Numerical evidence of regularized correlations in spin foam gravity

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ARTICLE INFO

ABSTRACT

We report on the numerical analysis of the area correlations in spin foam gravity on a single 4-simplex considered by Rovelli [C. Rovelli, Phys. Rev. Lett. 97 (2006) 151301]. We compare the asymptotics and confirm the inverse squared distance leading behaviour at large scales. This supports the recent advances on testing the semiclassical limit of the theory. Furthermore, we show that the microscopic discreteness of the theory dynamically suppresses and regularizes the correlations at the Planck scale.

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A fundamental theory of quantum gravity is expected to improve the UV behaviour of the non-renormalizable perturbative quantization of General Relativity. The latter should nonetheless emerge in the low-energy limit, where it can be considered as an effective field theory. Among the key features to be reproduced is the distance dependence of the free graviton propagator encoding Newton's law. In Loop Quantum Gravity (LQG) and its covariant version, the spin foam formalism [1], the UV behaviour is expected to be cured by the discreteness of spacetime at the Planck scale. While the regularizing effect of such discreteness is clear, it is rather non-trivial how the discreteness smooths out to a low energy limit given by the effective theory of gravitons. This is the problem of the semiclassical limit in spin foam gravity. In this Letter we give numerical evidence that a behaviour consistent with the graviton theory at low energies is dynamically suppressed and regularized at the Planck scale.

The study of the semiclassical limit has received a great deal of attention over the last few years and important results have been obtained [2]. In particular, in [3–7] a program was started to compute the graviton propagator using correlations between geometric quantities. Analytic results show that the leading order at large scales is consistent with that of the free propagator from the linearized quantum theory, thus providing an important piece of evidence for the correctness of the limit. Here we confirm this result numerically, supporting the approximations made to deal with the complexity of the calculations. Furthermore, the numerical approach also allows us to study the small scale structure of the propagator, where the non-perturbative effects of spin foams are dominant, and we show that the discrete structure does regularize the typically divergent behaviour of the correlations, suppressing

them at the Planck scale. Finally, we point out the limitations of the model used, and discuss the relevant developments to be made.

We consider the area correlations for 4d Riemannian quantum gravity defined and studied in [4]. These are correlations between fluctuations of the areas around a given background q of a 4-simplex, and correspond to some components of the graviton propagator $G_{\mu\nu\rho\sigma}(x, y) = \langle 0|h_{\mu\nu}(x)h_{\rho\sigma}(y)|0\rangle$. For extensive motivation and discussion, see [5]. We index the ten triangles of the 4-simplex by l, and the area eigenvalues are given by $A_l = \ell_P^2(2j_l + 1)$, with j_l half-integers. For simplicity, we choose the background to be a regular four-simplex, with all ten areas having the same value, $A_0 = \ell_P^2(2j_0 + 1)$. Given two triangles a and b, we consider the following area correlation,

$$W_{ab}(j_0) = \frac{1}{\mathcal{N}} \sum_{j_l} \hbar(j_a)\hbar(j_b)\Psi_q[j_l]K[j_l], \qquad (1)$$

where $\mathcal{N} = \sum_{j_l} \Psi_q[j_l]K[j_l]$ is the normalisation, Ψ_q is the boundary state and K the propagation kernel, or path integral amplitude. These are model-dependent quantities, and we describe our choices below. The quantity $\hbar(j_a) \equiv (A_a^2 - A_0^2)/A_0^2$ represents an area fluctuation, or equivalently, the fluctuation of the metric tensor $h_{\mu\nu}$ projected along the normal n_a^μ to the triangle. $W_{ab}(j_0)$ is the spin foam analogue of the projections $G_{ab} = n_a^\mu n_a^\nu n_b^\rho n_b^\sigma G_{\mu\nu\rho\sigma}(a, b)$ of the continuum graviton propagator around a flat background, with the two points taken to be the centers of the triangles. If the theory has the right semiclassical limit, the leading order of (1) should match the leading order of G_{ab}, namely the free propagator corresponding to the linearized theory, which we recall scales as the inverse squared distance between a and b.

