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REVIEW ARTICLE

Mathematical analysis of the Generalized Benjamin and Burger-Kdv Equations via the Extended Trial Equation Method

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Abstract In this paper, using the Extended Trial Equation (ETEM), we get new traveling wave solutions of the Generalized Benjamin, the Generalized Burger-Kdv Equations (GBE, GBKE). The obtained solutions not only constitute a novel analytical viewpoint in nonlinear complex phenomena, but they also form a new stand alone basis from which physical applications in this arena can be comprehended further, and moreover investigated. Furthermore, to concretely enrich this research production, we provide illustrative, Mathematica Release 7 based, 3D graphics of the gotten solutions, with well chosen, yet structure revealing parameters.

1. Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in many fields of applied sciences, such as chemistry, physics, and the engineering disciplines. Over the last few decades, seeking new traveling wave solutions of nonlinear partial differential equations, became a mandatory task, to significantly comprehend and describe complex phenomena. Well designed mathematical models accurately describing studied phenomena, can only enhance the chances of achieving analytical solutions, thereby yielding a better physical understanding of the phenomena. In the last decade, several techniques, such as sumudu transform method (Belgacem, 2006, 2010; Belgacem and Hussain, 2007; Katatbeh and Belgacem, 2011; Chaurasia et al., 2012; Gupta et al., 2011; Bulut et al., 2012), ansatz method, mapping method, three-wave method, the adomian decomposition method, soliton perturbation theory (Antonova and Biswas, 2009), Euler–Lagrange operator (Kara et al., 2013) He's variational principle (Girgis and Biswas, 2011) have been carried out for solving these differential equations (Mittal and Nigam, 2008; Achouri and Omrani, 2009) in

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terms of singular and soliton solutions (Razborova et al., 2013; Song et al., 2013; Ahmed and Biswas, 2013; Biswas, 2012).

In 2005, Liu (2005, 2006, 2010) initiated a different approach, which is now recognized as the Trial Equation Method, as an alternative. Recently, Gurefe et al. (Pandir et al., 2012, 2013a,b; Gurefe and Misirli, 2010; Gurefe et al., 2011, 2013) used the trial equation method and its extended version, to obtain new exact solutions of some generalized evolution equations. This was an important step towards the modelling of nonlinear complex phenomena.

In this study, we aim to provide elliptic functions and Jacobi elliptic functions solutions of the Generalized Benjamin Equation (GBE), and Generalized Burger-Kdv Equation (GBKE), as an application of the Extended Trial Equation Method (ETEM).

In Section 2, of this paper, we give the description of the ETEM, while in Section 3, pursuing suggestions and using the setup in Taghizadeh et al. (2012), we show how to get some exact solutions to the GBE,

$$u_{tt} + a(u^p u_x)_x + \beta u_{xxxx} = 0, \tag{1}$$

and the GBKE (Zhang et al., 2002),

$$u_t + au^p u_x + bu^{2p} u_x + \delta u_{xxx} = 0, \tag{2}$$

where $a, \quad b, \quad \beta, \quad \delta$ and p are arbitrary constants. In the discussion, we propose an even more generalized version of ETEM, and we will label it GETEM, for future reference.

2. The Extended Trial Equation Method

In this part of the manuscript, the Extended Trial Equation Method will be given. In order to apply this method to the GBE, we consider the following steps.

Step 1. We consider the Generalized Benjamin Equations (GBE), with dependent variable, u,

$$P(u_t, u_{tt}, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{3}$$

on which we apply the wave transformation, for $\lambda \neq 0$,

$$u(x,t) = u(\eta), \quad \eta = kx - \lambda t, \tag{4}$$

to get a nonlinear ordinary differential equation,

$$N(u, u', u'', u''', \ldots) = 0. \tag{5}$$

Step 2. Then, we take the trial equation,

$$u = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{6}$$

where, we have the rational polynomial setup, in $\Phi(\Gamma)$ and $\Psi(\Gamma)$

$$(\Gamma')^2 = \wedge(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\sum_{i=0}^{\theta} \xi_i \Gamma^i}{\sum_{j=0}^{\varepsilon} \zeta_j \Gamma^j} = \frac{\xi_0 + \xi_1 \Gamma + \cdots + \xi_\theta \Gamma^\theta}{\zeta_0 + \zeta_1 \Gamma + \cdots + \zeta_\varepsilon \Gamma^\varepsilon}. \tag{7}$$

Consequently, we get,

$$u' = \sqrt{\frac{\Phi(\Gamma)}{\Psi(\Gamma)}} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right) \quad \text{and} \quad (u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right)^2, \tag{8}$$

and,

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2} \right), \tag{9}$$

Substituting the relations Eqs. (7)–(9) into Eq. (5), yields polynomial equation in Γ,

$$\Omega(\Gamma) = \rho_s \Gamma^s + \cdots + \rho_1 \Gamma + \rho_0 = 0. \tag{10}$$

According to the balance principle, we can then compute some values of $\theta, \quad \varepsilon$ and δ.

Step 3. Letting the coefficients of $\Omega(\Gamma)$ be all null, yields a system of algebraic equations,

$$\rho_i = 0, \quad i = 0, \ldots, s. \tag{11}$$

Solving the algebraic system, helps specify the values of $\xi_0, \quad \xi_1, \ldots, \quad \xi_\theta$ and $\zeta_0, \quad \zeta_1, \ldots, \quad \zeta_\varepsilon$.

Step 4. Reducing Eq. (11) to the elementary integral form, we get,

$$\pm(\eta - \eta_0) = \int \frac{1}{\sqrt{\wedge(\Gamma)}} d\Gamma = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma. \tag{12}$$

Using a complete discrimination system for polynomial to classify the roots of $\Phi(\Gamma)$, we solve Eq. (12), with the help of Mathematica Release 7, and classify the exact solutions to Eq. (5). In addition, we can write the exact traveling wave solutions to Eq. (3), respectively.

3. Applications

In this section, we seek the exact solution of the GBE and GKDE by using ETEM. Subsequently, we drew 3D surfaces of the analytical solutions obtained by using ETEM.

Example 1. We consider the GBE, suggested in (Gurefe et al., 2011), given in Eq. (1). Many researchers have tried to get the approximate solutions of this equation by using a variety of methods. Let us consider the travelling wave solutions of Eq. (1), and we perform the transformation $u(x,t) = u(\eta)$, and $\eta = kx - \lambda t$, where k and λ are constants. Hence, integrating this equation with respect to η, and setting the integration constant to zero, we get,

$$\lambda^2 k^2 (p+1) v^2 + \alpha k p^2 v^3 + \beta k^4 (1 - p^2)(v')^2 + \beta k^4 p(1 + p) v v'' = 0. \tag{13}$$

Substituting, Eqs. (7)–(9) into Eq. (13), and using balance principle yields,

$$\theta = \varepsilon + \delta + 2. \tag{14}$$

When this resolution procedure is applied, we get the following cases.

