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Notes on the norm map between the Hecke algebras of the Gelfand–Graev representations of $GL(2,q^2)$ and $U(2,q)$

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ABSTRACT

Let $\tilde{G}$ be a connected reductive algebraic group defined over the field $\mathbf{F}_q$ and let F and F^* be two Frobenius maps such that $F^m=(F^*)^m$ for some integer m. Let $\tilde{G}^F, \tilde{G}^{F^*}$, and $\tilde{G}^{F^m}=\tilde{G}^{(F^*)^m}$ be the finite groups of fixed points. In this article we consider the case where $\tilde{G}=GL(2,\bar{F}_q)$, F is the usual Frobenius map so that $\tilde{G}^F=GL(2,q)$ and F^* is the twisted Frobenius map such that $\tilde{G}^{F^*}=U(n,q)$. In this case, $F^2=(F^*)^2$ and $\tilde{G}^{F^2}=\tilde{G}^{(F^*)^2}=GL(2,q^2)$. This article provides connections between the complex representation theory of these groups using the norm maps (see [C. Curtis, T. Shoji, A norm map for endomorphism algebras of Gelfand–Graev representations, in: Progr. Math., vol. 141, 1997, pp. 185–194]) from the Gelfand–Graev Hecke algebra of $GL(2,q^2)$ to the Gelfand–Graev Hecke algebras of both $GL(2,q)$ and $U(2,q)$.

1. Background information and notation

Let $\tilde{G}$ denote a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$ where q is a power of a prime and let $\alpha:\tilde{G}\to\tilde{G}$ be a Frobenius endomorphism. An α-stable maximal torus of $\tilde{G}$ will be denoted by $\tilde{T}$ and the unipotent radical of an α-stable Borel in $\tilde{G}$ will be denoted by $\tilde{U}$. For any subgroup $\tilde{A}$ of $\tilde{G}$, the group $\tilde{A}^\alpha$ of fixed points will be denoted by A. In particular let $G=\tilde{G}^\alpha$. That is, G is a finite group of Lie type. A maximal torus of G is a subgroup of G of the form $T=\tilde{T}^\alpha$. The *Gelfand–Graev characters* of G are the induced characters $\Gamma=Ind_U^G(\psi)$ where ψ is a nondegenerate linear character of $U=\tilde{U}^\alpha$. For the groups considered in this paper the center of $\tilde{G}$ is connected. Thus given a Frobenius endomorphism, there will be only one Gelfand–Graev character Γ (see [3, p. 519]).

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Given a nondegenerate linear character of U, ψ, let e denote the central primitive idempotent in $\mathbf{C}U$ corresponding to ψ:

$$e = |U|^{-1} \sum_{u \in U} \psi\left(u^{-1}\right) u.$$

Then $H = e\mathbf{C}Ge$ is called the *Hecke algebra* of the Gelfand–Graev representation of G [4, Section 11]. The Hecke algebra constructed using the Gelfand–Graev representation is anti-isomorphic to the $\mathbf{C}G$-endomorphism algebra of the induced $\mathbf{C}G$-module affording the Gelfand–Graev character of G. Thus, since the Gelfand–Graev character is multiplicity free, the Hecke algebra H is commutative.

The standard basis of H is constructed in the following way. Let $\{x_i \mid i \in I\}$ be a collection of double coset representatives of U in G and let $J = \{j \mid j \in I,\ {}^{x_j}\psi = \psi \text{ on } U \cap {}^{x_j}U\}$. (Here ${}^{x_j}\psi$ denotes the character of the conjugate group ${}^{x_j}U$ and is defined by ${}^{x_j}\psi({}^{x_j}u) = \psi(u)$ for $u \in U$.) Then $\{[U\colon {}^{x_j}U \cap U]ex_je \mid j \in J\}$ is the *standard basis* of H [4, Proposition 11.30]. In this section $[U\colon {}^{x}U \cap U]exe$ will be denoted by c_x and for the remainder of the article $[U\colon {}^{x}U \cap U]$ will be denoted by $ind(x)$.

There is a bijection from the set of irreducible characters χ of G such that $\langle \chi, \Gamma \rangle \neq 0$ to the set of all irreducible characters of H. Also the primitive central idempotents of H are $\{e\epsilon\}$ where ϵ is a primitive central idempotent of $\mathbf{C}G$ associated with a χ such that $\langle \chi, \Gamma \rangle \neq 0$. Since H is commutative, these idempotents are actually primitive idempotents. Thus they give us the simple module $\mathbf{C}Ge\epsilon$ which affords χ [4, Corollary 11.27].

Let $\tilde{T}_o$ denote a maximally split α-stable maximal torus of $\tilde{G}$. The $\tilde{G}$-conjugacy classes of α-stable maximal tori of $\tilde{G}$ are parametrized by the α-conjugacy classes of $N_{\tilde{G}}(\tilde{T}_o)/\tilde{T}_o$ where the α-conjugate of x by g is defined to be $gx\alpha(g)^{-1}$ [3]. In fact, given any α-conjugacy class $[x]$ of the Weyl group $N_{\tilde{G}}(\tilde{T}_o)/\tilde{T}_o$,

$$T_x = \left\{A \in \tilde{T}_o \;\middle|\; xAx^{-1} = \alpha(A)\right\}$$

is conjugate (in $\tilde{G}$) to a maximal torus of G. And conversely, every maximal torus of G is conjugate to some T_x [3]. Given an α-stable maximal torus $\tilde{T}$ of $\tilde{G}$, let $R^G_{T,\theta}$ denote the Deligne–Lusztig generalized character, where θ is an irreducible character of the torus $T = \tilde{T}^\alpha$ (see, for example [1, Chapter 7]). Let Q^G_T denote the Green function, which is defined for all unipotent elements $u \in G$ by $Q^G_T(u) = R^G_{T,\theta}(u)$ (see, for example [1, p. 212]). Given a pair $(\tilde{T}, \theta)$ there exists a unique irreducible character $\chi_{T,\theta}$ of G such that $\langle \chi_{T,\theta}, \Gamma \rangle \neq 0$ and $\langle \chi_{T,\theta}, R^G_{T,\theta} \rangle \neq 0$. Also any irreducible character χ of G such that $\langle \chi, \Gamma \rangle \neq 0$ coincides with a $\chi_{T,\theta}$ for some pair $(\tilde{T}, \theta)$ [3, Theorem 2.1]. Thus the irreducible characters of H can be indexed by the pairs $(\tilde{T}, \theta)$. Also two irreducible characters $f_{T,\theta}$ and $f_{T',\theta,}$ of H are equal if and only if $(\tilde{T}, \theta)$ and $(\tilde{T}', \theta')$ are geometrically conjugate [3, Theorem 3.1].

The following theorem is from [3, Theorem 4.2].

Theorem 1.1. *Given a Frobenius endomorphism α defined on $\tilde{G}$, let $\tilde{T}$ be an α-stable maximal torus of $\tilde{G}$ and let $T = \tilde{T}^\alpha$. Let θ be an irreducible character of T extended to $\mathbf{C}T$. Let $G = \tilde{G}^\alpha$ and let $\Gamma = Ind^G_U(\psi)$ denote the Gelfand–Graev representation where $\tilde{U}$ is the unipotent radical of an α-stable Borel in $\tilde{G}$, $U = \tilde{U}^\alpha$ and ψ is a nondegenerate linear character of U. Let H denote the Hecke algebra corresponding to G.*

(i) *There exists a unique homomorphism $f_T : H \to \mathbf{C}T$, independent of θ, which has the property that each character $f_{T,\theta} : H \to \mathbf{C}$ can be factored as $f_{T,\theta} = \theta \cdot f_T$.*
(ii) *$f_T(c_x) = \sum_{t \in T} f_T(c_x)(t)t$ where c_x is an element in the standard basis of H and the coefficients $f_T(c_x)(t)$ are given by*

$$f_T(c_x)(t) = \frac{[U\colon {}^{x}U \cap U]}{\langle Q^G_T, \Gamma \rangle |U||C_G(t)|} \sum_{\substack{g \in G, u \in U \\ (guxg^{-1})_s = t}} \psi\left(u^{-1}\right) Q^{C_G(t)}_T\left(\left(guxg^{-1}\right)_u\right).$$

Note that if α is a Frobenius endomorphism then α^m is also a Frobenius endomorphism for any nonnegative integer m. Denote by f_T^m the homomorphism f_T described in the previous theorem when $G = \tilde{G}^{\alpha^m}$. Similarly denote by H^m the Hecke algebra corresponding to $\tilde{G}^{\alpha^m}$. The following theorem is from [6, Theorem 1].

Theorem 1.2. *Using the notation in the previous theorem and the paragraph following that theorem, let N_T^m denote the extension of the usual norm map from $\tilde{T}^{\alpha^m}$ to $\tilde{T}^{\alpha}$ to a homomorphism of the group algebras. There exists a unique homomorphism of algebras $\Delta^m : H^m \to H$ that has the property $f_T \cdot \Delta^m = N_T^m \cdot f_T^m$ for all α-stable maximal tori $\tilde{T}$ of $\tilde{G}$.*

The map Δ^m will be called the norm map of the Hecke algebras H^m and H.

