Convergence of Convex Sets and of Solutions of Variational Inequalities*

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Introduction

The main object of this paper is to study convergence properties of solutions of variational inequalities such as

$$u \in K: \langle Tu, v - u \rangle \geqslant 0 \qquad \text{for all} \quad v \in K, \tag{1}$$

where T is a monotone hemicontinuous mapping from a real reflexive Banach space X to its dual X^* and K is a non-empty closed convex subset of the domain of T, when T and K are subjected to a perturbation.

We consider a sequence (T_n) of monotone hemicontinuous mappings from X to X^*, a sequence (K_n) of closed convex subsets of X, K_n contained in the domain of T_n, and for each n the variational inequality

$$u_n \in K_n: \langle T_n u_n , v - u_n \rangle \geqslant 0 \qquad \text{for all} \quad v \in K_n , \tag{1_n}$$

and we ask under what condition the solutions of (1_n) "converge" to the solutions of (1), as T_n "converges" to T and K_n "converges" to K. Real parametrized perturbations T_ϵ and K_ϵ would require only minor changes.

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It is well known that for any T and K as stated above, the solutions of (1) are a (possibly empty) closed convex subset of K. Therefore, to pose our problem more precisely, we need to specify: (*1*) the convergence of T_n to T as $n \to +\infty$, (*2*) the convergence of a sequence (S_n) of closed convex subsets of X to a closed convex subset S of X.

As for (*1*), we require that any point $\{v, Tv\}$ in the graph of T is the limit in the product strong topology of $X \times X^*$ of a sequence of points $\{v_n, T_n v_n\}$, each one in the graph of T_n, as $n \to +\infty$.

We introduce convergence (*2*) by means of the topological notion of $\underline{\text{Lim}}\ S_n$ in the strong topology of X and (sequential) $\overline{\text{Lim}}\ S_n$ in the weak topology of X; see Definition 1.1 of Lim S_n.

In Section 1 we give the main properties of Lim S_n and consider some examples. Moreover, by means of a notion of "local gap" between two closed convex sets, we relate such convergence with the Hausdorff metric convergence for closed sets and with the "gap" or "opening" convergence for linear subspaces.

In Section 2 we state our results in the case that the T_n's are uniformly coercive in X and the solution of (1) unique. Namely, we answer with Theorem A the following questions: weak convergence of u_n to u (as well as convergence of $\langle T_n u_n - Tu, u_n - u\rangle$ to 0); strong convergence of u_n to u; uniform boundedness of u_n.

In Section 5 we also deal with the degenerate case of T_n non-coercive and solution of (1) non-unique. The device that we shall use is the so called "elliptic regularization", that consists in adding to each T_n a coercive perturbation $n^{-\alpha}M$, $\alpha > 0$, which vanishes as $n \to +\infty$. Again we find that the approximate solutions converge to a solution of (1), provided T_n and K_n converge sufficiently fast to T and K, respectively, as $n \to +\infty$. This result is stated in Theorem C of Section 5.

We make a parallel (and equivalent) study for inequalities of type

$$u \in X: \langle Tu, v - u\rangle \geqslant f(u) - f(v) \qquad \text{for all} \quad v \in X, \tag{2}$$

where f is a lower-semicontinuous convex function from X to $(-\infty, +\infty]$, $f \not\equiv +\infty$. We reduce these inequalities to inequalities of type (1) in the space $X \times \mathbb{R}$ and we apply the theorems quoted above to prove Theorem B of Section 2 and Theorem D of Section 5. By taking $T = 0$, we obtain some results on the continuous dependence on f of the minimizing vector and the minimum value of a functional such as f. The convergence of a sequence (f_n) of convex functions is defined in terms of convergence in the space $X \times \mathbb{R}$ of the convex sets epi f_n, where epi f is the set of all $\{v, \beta\} \in X \times \mathbb{R}$ with $\beta \geqslant f(v)$.

In Section 3 we show that our results can be applied in two directions:

(1) The approximation of the solution of (1) by solutions of inequalities (1_n) relative to finite-dimensional spaces X_n, with K_n some kind of finite-dimensional approximation of K;

(2) To obtain results on the continuous dependence on the constraints of the solution of a variational problem such as (1); for example, of the problem

$$\begin{cases} u \in K \\ \langle Au - v', v - u \rangle \geqslant 0 \qquad \text{for all} \quad v \in K, \end{cases}$$

where A is a partial differential operator of type

$$Au = \sum_{|\alpha| \leqslant m} D^\alpha A_\alpha(x, u, \ldots, D^m u)$$

and K is a closed convex subset of the Sobolev space $W^{m,p}(\Omega)$ defined in terms of the boundary conditions imposed upon u.

The main properties of inequalities (1) and (2) are summarized in Section 0, where also some references to the literature can be found.

The results proved in this paper generalize and extend previous results obtained by the author in case T is a linear accretive operator in a Hilbert space (*34*). Some extensions to the nonlinear case were already stated, without proof, in (*35*).

0. Preliminary Remarks

1. *Notation*

We shall denote by X a real normed space, by X^* the dual space of X. We shall denote strong convergence in X, i.e., convergence in the strong topology of X, by s-lim or $\to$, and weak convergence in X, i.e. convergence in the weak topology of X, by w-lim or $\rightharpoonup$. We shall also use the same notation to denote convergence in the strong topology and the weak* topology of X^*.

The pairing between $v \in X$ and $v' \in X^*$ will be denoted by $\langle v', v \rangle$. Both the norm of v in X and the dual norm of v' in X^*, by $\| \cdot \|$.

2. *Some Definitions*

Let A be a mapping of a subset $D(A)$ of X to X^*:

A is *monotone* if

$$\langle Au - Av, u - v \rangle \geqslant 0 \qquad \text{for all} \quad u \text{ and } v \text{ in } D(A);$$

A is *strictly monotone* if A is monotone and

$$\langle Au - Av, u - v \rangle > 0 \qquad \text{whenever} \quad u \neq v;$$

A is *hemicontinuous* if $D(A)$ is convex and for any u and v in $D(A)$, the map $t \mapsto A(tu + (1 - t)v)$ of $[0, 1]$ to X^* is continuous for the natural topology of $[0, 1]$ and the weak topology of X^* (see T. Kato, (*21*) for a discussion of this and related continuity properties of monotone operators);

A is *coercive* (in X) on a subset K of $D(A)$, if there exists a function c: $(0, +\infty) \to [-\infty, +\infty]$, with $c(r) \to +\infty$ as $r \to +\infty$, such that

$$\| v \| \, c(\| v \|) \leqslant \langle Av, v \rangle \qquad \text{for all} \quad v \in K.$$

Thus, A is coercive on K whenever K is bounded, while A is coercive on an unbounded K if and only if

$$\frac{\langle Av, v \rangle}{\| v \|} \to +\infty \qquad \text{as} \qquad \| v \| \to +\infty, \qquad v \in K.$$

3. *Variational Inequalities for Convex Sets*

Let A be a map from X to X^*. If K is a (non-empty) subset of the domain $D(A)$ of A, we shall denote by

$$S(A, K)$$

the set of all vectors u of X such that

$$u \in K: \langle Au, v - u \rangle \geqslant 0 \qquad \text{for all} \quad v \in K. \tag{1}$$

The basic, though not the most general, results for inequality (1) can be summarized as follows:

$S(A, K)$ is a (possibly empty) subset of K, which is closed and convex, provided K is such and A is monotone and hemicontinuous;

If, in addition, X is a reflexive Banach space and A is coercive on K in X, then $S(A, K)$ is non-empty [existence of solutions of (1)];

If A is strictly monotone, then $S(A, K)$ consists at most of a single vector [uniqueness of the solution of (1)].

Let us notice a special case of (1). Let v' be a vector of X^* and $A - v'$ the map $v \mapsto Av - v'$ of $D(A)$ to X^*. Suppose $D(A)$ is a dense linear subspace of X. Then, the set

$$S(A - v', D(A))$$

coincides with the set of all solutions u in $D(A)$ of the equation

$$Au = v'.$$

4. *References for Inequalities (1)*

Inequalities such as (1) were introduced, and the existence theorem was proved, by G. Stampacchia, (*38*), for A an accretive linear operator in a Hilbert space, as a generalization to non-symmetric A and one-side constraints of the Euler-Lagrange equation for a variational problem. A further study of this special case of problem (1), also for non-coercive A, was done by J. L. Lions and G. Stampacchia in the joint papers, (*27*) and (*28*), with applications to elliptic and parabolic unilateral boundary value problems.

The existence theorem in the general form stated above (and its extension to semi-monotone operators) was obtained by F. E. Browder (*12*) and P. H. Hartman–G. Stampacchia (*20*) by using the "monotonicity" approach to nonlinear problems previously developed for operator equations in Hilbert space by E. H. Zarantonello (*41*), G. Minty (*31*) and F. E. Browder (*5*), (*6*) and for equations involving operators from a Banach space X to its dual X^* by F. E. Browder (*7*), (*8*), G. Minty (*32*) and J. Leray–J. L. Lions (*25*). A survey of the theory and further references to the literature can be found in F. E. Browder (*9*).

5. *Variational Inequalities for Convex Functions*

Now we show how inequality (1) can be written by replacing the subset K of X by a function on X with extended real values.

For any subset K of X, let δ_K (the *indicator function* of K) be the function defined on X by putting

$$\delta_K(v) = 0 \qquad \text{if} \quad v \in K,$$

$$\delta_K(v) = +\infty \qquad \text{if} \quad v \notin K.$$

Then, it is easy to verify that the vector u of K is a solution of (1) if and only if u is a vector of X such that

$$\langle Au, v - u\rangle \geqslant \delta_K(u) - \delta_K(v) \qquad \text{for all} \quad v \in X.$$

Therefore, we are led to consider, as a generalization of (1), inequalities of the following type:

$$u \in X\colon \langle Au, v - u\rangle \geqslant f(u) - f(v) \qquad \text{for all} \quad v \in X, \tag{2}$$

where f is an arbitrary function on X with values in $(-\infty, +\infty]$.

We shall discuss inequalities (2) in Subsection 7 below. First, we recall a few standard definitions from the theory of convex functions.

6. *Some More Definitions*

By *function* on X we mean a mapping f of X into $[-\infty, +\infty]$. A function f on X is *proper*, if $f(v) > -\infty$ for all $v \in X$. The *effective domain* of f is the subset of X

$$\operatorname{dom} f = \{v \in X\colon f(v) < +\infty\}.$$

The *epigraph* of f is the subset of $X \times \mathbb{R}$

$$\operatorname{epi} f = \{\{v, \beta\} \in X \times \mathbb{R}\colon f(v) \geqslant \beta\}.$$

A function f on X is *convex*, if $\operatorname{epi} f$ is a convex subset of $X \times \mathbb{R}$, that is, if for all u and v in X, we have

$$f(\lambda u + (1 - \lambda)v) \leqslant \lambda f(u) + (1 - \lambda) f(v)$$

for all λ with $0 < \lambda < 1$ (we assume $+\infty + (-\infty) = -\infty + (+\infty) = +\infty$). F is *stricly convex* if it is convex and, besides, one has

$$2f\left(\frac{u + v}{2}\right) < f(u) + f(v).$$

A convex function on X is *lower-semicontinuous* in X if $\operatorname{epi} f$ is a (convex) closed subset of $X \times \mathbb{R}$. By the convexity of $\operatorname{epi} f$, we can regard $X \times \mathbb{R}$ as endowed with the product topology of either the strong or the weak topology of X, and the natural topology of $\mathbb{R}$. Therefore, f is *lower-semicontinuous* in X if and only if we have

$$f(v) \leqslant \liminf f(v_n) \qquad \text{as} \qquad n \to +\infty,$$

for any sequence (v_n) converging (weakly or strongly) to v in X.

7. *Properties of inequalities* (2)

Let A be a map from X to X^*. If f is a proper function on X, with $\varnothing \neq \operatorname{dom} f \subset D(A)$, we shall denote by

$$S(A, f)$$

the set of all vectors u of $D(A)$ which are solution of inequality (2) above.

The properties of inequalities (2) are quite similar to those of inequalities (1). Namely, the following results hold:

$S(A, f)$ is a (possibly empty) subset of $\operatorname{dom} f$, which is closed and convex, provided f is lower semicontinuous and convex and A is monotone and hemicontinuous;

$S(A, f)$ is non-empty, if, in addition, X is a reflexive Banach space and either $\operatorname{dom} f$ is bounded or the following coerciveness condition is satisfied:

$$\{\langle Av, v\rangle + f(v)\}/\|v\| \to +\infty \qquad \text{as} \qquad \|v\| \to +\infty, \qquad v \in \operatorname{dom} f,$$

which is the case whenever A is coercive on $\operatorname{dom} f$.

If either A is strictly monotone or f is strictly convex, then $S(A, f)$ consists of, at most, a single vector.

[The uniqueness of the solution of (2) in case f is strictly convex, which seems not to have been noted explicitly in the literature, can be simply proved as follows: suppose u_1 and u_2 in $S(A, f)$, $u_1 \neq u_2$; we have

$$\langle Au_1, v - u_1\rangle \geqslant f(u_1) - f(v),$$
$$\langle Au_2, v - u_2\rangle \geqslant f(u_2) - f(v)$$

for all $v \in X$; putting $v = (u_1 + u_2)/2$ and adding, we find

$$\langle Au_1 - Au_2, u_2 - u_1\rangle \geqslant f(u_1) + f(u_2) - 2f\left(\frac{u_1 + u_2}{2}\right);$$

hence, since A is monotone

$$2f\left(\frac{u_1 + u_2}{2}\right) \geqslant f(u_1) + f(u_2),$$

which contradicts the strict convexity of f.]

A significant special case of (2) is obtained for $A = 0$. Then, $S(0, f)$ is the set of all u in X which minimize f on X (actually, on the effective domain of f).

8. *References for Inequalities (2)*

The inequalities (2) were introduced as a generalization of (1) by C. Lescarret (*26*), for A an accretive linear operator in a Hilbert space. This special case was also studied by J. L. Lions–G. Stampacchia (*28*).

The results stated in Subsection 7 above, are due to F. E. Browder (*13*) who has also considered non-coercive A (Ref. (*14*)), by making use of the duality mappings of X to X^* to obtain an "elliptic regularization" of A.

As we shall see below, any inequality such as (2) can be written as an inequality of type (1) in the space $X \times \mathbb{R}$. This makes it possible to deduce the properties of (2) from the corresponding properties of (1). A proof along this line of the existence theorem stated in Subsection 7 has been given by the author (*36*).

9. *Equivalence of Inequalities (1) and (2)*

We shall denote by $X \oplus \mathbb{R}$ the space $X \times \mathbb{R}$, normed by

$$\|\{v, \beta\}\| = (\| v \|^2 + | \beta |^2)^{1/2}.$$

We identify the dual $(X \oplus \mathbb{R})^*$ of $X \oplus \mathbb{R}$ with $X^* \oplus \mathbb{R}$, the pairing between $\{v, \beta\} \in X \oplus \mathbb{R}$ and $\{v', \beta'\} \in X^* \oplus \mathbb{R}$ being

$$\langle \{v', \beta'\}, \{v, \beta\} \rangle = \langle v', v \rangle + \beta' \beta.$$

For any map A of $D(A)$ in X to X^*, we shall denote by

$$A \oplus 1$$

the map $\{v, \beta\} \to \{Av, 1\}$ of $D(A \oplus 1) = D(A) \oplus \mathbb{R}$ in $X \oplus \mathbb{R}$ to $X^* \oplus \mathbb{R}$.

Clearly, $A \oplus 1$ is monotone, hemicontinuous, provided A is such.

Let A be given and let f be a proper function on X, with $\varnothing \neq \operatorname{dom} f \subset D(A)$. According to our notation of Subsection 3, $S(A \oplus 1, \operatorname{epi} f)$ is the set of all $\{u, \alpha\} \in \operatorname{epi} f$ such that

$$\langle A \oplus 1 \{u, \alpha\}, \{v, \beta\} - \{u, \alpha\} \rangle \geqslant 0 \qquad \text{for all} \quad \{v, \beta\} \in \operatorname{epi} f,$$

that is, the set of all $u \in X$, $\alpha \in \mathbb{R}$, with $\alpha \geqslant f(u)$, such that

$$\langle Au, v - u \rangle + \beta - \alpha \geqslant 0$$

for all $v \in X$ and $\beta \in \mathbb{R}$, with $\beta \geqslant f(v)$.

It follows that

$$\{u, \alpha\} \in S(A \oplus 1, \operatorname{epi} f)$$

if and only if

$$u \in S(A, f) \quad \text{and} \quad \alpha = f(u).$$

Further extensions and applications of the theory have been given by F. E. Browder (*15*) (where further references can be found) and G. Minty (*33*), who consider also multivalued maximal monotone operators, and by H. Brezis (*3*), who replaces the monotonicity assumption by suitable continuity properties of A.

In this paper we shall restrict our study to monotone (single-valued) mappings from a real reflexive Banach space to its dual. However, many of our results could be proved in the more general setting of linear spaces in duality.

1. Convergence of Convex Sets and Convex Functions

The classical Hausdorff definition of a metric for the space of closed subsets of a (compact) metric space has been generalized by many authors, who have introduced a topology, or a pseudo-topology, or simply a convergence, in the space of closed subsets of a topological space, see for instance L. Vietoris (*40*), C. Kuratowski (*24*), C. Choquet (*19*), and E. Michael (*30*).

However, in view of the applications given in this paper, we have found it more convenient to define a special convergence for *convex* closed subsets of a normed space X, in which both the strong and weak topologies of X are involved, see Definition 1.1 below. Let us notice, incidentally, that this convergence can be defined in any locally convex topological vector space.

As in Refs. (*24*) and (*19*), we have used the classical notions of lim inf and lim sup of sets (for these, see also C. Bouligand (*2*) and G. T. Whyburn (*39*)): the former relative to the strong topology of X, the latter to the weak one [actually, it suffices for our purposes to define lim sup in terms of weakly convergent *sequences* only].

In Subsection 4 we shall establish the connection between the convergence so defined and a convergence defined "locally" in terms of Hausdorff distance for closed sets, which generalize the "opening" convergence for linear subspaces of X (for this, see for instance T. Kato (*22*), where further references are given).

1. *Definition of Lim* S_n

Let (S_n) be a sequence of subsets of X. We shall denote by

$$\text{s-}\underline{\text{Lim}}\, S_n\,,$$

the set of all v in X, such that

$$v = \text{s-lim}\, v_n \qquad \text{in } X \text{ as} \qquad n \to +\infty,$$

for a sequence (v_n), with $v_n \in S_n$ for all large n.

We shall denote by

$$\text{w-}\overline{\text{Lim}}\, S_n\,,$$

the set of all v in X, such that

$$v = \text{w-lim}\, v_k \qquad \text{in } X \text{ as} \qquad k \to +\infty,$$

for a sequence (v_k), with $v_k \in S_{n_k}$ for every k and (S_{n_k}) a subsequence of (S_n).

Definition 1.1. A sequence (S_n) of subsets of X *converges* in X, if

$$\text{s-}\underline{\text{Lim}}\, S_n = \text{w-}\overline{\text{Lim}}\, S_n\,;$$

(S_n) *converges to* S in X, if (S_n) converges and S is a subset of X, such that

$$\text{s-}\underline{\text{Lim}}\, S_n = \text{w-}\overline{\text{Lim}}\, S_n = S.$$

If (S_n) converges to S, then we write either

$$S_n \to S,$$

or

$$S = \text{Lim}\, S_n\,.$$

Note that if $S_n \to S$ and $S \neq \varnothing$, then $S_n \neq \varnothing$ for every $n > n_1$, $n_1 > 0$. On the other hand, we may have $S_n \neq \varnothing$ for all n, while $S = \text{Lim } S_n = \varnothing$.

In case S_n consists, for each n, of a single vector v_n of X, then we have $S_n \to S$ in X and $S \neq \varnothing$, if and only if (v_n) converges strongly in X to a vector v of X and $S = \{v\}$.

2. *Definition of w-Lim S_n*

We shall also use a weaker limit of (S_n). Namely, let us denote by

$$\text{w-}\underline{\text{Lim}}\, S_n$$

the set of all v in X, such that

$$v = \text{w-lim } v_n \qquad \text{in } X \text{ as} \qquad n \to +\infty,$$

for a sequence (v_n) with $v_n \in S_n$ for all large n.

Then we give the following

Definition 1.2. A sequence (S_n) of subsets of X *converges weakly* in X to a subset S of X, if we have

$$\text{w-}\underline{\text{Lim}}\, S_n = \text{w-}\overline{\text{Lim}}\, S_n = S.$$

Then we write

$$S = \text{w-Lim } S_n\,.$$

Clearly, if $S_n = \{v_n\}$ for each n, then $S = \text{w-Lim } S_n \neq \varnothing$, if and only if v_n converges weakly to a vector v of X as $n \to +\infty$ and $S = \{v\}$.

3. *A Convergence for Convex Sets*

Let (S_n) be a sequence of closed convex subsets of X. If $S_n \to S$ in X, then clearly S is a closed convex subset of X. Moreover, we have $S_n \to S$ in X, if and only if

$$\text{(i)} \qquad S \subset \text{s-}\underline{\text{Lim}}\, S_n$$

$$\text{(ii)} \qquad \text{w-}\overline{\text{Lim}}\, S_n \subset S,$$

and (i) is trivially satisfied if $S \subset S_n$ for every n, while (ii) holds whenever $S_n \subset S$ for every n.

In particular,

(a) *If S is a closed convex subset of X and $S_n = S$ for every n, then (S_n) converges and $S = \mathrm{Lim}\, S_n$.*

If (S_k') is a subsequence of (S_n) we have the obvious inclusions

$$\text{s-}\underline{\mathrm{Lim}}\, S_n \subset \text{s-}\underline{\mathrm{Lim}}\, S_k'$$

$$\text{w-}\overline{\mathrm{Lim}}\, S_k' \subset \text{w-}\overline{\mathrm{Lim}}\, S_n .$$

Therefore, we have

(b) *If $S_n \to S$ and (S_k') is a subsequence of (S_n), then $S_k' \to S$.*

Furthermore, as we shall see below, the following property holds:

(c) *If any subsequence (S_k') of (S_n) contains a subsequence (S_h'') which converges to S in X, then (S_n) converges and $S = \mathrm{Lim}\, S_n$.*

Therefore, the mapping $(S_n) \mapsto \mathrm{Lim}\, S_n$, since (a), (b) and (c) are satisfied, gives to the family of all closed convex subsets of X a structure of space $\mathscr{L}^*$, in the terminology of Kuratowski (*23*).