Case 1. If we take $\varepsilon = 0, \quad \delta = 1$, and $\theta = 3$, then, according to Eqs. (7)–(9), with, $\zeta_0 \neq 0$. we get,

$$v' = \frac{\xi_0 + \xi_1\Gamma}{\zeta_0}, \tag{15}$$

$$v'' = \frac{\tau_1\left(\xi_1 + 2\xi_2\Gamma + 3\xi_3\Gamma^2\right)}{2\zeta_0}, \tag{16}$$

$$(v')^2 = \frac{\tau_1^2\left(\xi_0 + \xi_1\Gamma + \xi_2\Gamma^2 + \xi_3\Gamma^3\right)}{\zeta_0}. \tag{17}$$

Therefore, we have a system of algebraic equations from the coefficients of polynomial of Γ. After we substitute Eqs. (15)–(17) into Eq. (13), we get an algebraic equation system. When we solve this system by using Mathematica Release 7, we obtain the following relations;

$$\xi_0 = -\frac{\tau_0(\xi_2\tau_0 - 2\xi_1\tau_1)}{3\tau_1^2}, \quad \xi_1 = \xi_1, \quad \xi_2 = \xi_2,$$
$$\xi_3 = \frac{\tau_1(2\xi_2\tau_0 - \xi_1\tau_1)}{3\tau_0^2}, \tag{18}$$

$$\zeta_0 = -\frac{k^3\beta(p^2+3p+2)(2\xi_2\tau_0 - \xi_1\tau_1)}{6p^2a\tau_0^2}, \tag{19}$$

$$\lambda = \frac{\sqrt{6a\tau_0(\xi_2\tau_0 - \xi_1\tau_1)}}{\sqrt{k(p^2+3p+2)(\xi_1\tau_1 - 2\xi_2\tau_0)}}. \tag{20}$$

Substituting these coefficients into Eqs. (7) and (12), we have,

$$\pm(\eta - \eta_0) = A\int \frac{1}{\sqrt{B + C\Gamma + D\Gamma^2 + E\Gamma^3}}d\Gamma \tag{21}$$

where the coefficients, B, C, D and E are defined by,

$$A = \sqrt{\frac{k^3\beta\tau_1^2(p^2+3p+2)(\xi_1\tau_1 - 2\xi_2\tau_0)}{2p^2a}},$$
$$B = \tau_0^3(2\xi_1\tau_1 - \xi_2\tau_0), \quad C = 3\tau_1^2\tau_0^2\xi_1,$$
$$D = \tau_1^2 3\tau_0^2\xi_2, \quad E = \tau_1^3(2\xi_2\tau_0 - \xi_1\tau_1). \tag{22}$$

Integrating Eq. (21), we obtain the following equations ,

$$\pm(\eta - \eta_0) = -\frac{A}{v - \alpha_1}, \tag{23}$$

$$\pm(\eta - \eta_0) = \frac{A}{\sqrt{\alpha_1 - \alpha_2}}\ln\left|\frac{\sqrt{v-\alpha_2} + \sqrt{\alpha_1 - \alpha_2}}{\sqrt{v-\alpha_2} - \sqrt{\alpha_1 - \alpha_2}}\right|, \ \alpha_2 > \alpha_1, \tag{24}$$

$$\pm(\eta - \eta_0) = \frac{-2A}{\sqrt{\alpha_2 - \alpha_1}}F(m,n), \ \alpha_3 > \alpha_2 > \alpha_1, \tag{25}$$

where, $m = arc\sin\left(\sqrt{\frac{\alpha_2-\alpha_1}{v-\alpha_1}}\right)$, $n = \frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}$ and $F(m,n)$ is the elliptic function.Furthermore, the values, α_1, α_2 and α_3 are the roots of the polynomial equation,

$$B + C\Gamma + D\Gamma^2 + E\Gamma^3 = 0. \tag{26}$$

Therefore, we find the solutions,

$$u(x,t) = \left[\alpha_1 + \frac{4A^2}{(kx - \lambda t - \eta_0)^2}\right]^{1/p}, \tag{27}$$

$$u(x,t) = \left[\alpha_1 + (\alpha_2 - \alpha_1)\sec^2\left(\frac{(\eta - \eta_0)\sqrt{\alpha_2 - \alpha_1}}{2A}\right)\right]^{1/p}, \tag{28}$$

$$u(x,t) = \left[\alpha_3 + (\alpha_2 - \alpha_3)sn^2\left(\frac{(-\eta + \eta_0)\sqrt{\alpha_1 - \alpha_3}}{2A}, \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}\right)\right]^{1/p}, \tag{29}$$

where, λ, is then given by,

$$\lambda = \frac{\sqrt{6a\tau_0(\xi_2\tau_0 - \xi_1\tau_1)}}{\sqrt{k(p^2+3p+2)(\xi_1\tau_1 - 2\xi_2\tau_0)}}. \tag{30}$$

For simplicity, if we take $\eta_0 = 0$, then the solutions, Eqs. (27)–(29), are reduced to the rational and single kink solution, respectively,

$$u(x,t) = \left[\alpha_1 + 4A^2(kx - \lambda t)^{-2}\right]^{1/p}, \tag{31}$$

$$u(x,t) = \left[\alpha_1 + (\alpha_2 - \alpha_1)\sec^2\left(0.5(kx - \lambda t)A^{-1}\sqrt{\alpha_2 - \alpha_1}\right)\right]^{1/p}, \tag{32}$$

$$u(x,t) = [\alpha_3 + (\alpha_2 - \alpha_3)f(r,s)]^{1/p}, \tag{33}$$

where, we have, $r = \frac{-\sqrt{\alpha_1 - \alpha_3}}{2A}$, $s = \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}$ and $f(r,s) = sn^2(r(kx - \lambda t), s)$.

Remark 1. The solutions Eqs. (31)–(33) were obtained by using the Extended Trial Equation Method for Eq. (1), have been checked by Mathematica Release 7. To our knowledge, the rational function solution and single kink solution that we find in this paper, are new, and are not shown in the published literature. Consequently, we believe these results are new Jacobi-elliptic function solutions of Eq. (1).