We will consider the following specific set up. Let $\tilde{G} = GL(2, \bar{\mathbf{F}}_q)$. Let $F : \tilde{G} \to \tilde{G}$ be the Frobenius endomorphism given by $F(a_{ij}) = (a_{ij}^q)$. Let $F^* : \tilde{G} \to \tilde{G}$ be the Frobenius endomorphism given by $F^*(a_{ij}) = (a_{ji}^q)^{-1}$. Note that $\tilde{G}^{F^m} = GL(2, q^m)$ and $\tilde{G}^{F^*} = U(2, q)$. It will be convenient to take an isomorphic copy of this unitary group given by $w_0 \tilde{G}^{F^*} w_0^{-1}$ where

$$w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the remainder of this paper $U(2, q)$ will mean the group $w_0 \tilde{G}^{F^*} w_0^{-1}$.

Let B denote the upper triangular matrices of $GL(2, \bar{\mathbf{F}}_q)$ and let U denote its maximal unipotent subgroup. Let

$$U^{(m)} = U^{F^m} = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbf{F}_{q^m} \right\}$$

and

$$U^* = \left(w_0 U w_0^{-1}\right)^{F^*} = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbf{F}_q \right\}.$$

Note that $U^* = U^{(1)}$. Each nontrivial linear character, ψ_m of $U^{(m)}$ corresponds to a nontrivial linear character, χ_m, of the additive group of $\mathbf{F}_{q^m}$ by $\psi_m\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) = \chi_m(a)$. Fix a nontrivial linear character χ_1 of $U^{(1)}$ then choose (and fix) the nontrivial linear character of $U^{(m)}$, ψ_m, to be such that $\chi_m = \chi_1 \cdot Tr_{q^m,q}$. When $m = 1$, χ_m will be denoted by just χ.

In the following a diagonal matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ will be denoted by (a, b) and the unipotent element $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ will be denoted by $[a]$.

The maximal tori of $GL(2, q^m)$ are

$$T_0^{(m)} = T_e^{F^m} = \left\{ (t, u) \mid t, u \in \mathbf{F}_{q^m}^* \right\},$$
$$T_1^{(m)} = T_{(1,2)}^{F^m} = \left\{ \left(t, t^{q^m}\right) \mid t \in \mathbf{F}_{q^{2m}}^* \right\}.$$

The maximal tori of the unitary group $U(2, q)$ are

$$T_0^* = T_e^{F^*} = \left\{ \left(t, t^{-q}\right) \mid t \in \mathbf{F}_{q^2}^* \right\},$$
$$T_1^* = T_{(1,2)}^{F^*} = \left\{ (t, u) \mid t, u \in \mathbf{F}_{q^2}^*,\ t^{q+1} = u^{q+1} = 1 \right\}.$$

Note that in both $GL(2,q^m)$ and $U(2,q)$ the Weyl group is

$$W=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}=w_0\right\}\cong S_2.$$

When $\alpha=F^m$, we will denote H^m, f_T^m, N_T^m and Δ^m (defined in Theorem 1.1 and Theorem 1.2) by $H^{(m)}$, $f_T^{(m)}$, $N_T^{(m)}$ and $\Delta^{(m)}$ respectively. When $\alpha=F^*$, we will denote H and f_T by H^* and f_T^*. When $\alpha=F^*$ and $m=2$ we will denote N_T^m and Δ^m by N_T^* and Δ^* respectively.

2. Main results

This article provides some explicit descriptions of the maps $f_T^{(m)}$, f_T^*, $\Delta^{(m)}$ and Δ^* (defined in the previous section). In the last section of this article these descriptions lead to a direct comparison of the images of $\Delta^{(2)}$ and Δ^* of all the basis elements of $H^{(2)}$. In particular, in Theorem 10.4 it is shown the images of $\Delta^{(2)}$ and Δ^* are equal on the basis elements of $H^{(2)}$ with a slight adjustment. In addition, it is then shown in Theorem 10.7 that the images of Δ^* of certain basis elements of $H^{(2)}$ can be written in terms of Dickson polynomials.

The main results in Sections 9 and 10 describe the image under $\Delta^{(2)}$ and Δ^* of basis elements of the Hecke algebra $H^{(2)}$ as well as describe a certain subset of this basis in which these two maps are the same. In this way, the results in Sections 9 and 10 exhibit connections between the complex representation theory of $GL(2,q)$, $U(2,q)$ and $GL(2,q^2)$. Exhibiting connections between the representation theories of these groups has been investigated by many people. In particular, the connection between the representation theory of $GL(n,q)$ and $U(n,q)$ know as Ennola duality was shown to be true in all cases by Kawanaka [8] and connections between the representation theory of $GL(n,q)$ and $GL(n,q^m)$ were revealed by Shintani [9].

It would be an interesting problem to generalize the connections investigated in this article. That is, let α and α^* be any two Frobenius maps on a connected reduction algebraic group $\tilde{G}$ (as discussed in Section 1) such that $\alpha^m=(\alpha^*)^m$. Using the norm maps from [6] (described in Section 1 above), it would be interesting to describe the connections between the representation theory of $\tilde{G}^{\alpha}$, $\tilde{G}^{\alpha^*}$ and $\tilde{G}^{\alpha^m}=\tilde{G}^{(\alpha^*)^m}$.

3. The Hecke algebra for GL(2, q^m)

Let

$$e^{(m)}=\frac{1}{q^m}\sum_{u\in U^{(m)}}\psi_m\bigl(u^{-1}\bigr)u.$$

Then

$$H^{(m)}=\bigl\langle ind(n)e^{(m)}ne^{(m)}\bigm| n\in S,\ {}^n\psi_m=\psi_m \text{ on } U^{(m)}\cap {}^nU^{(m)}\bigr\rangle,$$

where S is a set of double coset representatives of $U^{(m)}$ in $GL(2,q^m)$. Note that

$$(u,v)w_0=\begin{pmatrix}0&-u\\v&0\end{pmatrix}.$$

Using the Bruhat decomposition, we see a set of double coset representatives of $U^{(m)}$ in $GL(2,q^m)$ is

$$S=\left\{\begin{pmatrix}u&0\\0&v\end{pmatrix},\begin{pmatrix}0&-u\\v&0\end{pmatrix}\Bigm| u,v\in\mathbf{F}_{q^m}\right\}.$$

In order to determine the standard basis of $H^{(m)}$ we first need to determine the set $\{n \mid n \in S,\ {}^{n}\psi_m = \psi_m \text{ on } U^{(m)} \cap {}^{n}U^{(m)}\}$.

Let $y \in U^{(m)}$ so $y = [b]$ for some $b \in \mathbf{F}_{q^m}$. Let $x \in T_0^{(m)} w_0$. Then $x = \left(\begin{smallmatrix} 0 & -u \\ v & 0 \end{smallmatrix}\right)$ for some $u, v \in \mathbf{F}_{q^m}^*$. Then

$$xyx^{-1} = \begin{pmatrix} 1 & 0 \\ -vu^{-1}b & 1 \end{pmatrix}.$$

So $xyx^{-1} \in U^{(m)}$ if and only if $-vu^{-1}b = 0$. But this is true if and only if $b = 0$ (since u and v are nonzero). Thus ${}^{x}U^{(m)} \cap U^{(m)} = I$ when $x \in T_o^{(m)} w_0$. For such an x, $ind(x) = |U^{(m)}| = q^m$. Note that the condition $\psi_m(y) = \psi_m(xyx^{-1})$ trivially holds for all $y \in U^{(m)} \cap {}^{x}U^{(m)} = I$.

Now suppose $x \in T_0^{(m)}$. Then $x = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)$ for some $u, v \in \mathbf{F}_{q^m}^*$. Note that $xU^{(m)}x^{-1} = U^{(m)}$ so $ind(x) = 1$. Also

$$xyx^{-1} = \begin{pmatrix} 1 & uv^{-1}b \\ 0 & 1 \end{pmatrix}.$$

The condition $\psi_m(y) = \psi_m(xyx^{-1})$ for all $y \in U^{(m)} \cap {}^{x}U^{(m)}$ implies $\chi_m(b) = \chi_m(uv^{-1}b)$ for all $b \in \mathbf{F}_{q^m}$. Thus $uv^{-1} = 1$. Thus $u = v$. Thus the standard basis for the Hecke algebra corresponding to $GL(2, q^m)$ is

$$\left\{ q^m e^{(m)} \begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix} e^{(m)}, \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} e^{(m)} \;\middle|\; u, v \in \mathbf{F}_{q^m}^* \right\}.$$

Denote $\left(\begin{smallmatrix} 0 & -u \\ v & 0 \end{smallmatrix}\right)$ by $x_{u,v}$ and denote the basis element $q^m e^{(m)} x_{u,v} e^{(m)}$ by $c_{u,v}^{(m)}$. Denote $\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right)$ by x_u and denote the basis element $e^{(m)} x_u e^{(m)}$ by $c_u^{(m)}$.

