JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 71, 580–589 (1979)

On Complex Strictly Convex Spaces, II

VASILE I. ISTRĂŢESCU

Submitted by Ky Fan

1. INTRODUCTION

This paper is a continuation of paper [12] and contains further results about complex strictly convex spaces.

The notion of complex strictly convex space is considered in the work of Thorp and Whitley [11] in which it is proved that $L^1(\Omega, B, P)$ is a complex strictly convex space. In this paper we consider also the problem of complex smoothness which was posed to the author by Professor G. Köthe and also the problem of isomorphism of a Banach space with a complex strictly convex space.

2. DEFINITIONS. SOME SUFFICIENT CONDITIONS FOR A POINT TO BE A COMPLEX EXTREME POINT

Let X be a complex Banach space and $S_X = \{x, x \in X, \|x\| = 1\}$.

DEFINITION 2.1. The point $u \in S_X$ is called a complex extreme point if whenever there exists $v \in X$ such that $\|u + sv\| \leq 1$, $|s| \leq 1$, then necessarily $v = 0$. X is called complex strictly convex if all points in S_X are complex extreme points.

DEFINITION 2.2. A point $u \in S_X$ is called a point of local uniform convexity (in short, l.u.c.) if

$$\tfrac{1}{2}(x_n + y_n) \to u,$$

where $x_n \in S_X$ and $y_n \in S_X$, then $x_n \to u$, $y_n \to u$.

PROPOSITION 2.3. *Every l.u.c. is a complex extreme point.*

Proof. The proof is very simple and is as follows: Suppose that u is not a

0022-247X/79/100580–10$02.00/0

complex extreme point and thus we find an element $v \in X$ such that $\| u + sv \| \leq 1$ for all s, $| s | \leq 1$. We take $s_n \to 1$ and define

$$x_n = u + s_n v, \qquad y_n = u - s_n v,$$

and since u is an l.u.c. we obtain that $x_n \to u$ and $y_n \to u$. But $x_n \to u + v$ and thus we obtain $v = 0$. This contradiction proves the proposition.

DEFINITION 2.4. A point $u \in S_X$ is called a point of weakly local uniform convexity if

$$\tfrac{1}{2}(x_n + y_n) \rightharpoonup u \qquad (\rightharpoonup \text{ denotes the weak convergence}),$$

where $x_n \in S_X$ and $y_n \in S_X$, then $x_n \to u$ and $y_n \to u$.

PROPOSITION 2.5. *Any point of weak local uniform convexity is a complex extreme point.*

Proof. Since it does not differ essentially from the above proof we omit it.

We present now a sufficient condition for a Banach space to be complex strictly convex. Our condition is related to a condition used for real Banach spaces in a recent announcement of Holub [4].

DEFINITION 2.6. For every normalized pair of linearly independent elements of X, $\{x_1, x_2\}$, let

$$[x_1, x_2] \text{ be the set of all } y = a_1 x_1 + a_2 x_2 \quad \text{with} \quad \operatorname{Im} a_1 = \operatorname{Im} a_2;$$
$$C[x_1, x_2] \text{ be the set of all } y = a_1 x_1 + a_2 x_2 \quad \text{with} \quad \operatorname{Re} a_1 \cdot \operatorname{Re} a_2 \geq 0.$$

THEOREM 2.7. *If X is a complex Banach space and for every normalized pair x_1, x_2 in X the points in $[x_1, x_2]$ equidistant from x_1 and x_2 are in $C[x_1, x_2]$, then X is a complex strictly convex space.*

Proof. Suppose that X is not a complex strictly convex space and thus we find a point x in S_X such that for some $y \in X$ and all s, $| s | \leq 1$, $\| x + sy \| \leq 1$.

Let s_1 and s_2 be two complex numbers such that (1) $| 2s_2 - s_1 | < 1$ and (2) $| s_i | < 1$ for $i = 1, 2$.