Case 2. If we take $\varepsilon = 0$, $\delta = 2$, and $\theta = 4$, then, according to Eqs. (7)–(9), we get the following,

$$v = \tau_0 + \tau_1\Gamma + \tau_2\Gamma^2, \tag{34}$$

$$v'' = (\tau_1 + \tau_2\Gamma)\frac{\xi_1 + 2\xi_2\Gamma + 3\xi_3\Gamma^2 + 4\xi_4\Gamma^3}{2\zeta_0} + 2\tau_2$$
$$\times\frac{\xi_0 + \xi_1\Gamma + \xi_2\Gamma^2 + \xi_3\Gamma^3 + \xi_4\Gamma^4}{\zeta_0}, \tag{35}$$

$$(v')^2 = (\tau_1 + \tau_2\Gamma)^2\frac{\xi_0 + \xi_1\Gamma + \xi_2\Gamma^2 + \xi_3\Gamma^3 + \xi_4\Gamma^4}{\zeta_0}, \tag{36}$$

where $\zeta_0 \neq 0$. Therefore, we have a system of algebraic equations from the coefficients of polynomial of Γ. Solving the algebraic equation system Eqs. (34)–(36) by using Mathematica Release 7, yields the following relations,

$$\xi_0 = -\frac{ap^2\zeta_0\tau_0^2}{2k^3(2+3p+p^2)\beta\tau_2},$$
$$\xi_1 = -\frac{ap^2\zeta_0\tau_1}{k^3(2+3p+p^2)\beta\tau_2},$$
$$\xi_2 = -\frac{ap^2\zeta_0\tau_2}{2k^3(2+3p+p^2)\beta},$$
$$\xi_3 = -\frac{ap^2\zeta_0\tau_1}{k^3(2+3p+p^2)\beta},$$
$$\xi_4 = -\frac{ap^2\zeta_0\tau_2}{2k^3(2+3p+p^2)\beta},$$
$$\lambda = \pm\frac{\sqrt{a(\tau_1^2 - 4\tau_0\tau_2)}}{\sqrt{2k\tau_2(p^2+3p+2)}}. \tag{37}$$

Substituting these coefficients into Eqs. (34)–(36), we have,

$$\pm(\eta-\eta_0)=\int\frac{1}{\sqrt{A+B\Gamma+C\Gamma^2+D\Gamma^3+E\Gamma^4}}d\Gamma, \tag{38}$$

where the coefficients, A, B, C, D and E, are defined by,

$$A=-\frac{ap^2\tau_0^2}{2k^3(2+3p+p^2)\beta\tau_2},$$

$$B=-\frac{ap^2\zeta_0\tau_1}{k^3(2+3p+p^2)\beta\tau_2},$$

$$C=-\frac{ap^2\tau_2}{2k^3(2+3p+p^2)\beta},$$

$$D=-\frac{ap^2\tau_1}{k^3(2+3p+p^2)\beta},$$

$$E=-\frac{ap^2\tau_2}{2k^3(2+3p+p^2)\beta}. \tag{39}$$

Integrating Eq. (38), we obtain the solutions to the Eq. (1), as follows,

$$\pm(\eta-\eta_0)=\frac{1}{\alpha_1-v}, \tag{40}$$

$$\pm(\eta-\eta_0)=\frac{1}{\alpha_1-\alpha_2}\log\left|\frac{v-\alpha_1}{v-\alpha_2}\right|,\ \alpha_1>\alpha_2, \tag{41}$$

$$\pm(\eta-\eta_0)=\frac{2}{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)}}F(m,n),\ \alpha_1>\alpha_2>\alpha_3>\alpha_4, \tag{42}$$

where, $m=arc\sin\left(\sqrt{\frac{(v-\alpha_2)(\alpha_1-\alpha_4)}{(v-\alpha_1)(\alpha_2-\alpha_4)}}\right)$, $n=\frac{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}$ and $F(m,n)$ is elliptic function. Furthermore, α_1, α_2, α_3 and α_4 are the roots of the polynomial equation,

$$A+B\Gamma+C\Gamma^2+D\Gamma^3+E\Gamma^4=0. \tag{43}$$

In consequence, we find the solutions,

$$u(x,t)=\left[\alpha_1\pm\frac{1}{\eta-\eta_0}\right]^{1/p}, \tag{44}$$

$$u(x,t)=\left[\alpha_2\pm\frac{\alpha_2-\alpha_1}{-1+e^{(\alpha_1-\alpha_2)(\eta-\eta_0)}}\right]^{1/p}, \tag{45}$$

$$u(x,t)=\left[\frac{-\alpha_2\alpha_4 sn^2(r,s)+\alpha_1\alpha_2(sn^2(r,s)-1)+\alpha_4}{-\alpha_2+(\alpha_1-\alpha_4)sn^2(r,s)+\alpha_4}\right]^{1/p}, \tag{46}$$

where, we have,

$$\eta=kx-\lambda t,\quad \lambda=\frac{\sqrt{6a\tau_0(\xi_2\tau_0-\xi_1\tau_1)}}{\sqrt{k(p^2+3p+2)(\xi_1\tau_1-2\xi_2\tau_0)}},$$

$$s=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)},\quad \text{and},$$

$$r=\frac{\pm(\eta-\eta_0)\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}.$$

For simplicity's sake, if we take $\eta_0=0$, solutions to Eqs. (44)–(46) are reduced to rational and single kink solutions, respectively;

$$u(x,t)=\left[\alpha_1\pm\frac{1}{kx-\lambda t}\right]^{1/p}, \tag{47}$$

$$u(x,t)=\left[\alpha_2\pm\frac{\alpha_2-\alpha_1}{-1+e^{\eta(\alpha_1-\alpha_2)}}\right]^{1/p}, \tag{48}$$

$$u(x,t)=\left[\frac{-\alpha_2\alpha_4 sn^2(r,s)+\alpha_1\alpha_2(sn^2(r,s)-1)+\alpha_4}{-\alpha_2+(\alpha_1-\alpha_4)sn^2(r,s)+\alpha_4}\right]^{1/p}. \tag{49}$$

The graphs that follow illustrate the solutions in Eqs. (47)–(49), with well chosen parameters to reveal the salient structures of each. It is feasible that other choices would reveal other views, and other features, and we invite the reader to delve into this dual analytical experimental approach corroborating one another. Should there be any interesting discoveries we hereby invite the interested readers to communicate and collaborate with us, towards hopefully, an even cumulative and better understanding of complex nonlinear phenomena.

Remark 2. The solutions to Eqs. (47)–(49) obtained by using the ETEM for Eq. (1), have been checked by Mathematica, Release 7. To our knowledge, the rational function solution and single kink solution that we find in this paper, are not found in the published literature to date, and hence make for new elliptic function solutions for Eq. (1).