4. The structure constants of $H^{(m)}$

In order to demonstrate all the structure constants for this Hecke algebra we will first need the following lemma.

Lemma 4.1. $c_{u,1}^{(m)} c_{v,1}^{(m)} = q^m c_{-u}^{(m)} \delta_{u,v} + \sum_{w \in \mathbf{F}_{q^m}^*} \chi_m(-wv^{-1} - uvw^{-1} - u^{-1}w) c_{w,w^{-1}uv}^{(m)}$.

Proof. Let I be an index set for the standard basis. Denote the basis elements by a_i, $i \in I$. Order these basis elements so that $a_i = x_{u_i} e^{(m)}$ for some $u_i \in \mathbf{F}_{q^m}^*$ for $i \leqslant q^m - 1$ and $a_i = q^m e^{(m)} x_{u_i, v_i} e^{(m)}$ for $i \geqslant q^m$. We have $c_{u,1}^{(m)} c_{v,1}^{(m)} = \sum_{k \in I} \mu_k a_k$ where $\mu_k = q^m \sum_{y \in D_1 \cap x_k D_2^{-1}} c_{u,1}^{(m)}(y) c_{v,1}^{(m)}(y^{-1} x_k)$. (See [4, Proposition 11.30]). In this sum $D_1 = U^{(m)} x_{u,1} U^{(m)}$ and $D_2^{-1} = U^{(m)} x_{v,1}^{-1} U^{(m)}$. Note that

$$D_1 = \left\{ [r] \begin{pmatrix} 0 & -u \\ 1 & 0 \end{pmatrix} [s] \;\middle|\; r, s \in \mathbf{F}_{q^m} \right\} = \left\{ \begin{pmatrix} r & -u + rs \\ 1 & s \end{pmatrix} \;\middle|\; r, s \in \mathbf{F}_{q^m} \right\}$$

and

$$D_2^{-1} = \left\{ [a] \begin{pmatrix} 0 & 1 \\ -v^{-1} & 0 \end{pmatrix} [b] \;\middle|\; a, b \in \mathbf{F}_{q^m} \right\} = \left\{ \begin{pmatrix} -av^{-1} & 1 - abv^{-1} \\ -v^{-1} & -bv^{-1} \end{pmatrix} \;\middle|\; a, b \in \mathbf{F}_{q^m} \right\}.$$

First consider the case when $k \leqslant q^m - 1$. Then $x_k = \left(\begin{smallmatrix} w & 0 \\ 0 & w \end{smallmatrix}\right)$ for some $w \in \mathbf{F}_{q^m}^*$. Then

$$x_k D_2^{-1} = \left\{ \begin{pmatrix} -awv^{-1} & w - abv^{-1}w \\ -wv^{-1} & -bv^{-1}w \end{pmatrix} \;\middle|\; a, b \in \mathbf{F}_{q^m} \right\}.$$

In order to determine the set $D_1 \cap\ x_k D_2^{-1}$ we need to find for which $a, b \in \mathbf{F}_{q^m}$ we have

$$\begin{pmatrix} r & -u+rs \\ 1 & s \end{pmatrix} = \begin{pmatrix} -awv^{-1} & w-abv^{-1}w \\ -wv^{-1} & -bv^{-1}w \end{pmatrix}.$$

A comparison of the 2, 1 entries shows that this intersection is empty unless $w = -v$. So assume $w = -v$. Then we have

$$\begin{pmatrix} r & -u+rs \\ 1 & s \end{pmatrix} = \begin{pmatrix} a & w+ab \\ 1 & b \end{pmatrix}.$$

Thus $a = r$ and $b = s$ and the intersection $D_1 \cap x_k D_2^{-1}$ is D_1 when $w = -u = -v$. Otherwise the intersection is empty. Thus assume $x_k = (-u, -u)$ and $u = v$. Let $y \in D_1$. Then $y = \left(\begin{smallmatrix} r & -u+rs \\ 1 & s \end{smallmatrix}\right)$ and $y^{-1}x_k = u^{-1}\left(\begin{smallmatrix} s & u-rs \\ -1 & r \end{smallmatrix}\right)\left(\begin{smallmatrix} -u & 0 \\ 0 & -u \end{smallmatrix}\right) = \left(\begin{smallmatrix} -s & -u+rs \\ 1 & -r \end{smallmatrix}\right)$. Thus $c_{u,1}^{(m)}(y) = \frac{1}{q^m}\chi_m(r+s)$ and $c_{v,1}^{(m)}(y^{-1}x_k) = \frac{1}{q^m}\chi_m(-r-s)$. Thus $c_{u,1}^{(m)}(y)c_{v,1}^{(m)}(y^{-1}x_k) = \frac{1}{q^{2m}}$. Thus $\mu_k = q^m \sum_{y \in D_1} \frac{1}{q^{2m}} = q^m$ when $x_k = (-u, -u)$ and $u = v$. Otherwise $\mu_k = 0$.

Now consider the case when $k \geqslant q^m$. Then $x_k = \left(\begin{smallmatrix} 0 & -w_1 \\ w_2 & 0 \end{smallmatrix}\right)$ for some nonzero $w_1, w_2 \in \mathbf{F}_{q^m}$. Then

$$x_k D_2^{-1} = \left\{ \begin{pmatrix} w_1 v^{-1} & b w_1 v^{-1} \\ -a w_2 v^{-1} & w_2 - ab w_2 v^{-1} \end{pmatrix} \,\middle|\, a, b \in \mathbf{F}_{q^m} \right\}.$$

In order to determine the set $D_1 \cap\ x_k D_2^{-1}$ we need to find for which $a, b \in \mathbf{F}_{q^m}$ we have

$$\begin{pmatrix} r & -u+rs \\ 1 & s \end{pmatrix} = \begin{pmatrix} w_1 v^{-1} & b w_1 v^{-1} \\ -a w_2 v^{-1} & w_2 - ab w_2 v^{-1} \end{pmatrix}.$$

A comparison of the 1, 1 and 2, 1 entries shows that this intersection is empty unless $r = w_1 v^{-1}$ and $a = -w_2^{-1} v$. So assume these two conditions. Then we have

$$\begin{pmatrix} w_1 v^{-1} & -u + w_1 v^{-1} s \\ 1 & s \end{pmatrix} = \begin{pmatrix} w_1 v^{-1} & b w_1 v^{-1} \\ 1 & w_2 + b \end{pmatrix}.$$

Setting the 1, 2 entries of both matrices equal to each other and the 2, 2 entries of both matrices equal to each other we have $s = w_2 + b$ and $s = b + w_1^{-1} uv$. Thus $w_2 + b = b + w_1^{-1} uv$. Thus b can be chosen to be any element in $\mathbf{F}_{q^m}$. Then s is determined by b and the intersection is empty unless $w_1 w_2 = uv$. Thus, for the case $x_k = \left(\begin{smallmatrix} 0 & -w_1 \\ w_2 & 0 \end{smallmatrix}\right)$, we have the set $D_1 \cap x_k D_2^{-1}$ has q^m elements when $w_1 w_2 = uv$ otherwise this intersection is empty. Assume $w_1 w_2 = uv$. Let y be an element in this intersection. Then

$$y = \begin{pmatrix} w_1 v^{-1} & s w_1 v^{-1} - u \\ 1 & s \end{pmatrix}.$$

Thus

$$y^{-1} x_k = u^{-1} \begin{pmatrix} s & -s w_1 v^{-1} + u \\ -1 & w_1 v^{-1} \end{pmatrix} \begin{pmatrix} 0 & -w_1 \\ w_2 & 0 \end{pmatrix} = \begin{pmatrix} -s + w_2 & -s u^{-1} w_1 \\ 1 & u^{-1} w_1 \end{pmatrix}.$$

Thus in this case $c_{u,1}^{(m)}(y) = \frac{1}{q^m}\chi_m(-w_1 v^{-1} - s)$ and $c_{v,1}^{(m)}(y^{-1}x_k) = \frac{1}{q^m}\chi_m(s - w_2 - u^{-1} w_1) = \frac{1}{q^m}\chi_m(s - uv w_1^{-1} - u^{-1} w_1)$. Thus $c_{u,1}^{(m)}(y)c_{v,1}^{(m)}(y^{-1}x_k) = \frac{1}{q^{2m}}\chi_m(-w_1 v^{-1} - uv w_1^{-1} - u^{-1} w_1)$. Thus

$\mu_k = q^m \sum_{y \in D_1 \cap\ {}^{x_k}D_2^{-1}} \frac{1}{q^{2m}} \chi_m(-w_1 v^{-1} - uvw_1^{-1} - u^{-1}w_1) = \chi_m(-w_1 v^{-1} - uvw_1^{-1} - u^{-1}w_1)$ when $x_k = \left(\begin{smallmatrix} 0 & -w_1 \\ w_2 & 0 \end{smallmatrix}\right)$ is such that $w_1 w_2 = uv$. Otherwise $\mu_k = 0$ when $k \geqslant q^m$.

