Note that Theorem 3.1 implies that a closed set K is recursive if and only if d_K is computable.

The assumption that K is a closed set is crucial in several of the implications. For example, we may define the distance function $d_K(x)$ as the infimum of $d(x,y)$ for $y\in K$ even for sets K which are not closed. Then d_K is simply $d_{cl(K)}$, so that d_K is computable if and only if $cl(K)$ is a recursive closed set. Similar statements hold for d_K being upper or lower semi-computable. Also, the condition that *range*(f) is dense in K for a computable function $f:\omega\to\mathfrak{R}$ could be modified to say that K is the closure of the range of such a function, and thus to imply that K is closed. As it is, of course any open rational interval has a dense countable subset of rationals, which can be given as the range of a computable function. In contrast, conditions (2)(d)–(f) all imply that K is a closed set, since any computable function must be continuous.

The characterization in terms of intervals which are met (or missed) likewise does not imply closure of itself. For example, consider the open interval $(0,1)$. $I_w = (p,q)$ meets $(0,1)$ if and only if either p or q is in $(0,1)$, which is clearly a recursive condition. Similarly, $\overline{I_w} = [p,q]$ misses $(0,1)$ if and only if either $p < q \leqslant 0$ or $1 \leqslant p < q$ which is again a recursive condition.

We note that such distinctions will be more important when we want to prove similar kinds of equivalence for closed sets which are the graph of the function F. In particular, we will want to find conditions on the graph of F which imply that F is computable without the assumption of continuity.

Since the space $[0,1]$ is a (recursive) closed subset of $\mathfrak{R}$ and the space $\{0,1\}^{\omega}$ is effectively homeomorphic to a recursive closed subset (the Cantor set) of $\mathfrak{R}$, the characterizations of Theorem 3.1 clearly carry over to these two spaces and their finite powers. The only difference is that for the space $\{0,1\}^{\omega}$, $I_w = \overline{I_w}$, which simplifies item (2b). Thus we focus on the space ω^{ω}. Recall that the distance $d(x,y)$ between two elements of ω^{ω} is defined to be 2^{-n} if n is the least such that $x(n) \neq y(n)$ and 0 if $x = y$.

Theorem 3.2. *Let $K \subset \omega^{\omega}$ be a closed set.*

Then the following statements are equivalent:

(a) *K is OIr.e.*

(b) *$\{w : K \cap I_w \neq \emptyset\}$ is recursively enumerable.*

(c) *d_K is upper semi-computable.*

(d) *$K = \emptyset$ or range(f) is dense in K for some computable $f : \omega \to \omega^{\omega}$.*

Proof. Of course, (a)$\Leftrightarrow$(b) is just the definition.

(b)$\Rightarrow$(d): Suppose that K is nonempty and that $\{w : K \cap I_w \neq \emptyset\}$ is recursively enumerable. Then let $I(\sigma_0), I(\sigma_1), \ldots$ be an effective enumeration of the intervals $I(\sigma)$ such that $K \cap I(\sigma) \neq \emptyset$.

We define a uniformly computable sequence x_n which will be dense in K by making each x_n the unique member of a decreasing intersection $I_{n,k}$ of intervals, defined as follows. $I_{n,0} = I(\sigma_n)$ and for each k, $I_{n,k+1} = I(\sigma_t)$, where $I(\sigma_t)$ is the first interval in the sequence $\{I(\sigma_i)\}_{i \in \omega}$ which is a proper subset of $I_{n,k}$. Note that since $K \cap I_{n,k} \neq \emptyset$, there must be such a proper subset in the enumeration. We should observe that since ω^{ω} is not compact, we cannot use compactness to see that $\bigcap_{k<\omega} I_{n,k} \neq \emptyset$ and in fact contains an element x_n of K. Nevertheless, the unique element x_n of $\bigcap_{k<\omega} I_{n,k} \neq \emptyset$ is defined and can be computed since each $I_{n,k} = I(\sigma_{n,k})$ for a computable list $\sigma_{n,k}$ of finite sequences such that $\sigma_{n,k+1}$ a proper extension of $\sigma_{n,k}$ for each k. Thus we simply define $x_n(k) = \sigma_{n,k+1}(k)$ for each k. Furthermore, for each k there is an element $y_{n,k} \in K \cap I_{n,k}$ which extends $\sigma_{n,k}$, so that $\lim_{k \to \infty} y_{n,k} = x_n$. Thus the assumption that K is closed is enough to ensure that $x_n \in K$.

It is clear that the sequence x_n is uniformly computable, so that there is a computable function $f:\omega\to\omega^\omega$ with $f(n)=x_n$. Since we have $x_n\in K\cap I(\sigma_n)$ for each n, it follows that $\{x_n\}_{n<\omega}$ is dense in K.

(d)$\Rightarrow$(c): Suppose that $\{x_n\}_{n<\omega}$ is a uniformly computable sequence which is dense in K. Then for any x and any rational q,

$$d(x,K) < q \Leftrightarrow (\exists n)d(x,x_n) < q \Leftrightarrow (\exists i,n)[x\lceil i = x_n\lceil i \,\&\, 2^{-1-i} \leqslant q].$$

It then easily follows that d_K is upper semi-computable.

(c)$\Rightarrow$(b): Suppose that d_K is upper semi-computable and let d_K be the limit of a decreasing sequence $\{f_n\}$ of computable functions from ω^ω into $[0,1]$. Observe that the set of triples $\langle\sigma,n,q\rangle$, with $\sigma\in\omega^{<\omega}$, $n<\omega$ and $q\in Q$, such that $f_n(\sigma^\frown 0^\omega)<q$, is recursively enumerable. This is because $f_n(\sigma^\frown 0^\omega)<q$ if and only if, at some stage in the computation of $f_n(\sigma^\frown 0^\omega)$, we have an estimate which is good enough to imply that the value of $f_n(\sigma^\frown 0^\omega)$ is $<q$. The result now follows from the fact that if $|\sigma|=\ell$, then

$$I(\sigma)\cap K \neq \emptyset \Leftrightarrow d_K(\sigma^\frown 0^\omega) < 2^{-1-\ell}.$$

To see this, suppose first that $I(\sigma)\cap K\neq\emptyset$, which is if and only if $\exists y\in K$ with $\sigma\prec y$. Thus the least n where such a y differs from $\sigma^\frown 0^\omega$ is at least ℓ. But this means that $d(y,\sigma^\frown 0^\omega)\leqslant 2^{-\ell}<2^{1-\ell}$. Each step of this argument may be reversed to get the other direction. $\square$

For the CIco-r.e. closed sets of ω^ω, only some of the statements are equivalent. Note that clause (f) is the usual definition of a Π^0_1 class.

Theorem 3.3. *Let $K\subset\omega^\omega$ be a closed set.*

(1) *The following statements are equivalent*:
  (a) *K is CIco-r.e.*
  (b) *$\{w : K\cap\overline{I_w}=\emptyset\}$ is recursively enumerable.*
  (c) *d_K is lower semi-computable.*

(2) *The following statements are equivalent*:
  (d) *$K=F^{-1}\{0^\omega\}$ for some computably continuous function $F:\omega^\omega\to\omega^\omega$.*
  (e) *$K=dom(G)$ for some partial computable function $G:\omega^\omega\to\omega$.*
  (f) *$\omega^\omega-K=\bigcup_{I(\sigma):\sigma\in B}$ for some recursively enumerable $B\subset\omega^{<\omega}$.*

(3) *Each of the first set of statements implies each of the second, but the converse does not hold.*

Proof. (1) Once again (a)$\Leftrightarrow$(b) is just the definition.

(c)$\Rightarrow$(b): Assume that d_k is lower semicomputable. It follows from our argument of the previous theorem that

$$\overline{I(\sigma)}\cap K = \emptyset \Leftrightarrow d_K(\sigma^\frown 0^\omega) > 2^{-\ell}.$$

Thus if d_K is lower semicomputable, then (b) holds.

(b) $\Rightarrow$ (c): Suppose that $W = \{w : K \cap \overline{I_w} = \emptyset\}$ is recursively enumerable. We can express the function d_K as the limit of a computable, increasing sequence of functions F_n as follows. Let $F_n(x) = 2^{-m}$ where m is the least k such that the code w for $I(x\lceil k)$ has been enumerated into W after n steps if there are such k and w and let $F_n(x) = 0$ otherwise. For each x, it is easy to see that $F_0(x), F_1(x), \ldots$ is an increasing sequence so that $\{F_n\}_{n\in\omega}$ is uniform increasing sequence of computable functions. It is clear that $\lim_{n\to\infty} F_n(x)$ will be 2^{-m}, where m is the least such that $K \cap I(x\lceil m) \neq \emptyset$ if there is such an m and is 0 otherwise. Thus $\lim_{n\to\infty} F_n(x) = d_K(x)$ and hence d_K is lower semicomputable.

(2) (d) $\Rightarrow$ (e): Let F be a computable function with computable representing function f such that $K = F^{-1}\{0^\omega\}$. Then the desired computable function G will have a partial computable representing function g where for all $\sigma \in \omega^{<\omega}$, $g(\sigma) = f(\sigma)$ if $f(\sigma)$ consist of all 0's and is undefined otherwise. Thus $G(x) = F(x)$ if $F(x) = 0^\omega$ and is undefined otherwise. Thus the domain of G equals K.

(e) $\Rightarrow$ (f): Suppose that $K = dom(G)$ for some partial computable function G with partial computable representing function g. Since g is partial computable, the set of σ such that $g(\sigma)$ is defined is r.e. and $\omega^\omega - K = \cup\{I(\sigma) : \sigma \notin dom(g)\}$.

(f) $\Rightarrow$ (d): Let $\sigma_1, \sigma_2, \ldots$ be a recursive list such that $\omega^\omega - K = \bigcup_n I(\sigma_n)$. Define a representing function f of a computable function F by setting $f(\sigma) = 0^n$ if $|\sigma| = n$ and there is no $i < n$ such that $\sigma_i \prec \sigma$ and setting $f(\sigma) = 0^j 1^{n-j}$ if j is the least k such that $\sigma\lceil k = \sigma_i$ for some $i < n$. It is clear that f is computable and satisfies the conditions needed to be representing function of a continuous function F. It is easy to check that our definition ensures that $K = F^{-1}\{0^\omega\}$.

(3) It is immediate that (b) implies (f). On the other hand, it is known from classical descriptive set theory that $\{\sigma : K \cap I(\sigma) \neq \emptyset\}$ can be a Σ^1_1 complete set for some Π^0_1 class K. For example, it was shown in the recent paper of Cenzer and Remmel [8], that the set of indices e such that P_e is nonempty is a Σ^1_1 complete set where $P_0, P_1, \ldots$ is a canonical enumeration of all Π^0_1 classes in ω^ω. Thus if we define the Π^0_1 class K so that $(e)^\frown x \in K$ if and only if $x \in P_e$, then $\{e : K \cap I((e)) = \emptyset\}$ is a Π^1_1 complete set and is therefore not recursively enumerable. $\square$

For the remainder of this section, we examine the possible implications between the four fundamental notions of effectively closed sets of reals, that is, open (closed) interval r.e. (co-r.e.).

There are just two positive results.

Theorem 3.4. *Let K be a closed subset of $\Re$.*
(a) *if K is open interval co-r.e., then K is closed interval co-r.e. and*
(b) *if K is closed interval r.e., then K is open interval r.e.*

Proof. (a) Suppose that K is OIco-r.e., i.e., that $\{w : I_w \cap K = \emptyset\}$ is recursively enumerable. Now to test whether $\overline{I_w} \cap K = \emptyset$, where $\overline{I_w} = [p, q]$, just observe that

$$[p,q]\cap K=\emptyset \Leftrightarrow (\exists n)\left(p-\frac{1}{n},q+\frac{1}{n}\right)\cap K=\emptyset.$$

Thus if $\{w: I_w\cap K=\emptyset\}$ is r.e., then $\{w:\overline{I_w}\cap K=\emptyset\}$ is r.e.

(b) Suppose that K is CIr.e., i.e., that $\{w:\overline{I_w}\cap K\neq\emptyset\}$ is recursively enumerable. The result now follows from the fact that

$$(p,q)\cap K\neq\emptyset \Leftrightarrow (\exists n)\left[p+\frac{1}{n},q-\frac{1}{n}\right]\cap K\neq\emptyset.$$

Thus if $\{w:\overline{I_w}\cap K\neq\emptyset\}$ is r.e., then $\{w: I_w\cap K\neq\emptyset\}$ is r.e. $\square$

It follows from Theorem 3.4 that there are at most two ways in which a closed set K of $\Re$ can satisfy exactly three of the four properties. Our next result will show that both of these possibilities can be realized.

Theorem 3.5. (a) *There is a closed set K of reals which is open interval decidable but is not closed interval r.e.*

(b) *There is a closed set K of reals which is closed interval decidable but is not open interval co-r.e.*

Proof. Let C be a set of natural numbers which is r.e. but not recursive and let $C=\bigcup_s C_s$, where $C_0=\emptyset$, express C as the union of a uniformly computable increasing sequence of recursive sets.

(a) Define K to contain 0 and let $2^{-n-1}\in K$ if and only if $n\notin C$ and to contain $2^{-n-1}+2^{-n-1-s-1}$ as long as $n\notin C^s$. Note that $2^{-n-1}+2^{-n-2}\in K$ for all n and that, if $2^{-n-1}+2^{-n-1-s-2}\in K$, then $2^{-n-1}+2^{-n-1-s-1}\in K$. K is not closed interval r.e. since

$$n\notin C \Leftrightarrow [2^{-n-1}-2^{-n-3},2^{-n-1}]\cap K\neq\emptyset$$

so that K being CIr.e. would imply that C is recursive.

K is open interval decidable since we can test whether $(p,q)\cap K=\emptyset$ by the following procedure. First, check to see whether (p,q) contains any points of the form $2^{-n-1}+2^{-n-2}$, in which case $(p,q)\cap K\neq\emptyset$. If not, then we must have $(p,q)\subset(2^{-n-2}-2^{-n-3},2^{-n-1}+2^{-n-2})$ for some n. Then if $q\leqslant 2^{-n-1}$, then $(p,q)\cap K=\emptyset$. Otherwise let s be the smallest t such that $2^{-n-1}+2^{-n-1-t-1}\in(p,q)$. Then we have

$$(p,q)\cap K\neq\emptyset \Leftrightarrow 2^{-n-1}+2^{-n-1-s-1}\in K \Leftrightarrow n\notin C^s.$$

(b) Define K to contain 0 plus 2^{-n-1} for all n. Moreover, we put $2^{-n-2}+2^{-n-2-s-1}$ in K if and only if $n\in C^{s+1}-C_s$. K is not open interval co-r.e. since

$$n\notin C \Leftrightarrow (2^{-n-2},2^{-n-1})\cap K=\emptyset$$

so that K being OIco-r.e. would imply that C is recursive.