The boundary state $\Psi_q[j_l]$ represents a dynamical coherent state peaked around both the (canonically conjugate) intrinsic and

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extrinsic 3-geometry of the classical background q chosen. Given the discrete 4-simplex considered, this means choosing a background configuration for the areas and their conjugate variables, the dihedral angles. Taking the simple choice of the equilateral configuration (j_0, θ), all triangle areas are given by $A_0 = \ell_P^2(2j_0 + 1)$ and all interior dihedral angles by $\theta = \arccos(\frac{1}{4})$.[1] The explicit form of such a state is not known in the full theory, but control can be gained by going to lowest order in the perturbative expansion. Based on analogies with the continuum linearized theory, in [4] the following Gaussian ansatz was made,

$$\Psi_q[j_l] = \exp\left\{ -\frac{1}{2j_0} \sum_{l,m} \alpha_{lm} \delta j_l \delta j_m + i\theta \sum_l (2j_l + 1) \right\}, \qquad (2)$$

where α_{lm} is a 10 by 10 constant matrix and $\delta j_l = j_l - j_0$. The matrix α_{lm} is non-diagonal, but the symmetries of the equilateral background reduce the number of independent entries to three (see discussion below). Having the squared width proportional to j_0 guarantees that in the large j_0 limit the state (2) is peaked around both conjugate variables of the background $q = (j_0, \theta)$ (e.g. [8]). The physical boundary state for the full theory is expected to have a lowest order contribution, corresponding to the free theory, given by (2) with a definite α_{lm}. So if we know the full state, we can fix α_{lm} by looking at its perturbative expansion.[2] In this Letter, we consider an asymptotic expansion for $j_0 \mapsto \infty$; as geometric areas are given by $\ell_P^2 j_0$, the limit $j_0 \mapsto \infty$ drives the ℓ_P expansion. Therefore, the parameter j_0 has a double role: on the one hand, it describes the background geometry of the boundary; on the other hand, it is the parameter of the asymptotic expansion.

The kernel depends on the spin foam model chosen. As in [4], we consider the Barrett–Crane (BC) model [9],

$$K[j_l] = \prod_l (2j_l + 1)^k \{10j\}, \qquad (3)$$

where the integer k parametrizes the choice of face weight in the measure, and the $10j$-symbol $\{10j\}$ is an SO(4)-invariant tensor constructed with Clebsch–Gordan coefficients. To study its asymptotic expansion, recall that when all spins are homogeneously large, i.e. $j_l = Nk_l$, $N \mapsto \infty$, the $10j$-symbol has a stationary phase contribution of the form [10]

$$\mu(j_l) \cos\left(S_R[j_l] + \phi \right), \qquad (4)$$

where $\mu(j_l)$ is a non-oscillating function scaling like $N^{-9/2}$, $S_R[j_l] = \sum_l (2j_l + 1)\theta_l(j_l)$ is the Regge action for a single 4-simplex with triangle areas $A_l = \ell_P^2(2j_l + 1)$ as independent variables, and ϕ is an irrelevant phase. It was shown in [11] (see also [12]) that this is masked by a non-oscillating contribution $D(j_l)$ that scales like N^{-2}. This dominant contribution corresponds to a degenerate geometry for the 4-simplex. However, as conjectured in [4] and proved in [6], $D(j_l)$ is negligible in the evaluation of quantities like (1), where its non-oscillating nature fails to properly match the phase of the boundary state (2). The work of this Letter supports these analytic calculations, and thus is the first (indirect) numerical evidence of (4). Therefore the asymptotic behaviour of (3) is effectively given by the kernel for area Regge calculus with measure $\prod_l (2j_l + 1)^k \mu(j_l)$.

The emergence of Regge calculus in the semiclassical limit of LQG and spin foam gravity has often been advocated [13], and would provide a solid bridge to low-energy physics: conventional

Regge calculus is a discrete representation of GR known to reproduce the correct linearized quantum theory [14], thus if in (1) we use Ψ_q and K from linearized Regge calculus, we expect to obtain the right free graviton propagator. This suggests a possible strategy to fix α_{lm} by evaluating the boundary state in linearized area Regge calculus [15]. In general, the matrix α_{lm} is non-diagonal, and this makes evaluating (1) extremely challenging. To simplify the numerical analysis we study the case where $\alpha_{lm} \equiv \alpha \delta_{lm}$ is diagonal. Indeed, using the formula for the $10j$-symbol as an integral over SO(4), this choice allows us to perform each sum over j_l separately, which simplifies the numerical task. The resulting 9-dimensional integrals are still very difficult to compute numerically, due to their oscillatory nature, but can be estimated using adaptive Monte Carlo methods (VEGAS algorithm) with between 10^9 and 10^{10} sample points.