Combining these two cases we have that

$$c_{u,1}^{(m)} c_{v,1}^{(m)} = q^m c_{-u}^{(m)} \delta_{u,v} + \sum_{w_1 \in \mathbf{F}_{q^m}^*} \chi_m\left(-w_1 v^{-1} - uvw_1^{-1} - u^{-1}w_1\right) c_{w_1, w_1^{-1}uv}^{(m)}. \qquad \square$$

The following proposition provides all the structure constants for $H^{(m)}$.

Proposition 4.2. *Let* $t, u, v, x, y \in \mathbf{F}_{q^m}^*$. *Then*

(i) $c_u^{(m)} c_v^{(m)} = c_{uv}^{(m)}$,
(ii) $c_u^{(m)} c_{t,v}^{(m)} = c_{ut,uv}^{(m)}$ *and*
(iii) $c_{t,u}^{(m)} c_{x,y}^{(m)} = q^m c_{-ty}^{(m)} \delta_{ty,ux} + \sum_{w \in \mathbf{F}_{q^m}^*} \chi_m(-x^{-1}yw - tu^{-1}xy^{-1}w^{-1} - t^{-1}uw) c_{uyw,txw^{-1}}^{(m)}$.

Proof. As x_u is central, parts (i) and (ii) of the proposition are clear. Note that $\left(\begin{smallmatrix} 0 & -a \\ b & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -ab^{-1} \\ 1 & 0 \end{smallmatrix}\right)$. Thus

$$c_{t,u}^{(m)} c_{x,y}^{(m)} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} c_{tu^{-1},1}^{(m)} c_{xy^{-1},1}^{(m)}.$$

Therefore, using Lemma 4.1, we have

$$\begin{aligned} c_{t,u}^{(m)} c_{x,y}^{(m)} &= \begin{pmatrix} uy & 0 \\ 0 & uy \end{pmatrix} \Bigg[q^m c_{-tu^{-1}}^{(m)} \delta_{tu^{-1},xy^{-1}} \\ &\quad + \sum_{w \in \mathbf{F}_{q^m}^*} \chi_m\left(-w\left(xy^{-1}\right)^{-1} - tu^{-1}xy^{-1}w^{-1} - \left(tu^{-1}\right)^{-1} w\right) c_{w,w^{-1}tu^{-1}xy^{-1}}^{(m)} \Bigg] \\ &= q^m c_{-ty}^{(m)} \delta_{ty,ux} + \sum_{w \in \mathbf{F}_{q^m}^*} \chi_m\left(-x^{-1}yw - tu^{-1}xy^{-1}w^{-1} - t^{-1}uw\right) c_{uyw,txw^{-1}}^{(m)}. \qquad \square \end{aligned}$$

5. The Hecke algebra for $U(2,q)$

Recall H^* denotes the Hecke algebra corresponding to $U(2,q)$. Let

$$e^* = \frac{1}{q} \sum_{u \in U^*} \psi_1\left(u^{-1}\right) u.$$

Note that $e^* = e^{(1)}$. Then

$$H^* = \left\langle ind(n) e^{(1)} n e^{(1)} \;\middle|\; n \in S^*,\ {}^n\psi_1 = \psi_1 \text{ on } U^{(1)} \cap {}^nU^{(1)} \right\rangle,$$

where S^* is a set of double coset representatives of $U^{(1)}$ in $U(2,q)$. Note that

$$S^* = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-q} \end{pmatrix}, \begin{pmatrix} 0 & -t \\ t^{-q} & 0 \end{pmatrix} \;\middle|\; t \in \mathbf{F}_{q^2}^* \right\}.$$

In order to determine the standard basis of H^* we first need to determine the set $\{n \in S^* \mid {}^n\psi_1 = \psi_1$ on $U^{(1)} \cap {}^nU^{(1)}\}$.

Let $y \in U^{(1)}$ so $y = [b]$ for some $b \in \mathbf{F}_q$. Let $x \in T_0^* w_0$. Then $x = \left(\begin{smallmatrix} 0 & -u \\ u^{-q} & 0 \end{smallmatrix}\right)$ for some $u \in \mathbf{F}_{q^2}^*$. Then

$$xyx^{-1} = \begin{pmatrix} 0 & -u \\ u^{-q} & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & u^q \\ -u^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -u^{-1-q}b & 1 \end{pmatrix}.$$

So $xyx^{-1} \in U^{(1)}$ if and only if $-u^{-1-q}b = 0$. But this is true if and only if $b = 0$ (since u is nonzero). Thus ${}^xU^{(1)} \cap U^{(1)} = I$ when $x \in T_0^* w_0$. For such an x, $ind(x) = |U^{(1)}| = q$. Note that the condition $\psi_1(y) = \psi_1(xyx^{-1})$ trivially holds for all $y \in U^{(1)} \cap {}^xU^{(1)} = I$.

Now suppose $x \in T_0^*$. Then $x = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-q} \end{smallmatrix}\right)$ for some $u \in \mathbf{F}_{q^2}^*$. Note that $xU^{(1)}x^{-1} = U^{(1)}$ so $ind(x) = 1$. Also

$$xyx^{-1} = \begin{pmatrix} u & 0 \\ 0 & u^{-q} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & u^q \end{pmatrix} = \begin{pmatrix} 1 & u^{q+1}b \\ 0 & 1 \end{pmatrix}.$$

The condition $\psi_1(y) = \psi_1(xyx^{-1})$ for all $y \in U^{(1)} \cap {}^xU^{(1)}$ implies $\chi_1(b) = \chi_1(u^{q+1}b)$ for all $b \in \mathbf{F}_q$. Thus $u^{q+1} = 1$. Thus the standard basis for the Hecke algebra corresponding to $U(2,q)$ is

$$\left\{ qe^*\begin{pmatrix} 0 & -v \\ v^{-q} & 0 \end{pmatrix}e^*, \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}e^* \;\middle|\; u, v \in \mathbf{F}_{q^2}^*, u^{q+1} = 1 \right\}.$$

Denote $\left(\begin{smallmatrix} 0 & -v \\ v^{-q} & 0 \end{smallmatrix}\right)$ by $x_{v,v^{-q}}$ and denote the basis element $qe^*x_{v,v^{-q}}e^*$ by $c^*_{v,v^{-q}}$. Denote $\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right)$ by x_u and denote the basis element x_ue^* by c^*_u.

6. The structure constants of *H**

The following proposition provides all the structure constants for the Hecke algebra H^*.

Proposition 6.1. *Let $u, v, x, z \in \mathbf{F}_{q^2}^*$ be such that $u^{q+1} = v^{q+1} = 1$. Then*

(i) $c^*_u c^*_v = c^*_{uv}$,
(ii) $c^*_u c^*_{x,x^{-q}} = c^*_{ux,ux^{-q}}$ *and*
(iii) $c^*_{x,x^{-q}} c^*_{z,z^{-q}} = qc^*_{-zx^{-q}}\delta_{x^{q+1},z^{q+1}} + \sum_{w \in \mathbf{F}_q^*} \chi_1(-wz^{-q-1} - w^{-1}x^{q+1}z^{q+1} - wx^{-q-1})c^*_{x^{-q}z^{-q}w,xzw^{-1}}$.

Proof. As in Proposition 4.2 parts (i) and (ii) are clear. Let I be an index set for the standard basis. Denote the basis elements by a_i. Order these basis elements so that $a_i = x_{u_i}e^*$ for some $u_i \in \mathbf{F}_{q^2}^*$ with $u_i^{q+1} = 1$ for $i \leqslant q+1$ and $a_i = qe^*x_{v_i,v_i^{-q}}e^*$ for some $v_i \in \mathbf{F}_{q^2}^*$ for $i \geqslant q+2$. We have $c^*_{x,x^{-q}}c^*_{z,z^{-q}} = \sum_{k \in I}\mu_k a_k$ where $\mu_k = q\sum_{y \in D_1 \cap {}^{x_k}D_2^{-1}} c^*_{x,x^{-q}}(y)c^*_{z,z^{-q}}(y^{-1}x_k)$. In this sum $D_1 = U^{(1)}x_{x,x^{-q}}U^{(1)}$ and $D_2^{-1} = U^{(1)}x^{-1}_{z,z^{-q}}U^{(1)}$. Note that

$$D_1 = \left\{\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -x \\ x^{-q} & 0 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \;\middle|\; r, s \in \mathbf{F}_q\right\} = \left\{\begin{pmatrix} rx^{-q} & -x + rsx^{-q} \\ x^{-q} & sx^{-q} \end{pmatrix} \;\middle|\; r, s \in \mathbf{F}_q\right\}$$

and

$$D_2^{-1} = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & z^q \\ -z^{-1} & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbf{F}_q\right\} = \left\{\begin{pmatrix} -az^{-1} & z^q - abz^{-1} \\ -z^{-1} & -bz^{-1} \end{pmatrix} \;\middle|\; a, b \in \mathbf{F}_q\right\}.$$