K is closed interval decidable since we can test whether $[p,q]\cap K=\emptyset$ by the following procedure. First, check to see whether $[p,q]$ contains any points of the form 2^{-n-1}, in which case $[p,q]\cap K\neq\emptyset$. If not, then we must have $(p,q)\subset(2^{-n-2},2^{-n-1})$ for some n. In this case, let s be the largest such that $2^{-n-2}+2^{-n-2-s-1}\in[p,q]$. Then we have

$$[p,q]\cap K\neq\emptyset\Leftrightarrow n\in C^{s+1}. \qquad \square$$

We note that Weihrauch [36] also gives examples of recursive closed sets, one of which is not closed interval r.e. and the other not open interval co-r.e.

Again Theorem 3.4 implies that there are only three possible ways that a closed set K of $\Re$ can have exactly two of our properties. Our next theorem will show that all three of these possibilities can be realized.

Theorem 3.6. (a) *There is a closed set K of reals which is open interval r.e. and closed interval r.e., but is neither closed interval co-r.e. nor open interval co-r.e.*

(b) *There is a closed set K of reals which is closed interval co-r.e. and open interval co-r.e., but is neither open interval r.e. nor closed interval r.e.*

(c) *There is a closed set K of reals which is recursive but is neither open interval co-r.e. nor closed interval r.e.*

Proof. Let C be a set of natural numbers which is recursively enumerable but not recursive. (a) Let $K=\{0\}\cup\{2^{-n-1}:n\in C\}$. To check whether $\overline{I_w}\cap K\neq\emptyset$, where $I_w=[p,q]$, there are two cases. If $p\leqslant 0$ and $q>0$, then automatically $\overline{I_w}\cap K\neq\emptyset$. Otherwise, if $p>0$, just let $\{n_1,\dots,n_t\}$ list the finite set of elements of $\overline{I_w}$ of the form 2^{-n-1} and check that $n_i\in C$ for some i. Thus K is CIr.e and hence K is automatically OIr.e.

On the other hand, suppose by way of contradiction that K is CIco-r.e. Then for any n, we have

$$n\notin C\Leftrightarrow[2^{-n-1},2^{-n-1}+2^{-n-2}]=\emptyset,$$

which would make C also co-r.e. and therefore recursive. Thus K is not CIco-r.e. and hence K is automatically not OIco-r.e.

(b) Let $K=\{0\}\cup\{2^{-n-1}:n\notin C\}$. To check whether $I_w\cap K=\emptyset$ where $I_w=(p,q)$ note that if $p\leqslant 0$ and $q>0$, then $I_w\cap K\neq\emptyset$ and otherwise, if $p>0$, there is a finite set $\{2^{-n_1},\dots,2^{-n_t}\}$ of elements of the form 2^{-n-1} in (p,q) so that $(p,q)\cap K=\emptyset$ if and only if $n_i\in C$ for each i. It follows that K is OIco-r.e. and hence K is automatically CIco-r.e.

On the other hand, suppose by way of contradiction that K is OIr.e. Then for any n, we have

$$n\notin C\Leftrightarrow(2^{-n-1}-2^{-n-2},2^{-n-1}+2^{-n-2})\neq\emptyset,$$

which would make C also co-r.e. and therefore recursive. Thus K is not OIr.e. and hence K is automatically not CIr.e.

(c) Let K_1 and K_2 be given by parts (a) and (b) of Theorem 3.5. Now define the desired set K to be $K_1 \cup \{x + \frac{1}{2} : x \in K_2 \cap [0, \frac{1}{2}]\}$. Since both K_1 and K_2 are recursive closed sets, it follows that K is also a recursive closed set. K is not closed interval r.e. since K_1 is not and K is not open interval co-r.e. since K_2 is not. □

Finally, we observe that there are two ways in which a set can satisfy exactly one of the properties. (Of course, it is clear that we may have a closed set which is not effectively closed in any sense.)

Theorem 3.7. (a) *There is a closed set K of reals which is open interval r.e. but is neither closed interval r.e. nor closed interval co-r.e.*

(b) *There is a closed set K of reals which is closed interval co-r.e. but is neither open interval r.e. nor open interval co-r.e.*

Proof. (a) By Theorem 3.6, let K_1 and K_2 be closed subsets of the real interval such that K_1 is open interval r.e. and closed interval r.e. but neither open interval co-r.e. nor closed interval co-r.e. and K_2 is closed interval co-r.e. and open interval r.e but not closed interval r.e. nor open interval co-re. We define the desired set K to be $K_1 \cup \{x + \frac{1}{2} : x \in K_2 \cap (0, 1]\}$. Since both K_1 and K_2 are open interval r.e., it follows that K is also open interval r.e. K is not closed interval r.e. since K_2 is not and K is not closed interval co-r.e. since K_1 is not.

(b) Proceed as in (a) where K_1 is the same and K_2 is open interval co-r.e. but is not open interval r.e. □

We can make a sharper distinction between the weaker notions of open interval r.e. and closed interval co-r.e. and the other notions by using index sets. Recall that given a standard enumeration of the partial recursive functions $\phi_0, \phi_1, \ldots$, we obtain a standard enumeration of all r.e. sets of natural numbers $W_0, W_1, \ldots$ by setting $W_e = \{n : \phi_e(n)\downarrow\}$. An index set is a set A of natural numbers such that whenever $W_a = W_b$, then $a \in A \Leftrightarrow b \in A$. An important type of question in computability theory is to determine the complexity of various index sets. Index sets which have been studied include the set *Fin* of indices for finite r.e. sets and the similar index sets *Inf*, *Cof* and *Coinf* for infinite, cofinite and coinfinite sets, as well as the set *Rec* of indices for recursive sets. We say that a given set A of natural numbers is Π^0_n (Σ^0_n) *complete* if it is a Π^0_n (Σ^0_n) set and for any other Π^0_n (Σ^0_n) set B, there is a recursive function ϕ such that, for all x, $x \in B \Leftrightarrow \phi(x) \in A$.

For our purposes, we shall need three index set results whose proofs can be found in [29].

Proposition 3.8. (a) *Inf is Π^0_2 complete.*

(b) *Coinf is Π^0_3 complete.*

(c) *Rec is Σ^0_3 complete.*

For more details about index sets, see Soare's book [29] or Cenzer and Remmel [6, 8].

We now consider index sets for closed sets which are CIco-r.e. and OIr.e. Recall from Theorem 3.1 that K is CIco-r.e if and only if the complement of K is the union of a recursively enumerable set of intervals. That is, K is a Π^0_1 class in the language of computability theory. Index sets for Π^0_1 classes of reals were studied by Cenzer and Remmel [8]. An enumeration of the CIco-r.e. closed subsets of $[0,1]$ may be given as follows. Let

$$K_e = [0,1] - \bigcup\{I_w : w \in W_e\}.$$

Thus

$$x \in K_e \Leftrightarrow (\forall w)(x \in I_w \rightarrow w \notin W_e).$$

Note that using compactness, we have that

$$\overline{I_w} \cap K_e = \emptyset \Leftrightarrow \exists(w_0,\dots,w_k)[\overline{I_w} \subset I_{w_0} \cup \ldots \cup I_{w_k} \,\&\, (\forall i < k+1) w_i \in W_e].$$

For an enumeration of the OIr.e. sets, note that if a set $V = \{w : I_w \cap K \neq \emptyset\}$ for some OIr.e. closed set of [0,1], then V has the following properties. First, if $I_w \subseteq I_z$ and $w \in V$, then $z \in V$. Second, if $w \in V$, then there is a proper subset I_z of I_w such that $w \in V$. Finally, if $z \in V$, and $I_z \subset I_v \cup I_w$, then either $v \in V$ or $w \in V$. On the other hand, if V satisfies these three properties, then $V = \{w : I_w \cap L \neq \emptyset\}$, where $L = [0,1] - \bigcup_{z \notin V} I_z$. This leads us to the following. For each e, let $L_e = [0,1] - \bigcup_{z \notin W_e} I_z$.

Theorem 3.9. *The set OICLOSED, which is the set of all e such that* $W_e = \{w : I_w \cap L_e \neq \emptyset\}$ *is a* Π^0_2 *complete set.*

Proof. As suggested above, we claim that e is in *OICLOSED* if and only if it satisfies the following (where we combine the first two properties into one):

(i) $(\forall w)[w \in W_e \Leftrightarrow (\exists z)(\overline{I_z} \subset I_w \,\&\, z \in W_e)]$.

(ii) $(\forall n, w, z)[w = \langle n, z_1, \dots, z_n\rangle \,\&\, (I_z \subset I_{z_1} \cup \cdots \cup I_{z_n} \,\&\, z \in W_e) \rightarrow (z_1 \in W_e \vee \cdots \vee z_n \in W_e)]$.

That is, as we observed above, if $W_e = \{w : I_w \cap L_e \neq \emptyset\}$, then W_e satisfies properties (i) and (ii). Next suppose that W_e satisfies properties (i) and (ii). It is clear that if $w \notin W_e$, then $L_e \cap I_w = \emptyset$. Now suppose that $L_e \cap I_w = \emptyset$. Then let $\overline{I_v} \subset I_w$. Then $\overline{I_v} \subseteq \bigcup_{z \notin W_e} I_z$. By compactness, there is a finite set $\{z_1, \dots, z_n\}$ with each $z_i \notin W_e$ such that $\overline{I_v} \subset I_{z_1} \cup \ldots \cup I_{z_n}$. By property (ii), it follows that $z \notin W_e$. Thus whenever $\overline{I_v} \subset I_w$, $v \notin W_e$ and hence by property (i), $v \notin W_e$. Thus $W_e = \{w : I_w \cap L_e \neq \emptyset\}$.

Since properties (i) and (ii) are Π^0_2 conditions, it follows that *OICLOSED* is a Π^0_2 set. To show that *OICLOSED* is Π^0_2 complete, we need only show that *Inf* is 1 : 1 reducible to *OICLOSED*. Given W_e, let $W_{f(e)}$ be the set of all w such that $\frac{1}{2} \in I_w$ and there is an n such that W_e has at least n elements and $|I_w| \geqslant 1/2^{n+2}$. It is easy to see that if W_e is infinite, then $W_{f(e)} = \{w : \frac{1}{2} \in I_w\}$. Thus $L_{f(e)} = \{\frac{1}{2}\}$

and $W_{f(e)} = \{w : I_w \cap L_{f(e)} \neq \emptyset\}$ and hence $f(e) \in OICLOSED$. Now if W_e is finite, say $|W_e| = k$, then all w such that $|I_w| < 1/2^{k+2}$ will not be in $W_{f(e)}$ so that $L_{f(e)} = \emptyset$. But then clearly $W_{f(e)} \neq \{w : I_w \cap L_{f(e)} \neq \emptyset\}$ so that $f(e) \notin OICLOSED$. Thus f shows that INF is $1:1$ reducible to $OICLOSED$ and hence $OICLOSED$ is Π_2^0 complete. □

Our goal is to show that the index sets $\{e : e \in OICLOSED \,\&\, L_e$ is CIr.e.$\}$ and $\{e : K_e$ is OIco-r.e.$\}$ are Σ_4^0 complete and that $\{e : K_e$ is CI-decidable$\}$ and $\{e : L_e \in OICLOSED$ & is OI-decidable$\}$ are Π_3^0 complete. To prove these results, we need to prove some new index set results plus a some coding results. We start with the new index set results.

A Π_2^0 set P may be defined from an r.e. set W by having $x \in P \Leftrightarrow (\forall n)\ (\langle x, n\rangle \in W)$. Thus we may define an enumeration of the Π_2^0 sets by defining

$$P_{2,e} = \{x : (\forall n)(\langle x, n\rangle \in W_e)\}.$$

Then an index set for Π_2^0 sets is just a set A such that whenever $P_{2,a} = P_{2,b}$, then $a \in A \Leftrightarrow b \in A$.

We can also enumerate the Σ_2^0 sets by defining

$$S_{2,e} = \omega - P_{2,e}$$

and similarly defining the notion of an index set for Σ_2^0 sets.

Theorem 3.10. *The following index sets are all Σ_4^0 complete*:
(a) $\{e : P_{2,e}$ is recursive$\}$;
(b) $\{e : P_{2,e}$ is *r.e.*$\}$;
(c) $\{e : P_{2,e}$ is $\Sigma_2^0\}$;
(d) $\{e : S_{2,e}$ is recursive$\}$;
(e) $\{e : S_{2,e}$ is *r.e.*$\}$;
(f) $\{e : S_{2,e}$ is $\Pi_2^0\}$;

Proof. It is easy to check that each index set is indeed a Σ_4^0 set. For example, $P_{2,e}$ is Σ_2^0 if and only if

$$(\exists a)(\forall x)[(\forall n)(\langle x, n\rangle \in W_e) \Leftrightarrow \neg((\forall n)(\langle x, n\rangle \in W_a))]$$

and $P_{2,e}$ is r.e. if and only if

$$(\exists a)(\forall x)[(\forall n)(\langle x, n\rangle \in W_e) \Leftrightarrow x \in W_a].$$

For the completeness of (a)–(c), let A be an arbitrary Σ_4^0 set. The fact that $Coinf$ is a complete Π_3^0 set implies that there is a recursive function f such that

$$x \in A \Leftrightarrow (\exists n)(f(x, n) \in Coinf).$$

Given x, we shall uniformly construct a Π_2^0 set B^x such that B^x is recursive if $\exists n(f(x, n) \in Coinf)$ and B^x is not Σ_2^0 if $\forall n(f(x, n) \notin Coinf)$. Let $T(e, n, c)$ be the recursive predicate which says that c codes a computation of eth Turing machine on input n which gives an output $\phi_e(n)$. Our indexing of Π_2^0 predicates is of the form $x \in P_{2,e} \Leftrightarrow \forall n \exists c T(e, \langle x,n \rangle, c)$. Moreover, it is easy to see that if we express a Π_2^0 set B in the form $x \in B \Leftrightarrow \forall n \exists m R(n, m, x)$ for some recursive predicate $R(m, n, x)$, then we can uniformly find a Π_2^0 index for B by writing out the predicate R in terms of the computation of some Turing machine using the predicate $T(e, n, c)$. Thus our construction of the B^x's will imply that there exists a recursive function g such that $P_{2,g(x)} = B^x$ for all x so that

$$\begin{aligned} x \in A &\Leftrightarrow P_{2,g(x)} \text{ is recursive} \\ &\Leftrightarrow P_{2,g(x)} \text{ is r.e.} \\ &\Leftrightarrow P_{2,g(x)} \text{ is } \Sigma_2^0. \end{aligned}$$

This given, we can define B^x as follows. Let $B^{x,e} = \{\langle e,n \rangle : \langle e,n \rangle \in B^x\}$. For all i, if $W_{f(x,i)}$ is cofinite, then we let n_i be the largest n such that $n \notin W_{f(x,i)}$ if there is such an n and $n_i = -1$ otherwise.