[(c) can be proved as follows:

First, suppose $S \neq \varnothing$. Note that for any v of X, we have $v \in \text{s-}\underline{\mathrm{Lim}}\, S_n$ if and only if $d(v, S_n) \to 0$ as $n \to +\infty$, where

$$d(v, U) = \inf\{\| v - u \| : u \in U\}$$

for any subset U of X, $U \neq \varnothing$. Now we prove that (i) holds. In fact, suppose there exists $v_0 \in S$ such that v_0 does not belong to $\text{s-}\underline{\mathrm{Lim}}\, S_n$. Then, $d(v_0, S_n) \not\to 0$ as $n \to +\infty$, hence there exists $\rho > 0$ and a subsequence (S_k') of (S_n), such that $d(v_0, S_k') > \rho$ for all k. On the other hand, since (S_k') contains a subsequence (S_h'') converging to S in X, there exists for each h a vector $v_h \in S_h''$, such that $v_0 = \text{s-lim}\, v_h$ in X as $h \to +\infty$. Hence we find a contradiction. Let us prove now that (ii) holds. If $v \in \text{w-}\overline{\mathrm{Lim}}\, S_n$, there exists a subsequence (S_k') of (S_n) such that $v = \text{w-lim}\, v_k$, with $v_k \in S_k'$ for each k. Thus, if (S_h'') is a subsequence of (S_k') which converges to S, we have $v \in \text{w-}\overline{\mathrm{Lim}}\, S_h$, hence $v \in S$. Therefore, we have proved that $S_n \to S$.

Now suppose $S = \varnothing$. To prove that $S_n \to S$, it suffices to prove that $\text{w-}\overline{\mathrm{Lim}}\, S_n = \varnothing$. On the contrary, it would exists a subsequence (S_k') of (S_n) and for each k a vector $v_k \in S_k'$, such that $v_k \rightharpoonup v$, with $v \in X$; whereas, since (S_k') has a subsequence, say, (S_h''), converging to S and $S = \varnothing$, we should have $\text{w-}\overline{\text{Lim}}\, S_h'' = \varnothing$. Hence, a contradiction.]

4. *The "Local Gap"*

We intend now to compare the convergence we have introduced in the previous subsection with a convergence defined in terms of the Hausdorff metric for closed sets. Let us notice that the remainder of the paper is independent of this subsection.

Following the definition of "gap", or "opening", between two closed linear subspaces of X—see for instance T. Kato (*22*)—we can define for each $R > 0$ a "local gap"

$$\sigma_R(S_1, S_2)$$

between two closed convex subsets S_1 and S_2 of X, by setting

$$\sigma_R(S_1, S_2) = \max\{\sigma(S_1{}^R, S_2), \sigma(S_2{}^R, S_1)\},$$

where

$$S^R = \{v \in X : v \in S, \| v \| \leqslant R\}$$

for any subset S of X, and

$$\sigma(U, V) = \sup\{d(u, V) : u \in U\}$$

for any couple of closed subsets of X, with the additional convention that $\sigma(U, V) = 0$ when both U and V are the empty set, while $\sigma(U, V) = +\infty$ if only one of them is $\varnothing$.

On every family of uniformly bounded non-empty closed convex sets, σ_R, for each R large enough, reduces to the classical Hausdorff metric.

On the other hand, since the "gap" $\hat{\delta}(M, N)$ between two closed linear subspaces M and N of X can be characterized as the maximum of the smallest η_1 and η_2, such that

$$d(v, N) \leqslant \eta_1 \| v \| \qquad \text{for all} \quad v \in M,$$
$$d(v, M) \leqslant \eta_2 \| v \| \qquad \text{for all} \quad v \in N$$

(see T. Kato, *loc. cit.*), then we have for every $R > 0$

$$\sigma_R(M, N) = R\hat{\delta}(M, N).$$

Therefore, for closed linear subspaces of X, the convergence according to $\mathcal{S}$ is equivalent to convergence according to σ_R for every R.

We send to the reference quoted above for a discussion of $\mathcal{S}$ and for further references on the subject.

Let (S_n) be a sequence of closed convex subsets of X, S a closed convex subset of X. For any $R > 0$, we have

$$\sigma_R(S_n, S) \to 0 \quad \text{as} \quad n \to +\infty,$$

if and only if for any $\rho > 0$, there exists $n_\rho > 0$ (possibly depending on R) such that, for all $n > n_\rho$, either $S_n = S = \varnothing$, or both the following conditions are satisfied

$$\text{(j)} \qquad \varnothing \neq S^R \subset I_\rho S_n,$$

$$\text{(jj)} \qquad \varnothing \neq S_n{}^R \subset I_\rho S,$$

where

$$I_\rho U = \{v \in X : d(v, U) \leqslant \rho\}$$

for any non-empty closed subset U of X.

[In fact,

$$\sigma_R(S_n, S) \leqslant \rho$$

is equivalent to

$$\sigma(S^R, S_n) \leqslant \rho, \qquad \sigma(S_n{}^R, S) \leqslant \rho,$$

hence either to $S_n = S = \varnothing$, or to

$$d(v, S_n) \leqslant \rho \quad \text{for all} \quad v \in S^R, \quad S^R \neq \varnothing$$

$$d(v, S) \leqslant \rho \quad \text{for all} \quad v \in S_n{}^R, \quad S_n{}^R \neq \varnothing,$$

which are the same as (j) and (jj).]

Lemma 1.1. *Let S be a non-empty closed convex subset of X, (S_n) a sequence of closed convex subsets of X. Then,*

(a) *If we have*

$$\sigma_R(S_n, S) \to 0 \quad \textit{as} \quad n \to +\infty, \tag{1.1}$$

for every $R > R_0$, $R_0 > 0$, then

$$S_n \to S \qquad \text{in } X \text{ as} \qquad n \to \infty, \tag{1.2}$$

according to Definition 1.1.

(b) *If X has finite dimension, then the converse of* (a) *is true.*

Remark 1.1. As we will show below by examples, in an infinite dimensional X the converse of (a) may be false, even if the S_n are uniformly bounded.

Proof of Lemma 1.1. Let us suppose that (1.1) holds for every $R > R_0$. Then, (j) and (jj) are satisfied for all $R > R_0$. Let us prove that (i) of Subsection 3 holds. Let $v \in S$ and $R > \max\{R_0, \| v \|\}$. For any $\rho > 0$, we have by (j) for all n large enough

$$v \in I_\rho S_n,$$

which is to say

$$d(v, S_n) \leqslant \rho.$$

Therefore, we have $d(v, S_n) \to 0$ as $n \to +\infty$, that is, $v \in \text{s-}\underline{\text{Lim}}\, S_n$. Thus (i) has been proved. Let us prove (ii) of Subsection 3. Let $v \in X$, $v_k \in S_k'$ for every k, with (S_k') a subsequence of (S_n), and suppose that

$$v = \text{w-lim}\, v_k \qquad \text{as} \qquad k \to +\infty.$$

There exists $R > R_0$ such that $\| v_k \| \leqslant R$ for all k, hence

$$v_k \in S_k'^R \qquad \text{for every} \quad k,$$

which implies, by (jj), that for any given $\rho > 0$ we have

$$v_k \in I_\rho S$$

for all k sufficiently large. Thus, since $I_\rho S$ is closed and convex, we find

$$v \in I_\rho S,$$

which implies, since ρ is an arbitary positive number, that $v \in S$. This proves (ii). Therefore, $S = \text{Lim}\, S_n$ and part (a) of the lemma has been proved.

Let us suppose now that X has finite dimension and prove that (1.2) implies that (1.1) holds for all $R > R_0$, for some $R_0 > 0$. Let $S^R \neq \varnothing$ for all $R > R_1$. Since S^R is compact, for any given $\rho > 0$ there exists a finite number of vector $v_1, \ldots, v_N$ of S^R, such that

$$S^R \subset \bigcup_{i=1}^{N} I_{\rho/2}\{v_i\}.$$

By (1.2), there exists n_ρ such that for all $n > n_\rho$, we have

$$d(v_i, S_n) \leqslant \rho/2 \qquad \text{for all} \quad i = 1, \ldots, N.$$

This implies

$$I_{\rho/2}\{v_i\} \subset I_\rho S_n, \qquad i = 1, \ldots, N,$$

hence

$$S^R \subset I_\rho S_n ;$$

thus (j) holds.

Now, let $S_n{}^R \neq \varnothing$ for all $R > R_2 > 0$ and all $n > n_2 > 0$. Let us suppose that there exists $\bar{R} > R_2$ and $\bar{\rho} > 0$, such that

$$S_k'^{\bar{R}} \not\subset I_{\bar{\rho}} S$$

for a subsequence (S_k') of (S_n). There exists then a sequence (v_k), with

$$v_k \in S_k'^{\bar{R}}, \qquad v_k \notin I_{\bar{\rho}} S$$

for all k, which is bounded in X, hence containes a subsequence (v_h') converging to a vector v of X as $h \to +\infty$. By (ii), we should have $v \in S$, whereas we have $v \notin I_\rho S$, $\rho < \bar{\rho}$. Therefore, also (jj) holds and part (b) of the lemma has been proved. ∎

Let us consider the Hilbert space l_2, of all sequences

$$v = (v^{(1)}, \ldots, v^{(h)}, \ldots), \qquad v^{(h)} \in \mathbb{R},$$

with

$$v = \left(\sum_{h=1}^{+\infty} | v^{(h)} |^2\right)^{1/2}.$$

Let us consider the following (uniformly bounded, closed convex) subsets of l_2:

$$S = B \cap \{v \in l_2 : 0 \leqslant v^{(h)} \leqslant 1 \text{ for all } h\},$$

$$S_n = B \cap \{v \in l_2 : 0 \leqslant v^{(h)} \leqslant 1 + n^{-\alpha} h \text{ for all } h\},$$

where $B = \{v \in l_2 : \| v \| \leqslant 2\}$, α is a given positive number and $n = 1, 2, \ldots$.

Then, $S_n \to S$ in l_2 as $n \to +\infty$, according to Definition 1.1, whereas it is false that for any $\rho > 0$ we have

$$S_n \subset I_\rho S$$

for all n sufficiently large.

Now let us take

$$S = B \cap C, \qquad S_n = B \cap C_n ,$$

where

$$C = \overline{\text{co}}\{(1, 0, \ldots), (0, 1, 0, \ldots), \ldots, (0, \ldots, 0, 1, 0, \ldots), \ldots\}$$

$$C_n = \overline{\text{co}}\{(1 + n^{-\alpha}, 0, \ldots), (0, 1 + 2n^{-\alpha}, 0, \ldots), \ldots, (0, \ldots, 0, 1 + hn^{-\alpha}, 0, \ldots), \ldots\}$$

Then, again we have $S_n \to S$ in l_2 as $n \to +\infty$, but it is not true that for any $\rho > 0$ we have

$$S \subset I_\rho S_n$$

for all large n.

5. *Examples*

In this subsection we collect some examples, of geometrical or functional nature, of sequences of closed convex subsets of a normed space X, which converge according to Definition 1.1.

Lemma 1.2. *Let (S_n) be an increasing sequence of closed convex subsets of X, $S_n \subset S_m$ if $n \leqslant m$. Then, (S_n) converges in X and*

$$\text{Lim}\, S_n = S,$$

where S is the closure of $\bigcup_n S_n$ in X.

Proof. S is a closed convex subset of X, hence S is weakly closed. Therefore (ii) of Subsection 3 holds. Moreover, (i) holds, for $d(v, S_n) \to 0$ as $n \to +\infty$ for each $v \in S$, because (S_n) is increasing. ∎

Lemma 1.3. *Let (S_n) be a decreasing sequence of closed convex subsets of X, $S_n \subset S_m$ if $n \geqslant m$. Then, (S_n) converges in X and*

$$\text{Lim}\, S_n = \bigcap_n S_n .$$

Proof. Put $S = \bigcap_n S_n$. Clearly, (i) holds. If w-$\overline{\text{Lim}}\, S_n = \varnothing$, the conclusion is trivial. Suppose there exists $v \in$ w-$\overline{\text{Lim}}\, S_n$ and that (S_k') is a subsequence of (S_n) such that $v =$ w-Lim v_k, with $v_k \in S_k'$ for each k. Since (S_k') is decreasing, for each $k_0 > 0$ we have $v_k \in S'_{k_0}$ for all $k > k_0$, hence, since S'_{k_0} is weakly closed, $v \in S'_{k_0}$. Therefore, $v \in \bigcap_k S_k'$, which implies $v \in S$. Thus w-$\overline{\text{Lim}}\, S_n \subset S$, that is (ii) holds. ∎

Lemma 1.4. *Let K be a closed convex subset of X, whose interior is nonempty, and (S_n) a sequence of closed convex subsets of X, such that $S_n \to S$ in X as $n \to +\infty$. Then, $K \cap S_n \to K \cap S$ in X as $n \to +\infty$.*

Remark 1.2. The lemma is trivially false, if we suppress the hypothesis that the interior of K is nonempty, as it can be seen by taking K consisting of a single vector v of l_2 with $v^{(h)} \neq 0$ for infinitely many h (or the one-dimensional linear subspace spanned by such a v), and S_n the n-dimensional linear subspace V_n of l_2, spanned by the first n vectors $(1, 0,\ldots), (0, 1, 0,\ldots),\ldots, (0,\ldots, 0, 1, 0,\ldots)$.

Proof of Lemma 1.4. First we prove that

$$\text{w-}\overline{\text{Lim}}\, K \cap S_n \subset K \cap S.$$

In fact, if $v =$ w-lim v_k, with $v_k \in K \cap S_k'$ and (S_k') a subsequence of (S_n), then $v \in K$, for K is weakly closed, and, besides, $v \in$ w-$\overline{\text{Lim}}\, S_n$. Hence $v \in K \cap S$. If $K \cap S = \varnothing$, it follows that $K \cap S_n \to K \cap S$. If $K \cap S \neq \varnothing$, it suffices to prove that

$$K \cap S \subset \text{s-}\underline{\text{Lim}}\, K \cap S_n .$$

Let $u_0 \in \text{int}\, K$, int K being the interior of K, and let $N(u_0)$ be a strong neighbourhood of u_0 contained in K. Let u be an arbitrary vector of $K \cap S$ and let C be the convex cone generated by u and $N(u_0)$. Clearly, $C \subset K$. Let $N(u)$ be any strong neighbourhood of u and take $u_1 \in \text{int}\, C \cap N(u)$, int C the interior of C. Such a vector u_1 exists, because it can be chosen of type $u_1 = \eta u_0 + (1 - \eta)u$ for $\eta > 0$ small enough. Let $N(u_1)$ be a strong neighborhood of u_1 contained in $C \cap N(u)$. Since $S_n \to S$, we have $S_n \cap N(u_1) \neq \varnothing$ for all $n > n_0$, $n_0 > 0$. Hence, we have $(K \cap S_n) \cap N(u) \neq \varnothing$ for all $n > n_0$. Thus, $u \in$ s-Lim $K \cap S_n$ and the proof is complete. ∎

Lemma 1.5. *Let X be a Hilbert space, K a bounded closed convex* *subset of X, and (P_n) a sequence of symmetric (linear) operators in X, with $D(P_n) = X$, such that*

$$P_n v \to v \quad \text{in } X \text{ as} \quad n \to +\infty \quad \text{for all} \quad v \in X. \tag{1.3}$$

Then,

$$P_n K \to K \quad \text{in } X \text{ as} \quad n \to +\infty,$$

where for each n, $P_n K = \{v \in X: v = P_n w, w \in K\}$.

Remark 1.3. The lemma is false if we omit the hypothesis that K is bounded. In fact, take $X = l_2$,

$$K = \overline{\text{co}}\{(1!, 0,\ldots), (0, 2!, 0,\ldots),\ldots, (0,\ldots, 0, h!, 0,\ldots),\ldots\}$$

and let P_n be for each n the orthogonal projection on the n-dimensional linear subspace V_n of l_2 considered in Remark 1.2. Then, $0 \in P_n K$ for all n, whereas $0 \notin K$.

That can happen even if K is a closed linear subspace. Indeed, let K be the closed linear subspace of l_2 which is spanned by the vectors $(1!, 2!, 0,\ldots), (0, 2!, 3!, 0,\ldots),\ldots, (0,\ldots, 0, h!, (h + 1)!, 0,\ldots)$. Then, $K \neq l_2$, while $P_n K = V_n$ for every n.

However, the hypothesis of boundedness of K can be obviously replaced by the assumption

$$P_n K \subset K \quad \text{for every} \quad n.$$

Proof of Lemma 1.5. The inclusion (i) of Subsection 3 is an immediate consequence of (1.3). It remains to prove that w-$\overline{\text{Lim}}\, P_n K \subset K$. Let us consider an arbitrary subsequence of (P_n), say still (P_n), and suppose that $v_n = P_n w_n$, with $w_n \in K$ for every n, and that

$$v_n \rightharpoonup v_0 \quad \text{in } X \text{ as} \quad n \to +\infty, \quad v_0 \in X.$$

We must prove that $v_0 \in K$.

Suppose $v_0 \notin K$. By the Hahn-Banach theorem, there exists a vector $v_0' \in X$, such that

$$(v_0', v_0) = 1,$$
$$(v_0', w) = 0 \quad \text{for all} \quad w \in K,$$

where (,) denotes the inner product in X. Therefore, since

$$0 = (v_0', w_n) = (v_0', v_n) + (v_0', w_n - v_n)$$

for every n, we find

$$\lim(v_0', v_n - w_n) = \lim(v_0', v_n) = (v_0', v_0) = 1$$

as $n \to +\infty$. On the other hand, we have for each n, by the symmetry of P_n,

$$(v_0', P_n w_n - w_n) = (P_n v_0' - v_0', w_n).$$

Since (w_n) is bounded in X, we have by (1.3),

$$\lim(v_0', v_n - w_n) = 0 \qquad \text{as} \qquad n \to +\infty,$$

hence a contradiction. ∎

If S is a subset and v a vector of X, we shall denote by

$$S + v$$

the set $\{z \in X\colon\ z = w + v,\ w \in S\}$ if $S \neq \varnothing$, and the empty set if $S = \varnothing$.

Lemma 1.6. *Let (S_n) be a sequence of subsets of X, such that $S_n \to S$ in X as $n \to +\infty$, (v_n) a sequence of vectors of X, such that $v_n \to v$ in X as $n \to +\infty$, v a vector of X. Then,*

$$S_n + v_n \to S + v \qquad \textit{in } X \textit{ as} \qquad n \to +\infty.$$

Proof. Clearly $S + v \subset \text{s-}\underline{\text{Lim}}\,(S_n + v_n)$. Let z be a vector of $\text{w-}\overline{\text{Lim}}\,(S_n + v_n)$, that is, $z = \text{w-lim}\, z_k$, with $z_k = w_k + v_k' \in S_k' + v_k'$ for each k, $(S_k' + v_k')$ being a subsequence of $(S_n + v_n)$. Since $v_k' \to v$ as $k \to +\infty$, then w_k converges weakly to $z - v$ in X as $k \to +\infty$. Hence $z - v \in \text{w-}\overline{\text{Lim}}\, S_n$, therefore $z - v \in S$, that is $z \in S + v$. Thus, $\text{w-}\overline{\text{Lim}}\,(S_n + v_n) \subset S + v$. ∎

The special case of Lemma 1.6 with $S_n = S$ for every n and S a closed convex subset of X, shows that

$$v_n \to v \qquad \text{in } X \text{ as} \qquad n \to +\infty,$$

implies

$$S + v_n \to S + v \qquad \text{in } X \text{ as} \qquad n \to +\infty.$$

A functional example of that will be considered in Lemma 1.7 below.

We recall before a few definitions relative to the Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$, that will be also used in Section 3.

Let $\mathbb{R}^s$ be the s-dimensional (real) Euclidean space. We denote by $x = (x_1, x_2, \dots, x_s)$ the general point in $\mathbb{R}^s$ and for any s-tple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$ of non-negative integers we put

$$D^\alpha = \prod_{i=1}^{s} \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}, \qquad |\alpha| = \sum_{i=1}^{s} \alpha_i .$$

Let Ω be a bounded open subset of $\mathbb{R}^s$ with a smooth boundary $\partial\Omega$, m a positive integer and p a real, with $1 < p < +\infty$.

$W^{m,p}(\Omega)$ is the space of all real functions $v \in L^p(\Omega)$, whose distribution derivatives $D^\alpha v$, with $|\alpha| \leqslant m$, also belong to $L^p(\Omega)$. With the norm

$$\| v \|_{m,p} = \Big(\sum_{|\alpha| \leqslant m} \| v \|^p_{L^p(\Omega)} \Big)^{1/p},$$

$W^{m,p}(\Omega)$ is a reflexive Banach space.

$W_0^{m,p}(\Omega)$ is the closure in $W^{m,p}(\Omega)$ of the linear subspace $C_0^\infty(\Omega)$ of all infinitely differentiable (real) functions on Ω with a compact support.

Following W. Littman–G. Stampacchia–H. T. Weinberger, (*29*), if $v \in W_0^{1,p}(\Omega)$ and E is a closed subset of Ω, we shall say that v is *non-negative* on E in the sense of $W_0^{1,p}(\Omega)$, and write

$$v \geqslant 0 \quad \text{on} \quad E,$$

if

$$v \in \overline{P_0(E, \Omega)},$$

where $\overline{P_0(E, \Omega)}$ is the closure in $W^{1,p}(\Omega)$ of the convex cone

$$P_0(E, \Omega) = \{\varphi \in C_0^\infty(\Omega) : \varphi \geqslant 0 \text{ on } E\}.$$

According to this definition, if u and v belong to $W_0^{1,p}(\Omega)$, we shall write

$$v \geqslant u \quad \text{on} \quad E$$

to mean that

$$v - u \in \overline{P_0(E, \Omega)}.$$

Finally, we recall that the *p-capacity* (relative to Ω) of a compact subset E of Ω, is defined by setting

$$p\text{-cap } E = \inf \left\{ \sum_{i=1}^{s} \| \varphi_{x_i} \|_{L_p(\Omega)}^{p} : \varphi \in C_0^\infty(\Omega), \varphi \geqslant 1 \text{ on } E \right\}, \tag{1.4}$$

where $\varphi_{x_i} = \partial\varphi/\partial x_i$; see Ref. (*29*), quoted above.