First consider the case when $k \leqslant q+1$. Then $x_k = (w, w)$ for some $w \in \mathbf{F}_{q^2}$ with $w^{q+1} = 1$. Then

$$x_k D_2^{-1} = \left\{ \begin{pmatrix} -awz^{-1} & wz^q - abwz^{-1} \\ -wz^{-1} & -bwz^{-1} \end{pmatrix} \,\middle|\, a, b \in \mathbf{F}_q \right\}.$$

In order to determine the set $D_1 \cap\ x_k D_2^{-1}$ we need to find for which $a, b \in \mathbf{F}_q$ we have

$$\begin{pmatrix} rx^{-q} & -x + rsx^{-q} \\ x^{-q} & sx^{-q} \end{pmatrix} = \begin{pmatrix} -awz^{-1} & wz^q - abwz^{-1} \\ -wz^{-1} & -bwz^{-1} \end{pmatrix}.$$

A comparison of the 2, 1 entries shows that this intersection is empty unless $w = -yx^{-q}$. So assume $w = -zx^{-q}$. Then we have

$$\begin{pmatrix} rx^{-q} & -x + rsx^{-q} \\ x^{-q} & sx^{-q} \end{pmatrix} = \begin{pmatrix} ax^{-q} & -z^{q+1}x^{-q} + abx^{-q} \\ x^{-q} & bx^{-q} \end{pmatrix}.$$

Thus $a = r$ and $b = s$ and the intersection is empty unless $w = -zx^{-q}$ and $x^{q+1} = z^{q+1}$. (Note that given these two conditions $(-zx^{-q})^{q+1} = z^{q+1}x^{-q^2-q} = -x^{q+1}x^{-1-q} = 1$. Thus $w^{q+1} = 1$ as required.) Thus assume $x_k = (-zx^{-q}, -zx^{-q})$ and $x^{q+1} = z^{q+1}$. Let $y \in D_1 \cap x_k D_2^{-1}$. Then $y = \left(\begin{smallmatrix} rx^{-q} & -x+rsx^{-q} \\ x^{-q} & sx^{-q} \end{smallmatrix}\right)$ and

$$\begin{aligned} y^{-1}x_k &= x^{q-1} \begin{pmatrix} sx^{-q} & x - rsx^{-q} \\ -x^{-q} & rx^{-q} \end{pmatrix} \begin{pmatrix} -zx^{-q} & 0 \\ 0 & -zx^{-q} \end{pmatrix} \\ &= \begin{pmatrix} sx^{-1} & x^q - rsx^{-1} \\ -x^{-1} & rx^{-1} \end{pmatrix} \begin{pmatrix} -zx^{-q} & 0 \\ 0 & -zx^{-q} \end{pmatrix} \\ &= \begin{pmatrix} -szx^{-q-1} & -z + rsx^{-q-1} \\ zx^{-q-1} & -rzx^{-q-1} \end{pmatrix} = \begin{pmatrix} -sz^{-q} & -z + rsy^{-q-1} \\ z^{-q} & -rz^{-q} \end{pmatrix}. \end{aligned}$$

Thus $c^*_{x,x^{-q}}(y) = q(\frac{1}{q^2}\chi(r+s))$ and $c^*_{z,z^{-q}}(y^{-1}x_k) = q(\frac{1}{q^2}\chi(-r-s))$. Thus $\mu_k = q\sum_{y\in D_1}\frac{1}{q^2}\chi(r+s-r-s) = q$ when $x_k = (-zx^{-q}, -zx^{-q})$ and $x^{q+1} = z^{q+1}$. Otherwise $\mu_k = 0$ for $k \leqslant q+1$.

Now consider the case when $k \geqslant q+2$. Then $x_k = \left(\begin{smallmatrix} 0 & -w \\ w^{-q} & 0 \end{smallmatrix}\right)$ for some nonzero $w \in \mathbf{F}_{q^2}$. Then

$$x_k D_2^{-1} = \left\{ \begin{pmatrix} wz^{-1} & bwz^{-1} \\ -aw^{-q}z^{-1} & z^q w^{-q} - abw^{-q}z^{-1} \end{pmatrix} \,\middle|\, a, b \in \mathbf{F}_q \right\}.$$

In order to determine the set $D_1 \cap\ x_k D_2^{-1}$ we need to find for which $a, b \in \mathbf{F}_q$ we have

$$\begin{pmatrix} rx^{-q} & -x + rsx^{-q} \\ x^{-q} & sx^{-q} \end{pmatrix} = \begin{pmatrix} wz^{-1} & bwz^{-1} \\ -aw^{-q}z^{-1} & z^q w^{-q} - abw^{-q}z^{-1} \end{pmatrix}.$$

A comparison of the 1, 1 and 2, 1 entries shows that this intersection is empty unless $r = wx^q z^{-1}$ and $a = -w^q x^{-q} z$. So assume these two conditions. Then we have

$$\begin{pmatrix} wz^{-1} & -x + wz^{-1}s \\ x^{-q} & sx^{-q} \end{pmatrix} = \begin{pmatrix} wz^{-1} & bwz^{-1} \\ x^{-q} & z^q w^{-q} + bx^{-q} \end{pmatrix}.$$

Setting the 1, 2 entries of both matrices equal to each other and the 2, 2 entries of both matrices equal to each other we have $s = x^q z^q w^{-q} + b$ and $s = b + w^{-1}xz$. Thus $x^q z^q w^{-q} + b = b + w^{-1}xz$. Thus b can be chosen to be any element in $\mathbf{F}_q$. Then s is determined by b and the intersection is empty unless $w^{1-q} = (xz)^{1-q}$. Thus, for the case $x_k = \left(\begin{smallmatrix} 0 & -w \\ w^{-q} & 0 \end{smallmatrix}\right)$, we have the set $D_1 \cap x_k D_2^{-1}$ has q

elements when $w^{1-q}=(xz)^{1-q}$ otherwise this intersection is empty. Assume $w^{1-q}=(xz)^{1-q}$. Let y be any element in this intersection. Since

$$y=\begin{pmatrix} rx^{-q} & -x+rsx^{-q} \\ x^{-q} & sx^{-q} \end{pmatrix}$$

we have $c^*_{x,x^{-q}}(y)=\frac{1}{q}\chi(-r-s)=\frac{1}{q}\chi(-wx^qz^{-1}-s)$. On the other hand,

$$\begin{aligned} y^{-1}x_k &= x^{q-1}\begin{pmatrix} sx^{-q} & -rsx^{-q}+x \\ -x^{-q} & rx^{-q} \end{pmatrix}\begin{pmatrix} 0 & -w \\ w^{-q} & 0 \end{pmatrix} \\ &= x^{q-1}\begin{pmatrix} -rsx^{-q}w^{-q}+xw^{-q} & -swx^{-q} \\ rw^{-q}x^{-q} & wx^{-q} \end{pmatrix}=\begin{pmatrix} -rsx^{-1}w^{-q}+w^{-q}x^q & -sx^{-1}w \\ rw^{-q}x^{-1} & x^{-1}w \end{pmatrix}. \end{aligned}$$

To determine $c^*_{y,y^{-q}}(y^{-1}x_k)$ we need to determine $c,d\in\mathbf{F}_q$ such that

$$[c]\begin{pmatrix} 0 & -z \\ z^{-q} & 0 \end{pmatrix}[d]=y^{-1}x_k.$$

Thus $cz^{-q}=-sz^{-q}+w^{-q}x^q$ and $dz^{-q}=x^{-1}w$. That is, $c=-s+w^{-q}x^qz^q=-s+w^{-1}xz$ and $d=z^qx^{-1}w$. Thus $c^*_{y,y^{-q}}(y^{-1}x_k)=\frac{1}{q}\chi(s-w^{-1}xz-wx^{-1}z^q)$. Thus $\mu_k=q\sum_{s\in\mathbf{F}_q}\frac{1}{q^2}\chi(-wx^qz^{-1}-s+s-w^{-1}xz-wx^{-1}z^q)=\chi(-wx^qz^{-1}-w^{-1}xz-wx^{-1}z^q)$ when $w^{q-1}=(xz)^{q-1}$. Otherwise $\mu_k=0$ for $k\geqslant q+2$.

Combining these two cases we have that

$$c^*_{x,x^{-q}}c^*_{z,z^{-q}}=qc^*_{-zx^{-q}}\delta_{x^{q+1},z^{q+1}}+\sum_{w\in\mathbf{F}^*_{q^2},\,w^{q-1}=(xz)^{q-1}}\chi\big(-wx^qz^{-1}-w^{-1}xz-wx^{-1}z^q\big)c^*_{w,w^{-q}}.$$

Make the change of variable $t=wx^qz^q$. Note the condition $w^{q-1}=(xz)^{q-1}$ implies $t^{q-1}=w^{q-1}x^{-q+1}z^{-q+1}=1$. So $t\in\mathbf{F}^*_q$. Thus

$$c^*_{x,x^{-q}}c^*_{z,z^{-q}}=-qc^*_{zx^{-q}}\delta_{x^{q+1},z^{q+1}}+\sum_{t\in\mathbf{F}^*_q}\chi_1\big(-tz^{-q-1}-t^{-1}x^{q+1}z^{q+1}-tx^{-q-1}\big)c^*_{tx^{-q}z^{-q},t^{-1}xz}. \qquad \square$$

7. The maps $f_{T_i^*}: H^*\to \mathrm{C}T_i^*$

In this section we will provide the image of $f^*_{T_i}$ on each standard basis element of H^*.