1. Let $\langle 0,n \rangle \in B^{x,0} \Leftrightarrow (\forall m \geqslant n)(\exists c)(T(f(x,0),m,c)$. Thus if $W_{f(x,0)}$ is coinfinite, then $B^{x,0} = \emptyset$ and if $W_{f(x,0)}$ is cofinite, then $B^{x,0} = \{\langle 0,n \rangle : n > n_0\}$.
2. For $e > 0$, let $\langle 2e,n \rangle \in B^{x,2e} \Leftrightarrow (\forall m \geqslant n)(\forall j \leqslant e)(\exists c)(T(f(x,j),m,c)$. Thus if there is a $j \leqslant e$ such that $W_{f(x,j)}$ is coinfinite, then $B^{x,2e} = \emptyset$. On the other hand, if $W_{f(x,i)}$ is cofinite for all $i \leqslant e$, then
$$B^{x,2e} = \{\langle 2e,n \rangle : n > \max\{n_0, \ldots, n_e\}\}.$$
3. For $e \geqslant 0$, let $\langle 2e+1,n \rangle \in B^{x,2e+1}$ if and only if
   (a) $(\forall m \geqslant n)(\forall j \leqslant e)(\exists c)(T(f(x,j),m,c)$ and
   (b) $(\forall j \leqslant n)(j \in P_{2,e}) \vee \exists y,t < m \forall c(\neg T(e, \langle t,y \rangle, c))$.
   Once again, if there is an $i \leqslant e$ such that $W_{f(x,e)}$ is coinfinite, then $B^{x,2e+1} = \emptyset$. On the other hand, suppose $W_{f(x,i)}$ is cofinite for all $i \leqslant e$. Then $B^{x,2e+1} = \{\langle 2e+1,n \rangle : n > \max\{n_0, \ldots, n_e\}\}$ if $S_{2,e} = \emptyset$. If $S_{2,e} \neq \emptyset$, then there is a least $y \in S_{2,e}$ and hence $\exists t \forall c(\neg T(e, \langle t,y \rangle, c))$. But in this case neither $y \in P_{2,e}$ nor $(\exists z,t < y)(\forall c)(\neg T(e, \langle t,y \rangle, c))$ holds. Thus $\langle 2e+1,y \rangle \notin B^{x,2e+1}$. Thus if $W_{f(x,i)}$ is cofinite for all $i \leqslant e$, then our construction ensures that $S_{2,e} \neq \{y : y \in B^{x,2e+1}\}$.

It is easy to see that our definition ensures that $\{B^x\}_{x \in \omega}$ is a uniform sequence of Π_2^0 sets. Moreover, if $\exists n(f(x,n) \in \mathit{Coinf})$, then for all but finitely many e, $B^{x,e} = \emptyset$ and if $B^{x,e} \neq \emptyset$, then $B^{x,e}$ is cofinite so that B^x is a recursive set. Now suppose that for all n, $W_{f(x,n)}$ is cofinite and that B^x is Σ_2^0. Then we can construct a uniform sequence of Σ_2^0 sets $\{C_e\}_{e \in \omega}$ by setting $C_e = \{y : \langle 2e+1,y \rangle \in B^x\}$. In particular, there will be a recursive function h such that $C_e = S_{2,h(e)} = \{y : \exists t(\langle t,y \rangle \notin W_{h(e)}\}$. But then by the recursion theorem, there is an e such that $W_e = W_{h(e)}$ so that $S_{2,e} = S_{2,h(e)} = \{y : \langle 2e+1,y \rangle \in B^{x,2e+1}\}$. Our construction ensures that in this case, $S_{2,e} \neq \{y : \langle 2e+1,y \rangle \in B^{x,2e+1}\}$ for all e so that B^x cannot be Σ_2^0.

For the completeness of (d)–(f), once again assume A is Σ_4^0 set such that

$$x \in A \quad \Leftrightarrow \quad (\exists n) f(x,n) \in \mathit{Coinf}.$$

Given x we shall uniformly construct a Σ_2^0 set C^x such that C^x is recursive if $\exists n W_{f(x,n)} \in \mathit{Coinf}$ and C^x is not Π_2^0 if $\forall W_{f(x,n)} \notin \mathit{Coinf}$. Our construction of the B^x's will imply that there exists a recursive function g such that $S_{2,g(x)} = B^x$ for all x so that

$$\begin{aligned} x \in A &\Leftrightarrow S_{2,g(x)} \text{ is recursive} \\ &\Leftrightarrow S_{2,g(x)} \text{ is r.e.} \\ &\Leftrightarrow S_{2,g(x)} \text{ is } \Pi_2^0. \end{aligned}$$

In this case, we can simply let $C^x = \omega - B^x$ for all x. By our previous construction, $\{C^x\}_{x\in\omega}$ is a uniform sequence of Σ_2^0 sets such that if $x \in A$, the C^x is recursive since B^x is recursive. Similarly, if $x \notin A$, then B^x is not Σ_2^0 and C^x is not Π_2^0. $\square$

Next we shall prove our required coding result.

Theorem 3.11. *There exist recursive functions f, g, h, k, and l such that $g(e)$ and $k(e)$ are in OICLOSED for all e and such that*
(a) *$P_{2,e}$ is r.e. if and only if $K_{f(e)}$ is OIco-r.e.*
(b) *$P_{2,e}$ is r.e. if and only if $L_{g(e)}$ is CIr.e.*
(c) *W_e is recursive if and only if $K_{h(e)}$ is CIr.e. so that W_e is recursive if and only if $K_{h(e)}$ is closed interval decidable.*
(d) *W_e is recursive if and only if $L_{k(e)}$ is OIco-r.e.*
(e) *$K_{l(e)}$ is always CI and OI decidable and W_e is recursive and only if $\{q \in \mathbb{Q} : q \notin K_{l(e)}\}$ is recursive.*

Proof. (a) First one can use the proof of the Π_2^0 completeness of *Inf* to show that there is a recursive function ϕ such that

$$x \in P_{2,e} \quad \Leftrightarrow \quad W_{\phi(e,x)} \text{ is infinite.}$$

This given, we construct $K_{f(e)}$ as follows. Let B_e be the following r.e. set of intervals. For each $n \in \omega$, we put

$$\left(\frac{1}{2n+1} + \frac{1}{2^{n+k+3}}, \frac{1}{2n} - \frac{1}{2^{n+k+3}}\right)$$

into B_e if and only if $|W_{\phi(e,n)}| \geqslant k$. Clearly, there is a recursive function f such that $W_{f(e)} = B_e$ for all e.

Now fix e. First observe that our construction ensures that 0 and all elements of the form $1/2^n$ for $n \in \omega$ are in $K_{f(e)}$. Thus if $I_w \cap K_{f(e)} = \emptyset$ where $I_w = (p,q)$, then we must have $(p,q) \subseteq (1/2^{n+1}, 1/2^n)$ for some n. Next observe that if $n \in P_{2,e}$, then $W_{\phi(e,n)}$

is infinite and hence

$$\left(\frac{1}{2n+1}+\frac{1}{2^{n+k+3}},\frac{1}{2n}-\frac{1}{2^{n+k+3}}\right)\cap K_e=\emptyset$$

for all k so that

$$\left(\frac{1}{2^{n+1}},\frac{1}{2^n}\right)\cap K_{f(e)}=\emptyset.$$

If $n\notin P_{2,e}$, then

$$\left[\frac{1}{2^{n+1}},\frac{1}{2^n}\right]-\left(\frac{1}{2^{n+1}}+\frac{1}{2^{n+k+3}},\frac{1}{2n}-\frac{1}{2^{n+k+3}}\right)=\left[\frac{1}{2^{n+1}},\frac{1}{2^n}\right]\cap K_{f(e)},$$

where $k=|W_{\phi(e,n)}|$. Thus $P_{2,e}=\{n:(1/2^{n+1},1/2^n)\cap K_{f(e)}=\emptyset\}$. Hence if $K_{f(e)}$ is OIco-r.e., then $P_{2,e}$ will be r.e. Vice versa, if $P_{2,e}$ is r.e., then we can enumerate the set $C_e=\{w:I_w\cap K_{f(e)}=\emptyset\}$ as follows. For each n, place an interval $(p,q)\subseteq(1/2^{n+1},1/2^n)$ into C_e if either $n\in P_{2,e}$ or if $|W_{\phi(e,n)}|\geqslant k$ and

$$(p,q)\subseteq\left(\frac{1}{2n+1}+\frac{1}{2^{n+k+3}},\frac{1}{2n}-\frac{1}{2^{n+k+3}}\right).$$

Thus $P_{2,e}$ is r.e. if and only if $K_{f(e)}$ is OIco-r.e.

(b) Again let ϕ be a recursive function such that

$$x\in P_{2,e}\iff W_{\phi(e,x)}\text{ is infinite.}$$

Given e, we construct a closed set L^e as follows. First put 0 into L^e and for all $n\in\omega$, put $1/2^{n+1}+1/2^{n+k+2}$ into L^e if and only if $|W_{\phi(e,n)}|\geqslant k$. Finally, we put $1/2^{n+1}$ into L^e if $W_{\phi(e,n)}$ is infinite. First we claim that L^e is OIr.e. That is, if I_w contains some element of the form $3/2^{n+2}$, then $I_w\cap L^e\neq\emptyset$. Similarly if $I_w\subseteq(3/2^{n+2},1/2^n)$ for some n, then $I_w\cap L^e=\emptyset$. Thus we are reduced to considering intervals I_w of the form (p,q) where $3/2^{n+3}<p$ and $1/2^{n+1}<q<3/2^{n+2}$. This given, there will be a least k such that $1/2^{n+1}+1/2^{n+k+2}\in(p,q)$ in which case $I_w\cap L^e\neq\emptyset$ if and only if $|W_e|\geqslant k$, which is an r.e. condition. It follows that L^e is OIr.e. and there is recursive function g such that $W_{g(e)}=\{w:I_w\cap L^e\neq\emptyset\}$ so that $L^e=L_{g(e)}$ for all e.

Next observe that for all n, $[3/2^{n+3}+1/2^{n+5},1/2^{n+1}]\cap L_{g(e)}\neq\emptyset$ if and only if $W_{\phi(e,n)}$ is infinite. Thus if $L_{g(e)}$ is *CIr.e.*, then $P_{2,e}$ is r.e. Vice versa, suppose that $P_{2,e}$ is r.e. Then for any w, $\overline{I_w}\cap L_{g(e)}\neq\emptyset$ if I_w contains an element of the form $3/2^{n+2}$. Similarly if $\overline{I_w}\subseteq(3/2^{n+2},1/2^n)$ for some n, then $\overline{I_w}\cap L^e=\emptyset$. Thus we are reduced to considering interval I_w of the form (p,q) where $3/2^{n+3}<p$ and $1/2^{n+1}\leqslant q<3/2^{n+2}$. This given, either $q=1/2^{n+1}$, in which case $\overline{I_w}\cap L_{g(e)}\neq\emptyset$ if and only if $n\in P_{2,e}$, or there is a least k such that $1/2^{n+1}+1/2^{n+k+2}\in[p,q]$, in which case $\overline{I_w}\cap L^e\neq\emptyset$ if and only if $|W_e|\geqslant k$. These are both r.e. conditions. Thus if $P_{2,e}$ is r.e., then $L_{g(e)}$ is CIr.e.

For (c), given W_e, let $K^e=\{0\}\cup\{1/2^{n+1}:n\notin W_e\}$. We claim that K^e is always CIco-r.e. That is, if $\overline{I_w}=[p,q]$, then if $p=0$, then $I_w\cap K^e\neq\emptyset$. Otherwise, there are only finitely many elements, $\{1/2^{n_1+1},\ldots,1/2^{n_{p+1}}$ of the form $1/2^{n+1}$ in $[p,q]$. Then

$I_w \cap K^e = \emptyset$ if and only if $\{n_1, \ldots, n_p\} \subseteq W_e$. Thus K^e is always CIco-r.e. and there is a recursive function h such that $K^e = K_{h(e)}$. Moreover, a similar argument shows that $K_{h(e)}$ is CIr.e. if and only if W_e is co-r.e. Thus W_e is recursive if and only if $K_{h(e)}$ is CI decidable.

For (d), given W_e, let $T^e = \{0\} \cup \{1/2^{n+1} : n \in W_e\}$. We claim that T^e is always OIr.e. That is, $I_w \cap T^e \neq \emptyset$ if and only if I_w contains 0 or some element of the form $1/2^{n+1}$ where $n \in W_e$. Thus T^e is always a closed *OIr.e.* set and there is a recursive function k such that $T^e = L_{k(e)}$. Moreover, it is easy to see that if $L_{k(e)}$ is OIco-.r.e. if and only if W_e is co-.r.e. Thus W_e is recursive if and only if $L_{h(e)}$ is OI decidable.

(e) We modify the example from (c) by adding $1/2^{n+1} \pm 1/2^{n+1+s_1}$ to K^e whenever $n \notin W_{e,s}$ where $W_{e,s}$ is the set of elements of W_e that have been enumerated into W_e by the end of stage s. Here is the procedure for checking whether K meets an interval $[p,q]$ and/or (p,q). If $p = 0$, then K meets both intervals. Since we have $1/2^{n+1} + 1/2^{n+2} \in K^e$ for all n, we know that if (p,q) or $[p,q]$ contains an element of the form $1/2^{n+1} + 1/2^{n+2}$, then the intervals have non-empty intersection with K^e. Thus, we may assume that $1/2^{n+2} + 1/2^{n+3} \leqslant p < q \leqslant 1/2^{n+1} + 1/2^{n+2}$ for some n and, for a closed interval that the inequalities are strict. Then K meets the interval if and only if $n \notin W_{e,s}$, where s is the least such that $1/2^{n+1} + 1/2^{n+1+s+1}$ or $1/2^{n+1} - 1/2^{n+1+s+1}$ is in the interval. Thus K^e is always both *CI* and *OI* decidable, so that there is a recursive function l such that $K_{l(e)} = K^e$. Of course, $K_{l(e)}$ is recursive set of rationals if and only if W_e is recursive. Thus a closed set of rationals K can be CI and OI decidable without being recursive. □

We can now combine our two previous results to prove the following.