Lemma 1.7. *Let Ω be a bounded open subset of $\mathbb{R}^s$ with a smooth boundary, E a closed subset of Ω and $1 < p < +\infty$. Let $u \in W_0^{1,p}(\Omega)$,*

$$K = \{v \in W_0^{1,p}(\Omega) : v \geqslant u \text{ on } E\},$$

(u_n) a sequence in $W_0^{1,p}(\Omega)$ and

$$K_n = \{v \in W_0^{1,p}(\Omega) : v \geqslant u_n \text{ on } E\}$$

for each n. Then, $K_n \to K$ in $W^{1,p}(\Omega)$ as $n \to +\infty$, provided u_n converges strongly to u in $W^{1,p}(\Omega)$ as $n \to +\infty$.

Proof: It suffices to apply the special case of Lemma 1.6 considered above, taking into account that $K = \overline{P_0(E, \Omega)} + u$ and $K_n = \overline{P_0(E, \Omega)} + u_n$ for every n. ▮

Lemma 1.8. *Let (E_n) be a sequence of compact subsets of Ω. Then, we have*[1]

$$W_0^{1,p}(\Omega) = \text{Lim } W_0^{1,p}(\Omega - E_n)$$

in the space $W^{1,p}(\Omega)$ according to Definition 1.1, *if and only if for any compact subset Ω' of Ω we have*

$$p\text{-cap } (E_n \cap \Omega') \to 0 \qquad as \qquad n \to +\infty.$$

Proof of Lemma 1.8. Let us prove first the "if" part of the lemma. Let v be an arbitrary function of $C_0^\infty(\Omega)$, Ω' a compact subset of Ω containing the support of v, and for each n let us put

$$E_n' = E_n \cap \Omega.$$

[1] For any closed subset E of Ω, we identify the space $W_0^{1,p}(\Omega - E)$ with the subspace of $W^{1,p}(\Omega)$ which is the closure in $W^{1,p}(\Omega)$ of all functions $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 0$ on E.

Since by our hypotesis p-cap $E_n' \to 0$ as $n \to +\infty$, there exists a sequence (φ_n) of functions of $C_0^\infty(\Omega)$, such that $\varphi_n \geqslant 1$ on E_n' for each n and

$$\sum_{i=1}^{s} \|(\varphi_n)_{x_i}\|^p_{L^p(\Omega)} \to 0 \qquad \text{as} \qquad n \to +\infty.$$

Obviously, we can suppose that $\varphi_n > 1$ on E_n'. By the Poincaré inequality, we also have $\| \varphi_n \|_{1,p} \to 0$ as $n \to +\infty$. It follows that the functions $\psi_n = \min\{\varphi_n , 1\}$, $n = 1, 2,...$ are such that for each n, $\psi_n = 1$ on a neighborhood of E_n' and, moreover, $\psi_n \in W^{1,p}(\Omega)$ and

$$\| \psi_n \|_{1,p} \to 0 \qquad \text{as} \qquad n \to +\infty.$$

By making a suitable regularization of ψ_n, we can find for every n a function $\psi_n^* \in C_0^\infty(\Omega)$ with $\psi_n^* = 1$ on E_n', such that

$$\| \psi_n - \psi_n^* \|_{1,p} \to 0 \qquad \text{as} \qquad n \to +\infty.$$

Now let us consider the function

$$w_n = v - \psi_n^* v.$$

Clearly, w_n belongs to $C_0^\infty(\Omega)$ and $w_n = 0$ on E_n, hence

$$w_n \in W_0^{1,p}(\Omega - E\ ).$$

Moreover, since

$$\begin{aligned} \| v - w_n \|_{1,p} &\leqslant \| \psi_n v \|_{1,p} + \|(\psi_n^* - \psi_n) v \|_{1,p} \\ &\leqslant \max_{i=1,\dots,s} \{\sup | v |, \sup | v_{x_i} |\}[\| \psi_n \|_{1,p} + \| \psi_n^* - \psi_n \|_{1,p}], \end{aligned}$$

it follows that

$$\| v - w_n \|_{1,p} \to 0 \qquad \text{as} \qquad n \to +\infty.$$

Therefore, we have proved

$$C_0^\infty(\Omega) \subset \text{s-}\underline{\text{Lim}}\ W_0^{1,p}(\Omega - E_n)$$

which implies, since s-$\underline{\text{Lim}}$ $W_0^{1,p}(\Omega - E_n)$ is closed in $W^{1,p}(\Omega)$, that

$$W_0^{1,p}(\Omega) \subset \text{s-}\underline{\text{Lim}}\ W_0^{1,p}(\Omega - E_n). \tag{1.5}$$

Since $W_0^{1,p}(\Omega - E_n) \subset W_0^{1,p}(\Omega)$ for every n, it follows

$$W_0^{1,p}(\Omega) = \operatorname{Lim} W_0^{1,p}(\Omega - E_n). \tag{1.6}$$

Conversely, suppose that (1.6) holds; hence (1.5) holds. Let Ω' be an arbitrary compact subset of Ω and α a function of $C_0^\infty(\Omega)$ such that $\alpha \equiv 1$ on Ω'.

Since $\alpha \in W_0^{1,p}(\Omega)$, there exists, by (1.5), for every n a function $\alpha_n \in W_0^{1,p}(\Omega - E_n)$, such that $\| \alpha - \alpha_n \|_{1,p} \to 0$ as $n \to +\infty$. Therefore, there exists also for each n a function $\beta_n \in C_0^\infty(\Omega)$, with $\beta_n = 0$ on E_n, such that $\| \alpha - \beta_n \|_{1,p} \to 0$ as $n \to +\infty$. Thus, if $\varphi_n = \alpha - \beta_n$ for each n, we have $\varphi_n \in C_0^\infty(\Omega)$, $\varphi_n = 1$ on $E_n' = E_n \cap \Omega'$ for every n, and

$$\| \varphi_n \|_{1,p} \to 0 \quad \text{as} \quad n \to +\infty.$$

Since p-cap $E_n \leqslant \| \varphi_n \|_{1,p}$ for every n, we find

$$p\text{-cap } E_n \to 0 \quad \text{as} \quad n \to +\infty. \qquad \blacksquare$$

6. "*Order* α" *Convergence of Convex Sets*

To study the dependence on the convex K of a solution of a variational inequality for a non-coercive mapping T, we need to control, as we already noticed in the Introduction, the rapidity of convergence of the approximate K_n to K as $n \to +\infty$. To this end, we shall use the following

Definition 1.3. Let (S_n) be a sequence of subsets of X and let $\alpha \geqslant 0$. We say that S_n *converges of order* $\geqslant \alpha$ in X to a subset S of X as $n \to +\infty$, and write

$$n^\alpha [S_n - S] \to 0 \quad \text{in} \quad X,$$

if (j) and (jj) below are satisfied:

(j) For any $v \in S$, we have

$$0 \in \text{s-}\underline{\operatorname{Lim}}\, n^\alpha (S_n - v) \quad \text{in} \quad X; \tag{1.7}$$

(jj) For any weakly convergent sequence (v_k) in X, with $v_k \in S_{n_k}$ for every k and (S_{n_k}) a subsequence of (S_n), we have [2]

$$0 \in \text{w-}\overline{\operatorname{Lim}}\, n_k^\alpha (v_k - S) \quad \text{in} \quad X. \tag{1.8}$$

[2] For any subset S of X, any vector $v \in X$ and any real c, we put

$$c(S - v) = \{z \in X : z = c(w - v), w \in S\}$$
$$c(v - S) = \{z \in X : z = c(v - w), w \in S\}.$$

According to our notation of Subsection 1, (1.7) means that there exists $v_n \in S_n$ (for all large n), such that

$$n^\alpha(v_n - v) \to 0 \qquad \text{in } X \text{ as} \qquad n \to +\infty,$$

while (1.8) means that there exists a sequence (w_h) in S, such that

$$n^\alpha_{k_h}(v_{k_h} - w_h) \rightharpoonup 0 \qquad \text{in } X \text{ as} \qquad h \to +\infty,$$

with (v_{k_h}) a subsequence of (v_k).

Let (S_n) be a sequence of convex closed subsets of X and S a non-empty convex closed subset of X:

Lemma 1.9. *We have $S = \text{Lim}\, S_n$ in X, according to Definition* 1.1, *if and only if S_n converges of order $\geqslant 0$ to S in X as $n \to +\infty$.*

Proof. It suffices to remark that

$$0 \in \text{s-}\underline{\text{Lim}}\,(S_n - v) \qquad \text{for every} \quad v \in S,$$

is equivalent to

$$S \subset \text{s-}\underline{\text{Lim}}\, S_n\,,$$

while

$$0 \in \text{w-}\overline{\text{Lim}}\,(v_k - S) \qquad \text{for any sequence such as} \quad (v_k),$$

is equivalent to

$$\text{w-}\overline{\text{Lim}}\, S_n \subset S. \qquad \blacksquare$$

It follows from Lemma 1.9 that if S_n converges of order $\geqslant\alpha$ to S, for some $\alpha \geqslant 0$, then $S = \text{Lim}\, S_n$.

Example. Let X be an inner product space, A a (linear) symmetric compact operator in X. Let $\beta > \alpha > 0$ and for any positive integer n let P_n be the orthogonal projection on the subspace of X, which is spanned by all eigenfunctions e_k of A corresponding to eigenvalues λ_k with $|\lambda_k| > n^{-\beta}$. Let H be a bounded subset of X, K a subset of AH, and, for each n, $K_n = P_nK$. Then, K_n converges of order $\geqslant\alpha$ to K in X as $n \to +\infty$. [In fact, this is a consequence of the inequality

$$n^\alpha\| Av - P_nAv \| \leqslant n^{-(\beta-\alpha)}\| v \| \qquad \text{for all} \quad v \in X \quad \text{and all} \quad n,$$

which implies that $n^{\alpha}(A - P_n A)$ converges to 0 as $n \to +\infty$ in the uniform topology of operators.] The hypothesis that H is bounded can be dropped if $P_n K \subset K$ for every n.

7. *A Convergence for Convex Functions*

We shall define below a convergence for (convex lower-semicontinuous) functions on X in terms of convergence of their epigraphs in $X \oplus \mathbb{R}$.

Notation and terminology in this and the following section are those of Subsections 6 and 9 of Section 0.

Definition 1.4. A sequence (f_n) of functions on X *converges* in X, if the sequence $(\operatorname{epi} f_n)$ of their epigraphs converges in $X \oplus \mathbb{R}$ according to Definition 1.1. We say that f_n *converges to* f in X as $n \to +\infty$, and write

$$f_n \to f, \qquad \text{or} \qquad f = \operatorname{Lim} f_n\,,$$

if (f_n) converges in X and f is a function on X, such that

$$\operatorname{epi} f = \operatorname{Lim} \operatorname{epi} f_n \quad \text{in} \quad X \oplus \mathbb{R},$$

according to Definition 1.1.

Remark 1.4. A sequence (S_n) of subsets of X converges to a subset S in X according to Definition 1.1, if and only if the sequence (δ_{S_n}) of the indicator functions of the S_n's converges to δ_S according to the definition above.

It is easy to show that if $(\operatorname{epi} f_n)$ converges in $X \oplus \mathbb{R}$ and $S = \operatorname{Lim} \operatorname{epi} f_n$, then there exists a function f on X, such that $S = \operatorname{epi} f$ It follows from Subsection 3 that $(f_n) \mapsto \operatorname{Lim} f_n$ is a convergence for the family of all convex lower-semicontinuous functions on X.

Some properties and examples of this convergence can be obtained along the lines of Subsection 5. A characterization of it is given by the following lemma.

Lemma 1.10. *Let (f_n) be a sequence of functions on X. Then we have*

$$f = \operatorname{Lim} f_n \quad \textit{in} \quad X,$$

if and only if (I) *and* (II) *below are satisfied:*

(I) *Any* $v \in X$ *is the limit in the strong topology of* X *of a sequence* (v_n) *in* X, *such that*

$$\limsup f_n(v_n) \leqslant f(v) \quad as \quad n \to +\infty; \tag{1.9}$$

(II) *Any subsequence* (f_k') *of* (f_n) *is such that for any* $v \in X$, *which is the limit in the weak topology of* X *of a sequence* (v_k) *in* X, *we have*

$$\liminf f_k'(v_k) \geqslant f(v) \quad as \quad k \to +\infty. \tag{1.10}$$

Proof. First we prove that (I) is equivalent to

$$\operatorname{epi} f \subset \text{s-}\underline{\mathrm{Lim}}\operatorname{epi} f_n \quad \text{in} \quad X \oplus \mathbb{R}. \tag{1.11}$$

Suppose $\operatorname{epi} f \neq \varnothing$ and take $\{v, \beta\} \in \operatorname{epi} f$, that is, $v \in X$, $\beta \in \mathbb{R}$ with $\beta \geqslant f(v)$. By (I), there exists a sequence (v_n) in X such that $v_n \to v$ as $n \to +\infty$ and (1.9) holds. If

$$\beta_n = \max\{f_n(v_n), \beta) \qquad \text{for each} \quad n,$$

then we have $\beta = \lim \beta_n$ as $n \to +\infty$. Therefore, $\{v_n, \beta_n\} \in \operatorname{epi} f_n$ for all n and $\{v_n, \beta_n\} \to \{v, \beta\}$ in $X \oplus \mathbb{R}$ as $n \to +\infty$. Thus (I) implies (1.11).

Conversely, let us suppose that (1.11) is satisfied. Let $v \in X$. Since (1.9) is trivial in case $f(v) = +\infty$, suppose $f(v) < +\infty$. Then, $\{v, f(v)\} \in \operatorname{epi} f$. Therefore, by (1.11), there exists $\{v_n, \beta_n\} \in \operatorname{epi} f_n$ for each n, such that $v_n \to v$ in X as $n \to +\infty$ and, moreover, $\beta_n \to f(v)$, hence

$$\limsup f_n(v_n) \leqslant f(v),$$

as $n \to +\infty$. Thus, (I) holds.

Now we prove that (II) is equivalent to

$$\text{w-}\overline{\mathrm{Lim}}\operatorname{epi} f_n \subset \operatorname{epi} f \quad \text{in} \quad X \oplus \mathbb{R}. \tag{1.12}$$

Let us suppose that (II) holds. Let (f_k') be a subsequence of (f_n) and $\{v, \beta\} \in X \oplus \mathbb{R}$ be the weak limit of a sequence $(\{v_k, \beta_k\})$ in $X \oplus \mathbb{R}$, with $\{v_k, \beta_k\} \in \operatorname{epi} f_k'$ for every k. By (II), since $v_k \rightharpoonup v$ in X as $k \to +\infty$, (1.10) holds. Since

$$\beta = \lim \beta_k \geqslant \liminf f_k'(v_k) \quad \text{as} \quad k \to +\infty,$$

we find $\beta \geqslant f(v)$. Thus (II) implies (1.12). Assume now that (1.12) holds.

Let $v_k \rightharpoonup v$ in X as $k \to +\infty$ and (f_k') be any subsequence of (f_n). If we have

$$\liminf f_k'(v_k) = -\infty \quad \text{as} \quad k \to +\infty,$$

then for any given $\delta < 0$, there exists a convergent sequence (β_h), with $\beta_h \geqslant f_h''(v_h')$ for every h, $(f_h''(v_h'))$ being a subsequence of $(f_k'(v_k))$, such that

$$\beta = \lim \beta_h < \delta \quad \text{as} \quad h \to +\infty.$$

By (1.12), we have $f(v) \leqslant \beta < \delta$. This implies $f(v) = -\infty$, hence (ll) holds. On the other hand, suppose

$$\beta = \liminf f_k'(v_k) > -\infty.$$

Clearly, we can also suppose $\beta < +\infty$. Hence, there exists a subsequence $(f_h''(v_h'))$ of $(f_k'(v_k))$, such that $\beta_h \geqslant f_h''(v_h')$ for every h, with $\beta_h \to \beta$ as $h \to +\infty$. Thus, $\{v_h', \beta_h\} \in \operatorname{epi} f_h''$ for every h, and $\{v_h', \beta_h\}$ converges weakly to $\{v, \beta\}$ in $X \oplus \mathbb{R}$ as $h \to +\infty$. Therefore, by (1.12), we have $\beta \geqslant f(v)$, that is (1.10) holds. ▮

Remark 1.5. It follows from Lemma 1.10, that, if $f = \operatorname{Lim} f_n$ in X, then any $v \in X$ is the limit in the strong topology of X of a sequence (v_n) in X, such that $f(v) = \lim f_n(v_n)$ as $n \to +\infty$.

Remark 1.6. Each f_n of a converging sequence (f_n) may be a proper function, without $f = \operatorname{Lim} f_n$ be such. On the other hand, if $f = \operatorname{Lim} f_n$ is proper and $f \not\equiv +\infty$, then any f_n, for all n large enough, is proper. [If not, there would exist a subsequence (f_k') of (f_n) and a sequence (v_k) of vectors of X, such that $f_k'(v_k) = -\infty$ for all k. Let $v_0 \in X$ with $f(v_0) < +\infty$. By (l), there exists a sequence (z_k) in X, such that $z_k \to v_0$ in X as $k \to +\infty$ and $f_k'(z_k) < +\infty$ for all $k > k_0$, $k_0 > 0$. Therefore, if

$$v_k' = \epsilon_k v_k + (1 - \epsilon_k) z_k, \qquad \text{with} \qquad \epsilon_k = k^{-1}(1 + \| v_k \|)^{-1},$$

we have $f_k'(v_k') = -\infty$ for all $k > k_0$ and $v_k' \to v_0$ in X as $k \to +\infty$. Thus, by (ll), we have $\liminf f_k'(v_k') \geqslant f(v_0)$ as $k \to +\infty$, hence $f(v_0) = -\infty$, which is a contradiction.] The statement above is false if $f \equiv +\infty$, as the following example shows: $X = \mathbb{R}$ and for each

$n = 1, 2, \ldots$, take f_n so defined: $f_n(r) = +\infty$ for all $r \neq n$, $f_n(r) = -\infty$ if $r = n$.

8. "*Order* α" *Convergence for Convex Functions*

Let (f_n) be a sequence of lower-semicontinuous convex functions on X and f a lower-semicontinuous function on X, $f \not\equiv +\infty$.

Definition 1.5. If $\alpha \geqslant 0$, we say that f_n *converges* of *order* $\geqslant \alpha$ to f in X as $n \to +\infty$, and write

$$n^{\alpha}[f_n - f] \to 0 \quad \text{in} \quad X,$$

if epi f_n converges of order $\geqslant \alpha$ to epi f in $X \oplus \mathbb{R}$ as $n \to +\infty$, according to Definition 1.3.

Remark 1.7. It follows from Lemma 1.9 and Definition 1.4, that f_n converges of order $\geqslant 0$ to f in X if and only if $f = \operatorname{Lim} f_n$. Thus, if f_n converges of order $\geqslant \alpha$ to f in X, $\alpha \geqslant 0$, then, in particular, $f = \operatorname{Lim} f_n$.

A characterization of the "order α" convergence is given by the following

Lemma 1.11. *If f is proper, and $\alpha \geqslant 0$, then we have*

$$n^{\alpha}[f_n - f] \to 0 \quad \textit{in} \quad X,$$

if and only if (m) *and* (mm) *below are satisfied*:

(m) *For any $v \in \operatorname{dom} f$, there exists a sequence (v_n) in X such that*

$$n^{\alpha}(v_n - v) \to 0 \quad \textit{in} \quad X \tag{1.13}$$

$$\limsup n^{\alpha}[f_n(v_n) - f(v)] \leqslant 0 \tag{1.14}$$

as $n \to +\infty$;

(mm) *For any subsequence (f_{n_k}) of (f_n) and any weakly convergent sequence (v_k) in X, with $\limsup f_{n_k}(v_k) < +\infty$ as $k \to +\infty$, there exists a subsequence $(f_h'(v_h'))$ of $(f_{n_k}(v_k))$, $f_h'(v_h') = f_{n_h'}(v_{k_h})$, $n_h' = n_{k_h}$ for every h, and a sequence (w_h) in X, such that*

$$n_h'^{\alpha}(v_h' - w_h) \rightharpoonup 0 \quad \textit{in} \quad X, \tag{1.15}$$

$$\liminf n_h'^{\alpha}[f_h'(v_h') - f(w_h)] \geqslant 0 \tag{1.16}$$

as $h \to +\infty$.

Proof. Let us prove that (m) is equivalent to:

$$0 \in \text{s-}\underline{\text{Lim}}\, n^{\alpha}(\text{epi} f_n - \{v, \beta\}) \quad \text{in } X \oplus \mathbb{R}, \qquad \text{for each} \quad \{v, \beta\} \in \text{epi} f \tag{1.17}$$

Let $v \in X$, $\beta \in \mathbb{R}$, with $\beta \geqslant f(v)$. By (m), there exists a sequence (v_n) in X, such that (1.13) and (1.14) hold. If

$$\beta_n = \max\{f_n(v_n), \beta\} \qquad \text{for every} \quad n,$$

then we have $n^{\alpha}(\beta_n - \beta) \to 0$ as $n \to +\infty$.[3] Thus (1.17) is satisfied. Conversely, suppose that (1.17) holds and let $v \in \text{dom} f$. Since f is proper, $\{v, f(v)\} \in \text{epi} f$, hence, by (1.17), there exists $\{v_n, \beta_n\} \in \text{epi} f_n$ for every n, such that

$$n^{\alpha}(\{v_n, \beta_n\} - \{v, f(v)\}) \to 0 \quad \text{in} \quad X \oplus \mathbb{R} \quad \text{as} \quad n \to +\infty.$$

Therefore, (1.13) holds and, moreover,

$$n^{\alpha}[\beta_n - f(v)] \to 0 \quad \text{as} \quad n \to +\infty,$$

which clearly implies (1.14), for $\beta_n \geqslant f_n(v_n)$ for every n. Thus (m) holds.