Proposition 7.1.

(i) $f^*_{T_i}(c^*_u)=(u,u)$ *for both* $i=0$ *and* $i=1$.

(ii)
$$f^*_{T_0}\big(c^*_{u,u^{-q}}\big)=\sum_{a\in\mathbf{F}^*_{q^2},\,a^{-q+1}=u^{-q+1}}\chi\big(-u^q\big(a+a^{-q}\big)\big)\big(a,a^{-q}\big).$$

(iii)
$$\begin{aligned} f^*_{T_1}\big(c^*_{u,u^{-q}}\big) &= \sum_{a,b\in\mathbf{F}^*_{q^2},\,u^{-q+1}=ab,\,a^{q+1}=b^{q+1}=1}\chi\big(-u^q(a+b)\big)(a,b) \\ &= \sum_{a\in\mathbf{F}^*_{q^2},\,a^{q+1}=1}\chi\big(-\big(au^q+a^{-1}u\big)\big)\big(a,a^{-1}u^{-q+1}\big). \end{aligned}$$

Proof. First we need to prove the following lemma.

Lemma 7.2. *Let $(t, t^{-q}) \in T_0^*$ and let $(t_1, t_2) \in T_1^*$. Let $[r] = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) \in U^{(1)}$ and let $n = (u, u^{-q})w_0$. Then $[r]n$ and (t, t^{-q}) have the same characteristic equation if and only if $ru^{-q} = t + t^{-q}$ and $u^{-q+1} = t^{-q+1}$. Also $[r]n$ and (t_1, t_2) have the same characteristic equation if and only if $ru^{-q} = t_1 + t_2$ and $u^{-q+1} = t_1 t_2$.*

Proof. This lemma is clear since

$$\det\bigl(xI - [r]n\bigr) = \det\left[\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -u \\ u^{-q} & 0 \end{pmatrix}\right]$$
$$= \det\begin{pmatrix} x - ru^{-q} & u \\ -u^{-q} & x \end{pmatrix} = x^2 - ru^{-q}x + u^{-q+1}$$

and $\det(xI - (t, t^{-q})) = x^2 - (t + t^{-q})x + t^{-q+1}$ and $\det(xI - (t_1, t_2)) = x^2 - (t_1 + t_2) + t_1 t_2$. $\square$

We will now prove part (i) of the proposition. In this proof $U(2, q)$ will be denoted by G and T will denote either maximal torus T_0^* of T_1^*. We have

$$f_T^*\bigl(c_v^*\bigr)(t) = \frac{[U^{(1)} \colon {}^{x_v}U^{(1)} \cap U^{(1)}]}{\langle Q_T^G, \Gamma\rangle |U^{(1)}||C_G(t)|} \sum_{\substack{g \in G,\ [r] \in U^{(1)} \\ (g[r]x_v g^{-1})_s = t}} \psi_1\bigl([r]^{-1}\bigr) Q_T^{C_G(t)}\bigl(\bigl(g[r]x_v g^{-1}\bigr)_u\bigr)$$
$$= \frac{1}{\pm q|C_G(t)|} \sum_{\substack{g \in G,\ [r] \in U^{(1)} \\ (g[r]x_v g^{-1})_s = t}} \psi_1\bigl([r]^{-1}\bigr) Q_T^{C_G(t)}\bigl(\bigl(g[r]x_v g^{-1}\bigr)_u\bigr).$$

If $t = (a, b)$ with $a \neq b$ then $(g[r]x_v g^{-1})_s = (gx_v g^{-1})_s = (v, v) \neq t$. So if $t = (a, b)$ with $a \neq b$ then $f_T^*(c_v^*)(t) = 0$.

If $t = (a, a)$ then $(g[r]x_v g^{-1})_s = (v, v)$ which equals t if and only if $a = v$. So assume $a = v$. Then

$$f_T^*\bigl(c_v^*\bigr)(t) = \frac{1}{\pm q|G|} \sum_{g \in G,\ [r] \in U^{(1)}} \psi_1\bigl([r]^{-1}\bigr) Q_T^G\bigl([r]\bigr)$$
$$= \frac{1}{\pm q} \sum_{[r] \in U^{(1)}} \psi_1\bigl([r]^{-1}\bigr) Q_T^G\bigl([r]\bigr)$$
$$= \pm q^{-1}(q + 1 - 1) = \pm 1.$$

(In the second to last equality we used the fact that $Q_T^G(I) = q + 1$ and $Q_T^G([r]) = 1$ for $[r] \neq I$, [7, Theorem 9.16].) This proves part (i) of the proposition.

We will now prove parts (ii) and (iii) of the proposition. As in the proof of part (i), $U(2, q)$ will be denoted by G and T denotes either T_0^* or T_1^*.

First suppose $t = (t_1, t_1)$ (with $t_1^{-q} = t_1$). Then

$$f_T^*\bigl(c_{v,v^{-q}}^*\bigr)(t_1, t_1) = \frac{q}{\pm q|G|} \sum_{\substack{g \in G,\ [r] \in U^{(1)} \\ (g[r]x_{v,v^{-q}} g^{-1})_s = t}} \psi_1\bigl([r]^{-1}\bigr) Q_T^G\bigl(\bigl(g[r]x_{v,v^{-q}} g^{-1}\bigr)_u\bigr).$$

But, by Lemma 7.2 the only $[r]$ such that $[r]x_{v,v^{-q}}$ and t have the same characteristic equation is

$$[r] = \begin{pmatrix} 1 & v^q(t_1+t_1) \\ 0 & 1 \end{pmatrix}.$$

Thus

$$f_T^*\big(c^*_{v,v^{-q}}\big)(t_1,t_1) = \frac{1}{\pm|G|} \sum_{\substack{g\in G \\ (g[2t_1v^q]x_{v,v^{-q}}g^{-1})_s = t}} \chi\big(-2t_1v^q\big) Q_T^G\big(\big(g\big[2t_1v^q\big]x_{v,v^{-q}}g^{-1}\big)_u\big).$$

The number of $g \in G$ such that $(g[2t_1v^q]x_{v,v^{-q}}g^{-1})_s = t$ is equal to $|C_G(t)|$ $(= |G|)$. Also note that the unipotent part of $[2t_1v^q]x_{v,v^{-q}} \neq I$. Thus $Q_T^G((g[2t_1v^q]x_{v,v^{-q}}g^{-1})_u) = 1$. Thus

$$f_T^*\big(c^*_{v,v^{-q}}\big)(t_1,t_1) = \pm\chi\big(-2t_1v^q\big).$$

Now suppose $t = (t_1, t_2)$ with $t_1 \neq t_2$. (If $T = T_0^*$ then $t_2 = t_1^{-q}$.) Then

$$\begin{aligned} f_T^*\big(c^*_{v,v^{-q}}\big)(t_1,t_2) &= \frac{q}{\pm q|T|} \sum_{\substack{g\in G,\ [r]\in U^{(1)} \\ (g[r]x_{v,v^{-q}}g^{-1})_s = t}} \psi_1\big([r]^{-1}\big) Q_T^T\big(\big(g[r]x_{v,v^{-q}}g^{-1}\big)_u\big) \\ &= \frac{1}{\pm|T|} \sum_{\substack{g\in G,[r]\in U^{(1)} \\ (g[r]x_{v,v^{-q}}g^{-1})_s = t}} \psi_1\big([r]^{-1}\big). \end{aligned}$$

As above the only $[r]$ such that $[r]x_{v,v^{-q}}$ and t have the same characteristic equation is

$$[r] = \big[v^q(t_1+t_2)\big].$$

Also the number of $g \in G$ such that $(g[(t_1+t_2)v^q]x_{v,v^{-q}}g^{-1})_s = t$ is equal to $|C_G(t)|$ $(= |T|)$. Thus $f_T^*(c^*_{v,v^{-q}})(t_1,t_2) = \pm\chi(-(t_1+t_2)v^q)$ when $t_1 \neq t_2$. Combining these two cases proves parts (ii) and (iii) of the proposition. □

Proposition 6.1 provided the structure constants for all the basis elements of H^* and Proposition 7.1 provides the images of the homomorphisms f_{T^*} on these basis elements of H^*. Without using the fact that $f_{T_1}^*$ is a homomorphism, but instead using Chang's Lemma [2, Lemma 1.2], it is now straightforward to verify that $f_{T_1}^*(c^*_{1,1}c^*_{1,1}) = f_{T_1}^*(c^*_{1,1})f_{T_1}^*(c^*_{1,1})$. It would be interesting to explore what other identities could be exhibited using the fact that $f_{T_i}^*(c^*_{u,u^{-q}}c^*_{v,v^{-q}}) = f_{T_i}^*(c^*_{u,u^{-q}})f_{T_i}^*(c^*_{v,v^{-q}})$.