Theorem 3.12. *(a) $S^1 = \{e : K_e$ is OIco-r.e.$\}$ is Σ_4^0 complete.*

(b) *$S^2 = \{e : L_e \in OICLOSED \,\&\, L_e$ is CIr.e.$\}$ is Σ_4^0 complete.*

(c) *$S^3 = \{e : K_e$ is closed interval decidable$\}$ is Σ_3^0 complete.*

(d) *$S^4 = \{e : L_e \in OICLOSED \,\&$ is open interval decidable$\}$ is Σ_3^0 complete.*

Proof. First it is easy to see that S^1 and S^2 are Σ_4^0 sets by simply writing out the definitions. That is

$$e \in S^1 \quad \Leftrightarrow \quad (\exists f)(\forall w)(I_w \cap K_e = \emptyset \quad \Leftrightarrow \quad w \in W_f).$$

Note that $I_w \cap K_e = \emptyset$ if and only if for all $x \in I_w$, there exists a w_1 in W_e such that $x \in I_{w_1}$. Thus both $w \in W_f$ and $I_w \cap K_e = \emptyset$ are Π_2^0 conditions so that S^1 is a Σ_4^0 set. Similarly,

$$e \in S^2 \quad \Leftrightarrow \quad e \in OICLOSED \,\&\, (\exists f)(\forall w)(\overline{I_w} \cap L_e \neq \emptyset \quad \Leftrightarrow \quad w \in W_f).$$

Now $e \in OICLOSED$ is a Π_2^0 condition. Moreover, if $e \in OICLOSED$, then $\overline{I_w} \cap L_e = \emptyset$ if and only if there exists $w_1, \ldots, w_k$ not in W_e such that $\overline{I_w} \subseteq \bigcup_{i=1}^k I_{w_i}$. Thus $I_w \cap L_e \neq \emptyset$ is Π_2^0 condition so that S^2 is a Σ_4^0 set.

To see that S^3 is a Σ^0_3 set, let p be a recursive function such that $W_{p(e)}$ is the set of all w such that there exist $w_1, \ldots, w_k \in W_e$ with $\overline{I_w} \subseteq \bigcup_{i=0}^{k} I_{w_i}$. Then $e \in S^3$ if and only if $W_{p(e)}$ is recursive which is a Σ^0_3 condition. Finally, to see that S^4 is a Σ^0_3 set, note that $e \in S^4$ if and only if $e \in OICLOSED$ and W_e is recursive set which is again a Σ^0_3 condition.

For the Σ^0_4 completeness of S^1 and S^2, note that the recursive functions f and g constructed in Theorem 3.11 show that the complete Σ^0_4 set $\{e : P_{2,e}$ is r.e$\}$ is $1:1$ reducible to S^1 and S^2. For the Σ^0_3 completeness of S^3 and S^4, note that the recursive functions h and k constructed in Theorem 3.11 show that the complete Σ^0_3 set Rec is $1:1$ reducible to S^3 and S^4. □

Finally, we should remark that the same index set results clearly hold for the reals $\Re$. For both $\{0,1\}^\omega$ and ω^ω, every basic interval is clopen so that there is no distinction between OIr.e. and CI.re. closed set and OIco-r.e. and CIco-r.e. closed sets. The analogue of Theorem 3.12 does hold and follows from index sets results on Π^0_1 classes proved by the authors in [6].

4. Graphs of continuous functions

For any Hausdorff topological spaces X and Y, the graph G of a continuous function F from X to Y must be a closed subset. It is a natural question for one of our four spaces X, whether the graph of a computably continuous function from X^n to X will be an effectively closed set and in what sense. We will determine the solution to this question for each of the spaces $\Re$, $[0,1]$, $\{0,1\}^\omega$ and ω^ω.

The reverse problem of whether a function with a closed graph is necessarily continuous is a general version of the well-known "automatic continuity" problem, usually studied for linear functions. In the spirit of Brattka and Weihrauch [2], we shall consider the effective analogue of this problem for effectively closed graphs. There are two versions of each problem. We first ask whether a function $F : X \to X$ with an effectively closed graph is necessarily computably continuous (or semicontinuous). Then we ask the same question when F is assumed to be continuous. These problems are considered for each of our four spaces $\Re$, $[0,1]$, $\{0,1\}^\omega$ and ω^ω. As in the previous section, the latter space leads to the most interesting results.

Theorem 4.1. *Let $F : X^n \to X$ be a computably continuous function where X is one of the spaces $\Re$, $[0,1]$, $\{0,1\}^\omega$ and ω^ω. Then the graph of F is an OIr.e. closed set.*

Proof. Let $F : X \to X$ be a computably continuous function, f be a computable representing function for F, and G be the graph of F. We claim that

$$G \cap (I_a \times I_b) \neq \emptyset \iff (\exists c)[I_c \subset I_a \,\&\, \overline{I_{f(c)}} \subset I_b].$$

To see this, suppose first that there is some pair $(x, y) \in G \cap (I_a \times I_b)$. Then $x \in I_a$ and $y = F(x) \in I_b$. Let $I_{e_1}, I_{e_2}, \ldots$ be a decreasing sequence of subintervals of I_a such that $\bigcap_k I_{e_k} = \{x\}$. Since f is a representing function for F, we have that for all k, $I_{f(e_k)} \supset I_{f(e_{k+1})}$ and $\bigcap_k I_{f(e_k)} = \{y\}$. It follows that some $I_{f(e_k)} \subseteq I_b$. Note that this does not depend on compactness, but only on metrizability. That is, let $B(y,r) = \{x \in X : d(x, y) < r\}$ be the open ball of radius r about y. Then each interval $I_{f(e_k)} \subseteq B(y;r_k)$ for some decreasing sequence $\{r_k\}$ of rationals with limit 0. On the other hand, there is some rational r such that $B(y;r) \subset I_b$ and hence some k such that

$$I_{f(e_k)} \subset B(y;r_k) \subset B(y:r) \subset I_b.$$

On the other hand, suppose that $I_c \subset I_a$ and $\overline{I_{f(c)}} \subset I_b$. Then for *any* $x \in I_c$, we have $F(x) \in I_b$, so that $G \cap (I_a \times I_b)$ is non-empty. □

We need local compactness to show that the graph of a computably continuous function is also CIco-.r.e.

Theorem 4.2. *Let $F : X^n \to X$ be a computably continuous function, where X is either $\Re$, $[0,1]$ or $\{0,1\}^\omega$. Then the graph of F is a CIco-r.e. closed set.*

Proof. With F, G and f as in the previous proof, we claim that $G \cap \overline{I_a \times I_b} = \emptyset$ if and only if

$$(\exists a_1,\ldots,a_k)[\overline{I_a} \subseteq I_{a_1} \cup I_{a_2} \cup \cdots \cup I_{a_k} \,\&\, (\forall i \leqslant k)((I_{a_i} \times I_{f(a_i)}) \cap G = \emptyset)]$$

and hence G is CIco-r.e.

To prove our claim, suppose first that $G \cap \overline{I_a \times I_b}$ is non-empty and contains some element (x,y) such that $y = F(x)$. Then for any $a_1,\ldots,a_k$ with $\overline{I_a} \subset I_{a_1} \cup \cdots \cup I_{a_k}$, we must have $x \in I_{a_j}$ for some j, so that $y \in I_{f(a_j)}$. It follows that $(x,y) \in (I_{a_i} \times I_{f(a_i)}) \cap G$ so that the right-hand condition cannot hold.

On the other hand, suppose that $G \cap \overline{I_a \times I_b} = \emptyset$. Then for any element $x \in \overline{I_a}$, $F(x) \notin \overline{I_b}$. It follows that there is some interval $I_{a(x)}$ containing x such that $\overline{I_{f(a(x))}} \cap \overline{I_b} = \emptyset$. Since the compact set $\overline{I_a}$ is covered by $\{I_{a(x)} : x \in \overline{I_a}\}$, there must be a finite subset $I_{a_1},\ldots,I_{a_k}$ of those intervals which cover $\overline{I_a}$. □

Corollary 4.3. *Let $F : X^n \to X$ be a computably continuous function, where X is one of the spaces $\Re$, $[0,1]$, and $\{0,1\}^\omega$. Then the graph of F is a recursive closed set.*

We saw in Section 3 that a recursive closed set need not be open interval co-r.e. or closed interval r.e. The following examples show that these types of results extend to graphs of computably continuous functions.

Theorem 4.4. *Let X be either $\Re$ or $[0,1]$.*

(a) *There is a computably continuous real function $F : X \to X$ such that the graph G of F is closed interval decidable but is not OIco-r.e.*

(b) *There is a computably continuous real function $F : X \to X$ such that the graph G of F is open interval decidable but is not CIr.e.*

(c) *There is a computably continuous real function* $F : X \to X$ *with a graph* G *which is neither OIco-r.e. nor CIr.e.*

Proof. Let C be a non-recursive r.e. set of natural numbers. Let g be a $1:1$ recursive function whose range is C and let $C_s = \{g(0), \ldots, g(s)\}$ for all s.

(a) Define $F : [0,1] \to [0,1]$ as follows (F can be extended to $\Re$ by setting $F(x) = 0$ for all other $x \notin [0,1]$). Let $F(0) = 0$ and let $F(2^{-n}) = 0$ for all n. Within $(2^{-n-1}, 2^{-n})$, there are two possibilities. If $n \notin C$, let $F(x) = 0$ for all x. If $n \in C^{s+1} - C^s$, let the graph of F consist of two line segments, the first from $\langle 1/2^{n+1}, 0\rangle$ to $\langle 1/2^{n+1} + 3/2^{n+2}, 1/2^{n+1+s+1}\rangle$ and the second from $\langle 1/2^{n+1} + 3/2^{n+2}, 1/2^{n+1+s+1}\rangle$ to $\langle 1/2^n, 0\rangle$. To compute $F(x)$ within $1/2^s$, it suffices to locate x in an interval $[1/2^{n+1}, 1/2^n]$ and check whether $n \in C_{s+1}$. If $n \notin C_{s+1}$, then $0 \leqslant F(x) \leqslant 1/2^{s+2}$. If $n \in C_{s+1}$, compute the least t such that $n \in C_{t+1}$. Then we can compute the value of $F(x)$ exactly given the definition above.

To see that the graph G of F is not OIco-r.e., just observe that

$$G \cap ((2^{-n-1}, 2^{-n}) \times (0,1)) = \emptyset \iff n \notin C.$$

To see that F is closed interval decidable, suppose that we are given a closed interval $A = \overline{I_a \times I_b} = [p_1, p_2] \times [q_1, q_2]$. Then we can test whether $G \cap A = \emptyset$ as follows. If $q_1 = 0$ and $[p_1, p_2]$ contains either 0 or some element of the form $1/2^n$, then of course $G \cap A \neq \emptyset$. Next suppose that $0 = q_1 < q_2$. Then find the least s such that $1/2^s \leqslant q_2$ and compute C_s. Then if $[p_1, p_2] \subseteq (1/2^{n+1}, 1/2^n)$ and $n \notin C_s$, it is easy to see that $G \cap A \neq \emptyset$. If $n \in C_s$, then we can compute F exactly on $[1/2^{n+1}, 1/2^n]$ so that we can easily decide if $G \cap A = \emptyset$. Finally suppose that $q_1 > 0$, then there is some n such that G cannot possibly meet $[1/2^n, 1] \times [q_1, q_2]$, since we always have $F(x) \leqslant 1/2^{n+2}$ on $[1/2^{n+1}, 1/2^n]$. Thus we may restrict $[p_1, p_2]$ to some finite set of subintervals of the form $[1/2^{n+1}, 1/2^n]$. On each interval $[1/2^{n+1}, 1/2^n]$, there is some s_n such that if $n \notin C_{s_n}$, then G cannot possibly meet A. Finally, if $n \in C_{s_n}$, then we can determine F exactly on $[1/2^{n+1}, 1/2^n]$ and thus test whether G meets A.

(b) Define $F : [0,1] \to [0,1]$ as follows (F can be extended to $\Re$ by setting $F(x) = \frac{1}{2}$ for all other $x \notin [0,1]$). Let $F(0) = \frac{1}{2}$ and let $F(2^{-n}) = \frac{1}{2}$ for all n. Within $(2^{-n-1}, 2^{-n})$, there are two possibilities. If $n \notin C$, let $F(x) = \frac{1}{2}$ for all x. If $n \in C^{s+1} - C^s$, let the graph of F consist of two line segments, the first from $\langle 1/2^{n+1}, \frac{1}{2}\rangle$ to $\langle 1/2^{n+1} + 3/2^{n+2}, \frac{1}{2} - 1/2^{n+1+s+1}\rangle$ and the second from $\langle 1/2^{n+1} + 3/2^{n+2}, \frac{1}{2} - 1/2^{n+1+s+1}\rangle$ to $\langle 1/2^n, \frac{1}{2}\rangle$. To compute $F(x)$ within $1/2^s$, it suffices to locate x in an interval $[1/2^{n+1}, 1/2^n]$ and check whether $n \in C_{s+1}$. If $n \notin C_{s+1}$, then $\frac{1}{2} \geqslant F(x) \geqslant \frac{1}{2} - 1/2^{s+2}$. If $n \in C_{s+1}$, compute the least t such that $n \in C_{t+1}$. Then we can compute the value of $F(x)$ exactly given the definition above.

To see that the graph G of F is not CIr.e., just observe that

$$G \cap \left[\frac{1}{2^{n+1}} + \frac{1}{2^{n+3}}, \frac{1}{2^n} - \frac{1}{2^{n+3}}\right] \times \left[\frac{1}{2}, 1\right] = \emptyset \iff n \notin C.$$

To see that F is open interval decidable we need only show that G is OIco-r.e. since it automatically OIr.e. Suppose that we are given a open interval $A=I_a\times I_b$. Then we can test whether $G\cap A=\emptyset$ as follows. If $I_b=(q_1,q_2)$ or $(q_1,1]$ where $q_1\geqslant\frac{1}{2}$, then clearly $G\cap A=\emptyset$. Thus, we can assume that $I_b=[0,q_2)$ or (q_1,q_2) where $0<q_1<\frac{1}{2}$. We now consider various cases.

Case 1: $I_b=[0,q_2)$ where $q_2>\frac{1}{2}$.

In this case is clear that $G\cap A\neq\emptyset$.

Case 2: $I_b=[0,q_2)$ where $q_2=\frac{1}{2}$.

Then consider I_a. If $I_a=[0,p_2)$ where $0<p_2$, then again $G\cap A\neq\emptyset$ since C is infinite so that there will be some $n\in C$ where $(1/2^{n+1},1/2^n)\subseteq[0,p_2)$ and hence $F(x)\in[0,\frac{1}{2})$ for some $x\in(1/2^{n+1},1/2^n)$. Otherwise $I_a=(p_1,p_2)$ where $0<p_1$. Let n the least m such that $1/2^{m+1}<p_1$. In this case, it is easy to see that $G\cap A\neq\emptyset$ if and only if $(p_1,p_2)\cap(1/2^{m+1},1/2^m)\neq\emptyset$ for some $m\in C\cap\{0,\dots,n\}$ which is an r.e. condition.