The elements $x_1 = x + s_1 y$ and $x_2 = x - s_2 y$ have the property $\| x_1 \| = \| x_2 \| = 1$. The point $z = (s_1 - s_2) y$ is equidistant from x_1 and x_2. Indeed,

$$x_1 - z = x + s_1 y - (s_1 - s_2) y = x + s_2 y,$$
$$x_2 - z = x + s_2 y - (s_1 - s_2) y = x + (2s_2 - s_1) y$$

and it follows that these are elements of S_X. Since $(s_1 - s_2) y = x_1 - x_2$ is not in $C[x_1, x_2]$ we have a contradiction.

3. Renorming X such that X and X^* are Complex Strictly Convex Spaces

As is well known, if X is a real Banach space such that there exist two equivalent norms, $\|\cdot\|_1$ and $\|\cdot\|_2$ such that $(X, \|\cdot\|_1)$ is a strictly convex space and $(X^*, \|\cdot\|_2^*)$ is a strictly convex space where $\|\cdot\|_2^*$ means the norm induced by $\|\cdot\|_2$ on the dual space, then there exists a new norm equivalent to the original one such that $(X, \|\cdot\|_3)$ possesses both of these properties.

In what follows we prove that the same result holds for the case of complex strict convexity.

Theorem 3.1. *Let X be a complex Banach space with an equivalent complex strictly convex norm (i.e., the space with this norm is complex strictly convex), and another norm whose dual norm on X^* is a complex strictly convex norm. Then there exists a new equivalent norm on X which possesses both of these properties.*

Proof. The proof is a consequence of a remark concerning Asplund's averaging norms: If f_0, g_0 are two functions, homogeneous of the second degree (i.e., $f_0(tx) = t_0^2 f(x)$, $g_0(t_0x) = t^2 g_0(x)$ for all $x \in X$ and $t > 0$), $g_0 \leqslant f_0 \leqslant (1 + c)\, g_0$, then the sequences

$$\{f_n\}, \qquad \{g_n\}$$

where

$$f_n(x) = \tfrac{1}{2}(f_{n-1}(x) + g_{n-1}(x)), \qquad g_n(x) = \inf_{y}\{\tfrac{1}{2}(f_{n-1}(x + y) + g_{n-1}(x - y))\}$$

define a function h such that

$$(1/1 - 2^{-n}C)\, h \leqslant g_n \leqslant h \leqslant f_n \leqslant (1/1 + 2^{-n}C)\, h,$$
$$g_n \leqslant f_n \leqslant (1/1 + 4^{-n}C)\, g_n .$$

Also, it is a consequence of these relations that for all n (see [3, p. 111])

$$2^{-n}[f_0(x) f_0(y) - 2f_0((x + y)/2) - C2^{-n}(f_0(x) + f_0(x))] \leqslant h(x) + h(y) - 2h((x + y)/2). \tag{*}$$

Suppose now that f_0 or g_0 has the property that if for some $y \in X$,

$$\sup_{|\xi| \leqslant 1}(f_0(x + \xi y) + f_0(x - \xi y)) = 2f_0(x), \tag{**}$$

then necessarily $y = 0$; then we show that h shares this property.

Indeed, let $\tilde{x} = x + \xi y$, $\tilde{y} = x - \xi y$ in (*). Then we have

$$2^{-n}[f_0(x + \xi y) + f_0(x - \xi y) - 2f_0(x) - C2^{-n}(f_0(x + \xi y) + f_0(x - \xi y)] \leqslant 0,$$

and this implies that

$$f_0(x + \xi y) + f_0(x - \xi y) - 2f_0(x) = 0.$$

But this last relation gives that $y = 0$. This proves our assertion.

We remark that the same conclusion is valid if we suppose that g_0 has property (**).

We remark that the function

$$\xi \to G(x + \xi y) + G(x - \xi y)$$

is a subharmonic function whenever G is a convex function and positive homogeneous of the second degree. Applying this remark to the above situation we conclude that

$$f_0(x + \xi y) = f_0(x - \xi y) = f_0(x).$$

Now if $f_0(x) = \frac{1}{2} \| x \|_1^2$, $g_0(x) = \frac{1}{2} \| x \|_2^2$ then we obtain using g^*, the conjugate of f_0, the functions h and h^* which are both complex strictly convex on X and X^*, respectively. Then

$$x \to [2h(x)]^{1/2}$$

is a norm which satisfies the theorem.