Example 2. In this application of the ETEM, we take into consideration the GBKE (Zhang et al., 2002). Let us consider the travelling wave solutions of Eq. (2) and we perform the transformation $u(x,t)=u(\eta)$, and $\eta=kx-\lambda t$, where k and λ are constants. Then, integrating this equation with respect to η, and setting the integration constant to zero, we get the following equation,

$$-\lambda v^2+\frac{ak}{p+1}v^3+\frac{bk}{2p+1}v^4+\frac{\delta k^3(1-p)}{p^2}(v')^2+\frac{\delta k^3}{p}vv''=0. \tag{50}$$

Substituting, Eqs. (7)–(9), into Eq. (50) and using the balance principle, yields for Eq. (50) $\theta=\varepsilon+2\delta+2$. Applying this resolution procedure, we design the following cases.

Case 1. If we take $\varepsilon=0$, $\delta=2$, and $\theta=6$, then, according to Eqs. (7)–(9), we get,

$$v'=\frac{\tau_1+2\tau_2\Gamma}{\sqrt{\zeta_0}}\times\sqrt{\xi_0+\xi_1\Gamma+\xi_2\Gamma^2+\xi_3\Gamma^3+\xi_4\Gamma^4+\xi_5\Gamma^5+\xi_6\Gamma^6}, \tag{51}$$

$$v''=\frac{(\xi_1+2\xi_2\Gamma+3\xi_3\Gamma^2+4\xi_4\Gamma^3+5\xi_5\Gamma^4+6\xi_6\Gamma^5)}{2\zeta_0}(\tau_1+\tau_2\Gamma)+\tau_2\frac{(\xi_0+\xi_1\Gamma+\xi_2\Gamma^2+\xi_3\Gamma^3+\xi_4\Gamma^4+\xi_5\Gamma^5+\xi_6\Gamma^6)}{2\zeta_0}, \tag{52}$$

$$(v')^2=\frac{(\xi_1+2\xi_2\Gamma+3\xi_3\Gamma^2+4\xi_4\Gamma^3+5\xi_5\Gamma^4+6\xi_6\Gamma^5)}{\zeta_0}(\tau_1+\tau_2\Gamma)^2, \tag{53}$$

where $\tau_1,\ \ \tau_2,\ \ \zeta_0 \neq 0$. Thus, we have a system of algebraic equations from the coefficients of the Γ polynomial. Solving the algebraic equation system via Mathematica Release 7, yields the following relations,

$$\xi_0 = \frac{18\xi_3\xi_5^3\xi_6^2 - \xi_5^5 - 1296\xi_1\xi_5\xi_6^4}{15552\xi_6^5}, \quad \xi_1 = \xi_1,$$
$$\xi_2 = \frac{\xi_3\xi_5}{6\xi_6} - \frac{\xi_5^4}{162\xi_6^3} + \frac{3\xi_1\xi_6}{\xi_5}, \quad \xi_3 = \xi_3,$$
$$\xi_4 = \frac{5\xi_5^2}{18\xi_6} + \frac{3\xi_3\xi_6}{2\xi_5}, \quad \xi_5 = \xi_5, \quad \xi_6 = \xi_6, \tag{54}$$

$$\zeta_0 = -\frac{bB^2(1+p)(2+p)^2\delta(5\xi_5^3 - 54\xi_3\xi_6^2)^2}{1296a^2p^2(1+2p)\xi_5^2\xi_6^3}, \tag{55}$$

$$\tau_0 = \frac{2a(1+2p)\xi_5^3}{b(2+p)\left(-5\xi_5^3+54\xi_3\xi_6^2\right)}, \quad \tau_1 = \frac{24a(1+2p)\xi_5^2\xi_6}{b(2+p)\left(-5\xi_5^3+54\xi_3\xi_6^2\right)}, \quad \tau_2 = \frac{72a(1+2p)\xi_5\xi_6^2}{b(2+p)\left(5\xi_5^3-54\xi_3\xi_6^2\right)}, \tag{56}$$

$$\lambda = -\frac{4ka^2(1+2p)\xi_5\left(7\xi_5^5 - 108\xi_3\xi_5^2\xi_6^2 + 3888\xi_1\xi_6^4\right)}{b(1+p)(2+p)^2\left(5\xi_5^3 - 54\xi_3\xi_6^2\right)^2}. \tag{57}$$

Substituting these coefficients into Eqs. (7) and (12), we have

$$\pm(\eta-\eta_0) = \int \frac{A}{\sqrt{B+\frac{\xi_1}{\xi_6}\Gamma + C\Gamma^2 + \frac{\xi_3}{\xi_6}\Gamma^3 + D\Gamma^4 + \frac{\xi_5}{\xi_6}\Gamma^5 + \Gamma^6}} d\Gamma, \tag{58}$$

where $B,\ \ C$ and D are defined by

$$A = \sqrt{\frac{bk^2(-1-p)(2+p)^2\delta(5\xi_5^3 - 54\xi_3\xi_6^2)^2}{1296a^2p^2(1+2p)\xi_5^2\xi_6^4}},$$
$$B = \frac{18\xi_3\xi_5^3\xi_6^2 - \xi_5^5 - 1296\xi_1\xi_5\xi_6^4}{15552\xi_6^6},$$
$$C = \frac{\xi_3\xi_5}{6\xi_6^2} - \frac{\xi_5^4}{162\xi_6^4} + \frac{3\xi_1\xi_6}{\xi_5\xi_6}, \quad D = \frac{5\xi_5^2}{18\xi_6^2} + \frac{3\xi_3\xi_6}{2\xi_5\xi_6}. \tag{59}$$

Integrating Eq. (58), we obtain the following solutions to Eq. (2),

$$\pm(\eta-\eta_0) = -\frac{A}{(v-\alpha_1)^2}, \tag{60}$$

$$\pm(\eta-\eta_0) = \frac{A(4v - 2\alpha_1 - 2\alpha_2)}{(\alpha_1-\alpha_2)^2\sqrt{(v-\alpha_1)(v-\alpha_2)}}, \ \alpha_1 > \alpha_2, \tag{61}$$

where α_1 and α_2 are the roots of the polynomial equation,

$$B + \xi_1\Gamma + C\Gamma^2 + \xi_3\Gamma^3 + D\Gamma^4 + \xi_5\Gamma^5 + \xi_6\Gamma^6 = 0. \tag{62}$$

Therefore, the solutions of Eq. (2) are given by,

$$u(x,t) = \left[\alpha_1 \pm \frac{\sqrt{A}}{\sqrt{2(\eta-\eta_0)}}\right]^{1/p}, \tag{63}$$

$$u(x,t) = \left[\frac{(\alpha_1+\alpha_2)E - (\alpha_1-\alpha_2)^3(\eta-\eta_0)F}{32A^2 - 2(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2}\right]^{1/p}, \tag{64}$$

where is $E = 16A^2 - (\alpha_2-\alpha_1)^4(\eta-\eta_0)^2$, $F = \sqrt{(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2 - 16A^2}$, and $\eta = kx - \lambda t$. For simplicity, if we take $\eta_0 = 0$, the solutions in Eqs. (63) and (64) reduce to the rational and single kink solutions, respectively;

$$u(x,t) = \left[\alpha_1 \pm \frac{\sqrt{A}}{\sqrt{2(kx-\lambda t)}}\right]^{1/p}, \tag{65}$$

$$u(x,t) = \left[\frac{(\alpha_1+\alpha_2)E + \eta(\alpha_1-\alpha_2)^3F}{32A^2 - 2\eta^2(\alpha_1-\alpha_2)^4}\right]^{1/p}, \tag{66}$$

where is $E = 16A^2 - \eta^2(\alpha_2-\alpha_1)^4$, and $F = \sqrt{(\alpha_1-\alpha_2)^4\eta^2 - 16A^2}$.