Case 3: $I_b=[0,q_2)$ where $q_2<\frac{1}{2}$. Then let s be the least t such $1/2^t<\frac{1}{2}-q_2$ and let m be the largest n such that $n\in C_{s+1}$. Then consider I_a. If $I_a\subseteq[0,1/2^m)$, then we know that $G\cap A=\emptyset$ since $F(x)\geqslant\frac{1}{2}-1/2^s>q_2$ for all $x\in[0,1/2^m)$. Otherwise, we need only consider $I_a\cap(1/2^{n+1},1/2^n)$ for $n\leqslant m$. The only way that $G\cap A\neq\emptyset$ is if there is some $n\leqslant m$ with $n\in C_s$ and an $x\in I_b\cap(1/2^{n+1},1/2^n)$ such that $F(x)<q_2$. But we can decide in this is the case since we can explicitly compute $F(x)$ for any $x\in(1/2^{n+1},1/2^n)$ if $n\in C_s$.

Case 4: $I_b=(q_1,q_2)$ where $q_1<\frac{1}{2}$ and $q_2>\frac{1}{2}$. In this case, let s be the least t such that $1/2^t<\frac{1}{2}-q_1$. Now if $I_a=[0,p_2)$ where $p_2>0$, then $G\cap A\neq\emptyset$ since $F(0)=\frac{1}{2}$ so that $\langle 0,\frac{1}{2}\rangle\in G\cap A$. Thus we can assume that $I_a=(p_1,p_2)$ where $0<p_1<p_2$. Then let n be the least m such that $1/2^m\leqslant p_1$. We can assume that there is some $m\leqslant n$ such that $(p_1,p_2)\subseteq(1/2^{m+1},1/2^m)$ since otherwise (p_1,p_2) contains some element of the form $1/2^t$ so that $\langle 1/2^t,\frac{1}{2}\rangle\in G\cap A$. Then if $n\notin C_s$, then we know that $F(x)\geqslant\frac{1}{2}-1/2^s>q_1$ for all $x\in(1/2^{m+1},1/2^m)$ so that $G\cap A\neq\emptyset$. If $m\in C_s$, then we can compute $F(x)$ for $x\in(1/2^{m+1},1/2^m)$ explicitly so that we can decide if $G\cap A\neq\emptyset$.

Case 5: $I_b=(q_1,q_2)$ where $q_1<\frac{1}{2}$ and $q_2=\frac{1}{2}$. In this case, let s be the least t such that $1/2^t<\frac{1}{2}-q_1$. Now if $I_a=[0,p_2)$ where $p_2>0$, then $G\cap A\neq\emptyset$ since C is infinite so that there will be some m such that $m\in C-C_s$ and $(1/2^{m+1},1/2^m)\subseteq[0,p_2)$ in which case $q_1<F(x)<\frac{1}{2}$ for all $x\in$ so that $G\cap A\neq\emptyset$. Thus we can assume that $I_a=(p_1,p_2)$ where $0<p_1<p_2$. Then let n be the least m such that $1/2^m\leqslant p_1$. Then $G\cap A\neq\emptyset$ if and only if there is an $m\in C$ with $m\leqslant n$ such that there is $x\in(p_1,p_2)\cap(1/2^{m+1},1/2^m)$ with $F(x)\in(q_1,q_2)$ which is an r.e. condition since we can explicitly calculate $F(x)$ for $x\in(1/2^{m+1},1/2^m)$ if $m\in C$.

Case 6: $I_b=(q_1,q_2)$ where $0<q_1<q_2<\frac{1}{2}$. In this case, let s be the least t such that $1/2^t<\frac{1}{2}-q_2$. There are only finitely many n such that $n\in C_{s+1}$ and then only possible x such that $F(x)<q_2$ must be in $\bigcup_{n\in C_s}(1/2^{n+1},1/2^n)$. However $F(x)$ is explicitly computable on $\bigcup_{n\in C_s}(1/2^{n+1},1/2^n)$ so that we can easily decide if $G\cap A\neq\emptyset$ in this situation.

(c) Define $F:[0,1]\to[0,1]$ as follows. Let $F(0)=0$ and let $F(1/2^n)=0$ for all n. Within $(1/2^{n+1},1/2^n)$, there are two possibilities. If $n\notin C$, the graph of F consist of two line segments, the first from $\langle 1/2^{n+1}, 0\rangle$ to $\langle 1/2^{n+1} + 3/2^{n+2}, \frac{1}{2}\rangle$ and the second from $\langle 1/2^{n+1} + 3/2^{n+2}, 0\rangle$. If $n \in C^{s+1} - C^s$, then the graph of F again consists of two line segments, one from $\langle 1/2^{n+1}, 0\rangle$ to $\langle 1/2^{n+1} + 3/2^{n+2}, \frac{1}{2} - 1/2^{n+1+s+1}\rangle$ and the second from $\langle 1/2^{n+1} + 3/2^{n+2}, \frac{1}{2} - 1/2^{n+1+s+1}\rangle$ to $\langle 1/2^n, 0\rangle$. It is easy to show that $F(x)$ is computably continuous via the same type of argument that was used in parts (a) and (b).

To see that the graph G of F is not CIr.e., just observe that

$$G \cap \left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \times \left[\frac{1}{2}, 1\right]\right) = \emptyset \quad \Leftrightarrow \quad n \notin C.$$

To see that the graph G is not OIr.e. observe that if $n \notin C$ then $F(5/2^{n+3}) = F(7/2^{n+3}) = \frac{1}{4}$ and hence $F(x) > \frac{1}{4}$ if $x \in (5/2^{n+3}, 7/2^{n+3})$. However if $n \in C$, then $F(5/2^{n+3}) = F(7/2^{n+3}) < \frac{1}{4}$. Thus

$$G \cap \left(\frac{5}{2^{n+3}}, \frac{7}{2^{n+3}}\right) \times \left[0, \frac{1}{4}\right) = \emptyset \Leftrightarrow n \notin C. \qquad \square$$

Theorem 4.4 essentially settles all the questions about which subsets of the four notions of effectively closed sets could be satisfied by a graph of a computably continuous function. That is, such a graph must be both OIr.e. and CIco-r.e. Theorem 4.4 show that there exists such graphs which have either one or none of the other two properties. Of course, the constant 0 function has all four properties.

The situation is a bit more complicated for the space ω^ω. Of course, in ω^ω, the basic intervals are clopen so that for a closed set K, K is OIr.e. if and only if K is CIr.e. and K is OIco-r.e. if and only if K is CIco-r.e. Our next result will show that the graph of a computably continuous function $f : \omega^\omega \to \omega^\omega$ is not always an OIco-r.e. closed set so that the graph of F is not a recursive closed set in the sense of Brattka and Weihrauch.

We say that two sets of natural numbers are *recursively isomorphic* if there is a recursive permutation $\phi : \omega \to \omega$ such that $a \in A \Leftrightarrow \phi(a) \in B$. Given $A, B \subseteq \omega$, we say that A is 1–1 reducible to B there is a $1:1$ recursive function $f : \omega \to \omega$ such that $x \in A \Leftrightarrow f(x) \in B$. We write that $A \leqslant_1 B$ if A is 1-1 reducible to B and we write $A \equiv_1 B$ if $A \leqslant_1 B$ and $B \leqslant_1 A$. It was shown by Myhill that A and B are recursively isomorphic whenever $A \equiv_1 B$. Recall that $\langle , \rangle$ is a recursive $1:1$ pairing function which maps $\omega \times \omega$ onto ω.

Theorem 4.5. *Let C be any infinite recursively enumerable subset of ω and let $D = \{2\langle n, m\rangle + 1 : n \in C \,\&\, m \in \omega\} \cup \{4n : n \in \omega\}$. Then there is a computably continuous function $F : \omega^\omega \to \omega^\omega$ such that $\{c : G \cap I_c \neq \emptyset\}$ is recursively isomorphic to D.*

Proof. Let $h : \omega \to \omega$ be a $1:1$ recursive function whose range is D. We let $D_s = \{h(0), \ldots, h(s)\}$ for all s.

The function F is defined by letting $F(x) = 1^\omega$ on the intervals $I((n, s))$ such that $n \in D_{s+1} - D_s$ and $F(x) = 0^\omega$ on the intervals $I((n, s))$ such that $n \notin D_{s+1} - D_s$. The value of $F(x)$ is completely determined by the first two values of x so that F is computably continuous.

Let $M=\{\langle a,b\rangle : G\cap(I_a\times I_b)\neq\emptyset\}$ where G is the graph of F. Let $g(n)$ be a $1:1$ recursive function such that $I_{g(n)}=I((n))$. Then D is 1–1 reducible to M since

$$n\in D \;\Leftrightarrow\; G\cap(I((n))\times I((1)))\neq\emptyset \;\Leftrightarrow\; \langle g(n),g(1)\rangle\in M.$$

On the other hand, M is also 1–1 reducible to D. That is, we can define a $1:1$ recursive function k such that $m\in M\Leftrightarrow k(m)\in D$ as follows. Given $m=\langle a,b\rangle$, let $\sigma,\tau\in\omega^\omega$ be such that $I_a\times I_b=I(\sigma)\times I(\tau)$. First suppose that $|\sigma|\geqslant 2$. Thus σ is of the form, $\sigma=(n,s,\sigma_3,\ldots,\sigma_p)$ for some n,s. Then $m\in M$ if and only if $I(\sigma)\times I(\tau)\cap G\neq\emptyset$. But by our definition of F, $(I(\sigma)\times I(\tau))\cap G\neq\emptyset$ if and only if $n\in D_{s+1}-D_s$ and $\tau\prec 1^\omega$ or $n\notin D_{s+1}-D_s$ and $\tau\prec 0^\omega$. Thus we let $k(m)=4m\in D$ if $(I(\sigma)\times I(\tau))\cap G\neq\emptyset$ and $k(m)=4m+2\notin D$ if $(I(\sigma)\times I(\tau))\cap G=\emptyset$. Next assume that $|\sigma|=1$ so that $\sigma=(n)$ for some n. Then if $\tau\prec 0^\omega$, then $(I(\sigma)\times I(\tau))\cap G\neq\emptyset$ so that we let $k(m)=4m$. If $\tau\prec 1^\omega$ and $|\tau|\geqslant 1$, then $(I(\sigma)\times I(\tau))\cap G\neq\emptyset$ if and only if $n\in D$ so that we let $k(m)=4m$ if $n\equiv 0 \bmod 4$, $k(m)=4m+2$ if $n\equiv 2\bmod 4$, and $k(m)=2\langle r,\langle k,m\rangle\rangle+1$ if n is of the form $2\langle r,k\rangle+1$. Finally if neither $\tau\prec 0^\omega$ nor $\tau\prec 1^\omega$, then $(I(\sigma)\times I(\tau))\cap G=\emptyset$ so that we let $k(m)=4m+2$. Finally suppose that $\sigma=\emptyset$. Then if either $\tau\prec 0^\omega$ or $\tau\prec 1^\omega$, then $(I(\sigma)\times I(\tau))\cap G\neq\emptyset$ so that we let $k(m)=4m$. Otherwise, we let $k(m)=4m+2$ since in that case $(I(\sigma)\times I(\tau))\cap G=\emptyset$ and $m\notin M$. □

Corollary 4.6. *There is a computably continuous function $F:\omega^\omega\to\omega^\omega$ such that the graph of F is not OIco-r.e.*

Proof. Just let C be an r.e. nonrecursive set and let F be constructed as in the proof of Theorem 4.5. Then if G is the graph of F, $\{\langle a,b\rangle : I_a\times I_b\cap G\neq\emptyset\}$ is an r.e. nonrecursive set so that the graph of F is not OIco-r.e. □

Next, we consider the question of whether the fact that the graph of a (continuous) function F is an effectively closed set implies that F is automatically computable. For the spaces $\Re$, $[0,1]$ and $\{0,1\}^\omega$, it follows from Corollary 4.3 above that the graph of any computable function must be a recursively closed set. Our next result shows that for these spaces, the assumption that the graph of a continuous function F is CIco-r.e. already implies that F must be computable.

Theorem 4.7. *Let $F:X^n\to X$ be a continuous function with a CIco-r.e. closed graph G, where X is either $\Re$, $[0,1]$ or $\{0,1\}^\omega$. Then F is computably continuous.*

Proof. We first give the argument for the compact spaces and then indicate how to extend the result to $\Re$. Assume that $F:X\to X$ is continuous function with graph G and that $\{\langle a,b\rangle : G\cap\overline{I_a\times I_b}=\emptyset\}$ is r.e. We will then define a computable representing function f for F.

First let

$$SS = \{\langle a,c\rangle : \overline{F(I_a)} \subset I_c\}.$$

Since $F(I_a)$ will be an open set, $\langle a,c\rangle \in SS$ means that $F(I_a)$ is properly included in I_c in the sense that the boundaries do not meet. We claim that SS is an r.e. set. Note that

$$\overline{F(I_a)} \subset I_c \quad \Leftrightarrow \quad G \cap (\overline{I_a} \times (X - I_c)) = \emptyset.$$

Moreover in either $[0,1]$ or $\{0,1\}^\omega$, $X - I_c$ may be effectively decomposed into a finite union of closed intervals. That is, in $[0,1]$, $X-(p,q)=[0,p] \cup [q,1]$. Similarly, in $\{0,1\}^\omega$, if $I_c = I(\sigma)$, then $X - I(\sigma) = \bigcup\{I(\tau) : |\sigma| = |\tau| \;\&\; \sigma \neq \tau\}$. Thus since G is CIco-r.e., SS is r.e.

As a first step to defining our desired representing function f for F, define a first approximation h to f by letting $h(a)=b$ where I_b is the intersection of all $c<a$ such that $\langle a,c\rangle \in SS$. h is not a representing function for F, but h does satisfy one of the two conditions for being a representing function of F. That is, let $a_1, a_2, \ldots$ be a sequence such that $I(a_{n+1}) \subset I(a_n)$ for all n and $\bigcap_n I(a_n) = \{x\}$ for some x. Let $y = F(x)$. First observe that for each n, $F(I_{a_n}) \subseteq I_{h(a_n)}$ so that $y = F(x)$ is an element of $I_{h(a_n)}$. Now for any interval I_c containing y, there is an interval I_{a_n} with $a_n > c$ such that $\overline{F(I_{a_n})} \subset I_c$ and therefore $I_{h(a_n)} \subset I_c$. It follows that $\bigcap_n I_{(h(a_n))} = \{y\}$.