4. Complex Smoothness

Let X be a complex Banach space and X^* its dual.

Definition 4.1. X is said to be complex smooth at x if whenever $f \in X^*$, $\| f \| = 1$ and $| f(x) - \xi g(x) | \leq 1$ for all ξ, $| \xi | \leq 1$ then necessarily $g = 0$.

The relation between complex smoothness and complex strict convexity is given in

Theorem 4.2. *If X^* is a complex smooth space then it is complex strictly convex.*

Proof. Suppose that X is not complex strictly convex, thus we find x and y such that

(1) $\| x + \xi y \| = 1$ for all ξ, $| \xi | \leq 1$,

(2) $y \neq 0$.

If g is an arbitrary element of $S(X^*)$ then, if F_x denotes the element x as element of X^{**}, we have

(1) $|F_x(g) + \xi F_y(g)| \leq 1$,

(2) $\| F_x + \xi F_y \| = 1$.

Since $y \neq 0$ we have that for some $g \in X^*$, $F_x(g) = 1$ and this implies that $F_y = 0$. This contradiction proves the theorem.

We do not know if the following assertion is true: If X^* is a complex strictly convex space then X is complex smooth. If the answer is "yes" then the notions of complex smooth and complex strictly convex spaces are in some sense dual notions.

5. A Necessary and Sufficient Condition for a Banach Space to Have an Equivalent Complex Strictly Convex Norm

If E is a complex Banach space we say that E has a complex strictly convex norm $\| \cdot \|$ if $\{E, \| \cdot \|\}$ is a complex strictly convex space.

As is well known for the strict convexity there exists a very general result about possible renormings of E such that with the new norm the space is strictly convex. We give now an extension of this result for the case of complex strict convexity. We mention that our proof depends on the theory of subharmonic functions.

THEOREM 5.1. *A complex Banach space E has an equivalence complex strictly convex norm iff there exists a complex strictly convex space F and a one-to-one lineat continuous operator $T: E \to F$.*

Proof. It is obvious that the condition is necessary, since in this case we can take T to be the identity and F is, of course, E with the corresponding complex strictly convex norm.

We prove that the condition is also sufficient. For this we remark that the functions,

$$\begin{aligned} \mu &\to \| x + \mu y \|, \\ \mu &\to \| T(x + \mu y) \|, \end{aligned} \tag{*}$$

are for each x and y fixed, subharmonic functions. We define the new norm on E by the relation,

$$x \to \{\tfrac{1}{2}(\| x \|^2 + \| Tx \|^2)\}^{1/2},$$

and note that it is equivalent with the original norm.

Suppose that $x \in E$ has the norm 1 in this new norm and thus,

$$\| x \|^2 + \| Tx \|^2 = 2. \tag{**}$$

If x is not a complex extreme point, we find $y \in E$ such that the norm of the elements $x + \mu y$, for $|\mu| \leqslant 1$, is equal to one. Thus for μ as above we have

$$\|x + \mu y\|^2 + \|T(x + \mu y)\|^2 = 2. \qquad (***)$$

Since the functions in (*) are subharmonic and satisfy conditions (**) and (***) for μ, $|\mu| \leqslant 1$, their values are constant.

Thus we have

$$\|x + \mu y\| = K_1 ,$$

$$\|T(x + \mu y)\| = K_2 .$$

The element Tx is a complex extreme point of the ball of radius equal to K_2 and this gives that Ty is the null element. Since T is one to one, it follows that y is the null element. Thus x is a complex extreme point with respect to this new norm.

6. Very Complex Strictly Convex Spaces

In this section we mention a possibly interesting class of complex Banach spaces suggested by some recent results concerning the higher duals of a Banach space [10].

Definition 6.1. A complex Banach space E is said to be very complex strictly convex if E as a subspace of E'' has the property that every element $x \in E$ is a complex extreme poit of the unit ball of E''.

It is easy to see that if E is very complex strictly convex then it is also complex strictly convex. Also every reflexive complex strictly convex space is very complex strictly convex.