We then have a model with a single free parameter α, determining the width of the Gaussian. For different values of α, we study the matrix (1) as a function of j_0, in particular to test its asymptotic behaviour and support the analytic calculations which appeared in [4], which we briefly recall here. First of all, $\prod_l (2j_l + 1)^k \mu(j_l)$ is a measure term in (1), thus it will affect only the higher order corrections [7]. The leading order is obtained by approximating (1) with (the second moment of) a Gaussian integral, like in the continuum linearized theory, with action $Q_{lm} \delta j_l \delta j_m$ given by the quadratic term in the boundary state and the second derivatives of the Regge action,

$$Q_{lm} = \frac{\alpha}{j_0} \delta_{lm} + i \frac{\partial^2 S_R}{\partial j_l \partial j_m}\bigg|_{j_0}. \qquad (5)$$

The Hessian matrix of the Regge action on the equilateral configuration was computed in [5], and it is a 10 by 10 matrix with a regular structure inherited from the regularity of the equilateral 4-simplex. Fixing a triangle a, there are only three distinguishable choices for b: the case when b is the same triangle, the six cases when b is an adjacent triangle sharing an edge with a, and the three cases when b is an opposite triangle sharing only a vertex with a. Correspondingly, for each row of the Hessian there are only three different entries,

$$\frac{\partial^2 S_R}{\partial j_l \partial j_m} = \frac{1}{j_0} H_{lm}, \quad H_{lm} = \frac{1}{2}\sqrt{\frac{3}{5}} f_{lm}, \qquad (6)$$

where, for each row, $f_{lm} = -9$ occurs once, $7/2$ six times, and -4 three times. We can then write

$$Q_{lm} = \frac{1}{j_0} A_{lm}, \quad A_{lm} \equiv (\alpha \delta_{lm} + i H_{lm}). \qquad (7)$$

Notice that H_{lm} is not invertible, just like the original continuum term, due to diffeomorphism invariance [15]. On the other hand, A_{lm} is invertible for any $\alpha > 0$, thus the boundary state effectively provides a gauge-fixing for the propagator.

Following the same procedure as [4], we obtain for the absolute value of the leading order

$$\left| W_{ab}^{\text{LO}}(\alpha) \right| = \frac{4}{j_0} \left| A_{ab}^{-1} \right|. \qquad (8)$$

If the theory has the right semiclassical limit, (8) should give the free graviton propagator of linearized quantum gravity.[3] Choosing the harmonic gauge, the latter is given by

$$G_{\mu\nu\rho\sigma}(a, b) = -\frac{1}{2} \frac{\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma} - \delta_{\mu\nu}\delta_{\rho\sigma}}{d(a, b)^2}. \qquad (9)$$

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[1]　Notice that for technical reasons we use here the interior angles, and not the exterior ones as in [4].

[2]　In this perspective, we stress that the Gaussian ansatz is not an arbitrary choice: any semiclassical physical state should be at leading order approximated in this way.