This given, we can define our desired representing function f of F by setting $f(a)=b$ where I_b is the intersection of all $I_{h(s)}$ such that $I_a \subset I_s$. The computability of f will immediately follow from the condition that whenever $I_a \subset I_s$, we must have $s \leqslant a$. It is not difficult to select a basis of intervals with this property. However since this is crucial to the argument, we give some details. For $\{0,1\}^\omega$, we enumerate the intervals $I(\sigma)$ in order first by length and then lexicographically, which will suffice since $I(\sigma) \subset I(\tau)$ implies that $|\tau| \leqslant |\sigma|$. For the real interval [0,1], we can revise our given effective list of basic intervals, if necessary, so that whenever $I_a \subset I_s$ with $s > a$, we replace I_s with a finite cover of I_s consisting of smaller intervals which do not include any previous interval. The revised list will still be a basis and will have the necessary property.

It is clear that the refined function f still satisfies the first condition for being a representing function for F. It satisfies the second condition, since whenever $I_a \subseteq I_e$, we have $e \leqslant a$, so that $I_{f(a)}$ is the intersection of a larger family of sets $I_{h(s)}$ and therefore $I_{f(a)} \subseteq I_{f(e)}$.

Now suppose that $F : \Re \to \Re$ is continuous and has a CIco-r.e. graph G. Let $A = \{\langle a, b\rangle : \overline{I_a \times I_b} \cap G = \emptyset\}$ and let A_s be the numbers enumerated into A by stage s. Without loss of generality, we may assume that $F(0)=0$. We claim that we can define a computable function $h : \mathbb{N} \to \mathbb{N}$ such that, for all n and all $x \in [-n,n]$, $|F(x)| < b(n)$. Here is the procedure for computing $b(n)$ from n. First let a be defined so that $I_a = (-n,n)$. Since F is continuous, there is some B such that $|F(x)| < B$ for all $x \in [-n,n]$. This implies that $G \cap ([-n,n] \times [B,B+1]) = G \cap ([-n,n] \times [-B-1,-B]) = \emptyset$. On the other hand, suppose that $G \cap ([-n,n] \times [B,B+1]) = G \cap ([-n,n] \times [-B-1,-B] = \emptyset$. Then it follows from the Intermediate Value Theorem that $|F(x)|<B$ for all $x\in[-n,n]$. We may thus compute an upper bound $b(n)$ by searching for the least stage s such that, for some rational B, $[-n,n]\times([B,B+1]\cup[-B-1,-B])$ is covered by intervals $I_{w_0},\dots,I_{w_k}$ such that $\langle a,w_i\rangle\in A_s$ for all $i\leqslant k$.

Given this computable bound $b(n)$, we can now construct a representing function f_n for $F:[-n,n]\to[-b(n),b(n)]$ by the same argument that we used to construct a representing function f for a continuous function $F:[0,1]\to[0,1]$ which had a CIco-r.e. closed graph. Since this can be done uniformly with respect to n, this allows us to compute $F(x)$ for any x. $\square$

Our previous results show that if G is the graph of a continuous function F and G is CIco-r.e., then G is automatically OIr.e. Next we consider the possibilities when F is continuous, but not computable. Hence G is not CIco-r.e. and therefore is not OIco-r.e. However our next examples will show that G may still be OIr.e.

Theorem 4.8. *Let X be any of the spaces $\Re$, $[0,1]$, $\{0,1\}^\omega$ and ω^ω, there exists a continuous but not computable function $F:X\to X$ with an OIr.e. closed graph.*

Proof. Let C be any r.e. set and $D=\{2\langle n,m\rangle+1:n\in C\,\&\,m\in\omega\}\cup\{4n:n\in\omega\}$. Let $h:\omega\to\omega$ be a $1:1$ recursive function whose range is D. We let $D_s=\{h(0),\dots,h(s)\}$ for all s.

Case 1: $X=\{0,1\}^\omega$. We define the function F as follows. First, let $F(0^\omega)=0^\omega$. For each n and s, we define $F(x)$ for the extensions of $0^{n-1}*1$ in two cases. If $n\notin D$, then $F(x)=0^\omega$ for all extensions of $0^{n-1}*1$. If $n\in D_{s+1}-D_s$, then we let $F(x)=0^{n-1}*1*0^\omega$ when x extends $0^{n-1}*1*0^s*1$ and we let $F(x)=0^\omega$, otherwise. It is clear that F is continuous.

We claim that D is 1–1 equivalent to $M=\{\langle a,b\rangle:G\cap I_a\times I_b\neq\emptyset\}$ where G is the graph of F. Thus G is OIr.e. but not OIco-r.e. Note that for $\{0,1\}^\omega$, G is OIco-r.e. if and only if G is CIco-r.e. Thus G is not CIco-r.e. and hence F is not computable by Theorem 4.7.

To see that $M\leqslant_1 D$, we define a $1:1$ recursive function k such that $m\in M\Leftrightarrow k(m)\in D$. Let $m=\langle a,b\rangle$ and let σ and τ be such that $I_a=I(\sigma)$ and $I_b=I(\tau)$. Then if τ has two or more 1's, then $(I(\sigma)\times I(\tau))\cap G=\emptyset$ so that $m\notin M$ and we can let $k(m)=4m+2$.

Thus suppose that τ is of the form $0^{n-1}*1*0^t$. If σ has two or more 1's and is not of the form $0^{n-1}*1*0^s*1*\gamma$ where $\gamma\in\{0,1\}^*$, then again $(I(\sigma)\times I(\tau))\cap G=\emptyset$ so that $m\notin M$ and we can let $k(m)=4m+2$. If σ is of the form $0^{n-1}*1*0^s*1*\gamma$ where $\gamma\in\{0,1\}^*$, then $(I(\sigma)\times I(\tau))\cap G\neq\emptyset$ if and only if $n\in D_{s+1}-D_s$ so that we can let $k(m)=4m$ if $n\in D_{s+1}-D_s$ and let $k(m)=4m+2$ otherwise. If σ has one 1 and is not of the form $0^{n-1}*1*0^t$ where $t\geqslant 0$, then again $(I(\sigma)\times I(\tau))\cap G=\emptyset$ so that $m\notin M$ and we can let $k(m)=4m+2$. Next suppose that σ is of the form $0^{n-1}*1*0^t$. Then $(I(\sigma)\times I(\tau))\cap G=\emptyset$ if $n\in D_{t+1}$ so that we can let $k(m)=4m+2$. If $n\notin D_t$, then $I(\sigma)\times I(\tau)\cap G\neq\emptyset$ if and only if $n\in D$ so that we can let $k(m)=4m$ if $n\equiv 0 \bmod 4$, $k(m)=4m+2$ if $n\equiv 2 \bmod 4$, and $k(m)=2\langle r,\langle k,m\rangle\rangle+1$ if n is of the form $2\langle r,k\rangle + 1$. Next suppose σ has no 1's so that $\sigma = 0^t$ for some $t \geqslant 0$. Then if $t \geqslant n$, then $(I(\sigma) \times I(\tau)) \cap G = \emptyset$ so that we can let $k(m) = 4m + 2$. If $t < n$, then $I(\sigma) \times I(\tau) \cap G \neq \emptyset$ if and only if $n \in D$ so that we can let $k(m) = 4m$ if $n \equiv 0 \bmod 4$, $k(m) = 4m+2$ if $n \equiv 2 \bmod 4$, and $k(m) = 2\langle r, \langle k,m\rangle\rangle + 1$ if n is of the form $2\langle r,k\rangle + 1$.

Finally suppose that τ has no 1's so that $\tau = 0^t$ for some $t \geqslant 0$. Next, suppose that σ has two or more 1's and $\sigma = 0^{n-1} * 1 * 0^s * 1 * \gamma$ where $\gamma \in \{0,1\}^*$. Then $(I(\sigma) \times I(\tau)) \cap G \neq \emptyset$ if $n \notin D_{s+1} - D_s$ so that we can let $k(m) = 4m$. If $n \in D_{s+1} - D_s$, then $(I(\sigma) \times I(\tau)) \cap G \neq \emptyset$ if and only if $t \leqslant n - 1$ so that we let $k(m) = 4m$ if $t \leqslant n-1$ and let $k(m) = 4m + 2$ if $t \geqslant n$. Finally it is easy to see that if σ at most one 1, then $(I(\sigma) \times I(\tau)) \cap G \neq \emptyset$ so that we can let $k(m) = 4m$.

Case 2: $X = \omega^\omega$. Define the function F so that $F(x) = 1^\omega$ if $n^s \prec x$ for some n and s such that $n \in D_{s+1}$ and let $F(x) = 0^\omega$ otherwise. Then F is clearly continuous, but is not computable since $n \in D \Leftrightarrow F(n^\omega)(0) = 1$. To see whether $G \cap (I(\sigma) \times I(\tau)) \neq \emptyset$, we first find the unique n and s such that $n^s \prec \sigma$. There are two cases. First, suppose that $n \notin C_{s+1}$. Then $G \cap (I(\sigma) \times I(\tau)) \neq \emptyset$ if $\tau(i) = 0$ for all i, or if $\tau(i) = 1$ for all i and $n \in C$ which is an r.e. condition. Second, suppose that $n \in C_{s+1}$. Then $G \cap (I(\sigma) \times I(\tau)) \neq \emptyset$ if and only if $\tau(i) = 1$ for all i. It follows that $\{\langle a,b\rangle : G \cap (I_a \times I_b) \neq \emptyset\}$ is r.e.

Case 3: $X = \mathfrak{R}$ or $X = [0,1]$. We will define a continuous function $F : [0,1] \to [0,1]$ with $F(0) = F(1) = 0$ such that F is not computable but the graph G of F is open interval recursively enumerable. Then F can be extended to the real line by setting $F(x) = 0$ for all $x \notin [0,1]$ to give an example for $\mathfrak{R}$.

Let the basic sets $I_a \times I_b \subset [0,1] \times [0,1]$ be enumerated as $B_1, B_2, \dots$. Then we define a continuous function F as a limit of uniformly computably sequence of piecewise linear functions F_s with graphs G_s such that $G_s \cap B_i \neq \emptyset$ implies that $G \cap B_i \neq \emptyset$ whenever $i < s$. (We will do this by selecting a point in B_i and keeping that point fixed thereafter.) Thus G will be open interval r.e., since

$$G \cap B_i \neq \emptyset \Leftrightarrow (\exists s > i)(G_s \cap B_i \neq \emptyset).$$

We will ensure that F is not computable by having $F(2^{-n}) = 0$ if and only if $n \in D$. We will make F continuous at $x \neq 0$ by ensuring that for each x, and all $x \in [2^{-n-1}, 2^{-n}]$, there is an s such that for all $t \geqslant s$, $F(x) = F_t(x)$. F will be continuous at 0 since we will have $F(x) \leqslant x$ for all x.

Initially $F_1(x) = 0$ for all x. After stage s, we will have a piecewise linear function F_s with graph G_s. We will also have a subset M_s of $\{0,1,\dots,s-1\}$ consisting of those $i < s$ such that $G_i \cap B_i \neq \emptyset$ and, for each $i \in M_s$, a point $\langle x_i, y_i\rangle \in B_i \cap G_s$, with x_i not of the form 2^{-n}. Let $n \in D_{s+1} - D_s$, then do the following.

Choose an interval (p,q) with diameter $< 2^{-n-3}$ containing 2^{-n-1} which does not contain any of the points $\langle x_i, y_i\rangle$ and does not contain any of the endpoints of the line segments making up G_s. Then define $F_{s+1}(x)$ on $[p,q]$ to consist of two line segments, from $\langle p, F_s(p)\rangle$ to $\langle 2^{-n-1}, 0\rangle$ and then from $\langle 2^{-n-1}, 0\rangle$ to $\langle q, F_s(q)\rangle$. For $x \notin [p,q]$, let $F_{s+1}(x) = x$. Let $M_{s+1} = \{i \leqslant s : G_{s+1} \cap B_i \neq \emptyset\}$ and choose for each $i \in M_{s+1} - M_s$, a point $\langle x_i, y_i\rangle \in G_{s+1} \cap B_i$ with $x_i \neq 2^{-n}$ for any n.

Let us check the conditions set out above, beginning with the continuity. First observe that $F_t(x)=x$ for all t and all $x \geqslant \frac{3}{4}$. Now fix n and let s be large enough so that $n \in D_s \Leftrightarrow n \in D$ and $n-1 \in D_s \Leftrightarrow n-1 \in D$. Then it follows from the construction that $F_t(x)=F_s(x)$ for all $t \geqslant s$ and all $x \in [2^{-n-1}, 2^{-n}]$. Thus F is continuous at all points $x \neq 0$.

By the construction, we have $F_0(x)=x$ and $F_{s+1}(x) \leqslant F_s(x)$ for all x and s so that $F(x) \leqslant x$ and thus F is continuous at $x=0$.

Next, we check that G is a recursively enumerable closed set. Let $M = \cup_s M_s$. We claim that $G \cap B_i \neq \emptyset \Leftrightarrow i \in M$. If $i \in M$, then by the construction we have selected $\langle x_i, y_i \rangle \in B_i$ and ensured that $y_i = F(x_i)$ so that $G \cap B_i \neq \emptyset$. Suppose that $G \cap B_i \neq \emptyset$ and let $\langle x, y \rangle \in G \cap B_i$. Then by the continuity argument, there is a stage s such that $F_t(x)=y$ for all $t \geqslant s$. Thus for some $t > i$, $G_t \cap B_i \neq \emptyset$ so that $i \in M$.

Finally, we check that F is not computable. By the construction, we have $F(2^{-n-1}) = 2^{-n-1}$ if $n \notin D$ and $F(2^{-n-1})=0$ if $n \in D$. If F were computable, then the set of zeroes of F would be a Π^0_1 class and therefore D would be co-r.e., contradicting the assumption that C and hence D is non-recursive. □

We end this section by considering the question of whether a function which has a graph that is OIr.e., CIr.e., OIco-r.e. or CIco-r.e. is necessarily continuous. For the compact spaces $[0,1]$ and $\{0,1\}^\omega$, it is easy to see that any function with a closed graph must be continuous. However, as we observed in Section 3, our definitions of OIr.e, CIr.e., etc., do not inherently imply that the set is closed.