Remark 3. The solutions to Eqs. (65) and (66) computed in Case1, were checked by means of Mathematica Release 7. We, once more, vouch that, in our current state of knowledge of the read literature, the gotten solutions are new traveling wave solutions of the GBKE in Eq. (2).

Case 2. If we take, $\varepsilon = 0, \quad \delta = 1, \theta = 4$, with, $\tau_1, \quad \zeta_0 \neq 0$, for Eqs. (7)–(9), we get the relations,

$$v' = \tau_1 \frac{\sqrt{\xi_0 + \xi_1\Gamma + \xi_2\Gamma^2 + \xi_3\Gamma^3 + \xi_4\Gamma^4}}{\sqrt{\zeta_0}}, \tag{67}$$

$$v'' = \tau_1 \frac{\xi_1 + 2\xi_2\Gamma + 3\xi_3\Gamma^2 + 4\xi_4\Gamma^3}{2\zeta_0}, \tag{68}$$

$$(v')^2 = \tau_1^2 \frac{\xi_0 + \xi_1\Gamma + \xi_2\Gamma^2 + \xi_3\Gamma^3 + \xi_4\Gamma^4}{\zeta_0}, \tag{69}$$

where $\tau_1, \quad \zeta_0 \neq 0$. Thus, we have a system of algebraic equations from the coefficients of polynomial of Γ. Solving the algebraic equation system Eqs. (12)–(14) by using Mathematica Release 7 yields the following,

$$\xi_0 = \xi_0, \quad \xi_1 = \frac{2\xi_4\tau_0^2(a + 2ap + b(2+p)\tau_0)}{b(2+p)\tau_1^3} + \frac{2\xi_0\tau_1}{\tau_0},$$

$$\xi_2 = \frac{\xi_4\tau_0(4a(1+2p) + 5b(2+p)\tau_0)}{b(2+p)\tau_1^1} + \frac{\xi_0\tau_1^2}{\tau_0^2},$$

$$\xi_3 = \frac{2\xi_4(a + 2ap + 2b(2+p)\tau_0)}{b(2+p)\tau_1}, \quad \xi_4 = \xi_4, \tag{70}$$

$$\zeta_0 = -\frac{B^2(1+p)(1+2p)\delta\xi_4}{bp^2\tau_1^2}, \quad \tau_0 = \tau_0, \quad \tau_1 = \tau_1, \tag{71}$$

$$\lambda = \frac{6k\xi_3\tau_0(a + 2ap + b(2+p)\tau_0) + \xi_2\tau_1(a + 2ap + 2b(2+p)\tau_0)}{(2 + 7p + 7p^2 + 2p^3)\xi_3}. \tag{72}$$

Substituting these coefficients into Eqs. (7) and (12), we have,

$$\pm(\eta-\eta_0) = A\int \frac{1}{\sqrt{\Gamma^4 + B\Gamma^3 + C\Gamma^2 + D\Gamma + E}} d\Gamma, \tag{73}$$

where, $B, \quad C, \quad D$ and E are defined by,

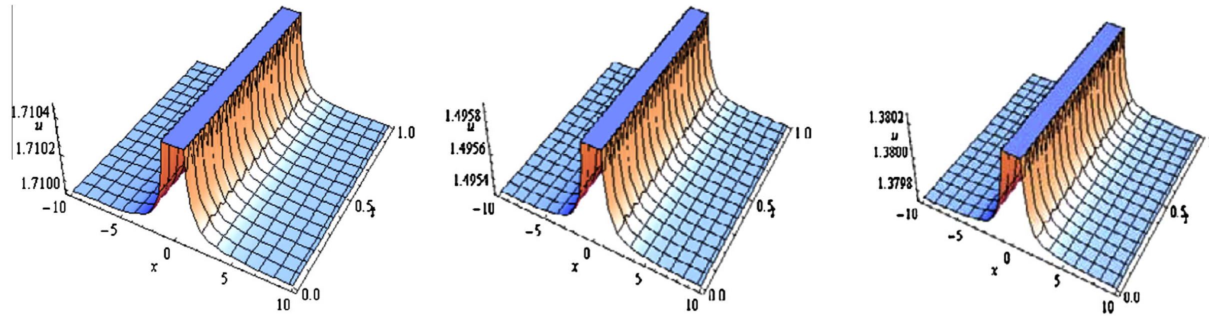

Figure 1 The 3D surfaces of the solution Eq. (31) corresponding to the values $p = -3$, $p = -4$, $p = -5$, from left to right, when $\xi_1 = \xi_2 = \tau_0 = \tau_1 = a = k = \alpha_1 = \beta = 0.2, \quad -10 < x < 10 \quad \text{and} \quad 0 < t < 1.$

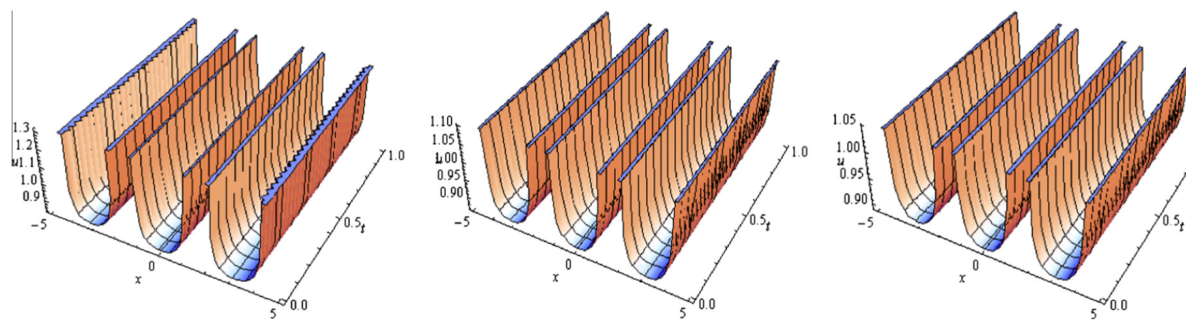

Figure 2 The 3D surfaces of the solution Eq. (32) corresponding to the values $p = -3$, $p = -4$, $p = -5$, from left to right, when $\xi_1 = \xi_2 = a = k = \alpha_1 = \beta = 2, \quad \tau_0 = \tau_1 = 3, \quad \alpha_2 = 0.3, \quad -5 < x < 5 \quad \text{and} \quad 0 < t < 1.$