Now let us prove that (mm) is equivalent to:

For any subsequence (f_{n_k}) of (f_n) and any weakly convergent sequence $(\{v_k, \beta_k\})$ in $X \oplus \mathbb{R}$, with $\{v_k, \beta_k\} \in \text{epi} f_{n_k}$ for every k, we have

$$0 \in \text{w-}\overline{\text{Lim}}\, n_k^{\alpha}(\{v_k, \beta_k\} - \text{epi} f) \quad \text{in} \quad X \oplus \mathbb{R}. \tag{1.18}$$

Let us suppose that (mm) holds and let $\{v_k, \beta_k\}$ be as stated above. Since $\limsup f_{n_k}(v_k) < +\infty$, there exists by (mm) a subsequence $(f_h'(v_h'))$ of $(f_{n_k}(v_k))$ and a sequence (w_h) in X such that (1.15) and (1.16) hold. Put $\beta_h' = \beta_{k_h}$ for every h. We have by (1.16)

$$\liminf n_h'^{\alpha}[\beta_h' - f(w_h)] \geqslant 0 \quad \text{as} \quad h \to +\infty,$$

hence

$$n_\ell''^{\alpha}[\beta_\ell'' - f(w_\ell')] \geqslant -\ell^{-1}, \qquad \ell = 1, 2, \ldots,$$

[3] In fact, $n^{\alpha}(\beta_n - \beta) \geqslant 0$ for all n, hence $\liminf n^{\alpha}(\beta_n - \beta) \geqslant 0$. On the other hand, $n^{\alpha}[f_n(v_n) - \beta] \leqslant n^{\alpha}[f_n(v_n) - f(v)]$ for all n, hence, by (1.14), $\limsup n^{\alpha}[f_n(v_n) - \beta] \leqslant 0$, which implies $\limsup n^{\alpha}(\beta_n - \beta) \leqslant 0$.

where $n_\ell'' = n_{h_\ell}'$, $\beta_\ell'' = \beta_{h_\ell}'$, $w_\ell' = w_{h_\ell}$ for every ℓ, (β_{h_ℓ}') being a subsequence of (β_h'). Let us set for each ℓ

$$\xi_\ell = \beta_\ell'' + \ell^{-1} n_\ell''^{-\alpha}.$$

We have $\xi_\ell \geqslant f(w_\ell')$ for every ℓ, and $n_\ell''^{\alpha}(\beta_\ell'' - \xi_\ell) \to 0$ as $\ell \to +\infty$. Thus (1.18) is satisfied.

Conversely, suppose (1.18) holds. Let (f_{n_k}) and (v_k) be as in (mm). First we note that, since f is proper, we cannot have $\liminf f_{n_k}(v_k) = -\infty$ as $k \to +\infty$. This is a consequence of (ll) of Lemma 1.10, that can be applied here, because our hypothesis implies $f = \operatorname{Lim} f_n$ (see Remark 1.7). [A direct proof can be given as follows: Suppose

$$\liminf f_{n_k}(v_k) = -\infty$$

and let $v_0 = \text{w-lim}\, v_k$ as $k \to +\infty$. Let $\delta < 0$ and (β_h) be a convergent sequence of reals, with $\beta = \lim \beta_h < \delta$ and $\beta_h \geqslant f_h'(v_h')$ for every h, $(f_h'(v_h'))$ being a subsequence of $((f_{n_k}(v_k))$. Then, by (1.18), there exists a sequence $(\{w_\ell, \xi_\ell\}) \subset \operatorname{epi} f$, such that

$$(v_\ell'' - w_\ell) \rightharpoonup 0 \quad \text{in } X, \qquad \text{hence} \quad w_\ell \rightharpoonup v_0 \quad \text{in } X;$$

$$(\beta_\ell - \xi_\ell) \to 0, \qquad \text{hence} \quad \xi_\ell \to \beta,$$

as $\ell \to +\infty$, with (v_ℓ'') a subsequence of (v_h'), $v_\ell'' = v_{h_\ell}'$ and $\beta_\ell' = \beta_{h_\ell}$ for every ℓ. Thus, by the lower-semincontinuity of f, we find as $\ell \to +\infty$

$$f(v_0) \leqslant \liminf f(w_\ell) \leqslant \lim \xi_\ell = \beta < \delta,$$

hence, since δ is arbitrary, $f(v_0) = -\infty$, which is a contradiction, for f is proper]. Therefore, we have

$$\liminf f_{n_k}(v_k) > -\infty \quad \text{as} \quad k \to +\infty.$$

Hence, there exists a convergent subsequence $(f_h'(v_h'))$ of $(f_{n_k}(v_k))$, $f_h'(v_h') = f_{n_h'}(v_{k_h})$, $n_h' = n_{k_h}$ for every h. Applying (1.18) to the sequence $\{v_h', f_h'(v_h')\}$, we find a sequence $(\{w_\ell, \xi_\ell\}) \subset \operatorname{epi} f$, such that

$$n_\ell''^{\alpha}(v_\ell'' - w_\ell) \rightharpoonup 0 \quad \text{in} \quad X,$$

$$n_\ell''^{\alpha}[f_\ell''(v_\ell'') - \xi_\ell] \to 0,$$

as $\ell \to +\infty$, for a subsequence $(f_\ell''(v_\ell''))$ of $(f_h'(v_h'))$, $f_\ell''(v_\ell'') = f_{h_\ell}''(v_{h_\ell}'')$

and $n_\ell'' = n'_{h_\ell}$ for every ℓ. Since $\xi_\ell \geqslant f(w_\ell)$ for all ℓ, the second limit above implies

$$\liminf n_\ell''^{\alpha}[f_\ell''(v_\ell'') - f(w_\ell)] \geqslant 0.$$

Thus (mm) holds. ∎

2. Coercive Mappings, Unique Solution

In the previous section we have introduced a convergence in the family of closed convex subsets of a normed space. In the remainder of this paper we shall use that notion to deal with the problem of the continuous dependence on the map T and the convex K of solutions of variational inequalities such as (1) and (2) of Introduction.

Let us rewrite below the variational inequality (1) associated with a given monotone map T from a Banach space X to its adjoint X^* and with a closed convex subset K of the domain $D(T)$ of T:

$$u \in K: \langle Tu, v - u\rangle \geqslant 0 \qquad \text{for all} \quad v \in K. \tag{1}$$

Thereafter X will be a *reflexive real* Banach space.

To begin with, let us consider the special case in which only the convex K is perturbed, while the map T is kept fixed. For sake of simplicity, we shall suppose that the perturbation of K can be described by a *sequence* of convex subsets of $D(T)$.[4]

Thus, let us assume that K_n is for each $n = 1, 2,\ldots$ a closed convex subset of $D(T)$ which converges to the given K in X as $n \to +\infty$ in the sense of Definition 1.1 of Section 1, and for any such K_n let us consider the variational inequality

$$u_n \in K_n: \langle Tu_n , v - u_n\rangle \geqslant 0 \qquad \text{for all} \quad v \in K_n . \tag{$1_n'$}$$

Then, under suitable assumptions on T, which guarantee the existence and uniqueness of the solutions of (1) and ($1_n'$), we shall prove as a corollary of Theorem A below that the solution u_n of ($1_n'$) *converges strongly* in X to the solution u of (1) as $n \to +\infty$. We have indeed the following

[4] However, as we have already noticed in the Introduction, the results below could be extended, with only sligth changes, to real parametrized perturbations K_ϵ of K or, more generally, to arbitrary indexed families of perturbed inequalities.

Corollary of Theorem A. *Let us suppose that*

(i) T *is a bounded*[5] *hemicontinuous map of* $D(T)$ *in* X *to* X^*, *with* $0 \in D(T)$, *such that*

$$\| v - u \| \gamma(\| v - u \|) \leqslant \langle Tv - Tu, v - u \rangle \qquad \text{for all} \quad u, v \in D(T),$$

where γ *is a continuous strictly increasing function from* $[0, +\infty)$ *to* $[0, +\infty]$, *with* $\gamma(0) = 0$ *and* $\gamma(r) \to +\infty$ *as* $r \to +\infty$;

(ii) K *and* K_n, $n = 1, 2, \ldots$, *are nonempty closed convex subsets of* $D(T)$, *such that*

$$K = \operatorname{Lim} K_n \quad \text{in} \quad X$$

according to Definition 1.1.

Then, there exists for each n *one and only one solution* u_n *of inequality* $(1_n')$ *and* u_n *converges strongly in* X *as* $n \to +\infty$ *to the unique solution* u *of inequality* (1).

In the following Section 3 we shall give some applications of this result by making use of the examples of converging sequences of convex sets considered in Section 1.

Below, we summarize the general results which hold in case of uniqueness of the solution and for a coercive T, for inequalities of type (1) or (2) of Introduction. The proofs are postponed to Section 4. Notation and definitions are those of Section 1.

More special results for non-coercive mappings and non-unique solutions will be given in Section 5.

1. *Inequalities* (1)

We shall denote the *graph* of a map A from X to X^*, by $G(A)$, that is,

$$G(A) = \{\{v, v'\} \in X \times X^* \colon v' = Av, v \in D(A)\}$$

where $D(A)$ is the domain of A.

Moreover, if (A_n) is a sequence of mappings from X to X^*, we say that they are *uniformly bounded* in X, if for any bounded subset B of X there exists a bounded subset B' of X^*, such that

$$A_n B_n \subset B' \qquad \text{for all} \quad n,$$

where $B_n = B \cap D(A_n)$ for each n.

[5] A map T from X to X is *bounded* if it carries bounded subsets of $D(T)$ into bounded subsets of X.

Let us make the following assumptions:

I. *T is a monotone hemicontinuous map of $D(T)$ in X to X^*; (T_n) is a sequence of monotone hemicontinuous mappings from X to X^*, which are uniformly bounded in X and satisfy*

$$G(T) \subset \text{s-}\underline{\text{Lim}}\, G(T_n) \quad \textit{in} \quad X \times X^*. \tag{2.1}$$

According to our notation of Subsection 1 of Section 1, (2.1) above means that for every $v \in D(T)$, there exists for each n a vector $v_n \in D(T_n)$, such that v_n converges strongly to v in X and $T_n v_n$ converges strongly to Tv in X^* as $n \to +\infty$.

II. *K is a non-empty closed convex subset of $D(T)$; (K_n) is a sequence of closed convex subsets of X, with $K_n \subset D(T_n)$ for every n, such that*

$$K = \text{Lim}\, K_n \quad \textit{in} \quad X,$$

in the sense of Definition 1.1.

Under the assumptions I and II above, we shall prove what follows:

If there exists a bounded sequence (u_n) of solutions of the inequalities (1_n), i.e., $u_n \in S(T_n, K_n)$ for each n, then the inequality (1) has a solution, that is, $S(T, K) \neq \varnothing$.

Furthermore, if the solution u of (1) is unique, i.e. $S(T, K) = \{u\}$, then $S(T_n, K_n)$ converges weakly to $\{u\}$ in X in the sense of Definition 1.2.

If, in addition to the existence of (u_n), we suppose also that condition III below is satisfied, then

$$S(T_n, K_n) \to \{u\} \quad \text{in } X, \qquad \text{as} \quad n \to +\infty,$$

in the sense of Definition 1.1. The condition is

III. *For any $u \in K$, there exists a continuous strictly increasing function $\beta : \overline{\mathbb{R}^+} \to [0, +\infty]$,[6] with $\beta(0) = 0$, such that*

$$\beta(\| v - u \|) \leqslant \liminf |\langle T_n v - Tu, v - u\rangle| \quad \textit{as} \quad n \to +\infty, \quad v \in D(T_n)$$

uniformly as v varies in a bounded subset of X. (2.2

Finally, we prove that there exists a bounded sequence (u_n) of

[6] We put $\mathbb{R}^+ = (0, +\infty)$, $\overline{\mathbb{R}}^+ = [0, +\infty]$.

solutions $u_n \in S(T_n, K_n)$, provided the T_n are *uniformly coercive on* K_n in X, in the following sense

$$\text{IV} \quad \begin{cases} \text{There exists a function } \alpha : \mathbb{R}^+ \to [0, +\infty], \text{ with } \alpha(r) \to +\infty \text{ as } r \to +\infty, \\ \text{such that} \\ \qquad \| v \| \, \alpha(\| v \|) \leqslant \langle T_n v, v \rangle \quad \text{for every} \quad n \qquad (2.3) \\ \text{and all } v \in K_n . \end{cases}$$

We have, indeed, the following theorem

Theorem A. *Under the assumptions* I *and* II, *the following results hold*:

(a) *If* $u_h \in S(T_{n_h}, K_{n_h})$ *for every* h, *with* $(S(T_{n_h}, K_{n_h}))$ *a subsequence of* $(S(T_n, K_n))$, *and* u_h *converges weakly to a vector* u *of* X *as* $h \to +\infty$, *then* $u \in S(T, K)$ *and*

$$\langle T_{n_h} u_h - Tu, u_h - u \rangle \to 0 \quad \text{as} \quad h \to +\infty. \tag{2.4}$$

Besides, if III holds, then u_h *converges strongly to* u *in* X.

(b) *If there exists a bounded subset* B *of* X *and* $n_0 > 0$ *such that*

$$S(T_n, K_n) \cap B \neq \varnothing \qquad \text{for all} \quad n > n_0, \tag{2.5}$$

then there exists at least one solution, u, *of inequality* (1). *Actually, we have*

$$\varnothing \neq \text{w-}\overline{\text{Lim}}\, S(T_n, K_n) \subset S(T, K). \tag{2.6}$$

Moreover, if the solution u *of* (1) *is unique, then* u *is the limit in the weak topology of* X *of any sequence* (w_h), *with* $w_h \in S(T_{n_h}, K_{n_h})$ *for every* h *and* $(S(T_{n_h}, K_{n_h}))$ *a subsequence of* $(S(T_n, K_n))$, *provided* (w_h) *is bounded in* X.

(c) *If the* T_n *are uniformly coercive on* K_n *in* X, *i.e.*, IV *holds, and* $0 \in \bigcap_n K_n$, *then there exists a bounded subset* B *of* X *and* $n_0 > 0$, *such that*

$$\varnothing \neq S(T_n, K_n) \subset B \qquad \text{for all} \quad n > n_0. \tag{2.7}$$

Remark 2.1. In part (c) of the theorem, the hypothesis that $0 \in K_n$ for all n can be replaced by the hypothesis that for any sequence (v_n), with $v_n \in K_n$ for each n, there exists a *bounded* sequence (z_n), $z_n \in K_n$ for each n, such that (4.8) holds. See indeed Proposition 4.1.

2. *Inequalities* (2)

By reducing the inequality (2) to an inequality of type (1) in the space $X \oplus \mathbb{R}$ and then applying Theorem A, we obtain the results which are summarized below.

We still assume the hypothesis I of Subsection 1. Besides:

II′ f *is a proper lower-semicontinuous convex function on* X, *with* $\operatorname{dom} f \neq \varnothing$; (f_n) *is a sequence of lower-semicontinuous convex functions on* X, *such that*

$$f = \operatorname{Lim} f_n \quad in \quad X,$$

in the sense of Definition 1.4.

Under the assumptions I and II′, if there exists a bounded sequence (u_n) of solutions $u_n \in S(T_n, f_n)$, then the inequality (2) has a solution. Moreover, if the solution of (2) is unique, which is the case if T is strictly monotone of f is strictly convex, then $S(T_n, f_n)$ converges weakly to $\{u\}$ in X as $n \to +\infty$, in the sense of Definition 1.2, and $u_n \rightharpoonup u$, $u_n \in S(T_n, f_n)$ for every n, implies $f_n(u_n) \to f(u)$, as $n \to +\infty$.

Furthermore, we have $S(T_n, f_n) \to \{u\}$ in X if the sense of Definition 1.1, provided the following condition is satisfied

III′ *For any* $u \in \operatorname{dom} f$, *there exists a continuous strictly increasing function* $\beta : \overline{\mathbb{R}^+} \to [0, +\infty]$, *with* $\beta(0) = 0$, *such that*

$$\beta(\| v - u \|) \leqslant \liminf\{|\langle T_n v - Tu, v - u\rangle| + |f_n(v) - f(u)|\}$$

as $n \to +\infty$, $v \in D(T_n)$, *uniformly as* v *varies in a bounded subset of* X.

Finally, if an uniform coerciveness hypothesis is satisfied by T_n and f_n, then there exists for each n a solution u_n in X of the inequality

$$\langle T_n u_n, v - u_n\rangle \geqslant f_n(u_n) - f_n(v) \qquad \text{for all} \quad v \in X$$

and the sequence (u_n) is bounded in X. The hypothesis is the following

IV′ *There exists a function* $\alpha : \mathbb{R}^+ \to [0, +\infty]$, *with* $\alpha(r) \to +\infty$ *as* $r \to +\infty$, *such that*

$$\| v \| \, \alpha(\| v \|) \leqslant \langle T_n v, v\rangle + f_n(v) \qquad \textit{for every} \quad n \tag{2.8}$$

and all $v \in X$.

The theorem that is obtained from Theorem A, is the following

Theorem B. *Let us suppose that* I *and* II′ *hold, with* $\operatorname{dom} f \subset D(T)$ *and* $\operatorname{dom} f_n \subset D(T_n)$ *for all n. Then*

(a) *If* $u_h \in S(T_{n_h}, f_{n_h})$ *for every h, with* $(S(T_{n_h}, f_{n_h}))$ *a subsequence of* $(S(T_n, f_n))$ *and* u_h *converges weakly to a vector u of X as* $h \to +\infty$, *then* $u \in S(T, f)$ *and*

$$f_{n_h}(u_h) \to f(u) \quad as \quad h \to +\infty, \tag{2.9}$$

$$\langle T_{n_h} u_h - Tu, u_h - u \rangle \to 0 \quad as \quad h \to +\infty. \tag{2.10}$$

Besides, u_h *converges strongly to u in X, provided the hypothesis* III′ *above is satisfied.*

(b) *If there exists a bounded subset B of X and* $n_0 > 0$, *such that*

$$S(T_n, f_n) \cap B \neq \varnothing \qquad for\ all \quad n > n_0,$$

then there exists at least one solution, u, of inequality (2) *and we have*

$$\varnothing \neq \text{w-}\overline{\operatorname{Lim}}\, S(T_n, f_n) \subset S(T, f).$$

Furthermore, if the solution u of (2) *is unique, then for any bounded sequence* (w_h) *in X, with* $w_h \in S(T_{n_h}, f_{n_h})$ *for every h,* $(S(T_{n_h}, f_{n_h}))$ *a subsequence of* $(S(T_n, f_n))$, *we have* $w_h \to u$ *in X and* $f_{n_h}(u_h) \to f(u)$ *as* $h \to +\infty$.

(c) *If the uniform coerciveness hypothesis* IV′ *is satisfied and* $f_n(0) = 0$ *for all n, then there exists a bounded subset B of X such that*

$$\varnothing \neq S(T_n, f_n) \subset B \qquad for\ all\ large\ n.$$

Remark 2.2. The hypothesis that $f_n(0) = 0$ for all n in part (c) of the theorem above can be dropped, provided the coerciveness condition IV′ is improved. See Proposition 4.2.

Corollary of Theorem B. *In addition to* II′, *suppose that f is strictly convex and that there exist two functions* α *and* β *as in* IV′ *and* III′ *above, such that*

$$\alpha(\| v \|) \leqslant f_n(v) \qquad for\ every\ n\ and\ all \quad v \in X \tag{2.11}$$

and, for each $u \in \operatorname{dom} f$

$$\beta(\| v - u \|) \leqslant \liminf |f_n(v) - f(u)| \quad \text{as} \quad n \to +\infty, \tag{2.12}$$

uniformly as v *varies in a bounded subset of* X.

Then, there exists for each n *a vector* $u_n \in X$ *minimizing* f_n *on* X *and* u_n *converges strongly in* X *as* $n \to +\infty$ *to the (unique) vector* $u \in X$ *which minimizes* f *on* X. *Moreover,* $f(u) = \lim f_n(u_n)$ *as* $n \to +\infty$.

3. Applications

Our main purpose in this section is to show what type of results can be obtained from the general theorems of Section 2. We shall not care in each special case for the maximum of generality. Therefore, the results given below can be somewhat improved or extended, and this will be done elsewhere.

1. *Finite-Dimensional Approximation*

We can use theorems A and B of Section 2 for solving a variational inequality by "discretization methods" of Ritz–Galerkin type, that is, by solving first an approximate problem in a finite-dimensional space and then letting the dimension $\to +\infty$.

(a) Let us suppose, first, that T is a bounded hemicontinuous map of a reflexive real Banach space X to its dual X^*, such that

$$\| v - u \| \gamma(\| v - u \|) \leqslant Tv - Tu, v - u\rangle \qquad \text{for all} \quad u, v \in X, \tag{3.1}$$

where $\gamma : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ is a continuous strictly increasing function, with $\gamma(0) = 0$ and $\gamma(r) \to +\infty$ as $r \to +\infty$. Moreover, let us suppose that K is a nonempty closed convex subset of X.

Now let (X_n^*) be a sequence of closed linear subspaces of X^* and for each n let us denote by Y_n^* the quotient Banach space

$$Y_n = X^*/X_n^*,$$

and by π_n^* the canonical homomorphism of X^* in Y_n^*,

$$\pi_n^* : X^* \to Y_n^*.$$

Let us denote by π_n the adjoint map of $\pi_n{}^*$. By the reflexivity of X, we have

$$\pi_n : Y_n \to X,$$

where

$$Y_n = (Y_n{}^*)^*$$

is the dual Banach space of $Y_n{}^*$. Clearly, $Y_n{}^*$ is the dual space of Y_n and $\pi_n{}^*$ is the adjoint map of π_n, thus our notation is consistent. Moreover, it is easy to show that π_n is an isomorphism of Y_n on the subspace

$$X_n = \pi_n Y_n$$

of X and

$$\| \pi_n y \| = \| y \|_n \qquad \text{for all} \quad y \in Y_n , \tag{3.2}$$

where $\| \cdot \|$ denotes, as usual, the norm in X, while $\| \cdot \|_n$ is the norm in Y_n— namely, the dual norm in $(Y_n{}^*)^*$ of the quotient norm in $Y_n{}^*$.

[In fact, since $\| \pi_n{}^* \| \leqslant 1$, then

$$\| \pi_n y \| \leqslant \| y \|_n \qquad \text{for all} \quad y \in Y_n .$$

Besides, for any $\sigma > 0$ and any $y' \in Y_n{}^*$, with $\| y' \|_{Y_n^*} \leqslant 1$, there exists a vector $v_\sigma' \in X^*$, such that $\pi_n{}^* v_\sigma' = y'$ and $1 \geqslant \| y' \|_{Y_n^*} \geqslant \| v_\sigma' \| - \sigma$. Therefore, we have

$$\begin{aligned} \| y \|_n &= \sup\{ |\langle y', y \rangle_n | : \| y' \|_{Y_n^*} \leqslant 1 \} \\ &\leqslant \sup\{ |\langle v', \pi_n y \rangle| : \| v' \| \leqslant 1 + \sigma \} \leqslant (1 + \sigma) \| \pi_n y \| \end{aligned}$$

which implies, since $\sigma > 0$ is arbitrary, that $\| y \|_n \leqslant \| \pi_n y \|$.]