Example 4.1. Define a function $F:[0,1] \to [0,1]$ by setting $F(x)=\frac{1}{4}$ for $x \neq \frac{1}{2}$ and $F(\frac{1}{2})=\frac{1}{2}$. Then F is not continuous but the graph G of F has the property that both $\{\langle a,b \rangle : G \cap (I_a \times I_b) \neq \emptyset\}$ and $\{\langle a,b \rangle : G \cap \overline{I_a \times I_b} = \emptyset\}$ are recursive sets. This is because

$$G \cap ((p,q) \times (r,s)) \neq \emptyset \;\Leftrightarrow\; r < \tfrac{1}{4} < s \vee (p < \tfrac{1}{2} < q \,\&\, r < \tfrac{1}{2} < s)$$

and

$$G \cap ([p,q] \times [r,s]) \neq \emptyset \;\Leftrightarrow\; r \leqslant \tfrac{1}{4} \leqslant s \vee (p \leqslant \tfrac{1}{2} \leqslant q \,\&\, r \leqslant \tfrac{1}{2} \leqslant s).$$

Example 4.2. Define a non-continuous map from $\{0,1\}^\omega$ to $\{0,1\}^\omega$ by setting $F(x) = 0^\omega$ if $x=0^\omega$ and $F(x)=1^\omega$ for $x \neq 0^\omega$. The graph G of F has the property that $\{\langle a,b \rangle : G \cap (I_a \times I_b) \neq \emptyset\}$ is a recursive set since $G \cap (I(\sigma) \times I(\tau)) \neq \emptyset$ if and only if

$$(\forall i < |\tau|)(\tau(i)=1) \vee [(\forall j < |\sigma|)(\sigma(j)=0) \,\&\, (\forall i < |\tau|)(\tau(i)=0)].$$

For the spaces $\mathfrak{R}$ and ω^ω, we can define non-continuous functions with closed graphs which have similar properties.

Example 4.3. Let $F:\mathfrak{R} \to \mathfrak{R}$ be defined by setting $F(x)=1/x^2$ for $x \neq 0$ and $F(0)=0$. The graph of G of F is clearly closed but it can be checked that both $\{\langle a,b \rangle : G \cap (I_a \times I_b) \neq \emptyset\}$ and $\{\langle a,b \rangle : G \cap \overline{I_a \times I_b} \neq \emptyset\}$ are recursive sets.

Example 4.4. Define a function $F:\omega^\omega \to \omega^\omega$ by setting $F(x)=0^\omega$ if $x=k^\omega$ for some k and setting $F(x)=(n+1)^\omega$ if n is the least such that $x(n+1)\neq x(n)$. The graph G of F is closed, by the following argument. Suppose that $\lim_t \langle x_t, y_t\rangle = \langle x, y\rangle$ for some sequence $\{\langle x_t, y_t\rangle\}$ with $F(x_t)=y_t$ and consider two cases. First, say that $y=0^\omega$. Then for all but finitely many t, $y_t(0)=0$, which implies that $x(t)=k_t^\omega$ for some k_t. Since $x=\lim_t x_t$, it follows that $\lim_t k_t = k$ for some k so that $y=F(x)$ as desired. Second, say that $y=(n+1)^\omega$ for some n. Since $y=\lim_t y_t$, it follows that for all but finitely many t, $y_t(0)=n+1$ so that n is the least such that $x_t(n+1)\neq x_t(n)$. Since $x=\lim_t x_t$, it then follows that n is the least such that $x(n+1)\neq x(n)$, so that $y=F(x)$ as desired.

G has the property that $\{\langle a,b\rangle : G\cap(I_a\times I_b)\neq\emptyset\}$ is a recursive set since $G\cap(I(\sigma)\times I(\tau))\neq\emptyset$ if and only if either σ is constant and τ is either constant 0 or is constant $n+1$ for some $n\geqslant|\sigma|$, or, for some n, n is the least such that $\sigma(n+1)\neq\sigma(n)$ and τ is constant $n+1$.

5. Complexity theory and closed sets

We note that the application of complexity theory in analysis is a well-developed subject (see Ko's book [18]). It is reasonable to ask whether there are natural complexity theoretic analogues of the results of Sections 2 and 3. The answer is that one can develop such a subject. The type of results that one obtains are typical of complexity theoretic analysis where some of the results continue to hold for polynomial time computable functions and closed sets, some of the results no longer hold, and some of the results are intimately connected with various separation questions in classical complexity theory, such as the P = NP question. We do not have the space to develop such a theory in this paper so that we will be content to simply give the basic definitions and a few results. We will give a more complete development in a future paper.

To give the definitions of various resource bounded versions of effectively closed sets, one requires some natural polynomial time enumeration of the basic intervals. We will illustrate this idea by considering the two easiest cases, namely, the unit interval $[0,1]$ and the space $\{0,1\}^\omega$ where such enumerations are relatively straightforward.

First consider the space $\{0,1\}^\omega$. The standard enumeration defined by letting $I_n = I(\sigma_n)$ where $bin(n+1)=1*\sigma_n$ is clearly a linear time enumeration and we can refer to $I(\sigma)$ rather than I_n in our calculations since $bin(n)$ and σ_n can be computed from each other in linear time.

For the real interval, we adapt the enumeration given by Cenzer and Remmel [8]. Here we take our basic open intervals to be the set of all dyadic open intervals $(i/2^{n+1},(i+2)/2^{n+1})$ for $n\geqslant 1$ and $i<2^{n+1}-1$, as well as the half-open intervals $[0,1/2^{n+1})$ and $(1-1/2^{n+1},1]$ and the entire space $[0,1]$. Let $[0,1]=I_0$, let $[0,1/2^{n+1})=I_{2^{2n}}$, let $(1-1/2^{n+1},1]=I_{2^{2n+1}}$, and let $(i/2^{n+1},(i+2)/2^{n+1})=I_{2^{n+1}+i+1}$. Then we can compute the end points of each interval I_k from $bin(k)$ and vice versa in linear time. One advantage of this collection of intervals is that we can split each interval into three overlapping subintervals in linear time. For example, $(0,\frac{1}{2})$ can split into $(0, \frac{1}{4})$, $(\frac{1}{8}, \frac{3}{8})$ and $(\frac{1}{4}, \frac{1}{2})$-the left half, middle half and right half of the interval. It follows that we can locate a real within 2^{-n} by successively finding intervals of radius $\frac{1}{2}, \frac{1}{4}, \ldots$, where each step requires checking at most 2 of the three subintervals.

We shall assume that the reader is familiar with the basic definition of polynomial time (P), non-deterministic polynomial time (NP), and co-non-deterministic polynomial time (CoNP) subsets of $\{0,1\}^*$. This given, we have the following natural complexity theoretic analogues of our various versions of effectively closed sets.

Definition 5.1. Let K be a closed set in the space X where X is either $\{0,1\}^\omega$ or $[0,1]$.

(i) K is *open interval* NP (OINP) if $\{bin(w) : I_w \cap K \neq \emptyset\}$ is in NP.
(ii) K is *open interval* CoNP (OICoNP) if $\{bin(w) : I_w \cap K = \emptyset\}$ is in NP.
(iii) K is *closed interval* NP (CINP) if $\{bin(w) : \overline{I_w} \cap K \neq \emptyset\}$ is in NP.
(iv) K is *closed interval* CoNP (CICoNP) if $\{bin(w) : \overline{I_w} \cap K = \emptyset\}$ is in NP.
(v) K is NP if K is both OINP and CICoNP.
(vi) K is *open interval polynomial time decidable* if $\{bin(w) : I_w \cap K \neq \emptyset\}$ is in P.
(vii) K is *closed interval polynomial time decidable* if $\{bin(w) : \overline{I_w} \cap K \neq \emptyset\}$ is in P.
(viii) K is *polynomial time decidable* if K is both open interval polynomial time decidable and closed interval polynomial time decidable.

The notion of polynomial time, NP and CoNP computable real functions is developed by Ko [18]. One can also give an equivalent formulation of polynomial time continuous computable function which mirrors our working definition of continuous computable function. Suppose we are given a computably continuous function F. Then F is said to be *polynomial time computable* if F has a representing function $f : \omega \to \omega$ such that f is polynomial time computable (relative to the binary representation of ω) and there is k such that for all m and t,

$$(*)\ diam(I_t) < 2^{-m^k} \to diam(I_{f(t)}) < 2^{-m}.$$

One can also define natural complexity theoretic analogues of upper and lower computable functions. Let $\mathbb{D}$ be the set of dyadic rationals in $\mathfrak{R}$. Here a string 0's and 1's, $\pm s_m \ldots s_0.t_1 \ldots t_n$, codes the diadic rational

$$d = \pm \sum_{i+0}^{m} s_i 2^i \pm \sum_{j=1}^{n} t_i 2^{-i}.$$

A function $F : X \to [0,1]$ is *upper (lower) polynomial time computable* if there is a polynomial time oracle Turing machine M such that M^x accepts $\{d \in \mathbb{D} : F(x) < d\}$ ($\{d : d < F(x)\}$). A function $F : X \to \mathfrak{R}$ is said to be *lower NP computable* if there is a nondeterministic polynomial time oracle Turing machine M such that M^x accepts $\{d \in \mathbb{D} : d < F(x)\}$. Similarly, F is *upper NP computable* if there is an NP oracle Turing machine M such that M^x accepts $\{d : d > F(x)\}$.

We shall give just a couple of results to show that one can also develop a complexity theoretic analogue of the results of this paper. For example, there are a number of results which relate the complexity of a closed set K to the complexity of the distance function d_K. The simplest example is the following.

Theorem 5.2. *Let K be a nonempty closed subset of $\{0,1\}^\omega$. Then*
(a) *K is OINP if and only if d_K is lower NP computable.*
(b) *K is OICoNP if and only if d_K is upper NP computable.*
(c) *K is OI polynomial time decidable if and only if d_K is polynomial time computable.*

Proof. Recall that we have a discrete distance function on the space $\{0,1\}^\omega$, that is, $d(x,y)=2^{-n}$ where n is the least such that $x(n)\neq y(n)$. Thus given $x\in\{0,1\}^\omega$ and $d\in\mathbb{D}$, we may assume that $d=2^{-n}$ for some n since those are the only possible distances in $\{0,1\}^\omega$. Part (a), (b), and (c) now easily follow from our definitions and the following observations.
(a) $d_K(x)<2^{-n}\Leftrightarrow I(x\lceil n+1)\cap K\neq\emptyset$.
(b) $d_K(x)>2^{-n-1}\Leftrightarrow\neg(d_K(x)<2^{-n})\Leftrightarrow I(x\lceil n+1)\cap K=\emptyset$.
(c) $d_K(x)=2^{-n}\Leftrightarrow d_K(x)<2^{-n+1}\,\&\,d_K(x)>2^{-n-1}$. □

One can also distinguish between the OI complexity of closed sets and the CI complexity of closed sets. We end this section with two simple examples of such a results. Recall that given sets A and B contained in $\{0,1\}^*$, we say that A is polynomial time 1–1 (m-1) reducible to B, $A\leqslant_1^p B$ ($A\leqslant_m^p B$), if there is a $1:1$ polynomial time (many-one polynomial time) function $f:\{0,1\}^*\to\{0,1\}^*$ such that for all $x\in\{0,1\}^*$, $x\in A\Leftrightarrow f(x)\in B$. We write $A\equiv_1^p B$ if both $A\leqslant_1^p B$ and $B\leqslant_1^p A$ and $A\equiv_m^p B$ if both $A\leqslant_m^p B$ and $B\leqslant_m^p A$.

First we give a result for the space $\{0,1\}^\omega$.

Theorem 5.3. *Let A be any subset of the binary representation of the natural numbers, $Bin(\omega)$. Then there is a closed set $K(A)\subseteq\{0,1\}^\omega$ such that $A\equiv_m^p\{bin(n):K(A)\cap I_n\neq\emptyset\}$.*

Proof. First if A is empty, then we can simply let $K(A)=\emptyset$ since in that case $A=\{bin(n):I_n\cap K(A)\neq\emptyset\}$. Thus suppose that we are given a non-empty set $A\subseteq Bin(\omega)$. For any n, let $bin(n+2)=1e_1\ldots e_k$ and let $I_{f(n)}=I(\sigma)$ where $\sigma=e_10e_20\ldots e_{k-1}0e_k1$ and $I_{g(n)}=I(\tau)$ where $\tau=e_10e_20\ldots e_{k-1}0e_k0$. Then define the closed set $K(A)$ as follows. First $K(A)$ contains all x such that $x(2n+1)=0$ for all n, that is, $(\{0,1\}\{0\})^\omega\subseteq K(A)$. Second $K(A)$ includes the interval $I_{f(n)}$ if and only if $bin(n)\in A$. It is easy to see that $K(A)$ is a closed set since $K(A)$ is the set of infinite paths through the tree T defined as follows. For any sequence ρ, $\rho\in T$ if and only if, either $\rho(2i+1)=0$ for all i, or else $\rho=e_10e_20\ldots e_{k-1}0e_k1^\frown\gamma$ for some $\gamma\in\{0,1\}^{<\omega}$ and $bin(n)\in A$ where $bin(n+2)=1e_1\ldots e_k$.

First we claim that

$$bin(n)\in A\quad\Leftrightarrow\quad bin(f(n))\in\{bin(m):\,K(A)\cap I_m\neq\emptyset\}.$$

That is, if $bin(n) \in A$, then $I_{f(n)} \subset K(A)$ by definition so that certainly $K(A) \cap I_{f(n)} \neq \emptyset$. On the other hand, suppose that $K(A) \cap I_{f(n)} \neq \emptyset$. Then there is a string $\rho = e_1 0 e_2 0 \ldots e_{k-1} 0 e_k 1^\frown \gamma$ in $K(A) \cap I_{f(n)}$ for some $\gamma \in \{0,1\}^\omega$. But the only way that $\rho \in K(A)$ is if $bin(m) \in A$ where $bin(m+2) = 1e_1 \ldots e_k$ and the only way that $\rho \in I_{f(n)}$ is if $m = n$. Thus if $K(A) \cap I_{f(n)} \neq \emptyset$, then $bin(n) \in A$. Since the function F defined by setting $F(bin(n)) = bin(f(n))$ is clearly polynomial time, it follows that $A \leqslant_m^p \{bin(n): I_n \cap K(A) \neq \emptyset\}$.

Next, we shall define a polynomial time function h such that $bin(m) \in \{bin(n) : I_n \cap K(A) \neq \emptyset\}$ if and only if $h(bin(m) \in A$. Let r_0 be the least n such that $bin(n) \in A$. Given $bin(n)$, first compute the finite sequence ρ such that $I_n = I(\rho)$.

Case I: If $\rho(2i+1) = 0$ for all i, then $K(A) \cap I_n \neq \emptyset$. In this case, let $bin(h(n)) = bin(r_0)$.

Case II: Otherwise, let i be the least such that $\rho(2i+1) = 1$. Then $K(A) \cap I_n \neq \emptyset$ if and only if $bin(m) \in A$, where $bin(m+2) = 1\rho(0)\rho(2)\ldots\rho(2i)$. In this case, let $h(bin(n)) = bin(m)$.