[3]　We take the absolute value for a better comparison with the linearized G_{ab}, because the spin foam kernel provides the complex exponential of the Regge action (4) even in Riemannian signature. The phase of (8) is irrelevant for our work.
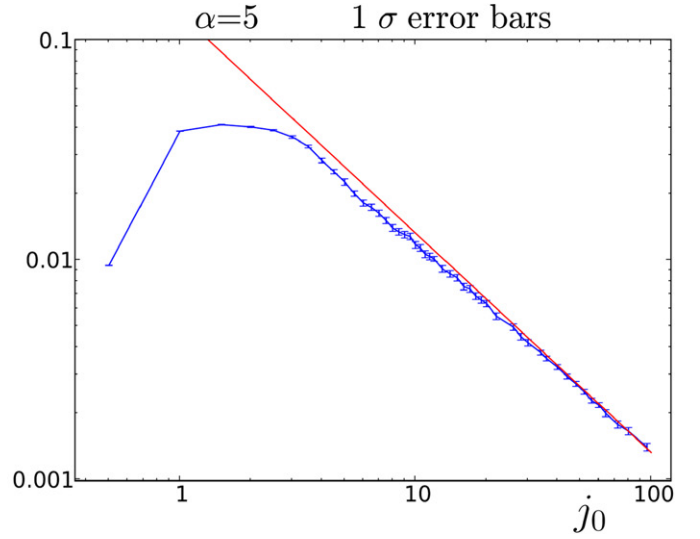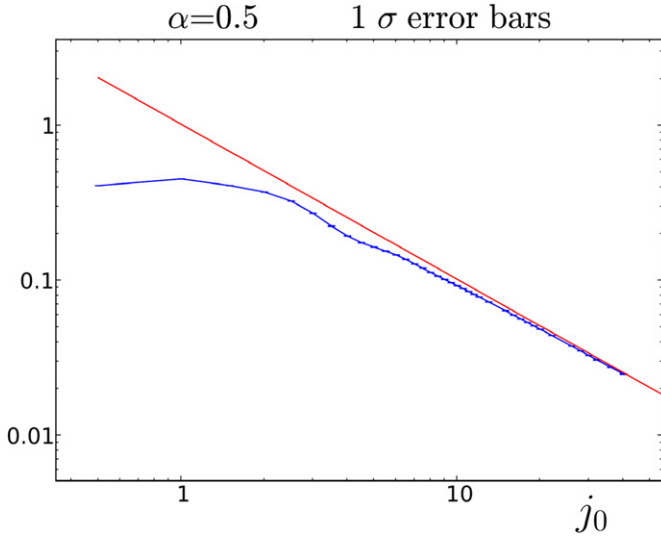
Fig. 1. Numerical study of (1) (dots), versus the analytic result of the leading order, on a log–log plot. Left panel: the case $\alpha = 0.5$. Right panel: the case $\alpha = 5$. Raw data and more plots are available at http://jdc.math.uwo.ca/graviton.

For the equilateral background we have chosen, there are only three independent projections G_{ab}, and furthermore the squared distances are all proportional to j_0. Both features are matched by (8), which has only the three independent entries discussed above and scales as $1/j_0$.[4] Note that, as a function of α, $|W^{LO}|$ scales as $1/\alpha$ for both $\alpha \gg 1$ and $\alpha \ll 1$.

To have explicit values, we choose a and b to be opposite triangles, and fix $\alpha = 0.5$ and $\alpha = 5$. From (8) we obtain

$$\left| W^{LO}_{opp}(0.5) \right| = \frac{1.02}{j_0}, \qquad \left| W^{LO}_{opp}(5) \right| = \frac{0.13}{j_0}. \tag{10}$$

In Fig. 1 we compare these analytic results with the numerical analysis of the full formula (1), where in the kernel (3) we choose the simple measure term with $k = 0$. The dots are the numerical evaluations of the absolute value of (1) for these cases, and the error bars are one σ. The plots shown in the figures required approximately 2 cpu years to produce on the SHARCNET clusters. Remarkably, good agreement is reached already at $j_0 \sim 50$. This is our first result: we numerically tested the inverse squared distance asymptotic behaviour of the area correlation in spin foam gravity. This supports the results of [4] and in particular the conjecture that the D term drops out.