Let T_n be for each n the map

$$T_n = \pi_n{}^* T \pi_n$$

of Y_n to $Y_n{}^*$ and H_n a closed convex subset of Y_n. Let us consider the variational inequality

$$x_n \in H_n : \ \langle T_n x_n , y - x_n \rangle_n \geqslant 0 \qquad \text{for all} \quad y \in H_n , \tag{3.3}$$

where $\langle \cdots \rangle_n$ denotes the pairing between Y_n and $Y_n{}^*$.

Proposition 3.1. *Let T and K be as stated above, $K_n = \pi_n H_n$ for every n and suppose that*

$$K = \operatorname{Lim} K_n \quad in \quad X, \tag{3.4}$$

in the sense of Definition 1.1. *Then, there exists for each n one and only one solution x_n of* (3.3) *and $\pi_n x_n$ converges strongly in X to the* (*unique*) *solution u of inequality* (1), *i.e., $u \in S(T, K)$.*

Proof. By the definition of T_n, we have that $x_n \in Y_n$ is a solution of (3.3) if and only if $u_n = \pi_n x_n$ and $u_n \in X$ is a solution of

$$u_n \in K_n : \langle Tu_n , v - u_n \rangle \geqslant 0 \qquad \text{for all} \quad v \in K_n$$

which is to say, $u_n \in S(T, K_n)$. Besides, K_n is by (3.2) a closed convex subset of X. Therefore Proposition 3.1 follows from Corollary of Theorem A. ∎

Proposition 3.1 can be used for a finite-dimensional approximation of the solution of inequality (1), whenever one can find a sequence of closed subspaces $X_n{}^*$ of X^*, each one of finite codimension, such that (3.4) is satisfied.

Corollary 1. *Suppose that the sequence $(X_n{}^*)$ is decreasing with n, with $\bigcap_n^\infty X_n{}^* = \{0\}$. Suppose, furthermore, that the interior of K is non-empty and that*

$$K_n = K \cap X_n \qquad \textit{for every} \quad n,$$

where $X_n = \pi_n Y_n$, $K_n = \pi_n H_n$. Then, the conclusion of Proposition 3.1 *holds.*

Proof. The sequence (X_n) is increasing with n and $\bigcup_n^\infty X_n$ is dense in X, as it can be seen by applying the Hahn–Banach theorem. By Lemma 1.2 and Lemma 1.4, we have

$$K = \operatorname{Lim} K \cap X_n \quad \text{in} \quad X,$$

thus the corollary follows from Proposition 3.1. ∎

Corollary 2. *Suppose that*

$$\pi_n H_n \subset K \qquad \textit{for every} \quad n,$$

and that there exists for each n a map ρ_n of X to Y_n, such that $\rho_n K \subset H_n$ for every n and moreover

$$\pi_n \rho_n v \to v \qquad \text{in } X \text{ as} \qquad n \to +\infty$$

for any $v \in K$. Then, the conclusion of Proposition 3.1 *holds.*

Proof. Again, it suffices to apply Proposition 3.1, for, by the hypotheses above, we have $K \subset \text{s-}\underline{\text{Lim}}\, \pi_n H_n$, hence, since $\pi_n H_n \subset K$ for every n, $K = \text{Lim}\, K_n$, where $K_n = \pi_n H_n$. ▮

Approximation methods of type of that furnished by Corollary 2 above, as well as methods for solving the discretized problems, have been given by C. Cea, (*18*), for equations involving an accretive linear operator T in a Hilbert space and J. P. Aubin, (*1*), for variational inequalities concerning such a T. Further extensions of these methods to equations involving monotone operators from a Banach space to its dual, have been considered by H. Brezis–M. Sibony (*4*).

(b) Let us suppose that X is a (real) Hilbert space and that there exists an increasing sequence of finite-dimensional subspaces of X, with $\bigcup_n^\infty X_n$ dense in X. The scalar product of X will be denoted by $(\cdot, \cdot)$. Let A be a bounded map of X into itself, which is continuous from the line segments of X to the weak topology of X and[7] satisfies the condition

$$\| v - u \| \gamma(\| v - u \|) \leqslant (Av - Au, v - u) \quad \text{for all} \quad u, v \in X$$

with γ a function as above, and let K be a nonempty closed convex subset of X.

Let P_n be for each n the orthogonal projection of X on X_n and put

$$A_n = P_n A P_n$$
$$K_n = P_n K.$$

Proposition 3.2. *In addition to the hypotheses above, suppose either that K is bounded or that $K_n \subset K$ for every n. Then, there exists for each n one and only one solution u_n of the inequality*

$$u_n \in K_n\colon \ (A_n u_n, v - u_n) \geqslant 0 \qquad \text{for all} \quad v \in K_n \tag{3.5}$$

[7] Actually, any such T, being also monotone, is demicontinuous on X, see T. Kato [*21*]. However, this is no more true in general under the hypothesis $D(T) = K$, that is what we really need in Proposition 3.2 below.

and such u_n *converges strongly in* X *as* $n \to +\infty$ *to the* (*unique*) *solution* u *of the inequality*

$$u \in K: \ (Au, v - u) \geqslant 0 \qquad \textit{for all} \quad v \in K. \tag{3.6}$$

Proof. Let J be the canonical isomorphism of X on its dual X^* and let $T = JA$. T is a bounded hemicontinuous map of X to X^* which satisfies condition (i) of Corollary of Theorem A. It is easy to show that u_n satisfies (3.5) if and only if $u_n \in S(T, K_n)$ and that u satisfies (3.6) if and only if $u \in S(T, K)$. Since $P_n v \to v$ in X as $n \to +\infty$, then it follows, trivially in case $K_n \subset K$ for all n and by applying Lemma 1.5 in case K is bounded, that $K_n \to K$ in X as $n \to +\infty$. Thus Proposition 3.2 follows from the Corollary quoted above. ▮

Let us remark that projection methods for solving equations involving non-linear operators in Banach spaces have been extensively investigated by W. V. Petryshyn (*37*), where further references are given, and by F. E. Browder, (*16*), (*17*). Proposition 3.2 generalizes (for a bounded A) the Hilbert space specialization of Theorem 8 of (*16*) and of Corollary 11 of (*37*).

(c) *Finite-dimensional approximation of minimum problems.* We shall consider below two special cases of Corollary of Theorem B, in which a given convex function f is approximated by functions f_n, whose effective domain is contained in a finite-dimensional subspace of X.

Let f be a proper, strictly convex lower-semicontinuous function on X, with $f(0) = 0$, such that

$$\alpha(\| v \|) \leqslant f(v) \qquad \text{for all} \quad v \in X$$

$$\beta(\| v - u \|) \leqslant | f(v) - f(u) | \qquad \text{for every} \quad u \in \operatorname{dom} f \qquad \text{and all} \quad v \in X$$

with α and β continuous strictly increasing functions $\overline{\mathbb{R}^+} \to [0, +\infty]$, with $\alpha(r) \to +\infty$ as $r \to +\infty$ and $\beta(0) = 0$.

Let us suppose that there exists an increasing sequence of finite dimensional subspaces of X, with $\bigcup X_n$ dense in X.

The proposition below is a formulation of the classical Ritz approximation method of the minimum of f.

Proposition 3.3. *Suppose that the interior of* dom f *is non-empty and for each* n, *let* f_n *be the function on* X *defined by*

$$f_n(v) = f(v) \qquad \textit{if} \quad v \in X_n\,,$$

$$f_n(v) = +\infty \qquad \textit{if} \quad v \notin X_n\,.$$

Then, there exists for each n one and only one vector $u_n \in X_n$ which minimizes f_n, and u_n converges strongly in X as $n \to +\infty$ to the (unique) vector $u \in X$ which minimizes f. Besides, $f_n(u_n) \to f(u)$ as $n \to +\infty$.

Proof. Let us consider the space $X \oplus \mathbb{R}$ and for each n the subspace $X_n \oplus \mathbb{R}$ of $X \oplus \mathbb{R}$. Clearly, we have

$$\operatorname{epi} f_n = \operatorname{epi} f \cap (X_n \oplus \mathbb{R}) \qquad \text{for every} \quad n.$$

Thus, by Lemma 1.4, $\operatorname{epi} f = \operatorname{Lim} \operatorname{epi} f_n$ in $X \oplus \mathbb{R}$. The proposition is then a consequence of Corollary of Theorem B. ▮

We recall that the *closure* of a convex function $\tilde{f}$ on X is the (convex lower-semicontinuous) function f on X, such that $\operatorname{epi} f$ is the closure of $\operatorname{epi} \tilde{f}$ in $X \oplus \mathbb{R}$.

Proposition 3.4. *Let us suppose that X is a Hilbert space and that* $\operatorname{dom} f$ *is bounded is X. Let (X_n) be as above and, for each n, let P_n be the orthogonal projection on X_n and f_n the closure of the function $\tilde{f}_n$ on X defined by*

$$\tilde{f}_n(v) = \inf\{f(w) \colon w \in X,\ P_n w = v\} \qquad \textit{if} \quad v \in X_n$$
$$\tilde{f}_n(v) = +\infty \qquad \textit{if} \quad v \notin X_n .$$

Then, there exists for each n a vector $u_n \in X_n$ which minimizes f_n and u_n converges strongly in X as $n \to +\infty$ to the (unique) vector $u \in X$ which minimizes f. Moreover, $f_n(u_n) \to f(u)$ as $n \to +\infty$.

Proof. It is easy to show that $\operatorname{epi} f_n$ is, for each n, the closure in $X \oplus \mathbb{R}$ of $(P_n \oplus I) \operatorname{epi} f$, where I is the identity on $\mathbb{R}$. It follows from Lemma 1.5 that[8]

$$\operatorname{epi} f = \operatorname{Lim}(P_n \oplus I) \operatorname{epi} f \quad \text{in} \quad X \oplus \mathbb{R},$$

hence also

$$\operatorname{epi} f = \operatorname{Lim} \operatorname{epi} f_n \quad \text{in} \quad X \oplus \mathbb{R}.$$[9]

Now let $\sigma > 0$. For any $v \in X_n$, there exists $w_{n,\sigma} \in X$, with $P_n w_{n,\sigma} = v$, such that

$$\tilde{f}_n(v) \geqslant f(w_{n,\sigma}) - \sigma;$$

[8] Note that $f(v) = \liminf \tilde{f}(w)$ as $w \to v$ in X.

[9] If (S_n) is a sequence of convex subsets of X and $S = \operatorname{Lim} S_n$ in X, then also $S = \operatorname{Lim} \bar{S}_n$ in X.

hence,

$$\tilde{f}_n(v) \geqslant \alpha(\| w_{n,\sigma} \|) - \sigma \geqslant \alpha(\| v \|) - \sigma \qquad \text{for all} \quad n,$$

which implies

$$f_n(v) \geqslant \alpha(\| v \|) \qquad \text{for all} \quad n.$$

Let $u \in \operatorname{dom} f$, $v \in X$. For any $\sigma > 0$, there exists $w_{n,\sigma} \in X$, with $P_n w_{n,\sigma} = v$, such that

$$f(w_{n,\sigma}) - \sigma \leqslant \tilde{f}_n(v) \leqslant f(w_{n,\sigma})$$

Therefore,

$$|\tilde{f}_n(v) - f(u)| \geqslant |f(w_{n,\sigma}) - f(u)| - \sigma$$
$$\geqslant \beta(\| w_{n,\sigma} - u \|) - \sigma \geqslant \beta(\| v - P_n u \|) - \sigma$$

which implies

$$|f_n(v) - f(u)| \geqslant \beta(\| v - P_n u \|),$$

hence

$$\liminf |f_n(v) - f(u)| \geqslant \beta(\| v - u \|) \quad \text{as} \quad n \to +\infty$$

uniformly with respect to v in a bounded set.

Therefore Proposition 3.4 follows from Corollary of Theorem B. ∎

Let us remark that these approximation results could be also generalized to variational inequalities of type (2), by applying the general form of Theorem B.

2. *Perturbation of Boundary Value Problems*

Theorem A can be applied to give a result on the continuous dependence on the constraints of the solution of a variational problem for a non-linear partial differential operator A in $\mathbb{R}^s$ of type

$$Au = \sum_{\alpha \leqslant m} D^\alpha A_\alpha(x, u, \ldots, D^m u). \tag{3.7}$$

With notation taken from Section 1, we assume that for each multi-index α, A_α is a real function of $x \in \Omega$ and $\xi \in \mathbb{R}^\ell$ (ℓ = number of derivations of order $\leqslant m$ in $\mathbb{R}^s$), which satisfies the following conditions:

I $\left\{ \begin{array}{l} A \text{ is measurable in } x \in \Omega \text{ for fixed } \xi \in \mathbb{R}^\ell \text{ and is continuous in } \xi \in \mathbb{R}^\ell \text{ for} \\ \text{fixed } x \in \Omega; \end{array} \right.$

$$\text{II}\quad \begin{cases} A_\alpha \textit{ is of polinomial growth in } \xi, \textit{ that is we have} \\ \qquad | A_\alpha(x, \xi)| \leqslant c(1 + |\xi|^{p-1}), \qquad x \in \Omega, \quad \xi \in \mathbb{R}^\ell \\ \textit{with } 1 < p < +\infty \textit{ and } c > 0. \end{cases}$$

Then, for each $u \in W^{m,p}(\Omega)$ we have for every α

$$A_\alpha(x, u, \ldots, D^m u) \in L^q(\Omega), \qquad \text{with} \quad q = p(p-1)^{-1}.$$

Therefore, the Dirichlet form

$$a(u, v) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \int A_\alpha(x, u, \ldots, D^m u)\, D^\alpha v\, dx$$

is well-defined for all u and v in $W^{m,p}(\Omega)$ and satisfies an inequality such as

$$| a(u, v)| \leqslant g(\| u \|_{m,p}) \| v \|_{m,p} \tag{3.8}$$

with $g(r)$ a continuous function of $r \in \mathbb{R}$.

Let us notice that the hypothesis II could be considerably weakened if the Sobolev embedding theorem is taken in account, see for instance F. E. Browder (7). In this paper, and in the paper of Leray–Lions quoted above, one can find an extensive discussion of the properties of the operator A, in connection with the monotone operators theory.

Now let X be a closed linear subspace of $W^{m,p}(\Omega)$, K a (nonempty) closed convex subset of X, (K_n) a sequence of (nonempty) closed convex subsets of X.

We can consider the following variational problems for the differential operator A:

$$\text{(p)} \qquad \begin{cases} u \in K, \\ a(u, v - u) \geqslant \langle f, v - u \rangle \qquad \text{for all} \quad v \in K; \end{cases}$$

$$(\text{p}_n) \qquad \begin{cases} u_n \in K_n, \\ a(u_n, v - u_n) \geqslant \langle f, v - u_n \rangle \qquad \text{for all} \quad v \in K_n, \end{cases}$$

$n = 1, 2, \ldots$, where f is a given element in the dual X^* of X.

The inequalities (p) and (p_n) can be written as variational inequalities of type (1), with respect to the map T of X to X^* defined, in virtue of (3.8), by

$$a(u, v) - \langle f, v \rangle = \langle Tu, v \rangle \qquad \text{for all} \quad u, v \in X.$$

As a consequence of assumptions I and II above, T is a (bounded) continuous map from the strong topology of X to the weak topology of

X^*. Indeed, $u \to A_\alpha(x, u, \ldots, D^m u)$ is then a continuous map of $W^{m,p}(\Omega)$ to $L^q(\Omega)$, see the papers quoted above.

As we know from Section 0, we have existence and uniqueness of the solutions u of (p) and u_n of (p_n), $n = 1, 2, \ldots$, provided T is strictly monotone and coercive in X. This is clearly the case if the differential operator A satisfies condition III below:

III $\left\{\begin{array}{l}\textit{There exists a continuous strictly increasing function } \gamma\colon [0, +\infty) \mapsto [0, +\infty] \\ \textit{with } \gamma(0) = 0 \textit{ and } \gamma(r) \to +\infty \textit{ as } r \to +\infty, \textit{ such that} \\ \qquad a(u, u - v) - a(v, u - v) \geqslant \| u - v \|_{m,p}\, \gamma(\| u - v \|_{m,p}) \\ \textit{for all } u, v \in X.\end{array}\right.$

Let us remark that, as far as existence and uniqueness of solutions is involved, condition III could be weakened in such a way that only the top order derivatives in A are affected by the monotonicity assumption. This corresponds to require that T is a semi-monotone operator, see again Refs. (*7*) and (*25*).

We are now in position to apply Theorem A of Section 2, taking Remark 2.1 into account, and we obtain

Proposition 3.5. *Under the assumptions* I, II, *and* III *above, problem* (p_n) *has for each* n *a unique solution* u_n, *and if* K_n *converges to* K *as* $n \to +\infty$ *according to Definition* 1.1, *then* u_n *converges strongly in* X *as* $n \to +\infty$ *to the* (*unique*) *solution* u *of problem* (p).

A first application of Proposition 3.5 is to variational boundary value problems for the operator A, with null boundary conditions corresponding to a closed linear subspace V of $W^{m,p}(\Omega)$, such that

$$W_0^{m,p}(\Omega) \subset V \subset X \subset W^{m,p}(\Omega).$$

That is, to the problem

$$(\mathrm{p}') \qquad \begin{cases} u \in V, \\ a(u, v) = \langle f, v \rangle \qquad \text{for all} \quad v \in V, \end{cases}$$

where f is a fixed element of X^*.

Let us recall that if

$$V = W_0^{m,p}(\Omega),$$

then problem (p′) is the variational formulation of the Dirichlet problem for the operator A, i.e.,

$$\text{(d)} \qquad \begin{cases} D^\beta u = 0 & \text{on} \quad \partial\Omega, \quad |\beta| \leqslant m - 1, \\ Au = f & \text{in} \quad \Omega. \end{cases}$$

[Here f is a distribution in Ω, whose derivatives of order $\leqslant m$ belong to $L^q(\Omega)$]. We shall call the solution u of (p′), the *variational solution* of the Dirichlet problem (d).

Now, let (Ω_n) be a sequence of bounded open subsets of $\mathbb{R}^s$ and suppose that for each n the space $W_0^{m,p}(\Omega_n)$ is isomorphic to a closed linear subspace of $W_0^{m,p}(\Omega)$. We identify $W^{m,p}(\Omega_n)$ with such subspace of $W^{m,p}(\Omega)$, with the norm induced by the norm of $W^{m,p}(\Omega)$.

> [For example, we may have
>
> $$\Omega_n = \Omega - E_n, \qquad n = 1, 2, \ldots,$$
>
> where (E_n) is a sequence of compact subsets of Ω. Then, $W_0^{m,p}(\Omega_n)$ can be obviously identified with the closure in $W^{m,p}(\Omega)$ of all $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 0$ on E_n.]

Let V_n be for each n a closed linear subspace of $W^{m,p}(\Omega)$, with

$$W_0^{m,p}(\Omega_n) \subset V_n \subset X \subset W^{m,p}(\Omega) \qquad \text{for every} \quad n,$$

and let us consider the variational boundary value problem for the operator A, with null boundary conditions corresponding to V_n, i.e., the problem

$$(\mathrm{p}_n') \qquad \begin{cases} u_n \in V_n, \\ a(u_n, v) = \langle f, v \rangle & \text{for all} \quad v \in V_n. \end{cases}$$

In case $V_n = W_0^{m,p}(\Omega_n)$, we have, as above, that u_n is the variational solution of the Dirichlet problem

$$(\mathrm{d}_n) \qquad \begin{cases} D^\beta u_n = 0 & \text{on} \quad \partial\Omega_n, \quad |\beta| \leqslant m - 1, \\ Au_n = f & \text{in} \quad \Omega_n. \end{cases}$$

Thus, applying the result stated above, we find

Corollary 1. *Under the assumptions* I, II, *and* III *above there exists for each n a unique solution u_n of problem* (p_n'), *and if $V = \operatorname{Lim} V_n$ in X, then u_n converges strongly in $W^{m,p}(\Omega)$ as $n \to +\infty$ to the (unique) solution u of* (p′).

In the special case in which A is of second order, i.e., $m = 1$, $V = W_0^{1,p}(\Omega)$ and, for each n, $V_n = W_0^{1,p}(\Omega - E_n)$, with E_n a compact subset of Ω, then, taking Lemma 1.8 into account, we obtain

Corollary 2. *Under the assumptions* I, II, *and* III *above for the operator* A, *suppose that for any compact subset* Ω' *of* Ω *we have*

$$p\text{-cap}(E_n \cap \Omega') \to 0 \quad \textit{as} \quad n \to +\infty.$$

Then, there exists for each n a unique variational solution of the Dirichlet problem (d_n), *where* $\Omega_n = \Omega - E_n$, *and* u_n *converges strongly in* $W_0^{1,p}(\Omega)$ *to the unique variational solution u of the Dirichlet problem* (d).

Another simple application of Proposition 3.5 arises in connection with a variational problem which has been studied by J. L. Lions–G. Stampacchia (*28*).

Suppose again that A is of second order and let v_0 be a fixed function of $W_0^{1,p}(\Omega)$, E a closed subset of Ω.

Let us consider the (closed convex) subset

$$K = \{v \in W_0^{1,p}(\Omega) \colon v \geqslant v_0 \text{ on } E\}$$

of $W_0^{1,p}(\Omega)$ and the problem

$$\text{(e)} \qquad \begin{cases} u \in K, \\ a(u, v - u) \geqslant \langle f, v - u \rangle \quad \text{for all} \quad v \in K, \end{cases}$$

where f is a given distribution in the dual of $W_0^{1,p}(\Omega)$.

Now let (v_n) be a sequence of functions of $W_0^{1,p}(\Omega)$ and for each n let us consider the problem

$$(\mathrm{e}_n) \qquad \begin{cases} u_n \in K_n\,, \\ a(u_n\,, v - u_n) \geqslant \langle f, v - u_n \rangle \quad \text{for all} \quad v \in K_n\,, \end{cases}$$

where

$$K_n = \{v \in W_0^{1,p}(\Omega) : v \geqslant v_n \text{ on } E\}.$$

By applying Proposition 3.5 and taking Lemma 1.7 into account, we find

Corollary 3. *Under the assumptions* I, II, *and* III, *if* v_n *converges strongly to* v_0 *in* $W_0^{1,p}(\Omega)$ *as* $n \to +\infty$, *then the* (*unique*) *solution* u_n *of problem* (e_n) *converges strongly in* $W_0^{1,p}(\Omega)$ *to the* (*unique*) *solution u of problem* (e) *as* $n \to +\infty$.