Again it is easy to see that h is polynomial time function so that $\{bin(n) : K(A) \cap I_n \neq \emptyset\} \leqslant_m^p A$. $\square$

Note that in $\{0,1\}^\omega$, every I_n is clopen. Thus, we automatically have that OINP = CINP and OICoNP = CICoNP. The question of whether there is a OINP closed which is not polynomial decidable or a OICoNP which is not polynomial time decidable is equivalent the question of whether P = NP. That is, an immediate application of Theorem 5.3 is the following.

Theorem 5.4. *The following are equivalent*:
- P = NP.
- *Every OINP closed subset of* $\{0,1\}^\omega$ *is polynomial time decidable.*
- *Every OICoNP closed subset of* $\{0,1\}^\omega$ *is polynomial time decidable.*

Proof. First we note that if P = NP, then it is immediate that every OINP closed set and every OICoNP set closed set is polynomial time decidable.

To show that P = NP, it is enough to show that every NP set $A \subseteq Bin(\omega)$ is in P or that every CoNP set $A \subseteq Bin(\omega)$ is in P. By Theorem 5.3, $A \equiv_m^p \{bin(n) : I_n \cap K(A) \neq \emptyset\}$. Thus A is in P if and only if $\{bin(n) : I_n \cap K(A) \neq \emptyset\}$ is in P and A is in NP if and only if $\{bin(n) : I_n \cap K(A) \neq \emptyset\}$ is in NP. Moreover $A \in$ CoNP if and only if $\{bin(n) : I_n \cap K(A) \neq \emptyset\}$ is in CoNP if and only $\{bin(n) : I_n \cap K(A) = \emptyset\}$ is in NP.

Now if every OINP closed set is polynomial time decidable, then given a NP set $A \subseteq Bin(\omega)$, $K(A)$ is in OINP and hence is automatically in P. But then A must be in P. Similarly, if every OICoNP closed set is polynomial time decidable, then for any CoNP set $B \subseteq Bin(\omega)$, $K(B)$ is OICoNP and hence is automatically in P. But then B is in P. $\square$

For the space $[0,1]$, the counterexamples given in section can be modified to give the following result.

Theorem 5.5. *Let C be any computable subset of $\{bin(2^n): n \in \omega\}$ and $D = \{bin(2^{3[(2^n)(2k+1)]+1}): bin(2^n) \in C\} \cup \{bin(2^{3n}): n \in \omega\}$. Then there is a closed subset K_D of $[0,1]$ such that K_D is OI polynomial time decidable and $D \equiv_1^p \{bin(n): K_D \cap \overline{I_n} = \emptyset\}$.*

Proof. Let K_D always contain 0 and put $1/2^{n+1}$ in K_D if and only if $bin(2^n) \in D$. Let M be a Turing machine which computes the characteristic function of D. For each n, let t_n be the unique t such that the computation of $\chi_D(bin(2^n))$ by M takes exactly t steps. We then put $c_n = 1/2^{n+1} + 1/2^{n+1+2^{t_n}} \in K_D$. We refer to c_n as the *checkpoint* for n.

Given an interval $I_e = (i/2^{m+1}, (i+2)/2^{m+1})$, we test whether K_D meets I_e as follows. There are two cases.

Case I: $1/2^{n+1} \in I_e$ for some n. This happens only if $I_e = ((2^k - 1)/2^{k+n+1}, (2^k + 1)/2^{n+k+1})$ for some k. In that case, $e = 2^{n+k+2} + 2^k - 1$ and $|bin(e)| \geqslant n + k + 1$. We then run the Turing machine M on $bin(2^n)$ for k steps. If the computation does not converge, then $t_n > k$ and therefore the checkpoint $c_n \in K_D \cap I_e$ so that $K_D \cap I_e \neq \emptyset$. If the computation converges, then $t_n \leqslant k$. Then $I_e \cap K_D \neq \emptyset$ if and only if $2^{t_n} > k$ or equivalently if and only if $t_n > |bin(k)|$. Since we can compute whether $t_n > |bin(k)|$ in k steps, we can decide if the checkpoint $c_n \in I_e$ in linear time in $|bin(e)|$. Now if $c_n \notin I_e$, then we have $K_D \cap I_e \neq \emptyset \Leftrightarrow bin(2^n) \in D$. But we have just computed whether $bin(2^n) \in D$ so once again we can determine the answer in linear time $|bin(e)|$. Thus we can decide if $I_e \cap K_D = \emptyset$ in linear time in $|bin(e)|$.

Case II: Not Case I. In this case, in linear time in $|bin(e)|$, we can find $s < m+1$ such that $2^s \leqslant i$, $i+2 \leqslant 2^{s+1}$ and hence $1/2^{m-s+1} \leqslant i/2^{m+1} < (i+2)/2^{m+1} \leqslant 1/2^{m-s}$. Then the only possible member of $K_D \cap I_e$ is the checkpoint for $m-s$. That is, if $i = 2^s + j$, then we must look for t_{m-s} such that

$$\frac{2^s + j}{2^{m+1}} < \frac{1}{2^{m-s+1}} + \frac{1}{2^{m-s+1+2^{t_{m-s}}}} < \frac{2^s + j + 2}{2^{m+1}}$$

or equivalently that

$$j < \frac{2^s}{2^{2^{t_{m-s}}}} < j + 2.$$

Thus there is a checkpoint in I_e if and only if j is of the form $j = 2^k - 1$ for some $k \leqslant s$ and $2^{t_{m-s}} = s - k$. But since $\log_2(k)$ and $\log_2(m-s)$ are both less than $|bin(e)|$, we can run our Turing machine M for s steps and determine if there is a checkpoint in I_e in linear time in $bin(e)$. Thus again we can determine if $I_e \cap K_D \neq \emptyset$ in linear time in $|bin(e)|$.

Now if I_e is not of the form $(i/2^{m+1}, (i+2)/2^{m+1})$, then either $e = 0$ and $I_e = [0,1]$, $e = 2^{2n}$, and $I_e = [0, 1/2^{n+1})$ or $e = 2^{2n+1}$ and $I_e = (1 - 1/2^{n+1}, 1]$. Clearly in each of these cases, we can decide if $I_e \cap K_D \neq \emptyset$ in linear time in $|bin(e)|$. It follows that K_D is OI polynomial time decidable.

Next, we want to show that $D \equiv_1^p U = \{bin(e) : \overline{I_e} \cap K_D \neq \emptyset\}$. To show that $U \leqslant_1^p D$, first observe that our proof of the fact that K_D was OI polynomial time decidable shows that $I_e \cap K_D \neq \emptyset$ only if I_e has one of the following forms:

1. $I_e = [0, 1]$ so that $e = 0$,
2. $I_e = [0, 1/2^{n+1})$ so that $e = 2^{2n}$,
3. $I_e = (2^k - 1)/2^{n+k+1}, (2^k + 1)/2^{n+k+1})$ for some k and n so that $e = 2^{n+k+1} + 2^k$, or
4. $I_e = ((2^s + 2^k - 1)/2^{m+1}, (2^s + 2^k + 1)/2^{m+1})$ for some $m > s > k$ so that $e = 2^{m+1} + 2^s + 2^k$.

Let T be the set of e of the form 2^{2n}, $2^n + 2^k$ for $k < n$ and $2^n + 2^s + 2^k$ where $n > s > k$. It is easy to see that the set $Bin(T) = \{bin(e) : e \in T\}$ is polynomial time isomorphic to the set $\{bin(2^n) : n \in \omega\}$. Thus let $\psi : \{0,1\}^* \to \{0,1\}^*$ be a $1:1$ polynomial time function such that $\psi(Bin(T)) = \{bin(2^n) : n \in \omega\}$.

We can now define a $1:1$ polynomial time function $f : \{0,1\}^* \to \{0,1\}^*$ which shows that $U \leqslant_1^p D$ as follows. First if $\sigma \notin Bin(\omega)$, then let $f(\sigma) = \sigma$ since σ is not in either U nor D. If $\sigma = bin(e)$ for some $e \in \omega$, then we first check if $I_e \cap K_D \neq \emptyset$. If $I_e \cap K_D \neq \emptyset$, then let $f(bin(e)) = bin(2^{3(5m)})$ where $\psi(bin(e)) = bin(2^m)$ so that $bin(e) \in U$ and $f(bin(e)) \in D$. If $I_e \cap K_D = \emptyset$, then our argument above shows that we can check in linear time in $|bin(e)|$ whether the endpoints of I_e are of the form $1/2^{n+1}$ or one of the checkpoints c_m for some m. If the endpoints are not a check point nor of the form $1/2^{n+1}$, then $bin(e) \notin U$ so that we can let $f(bin(e)) = bin(e)^\frown 1$ which is not in D. Otherwise, we have three cases.

Case A: One of the endpoints of I_e is a check point $c_n = 1/2^{n+1} + 1/2^{n+1+2^{t_n}}$. This means $bin(e) \in U$ and that either

(i) $I_e = ((2^{t_n} + 1)/2^{n+1+2^{t_n}}, (2^{t_n} + 3)/2^{n+1+2^{t_n}})$ or
(ii) $I_e((2^{t_n} - 1)/2^{n+1+2^{t_n}}, (2^{t_n} + 1)/2^{n+1+2^{t_n}})$. We then set $f(bin(e)) = 2^{3(5m+1)}$ in case (i) and $f(bin(e)) = 2^{3(5m+2)}$ in case (ii) where $\psi(bin(2^{n+1+2^{t_n}}) + 2^{t_n}) = bin(2^m)$.

Case B: Not Case A and $I_e = (2^k/2^{n+1+k}, (2^k + 2)/2^{n+1+k})$ for some n and k.
In this case, we are assuming that $I_e \cap K_D = \emptyset$ and neither endpoint is a check point. Then we have that

$$\overline{I_e} \cap K_D \neq \emptyset \Leftrightarrow \frac{1}{2^{n+1}} \in K_D \Leftrightarrow n \in D.$$

Now if $n = 3m$, then $n \in D$ and $bin(e) \in U$ so that we let $f(bin(e)) = bin(2^{3[5(\langle n,k \rangle)+3]})$ where $\langle , \rangle$ denote some $1:1$ polynomial time pairing function from $Bin(\omega) \times Bin(\omega) \to Bin(\omega)$. If $n = 3m + 2$, then $n \notin D$ and hence $bin(e) \notin U$ so that we can let $f(bin(e)) = bin(e)^\frown 1$ which not in D. Finally if $n = 3m + 1$, then we can write m in the form $(2^s)(2t + 1)$. Then $n \in D$ if and only if $s \in C$. Note that we can compute n, k, s and t in linear time in $|bin(e)|$. We then let $f(bin(e)) = bin(2^{3(2^s)(2\langle t,2k \rangle+1)+1}$. Then our definition ensures that $bin(e) \in U$ if and only if $f(bin(e)) \in D$ in this case.

Case C: Not Case A and $I_e = ((2^k - 2)/2^{n+1+k}, 2^k/2^{n+1+k})$ for some n and k.

Once again in this case, we are assuming that $I_e \cap K_D = \emptyset$ and neither endpoint is a check point. As in Case B, we have that

$$\overline{I_e} \cap K_D \neq \emptyset \Leftrightarrow \frac{1}{2^{n+1}} \in K_D \Leftrightarrow n \in D.$$

Now if $n = 3m$, then $n \in D$ and $bin(e) \in U$ so that we let $f(bin(e)) = bin(2^{3[5(\langle n,k \rangle)+4]})$. If $n = 3m + 2$, then $n \notin D$ and hence $bin(e) \notin U$ so that we can let $f(bin(e)) = bin(e)^\frown 1$ which not in D. Finally if $n = 3m + 1$, then we can write m in the form $(2^s)(2t + 1)$. Then $n \in D$ if and only if $s \in C$. Note that we can compute n, k, s and t in linear time in $|bin(e)|$. We then let $f(bin(e) = bin(2^{3(2^s)(2\langle t,2k+1 \rangle+1)+1}$. Then our definition ensures that $bin(e) \in U$ if and only if $f(bin(e)) \in D$ in this case.

It is now easy to check that f is a $1:1$ polynomial time function such that $\sigma \in U \Leftrightarrow f(\sigma) \in D$ for all $\sigma \in \{0,1\}^*$.

To see that $D \leqslant_1^p U$, first observe that

$$2^n \in D \Leftrightarrow \left[\frac{6}{2^{n+4}}, \frac{8}{2^{n+4}}\right] \cap K_D \neq \emptyset \Leftrightarrow 2^{n+4+1} + 7 \in \{e : I_e \cap K_D \neq \emptyset\}.$$

Also note that any string of the form $bin(e) = 111^\frown\sigma$ where $\sigma \in \{0,1\}^*$ corresponds to an interval $I_e \subseteq (\frac{1}{2}, 1]$ so that $I_e \cap K_D = \emptyset$. We can then define our desired $1:1$ polynomial time function g which shows that $D \leqslant_1^p U$ as follows. First if $\sigma \notin Bin(\omega)$, then we let $g(\sigma) = \sigma$. We define g on $Bin(\omega)$ by setting $g(bin(2^n)) = bin(2^{n+5} + 7)$ for all $n \in \omega$ and setting $g(bin(k)) = 111^\frown bin(k)$ if k is not of the for 2^n. $\square$

We should note that in Theorem 5.5, the fact that $\{bin(n) : K_D \cap \overline{I}_n = \emptyset\}$ is 1–1 polynomial time equivalent to what is essentially a tally set is an artifact of our coding of the intervals I_n. Under other natural codings, $\{bin(n) : K_D \cap \overline{I}_n = \emptyset\}$ will be polynomial time equivalent to essentially arbitrary computable subsets of $Bin(\omega)$ as C varies. However, these other codings do not have the same nice properties with respect to other complexity theoretic analogues of the results of this paper. We do not have the space to discuss such codings in full detail in this paper, but these issues will be addressed in a subsequent paper. However since 1–1 polynomial time equivalence preserves the properties of being NP, CoNP, or polynomial time, we have the following immediate corollary of Theorem 5.5.

Theorem 5.6. (1) *There is an OI polynomial time decidable closed set K of* $[0,1]$ *such that K is not* CINP *or* CICoNP.

(2) *If* P $\neq$ NP *for tally sets, then there is an OI polynomial time decidable closed set K of* $[0,1]$ *such that K is* CINP *but not CI polynomial time decidable and there is an OI polynomial time decidable closed set L of* $[0,1]$ *such that L is* CICoNP *but not CI polynomial time decidable.*

(3) *If* NP $\neq$ CoNP *for tally sets, then there is an OI polynomial time decidable closed set K of* $[0,1]$ *such that K is* CINP *but not* CICoNP *and there is an OI polynomial time decidable closed set L in* $[0,1]$ *such that L is* CICoN*P but not* CINP.

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