8. The maps $f_{T_i}^{(m)} : H^{(m)} \to CT_i^{(m)}$

The proof of the following proposition is analogous to the proof of the proposition in the previous section and is thus omitted. This proposition provides the image of the maps $f_{T_i}^{(m)}$ for all basis elements of $H^{(m)}$.

Proposition 8.1.

(i) $f_{T_i}^{(m)}(c_u^{(m)}) = (u,u)$ *for both* $i = 0$ *and* $i = 1$.

(ii)
$$f_{T_0}^{(m)}\big(c_{u,v}^{(m)}\big) = \sum_{a,b\in \mathbf{F}_{q^m}^*,\, ab=uv} \chi_m\big(-v^{-1}(a+b)\big)(a,b)$$
$$= \sum_{a\in \mathbf{F}_{q^m}^*} \chi_m\big(-\big(av^{-1}+a^{-1}u\big)\big)\big(a,a^{-1}uv\big).$$

(iii)
$$f_{T_1}^{(m)}\big(c_{u,v}^{(m)}\big) = \sum_{a\in \mathbf{F}_{q^{2m}}^*,\, a^{q^m+1}=uv} \chi_m\big(-v^{-1}\big(a+a^{q^m}\big)\big)\big(a,a^{q^m}\big).$$

9. The image of $\boldsymbol{\Delta^{(2)} : H^{(2)} \to H^{(1)}}$

Note $N_{T_0}^{(2)} : T_0^{(2)} \to T_0^{(1)}$ is given by $N_{T_0}^{(2)}(t_1,t_2) = (t_1^{1+q}, t_2^{1+q})$. Also $N_{T_1}^{(2)} : T_0^{(2)} \to T_1^{(1)}$ is given by $N_{T_1}^{(2)}(t_1,t_2) = (t_1 t_2^q, t_1^q t_2)$. In this section we will determine the image of some of the standard basis elements of $H^{(2)}$ under the norm map $\Delta^{(2)}$.

Let $P_m(x,y)$ be the polynomial:

$$P_m(x,y) = \sum_{j=0}^{[m/2]} (-1)^{m-j-1} \frac{m}{m-j} \binom{m-j}{j} x^{m-2j} y^j.$$

In [5] it was shown that

$$\Delta^{(m)}\big(c_{1,1}^{(m)}\big) = P_m\big(c_{1,1}^{(1)}, qc_{-1}^{(1)}\big). \tag{1}$$

Note that

$$P_2(x,y) = \sum_{j=0}^{1} (-1)^{1-j} \frac{2}{2-j} \binom{2-j}{j} x^{2-2j} y^j = -x^2 + 2y.$$

Thus identity (1) when $m=2$ becomes:

$$\Delta^{(2)}\big(c_{1,1}^{(2)}\big) = -\big(c_{1,1}^{(1)}\big)^2 + 2qc_{-1}^{(1)}. \tag{2}$$

The following three lemmas are extensions of identity (2). Note that this first lemma only applies for $u \in \mathbf{F}_q^*$ (not all of $\mathbf{F}_{q^2}^*$).

Lemma 9.1. *$\Delta^{(2)}(c_{u,1}^{(2)}) = P_2(c_{u,1}^{(1)}, qc_{-u}^{(1)})$ for all $u \in \mathbf{F}_q^*$.*

Proof. Note that by Proposition 8.1

$$N_{T_0}^{(2)} f_{T_0}^{(2)}\big(c_{u,1}^{(2)}\big) = N_{T_0}^{(2)}\Bigg(\sum_{\substack{x,y\in \mathbf{F}_{q^2}^* \\ xy=u}} \chi_2\big(-(x+y)\big)(x,y)\Bigg)$$
$$= \sum_{x\in \mathbf{F}_{q^2}^*} \chi_2\big(-\big(x+ux^{-1}\big)\big)\big(x^{q+1}, u^{q+1}x^{-q-1}\big).$$

Thus using $u \in \mathbf{F}_q^*$ we have

$$N_{T_0}^{(2)} f_{T_0}^{(2)}\bigl(c_{u,1}^{(2)}\bigr) = \sum_{x\in\mathbf{F}_{q^2}^*} \chi\bigl(-\bigl(x+x^q+ux^{-1}+ux^{-q}\bigr)\bigr)\bigl(x^{q+1}, u^2x^{-q-1}\bigr)$$
$$= \sum_{w\in\mathbf{F}_q^*} \sum_{\substack{x\in\mathbf{F}_{q^2}^* \\ x^{q+1}=w}} \chi\bigl(-\bigl(x+x^q+ux^q w^{-1}+uxw^{-1}\bigr)\bigr)\bigl(w, u^2w^{-1}\bigr).$$

Fix a $w\in\mathbf{F}_q^*$. The coefficient of (w, u^2w^{-1}) in the above equation is

$$\sum_{\substack{x\in\mathbf{F}_{q^2}^* \\ x^{q+1}=w}} \chi\bigl(-\bigl(x+x^q+ux^q w^{-1}+uxw^{-1}\bigr)\bigr)$$
$$= -\sum_{\substack{x\in\mathbf{F}_{q^2}^* \\ x^{q+1}=w}} \chi\bigl(\bigl(1+uw^{-1}\bigr)x+\bigl(1+uw^{-1}\bigr)x^q\bigr)$$
$$\overset{(\text{since } u,w\in\mathbf{F}_q)}{=} -\sum_{\substack{x\in\mathbf{F}_{q^2}^* \\ x^{q+1}=w}} \chi\bigl(\bigl(1+uw^{-1}\bigr)x+\bigl(1+uw^{-1}\bigr)^q x^q\bigr)$$
$$\overset{(\text{by [2, Lemma 1.2]})}{=} \sum_{b\in\mathbf{F}_q^*} \chi\bigl(b+\bigl(1+uw^{-1}\bigr)\bigl(1+uw^{-1}\bigr)wb^{-1}\bigr) - q\delta_{uw^{-1},-1}$$
$$= \sum_{b\in\mathbf{F}_q^*} \chi\bigl(b+\bigl(1+uw^{-1}+uw^{-1}+u^2w^{-2}\bigr)wb^{-1}\bigr) - q\delta_{u,-w}$$
$$= \sum_{b\in\mathbf{F}_q^*} \chi\bigl(b+wb^{-1}+2ub^{-1}+u^2w^{-1}b^{-1}\bigr) - q\delta_{u,-w}.$$

Thus we have

$$N_{T_0}^{(2)} f_{T_0}^{(2)}\bigl(c_{u,1}^{(2)}\bigr) = q(u,u) + \sum_{w,b\in\mathbf{F}_q^*} \chi\bigl(b+wb^{-1}+2ub^{-1}+u^2w^{-1}b^{-1}\bigr)\bigl(w, u^2w^{-1}\bigr).$$

On the other hand, note that using first Lemma 4.1 and then Proposition 8.1 we have

$$f_{T_0}^{(1)}\bigl(-\bigl(c_{u,1}^{(1)}\bigr)^2+2qc_{-u}^{(1)}\bigr) = f_{T_0}^{(1)}\Bigl(qc_{-u}^{(1)} - \sum_{t\in\mathbf{F}_q^*}\chi\bigl(-2u^{-1}t-u^2t^{-1}\bigr)c_{t,u^2t^{-1}}^{(1)} + 2qc_u^{(1)}\Bigr)$$
$$= q(u,u) - \sum_{t\in\mathbf{F}_q^*}\chi\bigl(2u^{-1}t+u^2t^{-1}\bigr) \sum_{\substack{w,v\in\mathbf{F}_q^* \\ wv=u^2}} \chi\bigl(-u^{-2}t(w+v)\bigr)(w,v)$$
$$= q(u,u) - \sum_{t\in\mathbf{F}_q^*}\chi\bigl(2u^{-1}t+u^2t^{-1}\bigr) \sum_{w\in\mathbf{F}_q^*} \chi\bigl(-u^{-2}t\bigl(w+u^2w^{-1}\bigr)\bigr)\bigl(w, u^2w^{-1}\bigr)$$
$$= q(u,u) + \sum_{t,w\in\mathbf{F}_q^*}\chi\bigl(2u^{-1}t+u^2t^{-1}+u^{-2}tw+tw^{-1}\bigr)\bigl(w, u^2w^{-1}\bigr).$$

Making the change of variable $b=t^{-1}u^2$ we get

$$f_{T_0}^{(1)}\bigl(-\bigl(c_{u,1}^{(1)}\bigr)^2+2qc_{-u}^{(1)}\bigr) = q(u,u) + \sum_{b,w\in\mathbf{F}_q^*}\chi\bigl(2ub^{-1}+b+b^{-1}w+u^2w^{-1}b^{-1}\bigr)\bigl(w, u^2w^{-1}\bigr).$$

Thus $N_{T_0}^{(2)} f_{T_0}^{(2)}(c_{u,1}^{(2)}) = f_{T_0}^{(1)}(-(c_{u,1}^{(1)})^2+2qc_{-u}^{(1)})$.