4. Proof of Theorems A and B

In this section we shall prove Theorem A and Theorem B of Section 2 and their corollaries.

1. *Proof of Theorem A*

Lemma 4.1 below is well known; however, we shall give its proof for sake of completeness.

Lemma 4.1. *Let T be a map of $D(T)$ in X to X^*; K a subset of $D(T)$. If T is monotone, then any solution u of inequality* (1) *is also a solution of the inequality*

$$\langle Tv, v - u\rangle \geqslant 0 \qquad \textit{for all} \quad v \in K. \tag{4.1}$$

Conversely, if T is hemicontinuous and K is convex, then any solution u in K of inequality (4.1) *is also a solution of inequality* (1).

Proof. The first part of the lemma is a trivial consequence of the monotonicity of T. Conversely, let $u \in K$ be a solution of (4.1) and v be an arbitrary vector of K. The vector

$$v_t = tv + (1 - t)u, \qquad 0 < t < 1,$$

belongs to K for all t, for K is convex. Hence, by (4.1)

$$\langle Tv_t , v_t - u\rangle \geqslant 0,$$

which is to say,

$$\langle Tv_t , v - u\rangle \geqslant 0.$$

Therefore, letting $t \to 0$, we find by the hemicontinuity of T,

$$\langle Tu, v - u\rangle \geqslant 0.$$

Thus u satisfies (1). ∎

Lemma 4.2. *Under the assumptions* I *and* II (*of Section* 2), *we have*

$$\text{w-}\overline{\text{Lim}}\, S(T_n , K_n) \subset S(T, K).$$

Proof. Let $v \in K$. Since $K = \text{Lim}\, K_n$, there exists for each n a vector $v_n \in K_n$, such that $v_n \to v$ in X as $n \to +\infty$. Moreover, since (2.1) holds, there exists for each n a vector $z_n \in D(T_n)$, such that $z_n \to v$ in X and $T_n z_n \to Tv$ in X^* as $n \to +\infty$. For each n and all $w_n \in S(T_n, K_n)$, we have

$$\langle T_n w_n, v_n - w_n \rangle \geqslant 0. \tag{4.2}$$

By the monotonicity of T_n we also have

$$\langle T_n z_n, z_n - w_n \rangle \geqslant \langle T_n w_n, z_n - v_n \rangle. \tag{4.3}$$

[In fact, we have

$$\langle T_n z_n, z_n - w_n \rangle \geqslant \langle T_n w_n, z_n - w_n \rangle$$
$$= \langle T_n w_n, v_n - w_n \rangle + \langle T_n w_n, z_n - v_n \rangle$$

hence (4.3) follows from (4.2)].

The lemma is trivial if w-$\overline{\text{Lim}}$ $S(T_n, K_n) = \varnothing$. In the opposite case, let $u \in$ w-$\overline{\text{Lim}}$ $S(T_n, K_n)$, that is,

$$u = w\text{-lim}\, u_h \quad \text{as} \quad h \to +\infty,$$

with $u_h \in S(T_{n_h}, K_{n_h})$ for any $h = 1, 2, \dots$. Since $u_h \in K_{n_h}$ and $K = \text{Lim}\, K_n$, we have $u \in K$. Moreover, by (4.3) we have

$$\langle T_{n_h} z_{n_h}, z_{n_h} - u_h \rangle \geqslant \langle T_{n_h} u_h, z_{n_h} - v_{n_h} \rangle,$$

Since (u_h) and (z_{n_h}) are bounded and the mappings T_n are uniformly bounded, we obtain letting $h \to +\infty$,

$$\langle Tv, v - u \rangle \geqslant 0.$$

Therefore, u is a solution in K of (4.1), thus, by Lemma 4.1, u is a solution of inequality (1), i.e., $u \in S(T, K)$. ∎

Lemma 4.3. *Assume* I *and* II (*of Section* 2). *Let* $v_h \in S(T_{n_h}, K_{n_h})$ *for every* h, *with* $(S(T_{n_h}, K_{n_h}))$ *a subsequence of* $(S(T_n, K_n))$. *Then,* $v \in X$ *and* $v_h \rightharpoonup v$ *in* X *as* $h \to +\infty$, *implies* $v \in D(T)$ *and*

$$\langle T_{n_h} v_h - Tv, v_h - v \rangle \to 0 \quad \textit{as} \quad h \to +\infty. \tag{4.4}$$

Proof. First we prove that

$$\limsup \langle T_{n_h} v_h - Tv, v_h - v \rangle \leqslant 0 \quad \text{as} \quad h \to +\infty. \tag{4.5}$$

Since $K = \operatorname{Lim} K_n$, we have $v \in K$ and, besides, there exists a vector $z_h \in K_{n_h}$, for every h, such that $z_h \to v$ in X as $h \to +\infty$. Moreover, since $v_h \in S(T_{n_h}, K_{n_h})$, we have

$$\langle T_{n_h} v_h, z_h - v_h \rangle \geqslant 0$$

for all h. Therefore,

$$\langle T_{n_h} v_h, v_h - v \rangle \leqslant \langle T_{n_h} v_h, z_h - v \rangle$$

which implies, by the uniform boundedness of T_n (note that (v_h) is bounded in X), that

$$\limsup \langle T_{n_h} v_h, v_h - v \rangle \leqslant 0 \quad \text{as} \quad h \to +\infty. \tag{4.6}$$

On the other hand, we have

$$\lim \langle Tv, v - v_h \rangle = 0 \quad \text{as} \quad h \to +\infty,$$

for $v_h \rightharpoonup v$ as $h \to +\infty$. Hence (4.5) holds.

To complete the proof, it suffices to apply the following Sublemma, that we state formally below because we shall need it later.

Sublemma. *Assume* I (*of Section* 2). *Let* $v_h \in D(T_{n_h})$ *for every* h, *with* (T_{n_h}) *a subsequence of* (T_n). *Then,* $v_h \rightharpoonup v$ *in* X *as* $h \to +\infty$, *implies*

$$\liminf \langle T_{n_h} v_h - Tv, v_h - v \rangle \geqslant 0 \quad \textit{as} \quad h \to +\infty. \tag{4.7}$$

Proof. We have $v \in D(T)$,

$$\langle T_{n_h} v_h - Tv, v_h - v \rangle = \langle T_{n_h} v_h, v_h - v \rangle + \langle Tv, v - v_h \rangle$$

and

$$\langle Tv, v - v_h \rangle \to 0 \quad \text{as} \quad h \to +\infty.$$

By (2.1) of I, there exists $z_h \in D(T_{n_h})$ for every h, such that $z_h \to v$ in X

and $T_{n_h} z_h \to Tv$ in X^* as $h \to +\infty$. Moreover, by the monotonicity of T_{n_h}, we have

$$\langle T_{n_h} v_h , v_h - v \rangle = \langle T_{n_h} v_h , v_h - z_h \rangle + \langle T_{n_h} v_h , z_h - v \rangle$$
$$\geqslant \langle T_{n_h} z_h , v_h - z_h \rangle + \langle T_{n_h} v_h , z_h - v \rangle.$$

Therefore, since $T_{n_h} z_h \to Tv$, $v_h - z_h \rightharpoonup 0$, $z_h \to v$ and the sequence $(T_{n_h} v_h)$ is bounded in X^*, we find

$$\liminf \langle T_{n_h} v_h , v_h - v \rangle \geqslant 0 \quad \text{as} \quad h \to +\infty.$$

Thus (4.7) holds. ∎

Proof of (a) of Theorem A. Let $u \in X$, $u = \text{w-lim}\, u_h$ in X, with $u_h \in S(T_{n_h}, K_{n_h})$ for every h. Since $u \in \text{w-}\overline{\text{Lim}}\, S(T_n , K_n)$, then, by Lemma 4.2, $u \in S(T, K)$. Moreover, (2.4) follows from Lemma 4.3. Now let us suppose that III of Section 2 holds. Since $u \in K$ and (u_h) is bounded, we have by III, for any $\sigma > 0$

$$\beta(\| u_h - u \|) \leqslant |\langle T_{n_h} u_h - Tu, u_h - u \rangle| + \sigma$$

for all $h > h_\sigma$, for some $h_\sigma > 0$. Therefore

$$\| u_h - u \| \leqslant \beta^{-1}(|\langle T_{n_h} u_h - Tu, u_h - u \rangle| + \sigma)$$

for all $h > h_\sigma$, where β^{-1} is the inverse function of β. Letting $h \to +\infty$, since then $\langle T_{n_h} u_h - Tu, u_h - u \rangle \to 0$ and β^{-1} is continuous on $\overline{\mathbb{R}^+}$, we find

$$\limsup \| u_h - u \| \leqslant \beta^{-1}(\sigma).$$

This implies $\| u_h - u \| \to 0$ as $h \to +\infty$. ∎

Proof of (b) of Theorem A. Since X is reflexive, then the hypothesis (2.5) implies

$$\text{w-}\overline{\text{Lim}}\, S(T_n , K_n) \neq \varnothing .$$

On the other hand, we have by (a) of Theorem A

$$\text{w-}\overline{\text{Lim}}\, S(T_n , K_n) \subset S(T, K).$$

Hence (2.6) holds. Now let us suppose that $S(T, K) = \{u\}$, u a vector of X. Let (w_h) be a bounded sequence in X, with $w_h \in S(T_{n_h}, K_{n_h})$ for all h, $(S(T_{n_h}, K_{n_h}))$ a subsequence of $(S(T_n, K_n))$. Then, again by the reflexivity of X, if follows from (2.6) that w_h converges weakly to u in X as $h \to +\infty$. ▮

Part (c) of Theorem A follows from the basic existence theorem for inequality (1) stated in Subsection 3 of Section 0, and the following

Proposition 4.1. *Let (T_n) be a sequence of uniformly bounded mappings from X to X^* and K_n, for each n, a subset of the domain $D(T_n)$ of T_n. Let us suppose that $S(T_n, K_n) \neq \varnothing$ for every n and that there exists a function $\gamma : \bar{\mathbb{R}}_+ \to (-\infty, +\infty]$, with $\gamma(r) \to +\infty$ as $r \to +\infty$, such that for any sequence (v_n) in X, with $v_n \in K_n$ for each n, there exists a bounded sequence (z_n) in X, with $z_n \in K_n$ for each n, such that*

$$\| v_n - z_n \| \gamma(\| v_n - z_n \|) \leqslant \langle T_n v_n - T_n z_n , v_n - z_n \rangle \qquad \textit{for all} \quad n. \tag{4.8}$$

Then, there exists a bounded subset B of X and $n_0 > 0$, such that

$$S(T_n, K_n) \subset B \qquad \textit{for all} \quad n > n_0 .$$

Proof. Clearly it suffices to prove that any sequence (v_n), with $v_n \in S(T_n, K_n)$ for every n, is bounded in X. In fact, we know by the hypothesis that, for any such (v_n), there exist a bounded sequence (z_n) in X, $z_n \in K_n$ for every n, such that (4.8) holds. On the other hand we have for all n,

$$\langle T_n v_n , v_n - z_n \rangle \leqslant 0;$$

hence

$$\begin{aligned}\langle T_n v_n - T_n z_n , v_n - z_n \rangle &\leqslant \langle T_n z_n , z_n - v_n \rangle \\ &\leqslant \| T_n z_n \| \| z_n - v_n \|.\end{aligned}$$

It follows by (4.8),

$$\gamma(\| v_n - z_n \|) \leqslant \| T_n z_n \| \qquad \text{for all} \quad v_n \neq z_n ;$$

hence (v_n) is bounded. ▮

Proof of (c) of Theorem A. We are supposing that the T_n are uniformly coercive on K_n in X, that is, that IV of Section 2 holds. Therefore, by (2.3) and assumptions I and II, T_n is, for each n, a coercive monotone hemicontinuous mapping of the non-empty closed convex subset K_n of $D(T_n)$ into X^*. Hence, by the existence theorem for inequalities (1) (see Subsection 3 of Section 0), we have

$$S(T_n, K_n) \neq \varnothing \qquad \text{for all} \quad n.$$

Now let $v_n \in K_n$ for each n. Since $0 \in K_n$ for each n, we can satisfy (4.8) of Proposition 4.1 by choosing $z_n = 0$ for all n. In fact, by (2.3), since $\| T_n 0 \|$ is uniformly bounded, we have

$$\| v_n \| \gamma(\| v_n \|) \leqslant \langle T_n v_n - T_n 0, v_n \rangle \qquad \text{for all} \quad n,$$

where

$$\gamma(r) = \alpha(r) - \sup_n \| T_n 0 \|, \qquad r \geqslant 0,$$

and $\gamma(r) \to +\infty$ as $r \to +\infty$. Therefore (c) follows from Proposition 4.1. ∎

Proof of Corollary of Theorem A. It suffices to apply Theorem A with $T = T_n$ for every n. Clearly, assumptions I and II are satisfied. Moreover, by our assumption on T, III is satisfied with β given by $\beta(r) = r\gamma(r)$, and besides, IV is satisfied, with $\alpha = \gamma - \| T0 \|$. Furthermore T is strictly monotone, hence $S(T, K)$ consists of a single vector u of K and each $S(T, K_n)$ of a single vector u_n of K_n. Therefore, by applying successively (c), (b), and (a) of Theorem A, and taking Remark 2.1 into account, we find that u_n converges strongly to u in X as $n \to +\infty$. ∎

2. *Proof of Theorem B*

To deduce Theorem B from Theorem A, we need the following lemma.

Lemma 4.4. *Let (T_n) be a sequence of uniformly bounded mappings from X to X^*, with $D(T_n)$ the domain of T_n. Let (f_n) be a sequence of functions on X, with* $\operatorname{dom} f_n \subset D(T_n)$ *for every n, which converges in the sense of Definition* 1.4 *to a proper function f on X, such that* $\operatorname{dom} f \neq \varnothing$.

If (v_n) is a bounded sequence in X, with $v_n \in S(T_n, f_n)$ for each n, then the sequence $(f_n(v_n))$ is bounded in $\mathbb{R}$.

Proof. Let $v_0 \in \operatorname{dom} f$. Since $f = \operatorname{Lim} f_n$ in X, there exists a sequence (z_n) in X, such that $z_n \to v_0$ in X and

$$\limsup f_n(z_n) \leqslant f(v_0) \quad \text{as} \quad n \to +\infty$$

(see Lemma 1.10). Since $v_n \in S(T_n, f_n)$ for every n, we have for all $v \in X$

$$\langle T_n v_n, v - v_n \rangle \geqslant f_n(v_n) - f_n(v)$$

hence, for all n large enough,

$$f_n(v_n) \leqslant T_n v_n, z_n - v_n \rangle + f_n(z_n).$$

Therefore, since (v_n) and (z_n) are bounded in X and, by the uniform boundedness of T_n, $(T_n v_n)$ is bounded in X^*, we find

$$\limsup f_n(v_n) < +\infty, \quad \text{as} \quad n \to +\infty,$$

for $f(v_0) < +\infty$. On the other hand, we have

$$\liminf f_n(v_n) > -\infty \quad \text{as} \quad n \to +\infty.$$

In fact, if $\liminf f_n(v_n) = -\infty$, there exists a subsequence $(f_h'(v_h'))$ of $(f_n(v_n))$ such that

$$f_h'(v_h') \to -\infty \quad \text{as} \quad h \to +\infty.$$

Since (v_h') is bounded in X and X is reflexive, there exists a subsequence (v'_{h_k}) of (v_h') which converges weakly in X to a vector w of X. Since $f_n \to f$ in X, we have, again by Lemma 1.10,

$$f(w) \leqslant \liminf f'_{h_k}(v'_{h_k}) \quad \text{as} \quad k \to +\infty,$$

hence $f(w) = -\infty$, which is a contradiction, for f is proper. ▌

Proof of Theorem B. Let us consider, with notation from Subsection 9 of Section 0:

The space $X \oplus \mathbb{R}$; the mappings $T \oplus 1$ and $T_n \oplus 1$, $n = 1, 2,...,$ from $X \oplus \mathbb{R}$ to $X^* \oplus \mathbb{R}$; the subsets epi f and epi f_n, $n = 1, 2,...,$ of $X \oplus \mathbb{R}$.

By the assumptions I and II′ of Theorem B, we know that $T \oplus 1$ is monotone and hemicontinuous; $(T_n \oplus 1)$ is a sequence of uniformly bounded monotone hemicontinuous mappings, such that

$$G(T \oplus 1) \subset \text{s-}\underline{\text{Lim}}\, G(T_n \oplus 1) \quad \text{in} \quad (X \oplus \mathbb{R}) \times (X^* \oplus \mathbb{R})$$

Moreover, epi f and all epi f_n, $n = 1, 2,...,$ are closed convex subsets of

$X \oplus \mathbb{R}$, with $\operatorname{epi} f \subset D(T \oplus 1)$ and $\operatorname{epi} f_n \subset D(T_n \oplus 1)$ for every n, and we have

$$\operatorname{epi} f = \operatorname{Lim} \operatorname{epi} f_n \quad \text{in} \quad X \oplus \mathbb{R},$$

in the sense of Definition 1.1.

Thus, we can apply Theorem A to the case at hand and we find what follows:

(a′) If $\{u_h, \alpha_h\} \in S(T_{n_h} \oplus 1, \operatorname{epi} f_{n_h})$ for every h, with

$$(S(T_{n_h} \oplus 1, \operatorname{epi} f_{n_h}))$$

a subsequence of $(S(T_n \oplus 1, \operatorname{epi} f_n))$, and $\{u_h, \alpha_h\} \rightharpoonup \{u, \alpha\}$ in $X \oplus \mathbb{R}$ as $h \to +\infty$, then $\{u, \alpha\} \in S(T \oplus 1, \operatorname{epi} f)$ and

$$\langle T_{n_h} \oplus 1\{u_h, \alpha_h\} - T \oplus 1\{u, \alpha\}, \{u_h, \alpha_h\} - \{u, \alpha\}\rangle \to 0 \quad \text{as} \quad h \to +\infty.$$

(b′) If there exists a bounded subset B_1 of $X \oplus \mathbb{R}$ and $n_0 > 0$, such that

$$S(T_n \oplus 1, \operatorname{epi} f_n) \cap B_1 \neq \varnothing \qquad \text{for all} \quad n > n_0,$$

then there exists at least one solution $\{u, \alpha\} \in S(T \oplus 1, \operatorname{epi} f)$, and we have

$$\varnothing \neq \text{w-}\overline{\operatorname{Lim}}\, S(T_n \oplus 1, \operatorname{epi} f_n) \subset S(T \oplus 1, \operatorname{epi} f).$$

Moreover, if $S(T \oplus 1, \operatorname{epi} f)$ consists of the single vector $\{u, \alpha\}$, then, for any sequence $(\{w_h, \beta_h\})$, with $\{w_h, \beta_h\} \in S(T_{n_h} \oplus 1, \operatorname{epi} f_{n_h})$ for every h, which is bounded in $X \oplus \mathbb{R}$, we have $\{w_h, \beta_h\} \rightharpoonup \{u, \alpha\}$ in $X \oplus \mathbb{R}$ as $n \to +\infty$.

At this point it is easy to show, taking the remarks of Subsection 9 of Section 0 and Lemma 4.4 into account, that (a′) and (b′) above are equivalent to (a) and (b) of Theorem B.

The proof of (c) of Theorem B follows, as in case of Theorem A, from the basic existence theorem for inequalities of type (2) that we have stated in Section 0 and the following proposition, which generalizes Proposition 4.1.

Proposition 4.2. *Let (T_n) be a sequence of uniformly bounded mappings from X to X^* and, for each n, let f_n be a function on X, with* $\operatorname{dom} f_n \subset D(T_n)$ *for every n. Let us suppose that $S(T_n, f_n) \neq \varnothing$ for every n, and that there exists a function $\gamma : \overline{\mathbb{R}}^+ \to (-\infty, +\infty]$, with $\gamma(r) \to +\infty$*

as $r \to +\infty$ *such that for any sequence* (v_n) *in* X, *with* $v_n \in \operatorname{dom} f_n$ *for every* n, *there exists a bounded sequence* (z_n) *in* X, $z_n \in D(T_n)$ *for each* n, *such that*

$$\| v_n - z_n \| \gamma(\| v_n - z_n \|) \leqslant \langle T_n v_n - T_n z_n , v_n - z_n \rangle + f_n(v_n) - f_n(z_n) \quad \textit{for all } n. \tag{4.9}$$

Then, there exists a bounded subset B *of* X, *such that*

$$S(T_n , f_n) \subset B \qquad \textit{for all } n \textit{ large enough.}$$

Proof. It suffices to prove that any sequence (v_n), with $v_n \in S(T_n , f_n)$ for every n, is bounded in X.

By the hypothesis, there exists a bounded sequence (z_n) in X, $z_n \in D(T_n)$, such that (4.9) holds. On the other hand, we have for each n, since $f_n(v_n) < +\infty$ and $f_n(z_n) < +\infty$,

$$\langle T_n v_n , v_n - z_n \rangle \leqslant f_n(z_n) - f_n(v_n)$$

hence

$$\langle T_n v_n - T_n z_n , v_n - z_n \rangle + f_n(v_n) - f_n(z_n) \leqslant \langle T_n z_n , z_n - v_n \rangle \leqslant \| T_n z_n \| \| z_n - v_n \|.$$

It follows, by (4.9), that

$$\gamma(\| v_n - z_n \|) \leqslant \| T_n z_n \| \qquad \text{whenever} \quad v_n \neq z_n ,$$

what implies that (v_n) is bounded. ▮

Proof of (c) of Theorem B. Since f is proper, $f \not\equiv +\infty$, and $f_n \to f$, then every f_n is proper for all n large enough, (see Remark 1.6). Moreover, condition IV′ implies that every couple T_n , f_n satisfies the coerciveness condition of the existence theorem for inequalities (2) stated in Subsection 7 of Section 0. Therefore, we have $S(T_n , f_n) \neq \varnothing$ for all large n. Furthermore, for any (v_n) as in Proposition 4.2, we can satisfy to (4.9) with a suitable γ, by choosing $z_n = 0$ for all n (recall that we are supposing $f_n(0) = 0$ for all n), because the T_n are uniformly bounded and IV′ holds. Therefore, we have by Proposition 4.2

$$S(T_n , f_n) \subset B$$

for some bounded subset B of X and all n large enough. ▮

Thus the proof of Theorem B is now complete.