Let us now focus on the short scale behaviour. We see from Figs. 1 and 2 that the correlations have a maximum. This peak is very close to the Planck scale, and its exact location depends on the value of α, but also on the measure term in (4). To show this, in Fig. 2 we plot again the case $\alpha = 5$, but this time with $k = 1$. Notice that the position of the peak is slightly pushed to the right, but the qualitative behaviour is the same. We interpret the presence of a peak as follows. The discrete microscopic structure of the theory provides a trivial regularization already of the leading order (8), where only half-integer steps in the distance between the two points are allowed. The non-trivial effect of the short scale dynamics is to introduce a suppression of the correlations, which instead of increasing monotonically at shorter distances, reach a maximum and then decrease (the effect being less noticeable for small values of α). In this sense, the divergent behaviour of the graviton correlations gets regularized at high energies by the discrete structure of spin foams. This shows how the full theory might enhance the ef-
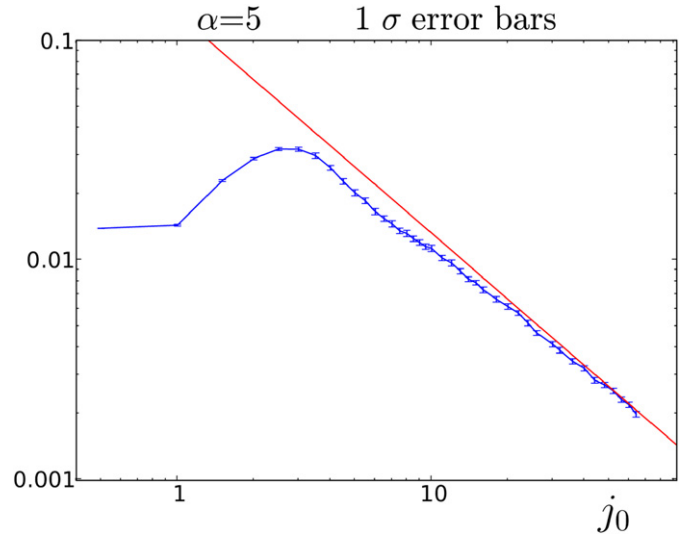

Fig. 2. The case $k = 1$, $\alpha = 5$. The leading order is the same as for $k = 0$, and the position of the peak shifted to the right.

fective field theory where the latter breaks down, and it confirms the intuition that spacetime cannot be considered as fluctuating around the flat metric at the Planck scale.

We want to conclude with an outlook for further developments to turn this picture into a concrete prediction of the theory. This will require enhancing both the boundary state and the kernel used here.

The first improvement would be to remove the ambiguity in α_{lm} by fixing it with a dynamical requirement. To obtain a quantitative matching with the free graviton G_{ab} we need the right boundary state, with the right non-diagonal structure and values of the entries, coming from the dynamics and a choice of gauge. As mentioned above, a possible way to obtain this quantity is to evaluate it in linearized area Regge calculus.

Furthermore, from analogies with conventional quantum field theory, we expect there to exist a procedure for extracting the full boundary state from the non-perturbative kernel. Extending this procedure in background independent quantum gravity (see for instance [16]) would be extremely useful. The full state will fix α_{lm} by looking at the perturbative expansion, but also contribute to the small scale structure, thus affecting the numerical analysis presented here. This will certainly modify numerical results such as

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[4] One of the three cases is the correlation of a triangle with itself. While the continuum graviton propagator between the same two points is divergent, this is not the case for W_{aa}. This should not be surprising, as it is among the regularizing effects of the discrete microscopic structure.

the exact location of the peak, but will most likely not change the qualitative picture of suppressed correlations at the Planck scale. This regularizing effect is very likely to survive in the full theory, because it is a facet of the discrete microscopic structure, more than of the details of the model.

On the other hand, the kernel itself needs improvement. Indeed the Barrett–Crane kernel does not capture the dynamics of quantum gravity in a fully consistent way. Here we used it in the restricted context of area correlations on a single 4-simplex, where it provides a sensible quantum gravity amplitude. To test the full tensorial structure of (9), we need to consider also projections that in the discrete picture would correspond to correlations between the dihedral angles of the boundary geometry. As pointed out in [5,17], these correlations cannot be studied using the BC vertex amplitude. Furthermore, the calculations presented here need to be extended to many 4-simplices, and the large spin limit (4) of the BC vertex precisely lacks the constraints which are needed to correctly treat the areas as independent variables. These issues have recently been addressed, and promising new models have been proposed [18]. It would be extremely interesting to apply the same analysis to these new models.

In conclusion, we have presented a spin foam model where correlations consistent with the graviton theory at low energies are dynamically suppressed and regularized at the Planck scale. This work shows that the spin foam framework for quantum gravity naturally provides the expected regularization of the corrections at short scales, i.e. high energies: the theory has a short length scale appearing dynamically, which also suggests that spacetime cannot be considered as fluctuating around the flat metric at the Planck scale. We stress that the model presented here is not predictive, and further work is needed before the correctness of the semiclassical limit of spin foam gravity can be claimed. Our results give a glimpse of what the qualitative picture of quantum gravity and its bridging to low-energy physics could be like, and suggest interesting new questions for further investigation.

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