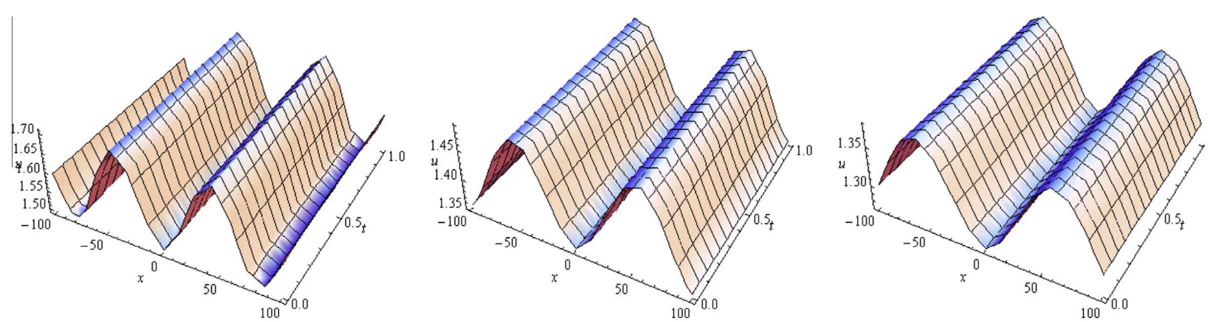

Figure 3 The 3D surfaces solutions for Eq. (33) corresponding to the values, $p = -3$, $p = -4$, $p = -5$, from left to right, $\xi_1 = \xi_2 = a = k = \beta = 2, \quad \tau_0 = \tau_1 = 3, \quad \alpha_1 = 0.1, \quad \alpha_2 = 0.2, \quad \alpha_3 = 0.3, \quad -100 < x < 100, \quad \textit{and} \quad 0 < t < 1.$

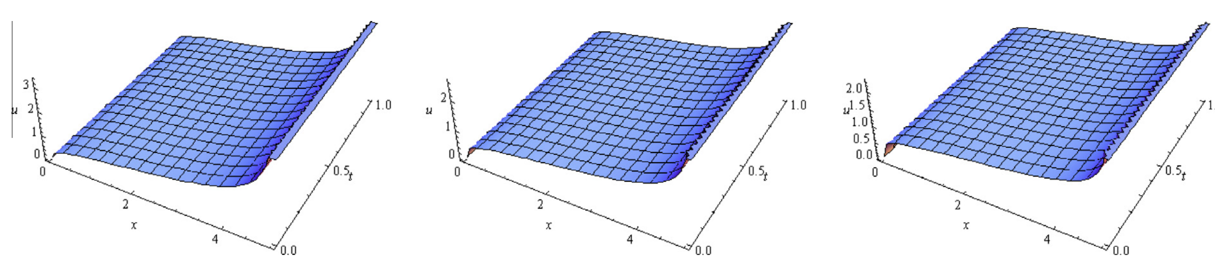

Figure 4 The 3D surfaces of the solution Eq. (47) corresponding to the values $p = -3$, $p = -4$, $p = -5$, from left to right, when $\xi_1 = \xi_2 = \tau_0 = \tau_1 = a = k = \beta = 0.2, \quad \alpha_1 = -1, \quad 0 < x < 5, \quad \text{and} \quad 0 < t < 1.$

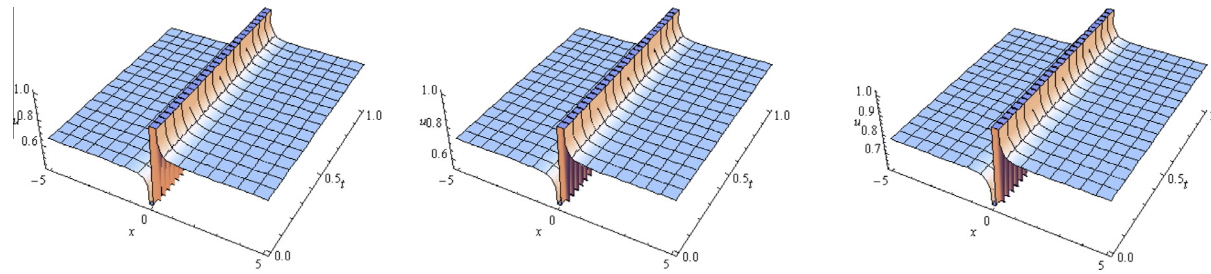

Figure 5 The 3D surfaces of the solution Eq. (48) corresponding to the values $p = -3$, $p = -4$, $p = -5$, from left to right, when $\xi_1 = \xi_2 = a = k = \alpha_1 = \beta = 2$, $\tau_0 = \tau_1 = 3$, $\alpha_2 = 0.3$, $-5 < x < 5$ and $0 < t < 1$.

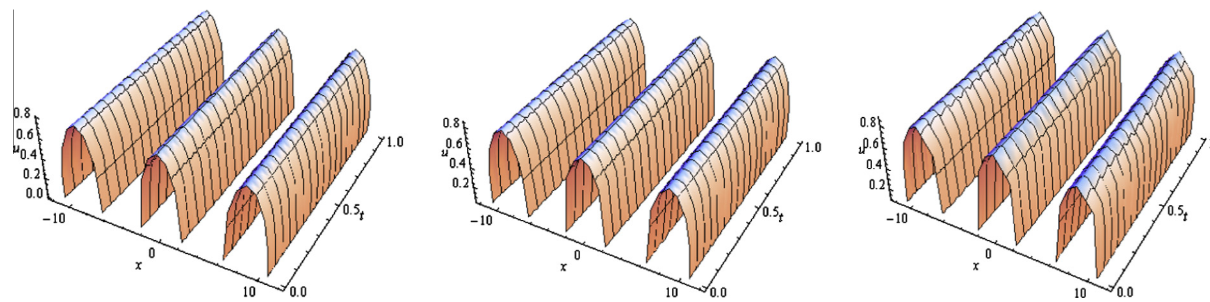

Figure 6 The 3D surfaces solution for Eq. (49) with values, $p = -3$, $p = -4$, $p = -5$, from left to right, $\xi_1 = \xi_2 = a = k = \beta = 2$, $\tau_0 = \tau_1 = 3$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_3 = 0.3$, $\alpha_4 = 0.4$, $-10 < x < 10$, and $0 < t < 1$.