Proof of Corollary of Theorem B. The hypothesis (2.11) implies by II′, that there exists a bounded closed convex subset B of X, with $\operatorname{dom} f \cap \operatorname{int} B \neq \varnothing$, such that

$$S(0, f_n) \subset B \qquad \textit{for all } n \textit{ large enough.} \tag{4.10}$$

We recall that $S(0, f_n)$ coincides for each n with the set of all vectors of X which minimize f_n on X, see Subsection 7 of Section 0.

Let us consider for each n the function $\tilde{f}_n$ that coincides with f_n on B and is $\equiv +\infty$ outside B. $\tilde{f}_n$ is, for each n large enough, a proper convex lower-semicontinuous function on X, for f_n is such. Moreover, $\operatorname{dom} \tilde{f}_n$ is bounded and non-empty. Thus, by the existence Theorem quoted in Subsection 7 of Section 0, $S(0, \tilde{f}_n)$ is non-empty and, by (4.10),

$$S(0, \tilde{f}_n) \subset B \qquad \textit{for all } n \textit{ large enough.}$$

Furthermore, by the assumption of the corollary, $\tilde{f}_n$ converges as $n \to +\infty$ to the function $\tilde{f}$, which coincides with f on B and is $\equiv +\infty$ outside B. The function $\tilde{f}$ is proper, strictly convex and lower-semicontinuous for f is such, and $\operatorname{dom} \tilde{f} \neq \varnothing$. Thus, $\tilde{f}$ and $\tilde{f}_n$ satisfy condition II′ of Section 2 and to prove the corollary it suffices to apply successively (b) and (a) of Theorem B, with $T_n = T = 0$ for every n, taking into account that $S(0, \tilde{f}) = S(0, f)$ consists of the single vector which minimizes f on X and that condition III′ of Section 2 specializes to the hypothesis (2.12) of the corollary. ∎

5. Non-Coercive Mappings, Non-Unique Solutions

In this section we extend the results of Section 2 to the case in which T may be non-coercive in X and the solution of problem (1) non-unique.

Let us remark, before, that when the hypotheses III and IV of Theorem A are not satisfied and we do not know that the solution of (1) is unique, then we can only conclude, on the basis of Theorem A, that any *bounded* sequence in X of approximating solutions u_n (i.e., $u_n \in S(T_n, K_n)$) has a *subsequence* which converges *weakly* in X to a solution of (1).

We shall improve this result, by making use of the so called "elliptic regularization", which is the standard device for dealing with the "degenerate" case at hand (see references mentioned in Subsections 4 and 8 of Section 0).

It consists in adding to each T_n a perturbation $n^{-\alpha}M$, with $\alpha > 0$ and M a coercive map of X to X^*, then solving for each n the problem

$$w_n \in K_n \colon \langle (T_n + n^{-\alpha}M)w_n , v - w_n \rangle \geqslant 0 \qquad \text{for all} \quad v \in K_n ,$$

and finally letting $n \to +\infty$.

We shall see that the method is successful, provided T_n converges to T and K_n to K rapidly enough, as $n \to +\infty$, in order that $T_n + n^{-\alpha}M$ still acts coercively in X while T_n approaches T and K_n approaches K.

Now we state our results with more details. Let us suppose that the following strenghtened version I_1 and II_1 of I and II are satisfied:

I_1 *T and T_n, $n = 1, 2,\ldots$, are as stated in I of Section 2. In addition, T is bounded and there exists $\alpha > 0$ such that for any $v \in D(T)$ we have*

$$0 \in \text{s-Lim inf}\, n^{\alpha}\{G(T_n) - \{v, Tv\}\} \quad \textit{in} \quad X \times X^*. \tag{5.1}$$

According to our notation of Subsection 1 of Section 1, (5.1) means that there exists $v_n \in D(T_n)$, for all $n > n_0$, $n_0 > 0$, such that

$$n^{\alpha}(v_n - v) \to 0 \quad \text{(strongly) in } X \text{ as} \quad n \to +\infty, \tag{5.1$'$}$$

$$n^{\alpha}(T_n v_n - Tv) \to 0 \quad \text{(strongly) in } X^* \text{ as} \quad n \to +\infty. \tag{5.1$''$}$$

II_1 *K and K_n, $n = 1, 2,\ldots$, are as stated in* II *of Section* 2. *In addition, there exists $\alpha > 0$ such that K_n converges to K of order $\geqslant \alpha$ in X as $n \to +\infty$, i.e.,*

$$n^{\alpha}[K_n - K] \to 0 \quad \textit{as} \quad n \to +\infty,$$

in the sense of Definition 1.3 *of Section* 1.

Clearly, we can assume that α in I_1 and II_1 is the same.

Now we suppose that

(m) *M is a bounded, monotone hemicontinuous map of X to X^*, such that if $S_0 = S(T, K)$, then $S(M, S_0)$ consists at most of a single vector (for example, let T or M be strictly monotone)*

without requiring, for the moment, that M be coercive.

Let us consider for each n the map

$$A_n = T_n + n^{-\alpha}M \tag{5.2}$$

from X to X^*, with domain $D(A_n) = D(T_n)$, and the problem

$$w_n \in K_n\colon \langle A_n w_n , v - w_n\rangle \geqslant 0 \qquad \text{for all} \quad v \in K_n . \tag{5.3$_n$}$$

We shall prove, in particular, that any bounded sequence of solutions w_n of $(5.3)_n$ converges weakly to the (unique) solution w_0 of inequality (1), such that

$$w_0 \in S(T, K)\colon \langle Mw_0 , v - w_0\rangle \geqslant 0 \qquad \text{for all} \quad v \in S(T, K).$$

Moreover,

$$\langle Mw_n - Mw_0 , w_n - w_0\rangle \to 0 \quad \text{as} \quad n \to +\infty,$$

hence w_n converges strongly to w_0 in X as $n \to +\infty$, whenever M satisfies the condition

III$_1$ *If (v_n) is a sequence in X which converges weakly to a vector $v \in X$, and, besides, is such that $\langle Mv_n - Mv, v_n - v\rangle \to 0$ as $n \to +\infty$, then v_n converges strongly to v in X as $n \to +\infty$.*

If, in addition, M is coercive in X, then such is every A_n, hence, by the existence theorem, there exists for each n a solution w_n of problem $(5.3)_n$. In part (c) of Theorem C below, we prove that if we know that inequality (1) has a solution, that is, $S(T, K) \neq \varnothing$, and IV$_0$, IV$_1$, IV$_2$ below are satisfied, then the sequence (w_n) is bounded in X. The conditions are as follows:

There exists a non-decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, with

$$\lim r/\varphi(r) < +\infty \quad \text{as} \quad r \to +\infty, \tag{5.4}$$

such that

IV$_0$ $\begin{cases} \langle Mv, v\rangle/\varphi(\| v \|) \to +\infty & \textit{as} \quad \| v \| \to +\infty \\ \| Mv \| \leqslant \varphi(\| v \|) & \textit{for all} \quad v \in X \end{cases}$

IV$_1$ *Either (5.1″) holds with $v_n = v$ for every n and $K \subset K_n$ for every n, or $\| T_n v \| \leqslant \varphi(\| v \|)$ for every n and all $v \in D(T_n)$*

IV$_2$ *For any sequence (w_n), $w_n \in K_n$ for each n and $\| w_n \| \to +\infty$ as $n \to +\infty$, there exists a sequence (z_n) in K, such that*

$$\limsup n^\alpha \| z_n - w_n \|/\varphi(\| w_n \|) < +\infty \quad \textit{as} \quad n \to +\infty$$

Clearly IV_2 is trivially satisfied if $K_n \subset K$ for every n (for then we can choose $z_n = w_n$).

All these results are given in the following

Theorem C. *Let us suppose that* I_1 *and* II_1 *hold,* M *satisfies* (*m*) *and* A_n *is given for each n by* (5.2). *Then we have*:

(a) *If* $w_h \in S(A_{n_h}, K_{n_h})$ *for every* h, *with* $(S(A_{n_h}, K_{n_h}))$ *a subsequence of* $(S(A_n, K_n))$, *and* w_h *converges weakly to a vector* w *of* X *as* $h \to +\infty$, *then* w *coincides with the* (*unique*) *solution* w_0 *of*

$$w_0 \in S(T, K): \langle Mw_0, v - w_0 \rangle \geqslant 0 \quad \textit{for all} \quad v \in S(T, K) \tag{5.5}$$

and, besides,

$$\langle Mw_h - Mw_0, w_h - w_0 \rangle \to 0 \quad as \quad h \to +\infty. \tag{5.6}$$

Thus, w_h *converges strongly to* w_0 *in* X, *whenever* M *has the property* III_1.

(b) *Suppose that there exists a bounded subset* B *of* X *and* $n_0 > 0$, *such that*

$$S(A_n, K_n) \cap B \neq \varnothing \qquad \textit{for all} \quad n > n_0.$$

Then, there exists w_0 *satisfying the inequality* (5.5) *and, furthermore, any bounded sequence* (w_j), *with* $w_j \in S(T_{n_j}, K_{n_j})$ *for every* j *and* $(S(T_{n_j}, K_{n_j}))$ *a subsequence of* $(S(T_n, K_n))$, *converges weakly to* w_0.

(c) *If* $S(T, K) \neq \varnothing$ *and conditions* IV_0, IV_1 *and* IV_2 *are satisfied, then there exists a bounded subset* B *of* X *and* $n_0 > 0$, *such that*

$$\varnothing \neq S(A_n, K_n) \subset B \qquad \textit{for all} \quad n > n_0.$$

Remark 5.1. Part (a) of Theorem C holds even if X is not reflexive, as it will be clear from the proof of the theorem that will be given in the following section. Part (c) of Theorem C can be somewhat generalized, see Proposition 6.1.*

Let us recall that a *duality mapping* of a Banach space X into X^*, with gauge function a given real-valued continuous strictly increasing function χ of $r \geqslant 0$, such that $\chi(0) = 0$ and $\chi(r) \to +\infty$ as $r \to +\infty$, is a map J of X into X^*, such that

$$\langle Jv, v \rangle = \| Jv \| \| v \|$$
$$\| Jv \| = \chi(\| v \|)$$

* Finally, let us note that assumption II′ requires, in particular, that $0 \in s\text{-}Lim\, n^{\alpha} (K_n - v)$ for all $v \in K$. Actually, as it will be clear from the proof of Theorem C, it is sufficient that the condition above only holds for every $v \in S(T, K)$. This can be useful whenever regularity properties of the solutions are known.

for all $v \in X$. We refer to (*10*) and (*11*) for a discussion of the properties of these mappings. We recall here that if X is uniformly convex and X^* is strictly convex, then there exists for each given gauge function χ one and only one duality mapping J of X into X^*, and such J is bounded, coercive, strictly monotone and continuous from the strong topology of X to the weak topology of X^*. Besides, J has the following property: If (v_n) is a sequence in X which converges weakly to a vector v of X and $\langle Jv_n - Jv, v_n - v\rangle \to 0$ as $n \to +\infty$, then (v_n) converges strongly to v in X.

By using this notion of duality mapping, we obtain from Theorem C the following

Corollary of Theorem C. *Let us suppose that* I_1 *and* II_1 *hold and that* X *is uniformly convex and* X^* *is strictly convex. Furthermore, let us suppose that* $S(T, K) \neq \varnothing$ *and that there exists a real-valued continuous strictly increasing function* φ *of* $r \geqslant 0$, *with* $\varphi(0) = 0$ *and*

$$\lim r/\varphi(r) < +\infty \quad as \quad r \to +\infty,$$

such that IV_1 *and* IV_2 *holds. Let* J *be the duality mapping of* X *into* X^* *with gauge function* φ. *Then, there exists for each* $n > n_0 > 0$ *a unique solution* w_n *of*

$$w_n \in K_n\colon \ \langle (T_n + n^{-\alpha}J)w_n, v - w_n\rangle \geqslant 0 \qquad \text{for all} \quad v \in K_n. \qquad (5.7)_n$$

Such a w_n *converges strongly in* X *as* $n \to +\infty$ *to the unique solution* w_0 *of*

$$w_0 \in S(T, K)\colon \ \langle Jw_0, w - w_0\rangle \geqslant 0 \qquad \text{for all} \quad w \in S(T, K). \qquad (5.8)$$

Moreover,

$$\langle Jw_n - Jw_0, w_n - w_0\rangle \to 0 \quad as \quad n \to +\infty. \qquad (5.9)$$

Now we state the analogue of Theorem C for inequalities of type (2). While I_1 is unchanged, II_1 must be replaced by

II_1' *f and (f_n) are as stated in* II′ *of Section* 2. *In addition, there exists* $\alpha > 0$ *such that*

$$n^\alpha[f_n - f] \to 0 \quad as \quad n \to +\infty$$

in the sense of Definition 1.5.

We shall assume that the α in I_1 and II_1' is the same.

The inequality $(5.3)_n$ are now replaced by

$$w_n \in X : \langle A_n w_n , v - w_n \rangle \geqslant f_n(w_n) - f_n(v) \qquad \text{for all} \quad v \in X;$$

i.e., $w_n \in S(A_n, f_n)$, where again A_n is given for each n by (5.2) and M satisfies

(m′) *M is a bounded monotone hemicontinuous map of X to X^*, such that $S(M, S_1)$ consists at most of a single vector, where $S_1 = S(T, f)$.*

Clearly the last condition in (m′) is satisfied whenever M is strictly monotone, or when either T is strictly monotone or f is strictly convex.

Condition III_1 is replaced by

III_1' *If (v_n) is a sequence in X which converges weakly to a vector v of X and, besides, $f_n(v) \to f(v)$ and $\langle Mv_n - Mv, v_n - v \rangle \to 0$ as $n \to +\infty$, then (v_n) converges strongly to v in X.*

Finally, condition IV_0 remains unchanged, again with φ a non-decreasing function of $\mathbb{R}^+$ in $\mathbb{R}^+$ which satisfies (5.4), while conditions IV_1 and IV_2 must be replaced by

IV_1' *Either (5.1″) holds with $v_n = v$ for every n and* epi $f \subseteq$ epi f_n *for all n, or $\| T_n v \| \leqslant \varphi(\| v \|)$ for every n and all $v \in D(T_n)$.*

IV_2' *For any sequence (v_n) in X with $\| v_n \| \to +\infty$ as $n \to +\infty$, and any $v \in D(T)$, there exists a sequence (z_n) in X, such that*

$$\limsup n^{\alpha}\{\langle Tv, z_n - v_n \rangle + f(z_n) - f_n(v_n)\}/\varphi(\| v_n \|) < +\infty$$

as $n \to +\infty$.

Clearly IV_2' is trivially satisfied, with $z_n = v_n$ for every n, if $\operatorname{dom} f_n \subset \operatorname{dom} f$ for all n.

Theorem D. *Let us suppose that T and (T_n) satisfy I_1, f and (f_n) satisfy II_1', M is as stated in (m′) and that A_n is given for each n by (5.2). Then we have*

(a) *If $w_h \in S(A_{n_h}, f_{n_h})$ for every h, with $(S(A_{n_h}, f_{n_h}))$ a subsequence of $(S(A_n, f_n))$, and w_h converges weakly to a vector w of X as $h \to +\infty$, then w coincides with the (unique) solution w_0 of*

$$w_0 \in S(T, f) \colon \langle Mw_0 , w - w_0 \rangle \geqslant 0 \qquad \textit{for all} \quad w \in S(T, f) \tag{5.10}$$

and, besides,

$$f_{n_h}(w_h) \to f(w_0) \tag{5.11}$$

$$\langle Mw_h - Mw_0, w_h - w_0 \rangle \to 0 \tag{5.12}$$

as $h \to +\infty$. Thus, w_h converges strongly to w_0 in X, provided property III_1' *holds.*

(b) *Suppose that there exists a bounded subset B of X and $n_0 > 0$, such that*

$$S(A_n, f_n) \cap B \neq \varnothing \qquad \textit{for all} \quad n > n_0. \tag{5.13}$$

Then, there exists w_0 satisfying (5.10) *and, furthermore, any bounded sequence (w_j), with $w_j \in S(A_{n_j}, f_{n_j})$ for every j and $(S(A_{n_j}, f_{n_j}))$ a subsequence of $(S(A_n, f_n))$, converges weakly to w_0.*

(c) *If $S(T,f) \neq \varnothing$ and conditions* IV_0, IV_1' *and* IV_2' *are satisfied, then there exists a bounded subset B of X and $n_0 > 0$, such that*

$$\varnothing \neq S(A_n, f_n) \subset B \qquad \textit{for all} \quad n > n_0.$$

Remark 5.2. The hypothesis of reflexivity of X is unnecessary in part (a) of Theorem D. Moreover, (c) can be generalized to Proposition 6.1 of the following section, see Remark 5.1.

Remark 5.3. When $T_n = T$ and $f_n = f$ for all n, $\varphi(r) \equiv r$ and M is a map of improvability, see Ref. (*14*), then Theorem C specializes to Theorem 1 of Ref. (*14*) (for a bounded T).

Corollary of Theorem D. *Let us suppose that X is uniformly convex, X^* is strictly convex, f and (f_n) satisfy* II_1' *and* IV_2' *with $\varphi(r) \equiv r$ and $T = 0$. Moreover, let f have a minimum in X and let J be the duality mapping of X into X^* with gauge function $\varphi(r) \equiv r$. Then, there exists for each n one and only one solution $w_n \in X$ of*

$$f_n(w_n) \leqslant f_n(v) + n^{-\alpha}\langle Jw_n, v - w_n \rangle \qquad \textit{for all} \quad v \in X.$$

Such a w_n converges strongly in X to the vector w_0 of X, uniquely determined, which minimizes f on X and satisfies

$$\langle Jw_0, w - w_0 \rangle \geqslant 0$$

for all vectors w of X minimizing f on X. Moreover, $f_n(w_n) \to f(w_0)$ and $\langle Jw_n - Jw_0, w_n - w_0 \rangle \to 0$ as $n \to +\infty$.

6. Proofs of Theorems C and D

We shall prove below Theorem C and Theorem D of Section 5 and their corollaries.

Proof of (a) of Theorem C. The mappings T and (A_n) satisfy the assumption I of Theorem A. Indeed, we have for each n,

$$A_n = T_n + n^{-\alpha}M, \qquad D(A_n) = D(T_n), \tag{6.1}$$

where $\alpha > 0$, T and (T_n) are as stated in I of Section 2 and M is as stated in (m) of Section 5. Therefore, (A_n) is a sequence of monotone hemicontinuous mappings from X to X^*, which are uniformly bounded in X and, by (5.1) satisfy

$$G(T) \subset \text{s-}\underline{\text{Lim}}\, G(A_n) \quad \text{in} \quad X \times X^*.$$

Since, in virtue of II_1, the assumption II of Theorem A is also satisfied, we can apply Theorem A to the mappings T and (A_n) and to the sets K and (K_n). It follows, by (a) of that theorem, that if w and (w_h) are as stated in (a) of Theorem C, then

$$w \in S(T, K).$$

We shall prove below that for any $v \in S(T, K)$,

$$\limsup \langle Mw_h , w_h - v\rangle \leqslant 0 \quad \text{as} \quad h \to +\infty. \tag{6.2}$$

Once (6.2) is achieved, the proof of (a) of Theorem C can be concluded as follows.

By the monotonicity of M, (6.2) implies

$$\limsup \langle Mv, w_h - v\rangle \leqslant 0 \quad \text{as} \quad h \to +\infty,$$

hence, since $w_h \rightharpoonup w$ as $h \to +\infty$,

$$\langle Mv, v - w\rangle \geqslant 0$$

for all $v \in S(T, K)$. Since M is hemicontinuous and $S(T, K)$ is convex (see Subsection 3 of Section 0), it follows by Lemma 4.1,

$$\langle Mw, v - w\rangle \geqslant 0,$$

thus, by the last property of M stated in (m), w coincides with the unique solution w_0 of (5.5).

To prove the remaining part of (a), let us put $v = w_0$ in (6.2). Since $w_h \rightharpoonup w_0$ as $h \to +\infty$, we find

$$\limsup\langle Mw_h - Mw_0, w_h - w_0\rangle \leqslant 0 \quad \text{as} \quad h \to +\infty$$

hence, by the monotonicity of M, (5.6) holds, what implies in turn, that w_h converges strongly to w_0 in X as $h \to +\infty$, provided M has the property III_1.

Therefore, it remains to prove that (6.2) holds for every $v \in S(T, K)$. Let us show that we are led to a contradiction, by assuming that there exists a vector $v_0 \in S(T, K)$ and a subsequence (w_j') of (w_h), with $w_j' = w_{h_j}$ for every j, such that

$$\lim\langle Mw_j', w_j' - v_0\rangle > 0 \quad \text{as} \quad j \to +\infty. \tag{6.3}$$

Since K_n converges of order $\geqslant\alpha$ to K as $n \to +\infty$, there exists for every j a vector $v_j \in K_{\tilde{j}}$, where $\tilde{j} = n_{h_j}$, such that

$$\tilde{j}^{\alpha}(v_j - v_0) \to 0 \quad \text{in } X \text{ as} \quad j \to +\infty. \tag{6.4}$$

Moreover, we have

$$\langle A_{\tilde{j}} w_j', v_j - w_j'\rangle \geqslant 0 \qquad \text{for all} \quad j,$$

for $w_j' \in S(A_{\tilde{j}}, K_{\tilde{j}})$. Therefore,

$$\langle A_{\tilde{j}} w_j', v_0 - w_j'\rangle + \langle A_{\tilde{j}} w_j', v_j - v_0\rangle \geqslant 0,$$

which is to say, by (6.1),

$$\langle T_{\tilde{j}} w_j', w_j' - v_0\rangle + \tilde{j}^{-\alpha}\langle Mw_j', w_j' - v_0\rangle \leqslant \langle A\, w_j', v_j - v_0\rangle, \tag{6.5}$$

for all j.