An interesting problem appears in connection with Köthe's question translated to this setting: What is the connection between E and quotient spaces with respect to the notion considered in Definition 6.1 ? We intend to give, in a future paper, results in this direction as well as some related results concerning Köthe's question with respect to the rotundity and smoothness.

7. An Example of a Complex Normed Space not Isomorphic to Complex Strictly Convex Space

In [2] Day gives the first example of a normed space not isomorphic to strictly convex or smooth spaces. In what follows we show that the example of Day can be used to give an example of a normed space (complex normed space) which is not isomorphic to a complex strictly convex space.

Let I be any uncountable index set and $m(I)$ be the space of all bounded complex-valued functions on I with the norm,

$$\| x \| = \operatorname*{lub}_{i \in I} | x(i) | ,$$

and $m_0(I)$ be the subspace of all those x in $m(I)$ which vanish except on a countable set.

THEOREM 7.1. *The space $m_0(I)$ is not isomorphic to any complex strictly convex space.*

Proof. Let U be the unit sphere in $m_0(I)$ and $| \ |$ the new norm equivalent to the usual norm of $m_0(I)$. We can suppose without loss of generality that for all $x \in m_0(I)$

$$\| x \| \leqslant | x | \leqslant k \| x \|$$

for some $k \geqslant 1$. The facet of U determined by x is the set of all $y \in U$ such that $y(i) = x(i)$ at every point where $x(i) \neq 0$.

If we define

$$M_x = \sup\{ | y | , y \in F_x \text{, the facet determined by } x\},$$

and

$$m_x = \inf\{ | y | , y \in F_x\},$$

then in Day's paper it is proved that

$$M_x + m_x \geqslant 2 | x |$$

and also a sequence $\{x_n\}$ is constructed. With this sequence an element of $m_0(I)$ is defined: If there is an n such that $x_n(i) \neq 0$ let $x(i) = x_n(i)$; if all $x_n(i) = 0$, let $x(i) = 0$. One the sphere of radius $| x |$ we can find elements y such that $| x + \mu y | \leqslant 1$ since the construction of x involves only a countable set, and this proves the theorem.

8. COMPLEX POLYHEDRAL BANACH SPACES

We consider now a special class of Banach spaces connected with the notion of complex extreme points and with complex strictly convex spaces. For any complex Banach space X, $\mathrm{Ext}_c(K)$ denotes the set of all complex extreme points of the set K.

DEFINITION 8.1. A complex Banach space X is said to be complex polyhedral iff for every finite-dimensional subspace E of X, $S(E)$ has only finitely many complex extreme points.

We give now some examples.

EXAMPLE 8.2. Let c_0 be the space of all sequences of complex numbers converging to zero. We prove that any subspace of c_0 with the property that its unit ball has a complex extreme point is finite-dimensional.

First we prove that for any complex extreme point p of $S(E)$

$$I = \{i: | x_i | = 1\}, \qquad p = (x_i),$$

and $x \in c_0 \cap E$ with $x = 0$ for $i \in I$ implies $x = 0$.

Indeed for any complex number ξ, $| \xi | \leqslant 1$ we have

$$\| p + \xi x \| \leqslant 1$$

since for each i

$$| p + \xi x | \leqslant 1 \tag{*}$$

because we can suppose, without loss of generality, that $\| x \| \leqslant 1$.

The relation (*) is clear for any $i \in I$ and also for $i \notin I$. But p is a complex extreme point and this gives that $x = 0$ and thus our assertion is proved.

Since the unit ball of E has a complex extreme point p, we define the set I as above. For each $y \in E$ we define the map

$$T: E \rightarrow C^I \qquad (C \text{ is the field of complex numbers}),$$

by the relation

$$Ty = (y/I) \in C^I.$$

Obviously T is a linear mapping into the finite-dimensional complex space C^I.

From the above assertion about complex extreme points it follows, since the kernel of T is zero, that E is finite dimensional.

We prove now that in fact E is a complex polyhedral space.