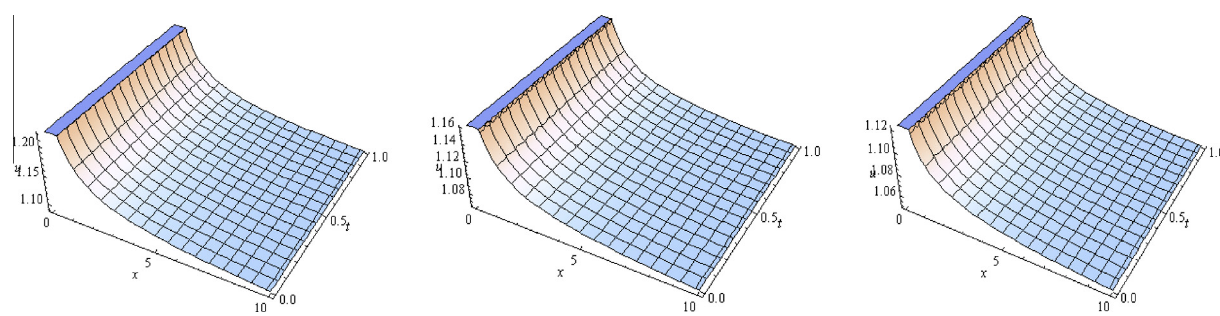

Figure 7 The 3D surfaces of the solution Eq. (65) with values $p = 3$, $p = 4$, $p = 5$, from left to right, when $\xi_1 = \xi_2 = a = k = \beta = 2$, $\tau_0 = \tau_1 = 3$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\delta = -3$, $\alpha_3 = 0.3$, $\alpha_4 = 0.4$, $0 < x < 10$, and $0 < t < 1$.

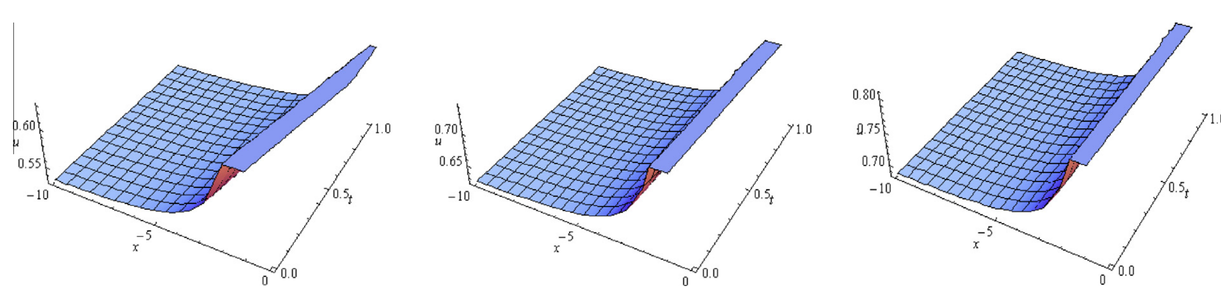

Figure 8 The 3D surfaces solution to Eq. (66) with values $p = -3$, $p = -4$, $p = -5$, from left to right, when $\xi_1 = 1$, $\xi_2 = 2$, $\xi_3 = 3$, $\xi_4 = 4$, $\xi_5 = 5$, $\xi_6 = 6$, $a = b = k = \beta = \delta = 0.2$, $\alpha_1 = 1$, $\alpha_2 = 3$, $-10 < x < 0$, and $0 < t < 1$.

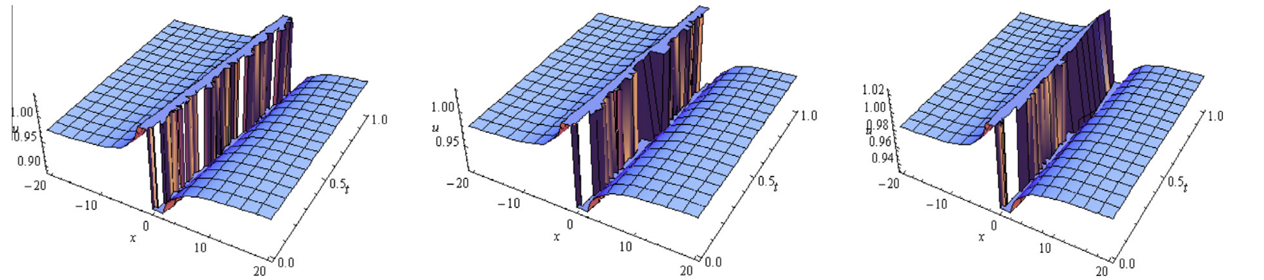

Figure 9 The 3D surfaces solution Eq. (82) with values $p = -3, p = -4, p = -5$, from left to right, for $\xi_0 = 0.1$, $\xi_1 = 13.0889$, $\xi_2 = 166.767$, $\xi_3 = 16.88$, $\xi_4 = 2$, $\tau_0 = \tau_1 = a = k = b = \beta = 3$, $\alpha_1 = 1.12$, $\delta = -3$, $-20 < x < 20$, and $0 < t < 1$.

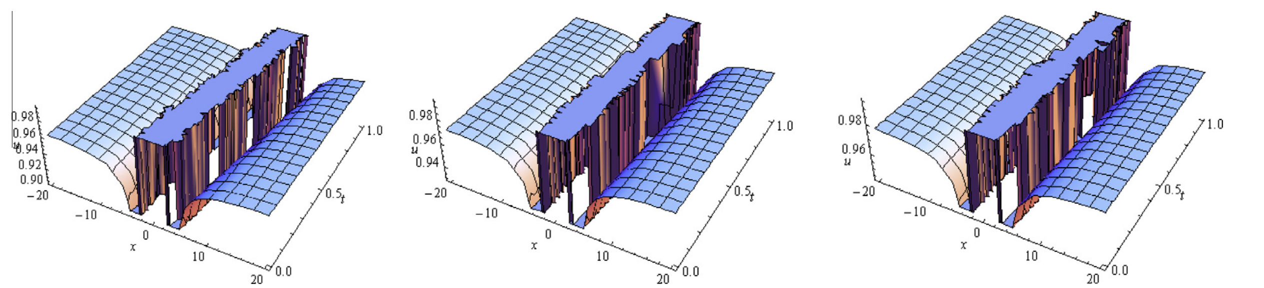

Figure 10 The 3D surfaces of the solution Eq. (83) corresponding to the values$p = -3$, $p = -4$, $p = -5$, from left to right, when $\xi_0 = 0.1$, $\xi_1 = 13.0889, \xi_2 = 166.767$, $\xi_3 = 16.88, \xi_4 = 2$, $\tau_0 = \tau_1 = a = k = b = \beta = 3$, $\alpha_1 = 1.12$, $\alpha_2 = 0.8$, $\delta = -3$, $-20 < x < 20$, and $0 < t < 1$.