On the other hand, again since K_n converges of order $\geqslant\alpha$ to K as $n \to +\infty$, there exist a sequence (z_ℓ) of vectors of K and a subsequence (w'_{j_ℓ}) of (w_j'), such that

$$\tilde{j}_\ell^{\alpha}(z_\ell - w'_{j_\ell}) \rightharpoonup 0 \quad \text{in } X \text{ as} \quad \ell \to +\infty. \tag{6.6}$$

Moreover, we have

$$\langle Tv_0, z_\ell - v_0\rangle \geqslant 0 \qquad \text{for every} \quad \ell,$$

for $v_0 \in S(T, K)$. Therefore,

$$\langle Tv_0, v_0 - w'_{j_\ell}\rangle \leqslant \langle Tv_0, z_\ell - w'_{j_\ell}\rangle \qquad \text{for every} \quad \ell. \tag{6.7}$$

Furthermore, by (5.1) of assumption I_1, there exists for each ℓ a vector $x_\ell \in D(T_{j_\ell})$, such that

$$j_\ell^{\alpha}(x_\ell - v_0) \to 0 \quad \text{in } X \text{ as} \quad \ell \to +\infty, \tag{6.8}$$

$$j_\ell^{\alpha}(T_{j_\ell} x_\ell - T v_0) \to 0 \quad \text{in } X^* \text{ as} \quad \ell \to +\infty. \tag{6.9}$$

Now, by (6.7) we have

$$\langle T_{j_\ell} x_\ell , x_\ell - w'_{j_\ell} \rangle + \langle T v_0 - T_{j_\ell} x_\ell , x_\ell - w'_{j_\ell} \rangle + \langle T v_0 , v_0 - x_\ell \rangle$$
$$\leqslant \langle T v_0 , z_\ell - w'_{j_\ell} \rangle \tag{6.10}$$

and by (6.5),

$$\langle T_{j_\ell} w'_{j_\ell} , w'_{j_\ell} - x_\ell \rangle + \langle T_{j_\ell} w'_{j_\ell} , x_\ell - v_0 \rangle + j_\ell^{-\alpha} \langle M w'_{j_\ell} , w'_{j_\ell} - v_0 \rangle$$
$$\leqslant \langle A_{j_\ell} w'_{j_\ell} , v_{j_\ell} - v_0 \rangle \tag{6.11}$$

for every ℓ.

Adding (6.10) to (6.11) and taking the monotonicity of T_{j_ℓ} into account, we find

$$\langle M w'_{j_\ell} , w'_{j_\ell} - v_0 \rangle \leqslant j_\ell^{\alpha} \{ \langle T v_0 - T_{j_\ell} x_\ell , w'_{j_\ell} - x_\ell \rangle + \langle T v_0 - T_{j_\ell} w'_{j_\ell} , x_\ell - v_0 \rangle$$
$$+ \langle T v_0 , z_\ell - w'_{j_\ell} \rangle + \langle A_{j_\ell} w'_{j_\ell} , v_{j_\ell} - v_0 \rangle \}. \tag{6.12}$$

Letting $\ell \to +\infty$, we find that the first term on the right-hand side of the inequality above goes to zero, because of (6.9) and the boundedness in X of (w'_{j_ℓ}) and (x_ℓ); the second term goes to zero, because of (6.8) and the boundedness in X^* of $(T_{j_\ell} w'_{j_\ell})$; the third term goes to zero, because of (6.6); finally, the last term also goes to zero, because of (6.4) and the boundedness of $(A_{j_\ell} w'_{j_\ell})$ in X^*. Thus,

$$\limsup \langle M w'_{j_\ell} , w'_{j_\ell} - v_0 \rangle \leqslant 0 \quad \text{as} \quad \ell \to +\infty,$$

which contradicts (6.3). ∎

Proof of (b) of Theorem C. By the hypothesis of (b) and the reflexivity of X, there exists a sequence (w_h) satisfying the hypothesis of (a) of Theorem C. Hence, by (a), there exists a (unique) vector w_0 which is a solution of (5.5). Moreover, it is easy to show that the last conclusion of (b) is again a consequence of (a) and the reflexivity of X. ∎

Proof of (c) of Theorem C. If M is coercive, such is for each n the (monotone hemicontinuous) map A_n given by (6.1). Since K_n is for every n large enough a non-empty closed convex subset of $D(A_n) = D(T_n)$ and X is reflexive, it follows by the existence theorem for inequality (1), that

$$S(A_n, K_n) \neq \varnothing \qquad \text{for all} \quad n \text{ large enough.}$$

It remains to prove that, in consequence of the hypotheses $S(T, K) \neq \varnothing$, IV_0, IV_1 and IV_2, there exists a bounded subset B of X and $n_0 > 0$, such that

$$S(A_n, K_n) \subset B \qquad \text{for all} \quad n > n_0 .$$

Clearly, it suffices to show that we find a contradiction if we suppose that there exists a sequence (w_h), with $w_h \in S(A_{n_h}, K_{n_h})$ for every h, $(S(A_{n_h}, K_{n_h}))$ a subsequence of $(S(A_n, K_n))$ such that

$$\| w_h \| \to +\infty \quad \text{as} \quad h \to +\infty. \tag{6.13}$$

Let us choose a vector $w_0 \in S(T, K)$. By II_1, there exists for each h a vector $v_h \in K_{n_h}$, with $v_h = w_0$ if $K \subset K_n$ for all n, such that

$$n_h^{\alpha} \| v_h - w_0 \| \to 0 \quad \text{as} \quad h \to +\infty \tag{6.14}$$

and, by IV_2, there exists a sequence (z_h) in K, such that

$$\limsup n_h^{\alpha} \| z_h - w_h \| / \varphi(\| w_h \|) < +\infty \quad \text{as} \quad h \to +\infty.$$

Moreover, by I_1, there exists for each h a vector $x_h \in D(T_{n_h})$, such that

$$n_h^{\alpha} \| x_h - w_0 \| \to 0 \quad \text{as} \quad h \to +\infty, \tag{6.15}$$

$$n_h^{\alpha} \| T_{n_h} x_h - T w_0 \| \to 0 \quad \text{as} \quad h \to +\infty \tag{6.16}$$

[with, possibly, $x_h = w_0$ for every h, in case (5.1″) holds with $v_n = v$ for every n].

By an argument quite similar to the one used to prove (6.12) in Proof of (a) of Theorem C, one finds

$$\langle M w_h, w_h - w_0 \rangle \leqslant n_h^{\alpha} \{ \langle T w_0 - T_{n_h} x_h, w_h - x_h \rangle + \langle T w_0 - T_{n_h} w_h, x_h - w_0 \rangle$$
$$+ \langle T w_0, z_h - w_h \rangle + \langle A_{n_h} w_h, v_h - w_0 \rangle \};$$

hence

$$\begin{aligned}\langle Mw_h, w_h\rangle/\varphi(\| w_h \|) \leqslant n_h{}^\alpha \| Tw_0 - T_{n_h} x_h \| \, [\| w_h \|/\varphi(\| w_h \|) + \| x_h \|/\varphi(\| w_h \|)] \\ + [\| Tw_0 \|/\varphi(\| w_h \|) + \| T_{n_h} w_h \|/\varphi(\| w_h \|)] n_h{}^\alpha \| x_h - w_0 \| \\ + \| Tw_0 \| \, n_h{}^\alpha \| z_h - w_h \|/\varphi(\| w_h \|) \\ + [\| T_{n_h} w_h \|/\varphi(\| w_h \|)] \, n_h{}^\alpha \| v_h - w_0 \| \\ + [\| Mw_h \|/\varphi(\| w_h \|)] \, \| v_h \|. \end{aligned} \tag{6.17}$$

Letting $h \to +\infty$, one finds that the first term on the right-hand side of the inequality above, goes to zero in virtue of (6.16), the boundedness of $(\| x_h \|)$ and the property (5.4) of φ; the second term also goes to zero, because of (6.15), (5.4) and IV_1; the third term is bounded (as $h \to +\infty$), in virtue of IV_2; the fourth term goes to zero, because of (6.14) and IV_1; finally, the last term also goes to zero, because of (6.14) and IV_0. It follows

$$\limsup \langle Mw_h, w_h\rangle/\varphi(\| w_h \|) < +\infty \quad \text{as} \quad h \to +\infty,$$

and this, by (6.13), contradicts IV_0. ∎

Proof of Corollary of Theorem C. By the properties of J that we have summarized before the Corollary of Theorem C in Section 5, we can apply Theorem C, with $M = J$ satisfying (m), III_1 and IV_0. Therefore, since $A_n = T_n + n^{-\alpha} J$ is strictly monotone for every n, it follows from (c) of that theorem that there exists for each n one and only one solution w_n of $(5.7)_n$ and the sequence (w_n) is bounded in X. Thus, by (b), w_n converges weakly in X as $n \to +\infty$ to the (unique) solution w_0 of (5.8), hence, by (a), w_n converges strongly in X to w_0 as $n \to +\infty$ and (5.9) holds. ∎

Proof of Theorem D. We shall deduce (a) and (b) of Theorem D from (a) and (b) of Theorem C by the same argument that we have used in Section 4 to deduce Theorem B from Theorem A.

Under the assumption I_1 and II_1', it is easy to show that the mappings $T \oplus 1$, $M \oplus 0$ and

$$A_n \oplus 1 = T_n \oplus 1 + n^{-\alpha} M \oplus 0, \qquad n = 1, 2, \ldots,$$

satisfy the hypothesis I_1 of Theorem C, with X replaced by $X \oplus \mathbb{R}$ (notation of Section 0). Let us only notice that, by (m′), $M \oplus 0$ is a bounded monotone hemicontinuous map of $X \oplus \mathbb{R}$ to $X^* \oplus \mathbb{R}$, such that

$$S(M \oplus 0, S_0),$$

where $S_0 = S(T \oplus 1, \operatorname{epi} f)$, consists at most of a single vector. [In fact,

$$\{w, \alpha\} \in S(T \oplus 1, \operatorname{epi} f) \quad \text{and} \quad \langle M \oplus 0 \{w, \alpha\}, \{v, \beta\} - \{w, \alpha\}\rangle \geqslant 0$$

for all $\{v, \beta\} \in S(T \oplus 1, \operatorname{epi} f)$, is equivalent to

$$w \in S(T, f), \qquad \alpha = f(w) \qquad \text{and} \qquad \langle Mw, v - w\rangle \geqslant 0$$

for all $v \in S(T, f)$, hence, by (m′), such a vector $\{w, \alpha\}$ is uniquely determined].

Moreover, $\operatorname{epi} f$ and $\operatorname{epi} f_n$ satisfy, by II_1', the hypothesis II_1 for the case at hand.

Therefore, we can apply Theorem C, with X replaced by $X \oplus \mathbb{R}$, T by $T \oplus 1$, A_n by $A_n \oplus 1$, K by $\operatorname{epi} f$ and K_n by $\operatorname{epi} f_n$. Thus (a) and (b) of Theorem D can be obtained, by taking Lemma 4.4 into account, from (a) and (b) of Theorem C, respectively, as we shall show below with more details.

Proof of (a) of Theorem D. Let (w_h) be a sequence in X satisfying the hypothesis of (a). Since (w_h) is bounded, then, by Lemma 4.4, $(f_{n_h}(w_h))$ is also bounded; hence there exists a subsequence (w_{h_j}) of (w_h) such that $w_{h_j} \rightharpoonup w$ and

$$f_{n_{h_j}}(w_{h_j}) \to \alpha,$$

$\alpha \in \mathbb{R}$, as $j \to +\infty$. Since

$$\{w_{h_j}, f_{n_{h_j}}(w_{h_j})\} \in S(A_{n_{h_j}} \oplus 1, \operatorname{epi} f_{n_{h_j}})$$

for every j, by applying (a) of Theorem C one finds that $\{w, \alpha\}$ coincides with the unique solution $\{w_0, \alpha_0\} \in S(T \oplus 1, \operatorname{epi} f)$ of the inequality

$$\langle M \oplus 0 \{w_0, \alpha_0\}, \{v, \beta\} - \{w_0, \alpha_0\}\rangle \geqslant 0 \qquad \text{for all} \quad \{v, \beta\} \in S(T \oplus 1, \operatorname{epi} f)$$

which is to say, $w = w_0 \in S(T, f)$, $\alpha = \alpha_0 = f(w_0)$ and

$$\langle Mw_0, v - w_0\rangle \geqslant 0 \qquad \text{for all} \quad v \in S(T, f).$$

Thus, w coincides with the solution w_0 of (5.10). Besides, by the uniqueness of w_0, we have $f_{n_h}(w_h) \to f(w_0)$ as $h \to +\infty$. Finally, (5.12) follows trivially from (5.6) of Theorem C in the case at hand, and the last assertion of (a) is obvious. ▮

Proof of (b) of Theorem D. By the hypothesis (5.13) of (b) and Lemma 4.4, there exists a bounded subset B_1 of $X \oplus \mathbb{R}$ and $n_0 > 0$, such that

$$S(A_n \oplus 1, \operatorname{epi} f_n) \cap B_1 \neq \varnothing \qquad \text{for all} \quad n > n_0 .$$

Hence (b) of Theorem D follows from (b) of Theorem C. ▮

Part (c) of Theorem D is a special case of the following

Proposition 6.1. *Let T be a map from a normed space X to X^*, f a proper function on X with* $\operatorname{dom} f \neq \varnothing$, *and suppose*

$$S(T, f) \neq \varnothing .$$

Let (T_n) be a sequence of monotone mappings from X to X^*, M a map of X into X^* and for a given $\alpha > 0$ let for each n,

$$A_n = T_n + n^{-\alpha} M,$$

with $D(A_n) = D(T_n)$. Moreover, let f_n be for every n a proper function on X with $\operatorname{dom} f_n \neq \varnothing$, *and suppose that*

$$S(A_n, f_n) \neq \varnothing \qquad \textit{for all} \quad n.$$

Let us suppose that there exists a non-negative function φ of $r > 0$, with

$$\lim r/\varphi(r) < +\infty \quad \textit{as} \quad r \to +\infty, \tag{6.18}$$

such that for each $w \in S(T, f)$ and any sequence (w_n), with

$$w_n \in D(T_n) \cap \operatorname{dom} f_n$$

for every n and $\| w_n \| \to +\infty$ as $n \to +\infty$, the following conditions $\widetilde{\mathrm{IV}}_0, \ldots, \widetilde{\mathrm{IV}}_2$ hold:

$\widetilde{\mathrm{IV}}_0$ $\left\{ \langle M w_n, w_n - v_n \rangle / \varphi(\| w_n \|) \to +\infty \right.$ *as* $n \to +\infty$ *for any bounded sequence (v_n) in X.*

$$\widetilde{\mathrm{IV}}_1' \begin{cases} \textit{Either } w \in D(T_n) \textit{ for all } n \textit{ and} \\ \qquad \limsup n^\alpha \| T_n w - Tw \| < +\infty \quad \textit{as} \quad n \to +\infty, \\ \textit{or there exists } x_n \in D(T_n) \textit{ for every } n, \textit{ such that} \\ \qquad \limsup n^\alpha \| x_n - w \| < +\infty \\ \qquad \limsup n^\alpha \| T_n x_n - Tw \| < +\infty \\ \textit{and, besides,} \\ \qquad \limsup \| T_n w_n \| / \varphi(\| w_n \|) < +\infty \\ \textit{as } n \to +\infty. \end{cases}$$

$$\widetilde{\mathrm{IV}}_1'' \begin{cases} \textit{There exists a bounded sequence } (v_n) \textit{ in } X, \textit{ such that} \\ \quad \limsup n^\alpha \{ \langle T_n w_n, v_n - w \rangle + f_n(v_n) - f(w) \} / \varphi(\| w_n \|) < +\infty \\ \textit{as } n \to +\infty. \end{cases}$$

$$\widetilde{\mathrm{IV}}_2 \begin{cases} \textit{There exists a sequence } (z_n) \textit{ in } X, \textit{ such that} \\ \quad \limsup n^\alpha \{ \langle Tw, z_n - w_n \rangle + f(z_n) - f_n(w_n) \} / \varphi(\| w_n \|) < +\infty \\ \textit{as } n \to +\infty. \end{cases}$$

Then, there exists a bounded subset B of X and $n_0 > 0$, such that

$$S(A_n, f_n) \subset B \qquad \textit{for all} \quad n > n_0 .$$

Remark 6.1. The hypothesis $\widetilde{\mathrm{IV}}_1''$ is trivially satisfied if

$$\operatorname{epi} f \subset \operatorname{epi} f_n \qquad \text{for all} \quad n,$$

for then we can take $v_n = w$ for every n. On the other hand, the hypothesis $\widetilde{\mathrm{IV}}_2$ is satisfied whenever

$$\operatorname{epi} f_n \subset \operatorname{epi} f \qquad \text{for all} \quad n,$$

by taking $z_n = w_n$ for every n.

Proof. Let us suppose that there exists a sequence (w_h) in X, with $w_h \in S(A_{n_h}, f_{n_h})$ for every h and $(S(A_{n_h}, f_{n_h}))$ a subsequence of $(S(A_n, f_n))$, such that $\| w_h \| \to +\infty$ as $h \to +\infty$, and let us show that this leads to a contradiction.

Let us choose $w \in S(T, f)$ and let (v_n) be a (bounded) sequence in X, such that $\widetilde{\mathrm{IV}}_1''$ holds. We have for every h,

$$\langle A_{n_h} w_h , v_{n_h} - w_h \rangle \geqslant f_{n_h}(w_h) - f_{n_h}(v_{n_h}),$$

hence also,

$$\langle T_{n_h} w_h , w - w_h \rangle + \langle T_{n_h} w_h , v_{n_h} - w \rangle + n_h^{-\alpha} \langle M w_h , v_{n_h} - w_h \rangle \geqslant f_{n_h}(w_h) - f_{n_h}(v_{n_h}). \tag{6.19}$$

Moreover, let (z_n) satisfy $\widetilde{\mathrm{IV}}_2$. We have for every h,

$$\langle Tw, z_{n_h} - w \rangle \geqslant f(w) - f(z_{n_h});$$

hence also,

$$\langle Tw, w_h - w \rangle + \langle Tw, z_{n_h} - w_h \rangle \geqslant f(w) - f(z_{n_h}). \tag{6.20}$$

By $\widetilde{\mathrm{IV}}_1'$, there exists $x_n \in D(T_n)$ such that

$$\limsup n^\alpha \| x_n - w \| < +\infty, \tag{6.21}$$

$$\limsup n^\alpha \| T_n x_n - Tw \| < +\infty \tag{6.22}$$

as $n \to +\infty$. We assume $x_n = w$ for all n, if $w \in D(T_n)$ for all n. From (6.19) we obtain, since both $f_{n_h}(w_h)$ and $f_{n_h}(v_{n_h})$ are finite,

$$\langle T_{n_h} w_h , w_h - x_{n_h} \rangle + \langle T_{n_h} w_h , x_{n_h} - w \rangle + \langle T_{n_h} w_h , w - v_{n_h} \rangle + n_h^{-\alpha} \langle M w_h , w_h - v_{n_h} \rangle \leqslant f_{n_h}(v_{n_h}) - f_{n_h}(w_h) \tag{6.23}$$

and from (6.20), since $f(w) < +\infty$ and $f(z_{n_h}) < +\infty$,

$$\langle T_{n_h} x_{n_h} , x_{n_h} - w_h \rangle + \langle Tw - T_{n_h} x_{n_h} , x_{n_h} - w_h \rangle + \langle Tw, w - x_{n_h} \rangle + \langle Tw, w_h - z_{n_h} \rangle \leqslant f(z_{n_h}) - f(w). \tag{6.24}$$

Adding (6.23) to (6.24) we find by the monotonicity of T_{n_h},

$$\langle M w_h , w_h - v_{n_h} \rangle \leqslant n_h^{\alpha} \{ \langle Tw - T_{n_h} x_{n_h} , w_h - x_{n_h} \rangle + \langle Tw - T_{n_h} w_h , x_{n_h} - w \rangle + \langle T_{n_h} w_h , v_{n_h} - w \rangle + f_{n_h}(v_{n_h}) - f(w) + \langle Tw, z_{n_h} - w_h \rangle + f(z_{n_h}) - f_{n_h}(w_h) \};$$

hence,

$$\begin{aligned}\langle Mw_h, w_h - v_{n_h}\rangle/\varphi(\| w_h \|) &\leqslant n_h{}^\alpha \| Tw - T_{n_h}x_{n_h} \|(\| w_h \| + \| x_{n_h} \|)/\varphi(\| w_h \|)\\ &+ n_h{}^\alpha \| x_{n_h} - w \|(\| Tw \| + \| T_{n_h}w_h \|)/\varphi(\| w_h \|)\\ &+ n_h{}^\alpha[\langle T_{n_h}w_h, v_{n_h} - w\rangle + f_{n_h}(v_{n_h}) - f(w)]/\varphi(\| w_h \|)\\ &+ n_h{}^\alpha[\langle Tw, z_{n_h} - w_h\rangle + f(z_{n_h}) - f_{n_h}(w_h)]/\varphi(\| w_h \|).\end{aligned}$$

Now we let $h \to +\infty$ in the inequality above. Then, both the first and second term of the right member are bounded by (6.22), (6.21) and (6.18); the third term is also bounded by $\widetilde{\mathrm{IV}}_1''$; finally, the last term is bounded by $\widetilde{\mathrm{IV}}_2$. Thus, we find

$$\limsup \langle Mw_h, w_h - v_{n_h}\rangle/\varphi(\| w_h \|) < +\infty$$

as $h \to +\infty$, which contradicts $\widetilde{\mathrm{IV}}_0$. ∎

Proof of (c) of Theorem D. The map M of Theorem D satisfies, by IV_0, the hypothesis $\widetilde{\mathrm{IV}}_0$ of Proposition 6.1, and the mapping T and T_n satisfy, by I_1 and IV_1', the hypothesis $\widetilde{\mathrm{IV}}_1'$ of that proposition.[10] Since by II_1' we have $n^\alpha[f_n - f] \to 0$ as $n \to +\infty$, then there exists, by Lemma 1.11, a sequence (v_n) in X, such that

$$n^\alpha \| v_n - w \| \to 0$$

$$\limsup n^\alpha[f_n(v_n) - f(w)] \leqslant 0$$

as $n \to +\infty$; besides, we have, by IV_1',

$$\limsup \| T_n w_n \|/\varphi(\| w_n \|) < +\infty$$

as $n \to +\infty$. Thus the hypothesis $\widetilde{\mathrm{IV}}_1''$ of Proposition 6.1 is also satisfied. Finally, IV_2' clearly implies $\widetilde{\mathrm{IV}}_2$. Therefore (c) of Theorem D follows from Proposition 6.1 and the existence theorem stated in Subsection 7 of the Introduction. ∎

Proof of Corollary of Theorem D. It suffices to apply Theorem D with $T_n = T = 0$ for every n and $M = J$. ∎

[10] The hypothesis $\widetilde{\mathrm{IV}}_1''$ is trivially satisfied in case $\operatorname{epi} f \subset \operatorname{epi} f_n$ for all n (see Remark 6.1).

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