Indeed, since E is finite dimensional we find a basis, say $\{e_1 ,..., e_n\}$, and put $V = \text{conv}[e_i , -e_i]$, where

$$[e_i , -e_i] = \{x, x = -r \cdot e_i + (1 - r) e_i , 0 < r < 1\}.$$

Clearly, V is a convex body in E, 0 is in the interior of V, and of course $S_{(E)}$ the unit ball of E is compact. We thus find an integer m such that $S_{(E)}$ is in mV. Thus if $v \in V$ and $v = (v^i)$ we have $| v^i | < 1/m$ and similarly for $u = (u^i)$ since from the property of $S_{(E)}$ we find an integer N such that $| q_i^j | < 1/m$ for $1 \leqslant i \leqslant r$ and all $j > N$.

In this case for each complex extreme point p we must have a subset $A_p = \{i, p_i = 1\}$ and a subset $B_p = \{j, p_j = i\}$ and similarly for the sets -1 and $-i$, denoted by C_p and D_p, respectively.

If $A_p = A_{p_1}, \dots, D_p = D_{p_1}$ it follows by an application of the property proved at the beginning of the example that $p = p_1$. This proves our assertion.

We give now a characterization of finite-dimensional spaces which are complex polyhedral.

THEOREM 8.3. *A necessary and sufficient condition for a finite-dimensional space to be complex polyhedral is that all 2-dimensional subspaces will be complex polyhedral.*

Proof. The condition is obviously necessary. Suppose that it is satisfied. We prove the assertion by induction. Argument:

Suppose that the assertion is true for all $i < n$ and we prove it for $n + 1$. Let F be a subspace such that $\dim F = n + 1$. If $\{e_i\}$ is the set of all complex extreme points then it is clear that the number of linearly independent complex extreme points is less than or equal to $n + 1$. We can decompose the set $\{e_i\}$ in two subsets, each of them containing a number of linearly independent points which is less than or equal to n. In this case it follows that the set of all complex extreme points is finite, and the theorem is proved.

Remark 8.4. We think that the following assertion is true: A Banach space is complex polyhedral if there does not exist a sequence $\{x_n\}$ in the Banach space such that for every choice of signs

$$\| x_n \pm x_m \| \leqslant \| x_n \| + \| x_m \| - 1, \qquad (*)$$

ACKNOWLEDGMENT

This work was done while the author was an Alexander von Humboldt fellow at the University of Frankfurt am Main. The author thanks Professor G. Köthe for his warm encouragement and useful discussions. The author thanks A. Humboldt Foundation for some financial support.

REFERENCES

1. E. ASPLUND, Averaged norms, *Israel J. Math.* **5** (1967), 227–233.
2. M. M. DAY, Strict convexity and smoothness of normed spaces, *Trans. Amer. Math. Soc.* **78** (1955), 516–528.
3. J. DIESTEL, "Geometry of Banach Spaces–Selected Topics," Lecture Notes in Mathematics No. 485, Springer–Verlag, Berlin/New York, 1975.
4. J. R. HOLUB, Rotundity, orthogonality, and characterizations of inner product spaces, *Bull. Amer. Math. Soc.* **81** (1975), 1087–1089.
5. V. KLEE, Some new results on smoothness and rotundity in normed linear spaces, *Math. Ann.* **139** (1959), 51–63.

6. V. KLEE, Mappings into normed linear spaces, *Fund. Math.* **49** (1960), 25–34.
7. G. KÖTHE, "Topological Vector Spaces," Springer-Verlag, Berlin/New York, 1960.
8. K. FAN AND I. GLICKSBERG, Some geometrical properties of spheres in a normed space, *Duke Math. J.* **25** (1958), 553–568.
9. J. LINDENSTRAUS, Extension of compact operators, *Mem. Amer. Math. Soc.* No. 48.
10. F. SULLIVAN, Geometrical properties determined by the higher duals of a Banach space, *Illinois J. Math.*, in press.
11. E. THORP AND R. WHITLEY, The strong maximum modulus theorem for analytic functions into a Banach space, *Proc. Amer. Math. Soc.* **18**, No. 4 (1967), 640–646.
12. I. ISTRĂŢESCU AND V. ISTRĂŢESCU, On complex strictly convex spaces, I, *J. Math. Anal. Appl.* **70** (1979), 423–429.