$$A = \frac{1}{\sqrt{\xi_4}}, \quad B = \frac{2(a + 2ap + 2b(2+p)\tau_0)}{b(2+p)\tau_1},$$
$$C = \frac{\tau_0(4a(1+2p) + 5b(2+p)\tau_0)}{b(2+p)\tau_1^1} + \frac{\xi_0\tau_1^2}{\xi_4\tau_0^2},$$
$$D = \frac{2\tau_0^2(a + 2ap + b(2+p)\tau_0)}{b(2+p)\tau_1^3} + \frac{2\xi_0\tau_1}{\xi_4\tau_0}, \quad E = \frac{\xi_0}{\xi_4}. \tag{74}$$

Integrating Eq. (73), we obtain the following solutions to Eq. (2),

$$\pm(\eta - \eta_0) = \frac{2A}{\alpha_1 - v}, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4, \tag{75}$$

$$\pm(\eta - \eta_0) = \frac{4A}{\alpha_1 - \alpha_3}\sqrt{\frac{v - \alpha_1}{v - \alpha_3}}, \quad \alpha_1 > \alpha_2 = \alpha_3 = \alpha_4, \tag{76}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}} \times \log\left(\frac{\sqrt{(v - \alpha_2)(\alpha_1 - \alpha_3)} - \sqrt{(v - \alpha_3)(\alpha_1 - \alpha_2)}}{\sqrt{(v - \alpha_2)(\alpha_1 - \alpha_3)} + \sqrt{(v - \alpha_3)(\alpha_1 - \alpha_2)}}\right), \tag{77}$$

where α_1 and α_2 are the roots of the polynomial equation,

$$\Gamma^4 + B\Gamma^3 + C\Gamma^2 + D\Gamma + E = 0. \tag{78}$$

Therefore, we find the following solutions for Eq. (2),

$$u(x, t) = \left[\alpha_1 \pm \frac{2A}{\eta - \eta_0}\right]^{1/p}, \tag{79}$$

$$u(x, t) = \left[\frac{16A^2\alpha_2 - \alpha_1(\alpha_1 - \alpha_2)^2(\eta - \eta_0)^2}{16A^2 - (\alpha_1 - \alpha_2)^2(\eta - \eta_0)^2}\right]^{1/p}, \tag{80}$$

$$u(x, t) = \alpha_1 - \frac{2(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}{2\alpha_1 - \alpha_2 - \alpha_3 + \cos h\left(\frac{\eta\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}}{2A}\right)(\alpha_3 - \alpha_2)}, \tag{81}$$

where, $\eta = kx - \lambda t$. For the case, $\eta_0 = 0$, the solutions to Eqs. (79)–(81), are reduced to the rational and single kink solutions, respectively;

$$u(x, t) = \left[\alpha_1 \pm \frac{2A}{\eta}\right]^{1/p}, \tag{82}$$

$$u(x, t) = \left[\frac{16A^2\alpha_2 - \alpha_1(\alpha_1 - \alpha_2)^2\eta^2}{16A^2 - (\alpha_1 - \alpha_2)^2\eta^2}\right]^{1/p}, \tag{83}$$

$$u(x, t) = \alpha_1 - \frac{2(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}{2\alpha_1 - \alpha_2 - \alpha_3 + (\alpha_3 - \alpha_2)\cos h\left(\frac{\eta\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}}{2A}\right)}. \tag{84}$$

Remark 4. The new rational and hyperbolic function solutions for Eq. (2), via Eqs. (82)–(84) with respect to Case 2., have been checked by Mathematica Release 7. We believe that these solutions have not appeared in the published literature.

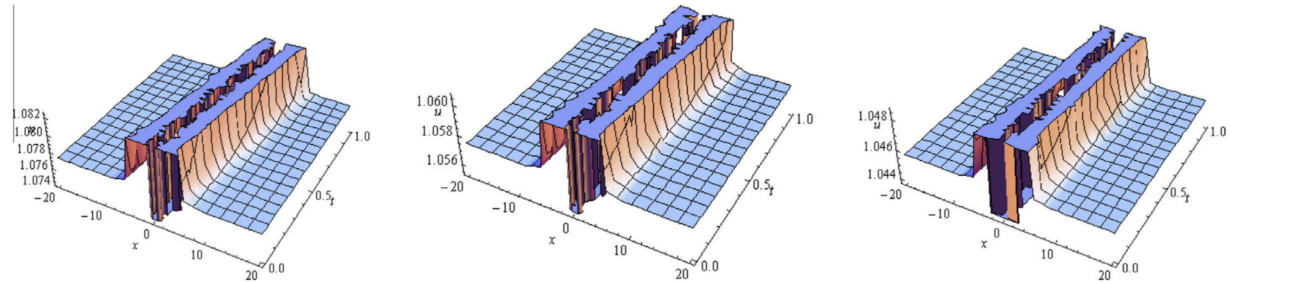

Figure 11 The 3D surfaces of the solution Eq. (84) corresponding to the values $p = -3$, $p = -4$, $p = -5$, from left to right, when $\xi_0 = 0.1$, $\xi_1 = 13.0889$, $\xi_2 = 166.767$, $\xi_3 = 16.88$, $\xi_4 = 2$, $\tau_0 = \tau_1 = a = k = b = \beta = 3$, $\alpha_1 = 1.12$, $\alpha_2 = 0.8$, $\alpha_3 = 2$, $\delta = -3$, $-20 < x < 20$ and $0 < t < 1$.

4. Conclusions

In this paper, the ETEM has been applied to the equation pair Generalized Benjamin Equation and Generalized Burger-Kdv Equation (GBE, GBKE), in a respective manner, only to obtain new analytical solutions. The new solutions are found in terms of logaritmic functions, rational, elliptic and Jacobi elliptic functions. Moreover, when we consider all Figs. 1–11, and Mathematica Release 7, checked computaions, we conclude that our method is reliable, and yields an effective approach for finding solutions of nonlinear equations, arising in applied physics and engineering.

To our current state of knowledge, we do believe that the obtained analytical solutions are new and have not appeared in the literature previously. Therefore, they can be used to serve and enhance our state of knowledge, in the realm of nonlinear complex phenomena.

Furthermore, the ideas introduced in this paper, and the applications provided, may well serve as a guide to us and to research scholars treading the path of nonlinear differential equations.

Competing interests

The authors declare that they have **no** competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved of the final draft.

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