By an analogous proof it also follows that $N^{(1)}_{T_1} f^{(2)}_{T_1}(c^{(2)}_{u,1}) = f^{(1)}_{T_1}(-(c^{(1)}_{u,1})^2 + 2qc^{(1)}_{u})$. Thus

$$N_T f^{(2)}_T\big(c^{(2)}_{u,1}\big) = f^{(1)}_T\big(-\big(c^{(1)}_{u,1}\big)^2 + 2qc^{(1)}_u\big)$$

for all maximal tori T and all $u \in \mathbf{F}^*_q$. That is,

$$N_T f^{(2)}_T\big(c^{(2)}_{u,1}\big) = f^{(1)}_T\big(P_2\big(c^{(1)}_{u,1}, qc^{(1)}_{-u}\big)\big)$$

for all maximal tori T and all $u \in \mathbf{F}^*_q$. Thus $\Delta^{(2)}(c^{(2)}_{u,1}) = P_2(c^{(1)}_{u,1}, qc^{(1)}_{-u})$. □

Lemma 9.2. $\Delta^{(2)}(c^{(2)}_u) = c^{(1)}_{u^{q+1}}$ *for all* $u \in \mathbf{F}^*_{q^2}$.

Proof. This follows from the fact that $N^{(2)}_T f^{(2)}_T(c^{(2)}_u) = (u^{q+1}, u^{q+1}) = f^{(1)}_T(c^{(1)}_{u^{q+1}})$ for all maximal tori T and all $u \in \mathbf{F}^*_{q^2}$. □

Lemma 9.3. $\Delta^{(2)}(c^{(2)}_{u,v}) = c^{(1)}_{v^{q+1}} \Delta^{(2)}(c^{(2)}_{v^{-1}u,1})$ *for all* $u, v \in \mathbf{F}^*_{q^2}$.

Proof. Note that the basis element

$$\begin{aligned} c^{(2)}_{u,v} &= q^2 e^{(2)} \begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix} e^{(2)} \\ &= q^2 e^{(2)} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & -uv^{-1} \\ 1 & 0 \end{pmatrix} e^{(2)} \\ &= c^{(2)}_v c^{(2)}_{v^{-1}u,1}. \end{aligned}$$

Thus, using that $\Delta^{(2)}$ is a homomorphism, $\Delta^{(2)}(c^{(2)}_{u,v}) = \Delta^{(2)}(c^{(2)}_v c^{(2)}_{v^{-1}u,1}) = \Delta^{(2)}(c^{(2)}_v)\Delta^{(2)}(c^{(2)}_{v^{-1}u,1}) =$ (by Lemma 9.2) $c^{(1)}_{v^{q+1}} \Delta^{(2)}(c^{(2)}_{v^{-1}u,1})$. □

Note that (unlike Lemma 9.1) the following lemma holds for all $v \in \mathbf{F}^*_{q^2}$.

Proposition 9.4. $\Delta^{(2)}(c^{(2)}_{v,v^{-q}}) = c^{(1)}_{v^{-q-1}} P_2(c^{(1)}_{v^{q+1},1}, qc^{(1)}_{-v^{q+1}})$ *for all* $v \in \mathbf{F}^*_{q^2}$.

Proof. By Lemma 9.3 $\Delta^{(2)}(c^{(2)}_{v,v^{-q}}) = c^{(1)}_{v^{-q-1}} \Delta^{(2)}(c^{(2)}_{v^{q+1},1})$. But v^{q+1} is an element of $\mathbf{F}^*_q$ so we can apply Lemma 9.1 to get $\Delta^{(2)}(c^{(2)}_{v,v^{-q}}) = c^{(1)}_{v^{-q-1}} P_2(c^{(1)}_{v^{q+1},1}, qc^{(1)}_{-v^{q+1}})$. □

10. The image of $\Delta^* : H^{(2)} \to H^*$

Let $\mathcal{B}$ denote the standard basis of $H^{(2)}$. Thus

$$\mathcal{B} = \big\{c^{(2)}_u, c^{(2)}_{u,v} \bigm| u, v \in \mathbf{F}^*_{q^2}\big\} = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} e^{(2)},\ q^2 e^{(2)} \begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix} e^{(2)} \biggm| u, v \in \mathbf{F}^*_{q^2} \right\}.$$

Let

$$\mathcal{B}^{F^*} = \big\{c^{(2)}_u, c^{(2)}_{v,v^{-q}} \bigm| u, v \in \mathbf{F}^*_{q^2}, u^{q+1} = 1\big\}.$$

That is, $\mathcal{B}^{F^*}$ is the subset of $\mathcal{B}$ of elements which are constructed using the matrices $\left(\begin{smallmatrix} 0 & -u \\ v & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right)$ which are also used in the construction of the basis of H^*. Note that the previous section provides the image under $\Delta^{(2)}$ of all the elements in $\mathcal{B}^{F^*}$. In this section we will determine the image under $\Delta^* : H^{(2)} \to H^*$ of all elements in $\mathcal{B}^{F^*}$. Furthermore, it will be shown that the norm maps $\Delta^{(2)}$ and Δ^* are equal on a certain subset of $\mathcal{B}^{F^*}$.

To simplify notation, in this section we will denote e^* $(= e^{(1)})$ by e. Since $H^{(1)} = \langle qe(u,v)w_0e, (u,u)e \mid u, v \in \mathbf{F}_q\rangle$ and $H^* = \langle qe(x, x^{-q})w_0e, (y,y)e \mid x, y \in \mathbf{F}_{q^2},\ y^{q+1} = 1\rangle$, the intersection $H^{(1)} \cap H^*$ is nonempty. Let $\mathcal{C}$ denote the intersection of these standard bases of $H^{(1)}$ and H^*. Then

$$\mathcal{C} = \left\{c^{(1)}_{\pm 1}, c^{(1)}_{v,v^{-1}} \;\middle|\; v \in \mathbf{F}_q^*\right\} = \left\{c^*_{\pm 1}, c^*_{v,v^{-1}} \;\middle|\; v \in \mathbf{F}_q^*\right\}.$$

Lemma 10.1. *Let $c \in \mathcal{C}$, then $f^*_{T_i}(c) = f^{(1)}_{T_i}(c)$ for $i = 0, 1$.*

Proof. If $c = c^{(1)}_{\pm 1}$ then $f^*_{T_i}(c) = (\pm 1, \pm 1) = f^{(1)}_{T_i}(c)$.

Now suppose $c = c^{(1)}_{u,u^{-1}}$ for some $u \in \mathbf{F}_q^*$. Then, by Proposition 7.1

$$\begin{aligned} f^*_{T_0}\left(c^*_{u,u^{-1}}\right) = f^*_{T_0}\left(c^*_{u,u^{-q}}\right) &= \sum_{a \in \mathbf{F}^*_{q^2}, a^{-q+1} = u^{-q+1}} \chi\left(-u^q\left(a + a^{-q}\right)\right)\left(a, a^{-q}\right) \\ &= \sum_{a \in \mathbf{F}_q^*} \chi\left(-u\left(a + a^{-1}\right)\right)\left(a, a^{-1}\right), \end{aligned}$$

since $u \in \mathbf{F}_q^*$ and thus $a^{-q+1} = u^{-q+1} = 1$ implies $u \in \mathbf{F}_q^*$. On the other hand, by Proposition 8.1

$$\begin{aligned} f^{(1)}_{T_0}\left(c^{(1)}_{u,u^{-1}}\right) &= \sum_{a,b \in \mathbf{F}_q^*, ab = 1} \chi\left(-u(a+b)\right)(a,b) \\ &= \sum_{a \in \mathbf{F}_q^*} \chi\left(-u\left(a + a^{-1}\right)\right)\left(a, a^{-1}\right). \end{aligned}$$

Thus $f^*_{T_0}(c^*_{u,u^{-q}}) = f^{(1)}_{T_0}(c^{(1)}_{u,u^{-1}})$ for all $u \in \mathbf{F}_q^*$. Similarly by Proposition 7.1

$$\begin{aligned} f^*_{T_1}\left(c^*_{u,u^{-q}}\right) &= \sum_{\substack{a,b \in \mathbf{F}^*_{q^2} \\ ab = u^{-q+1}, a^{q+1} = b^{q+1} = 1}} \chi\left(-u(a+b)\right)(a,b) \\ &= \sum_{a \in \mathbf{F}^*_{q^2}, a^{q+1} = 1} \chi\left(-u\left(a + a^{-1}\right)\right)\left(a, a^{-1}\right), \end{aligned}$$

since $u^{-q+1} = 1$. On the other hand, by Proposition 8.1

$$f^{(1)}_{T_1}\left(c^{(1)}_{u,u^{-1}}\right) = \sum_{a \in \mathbf{F}^*_{q^2}, a^{q+1} = 1} \chi\left(-u\left(a + a^{-1}\right)\right)\left(a, a^{-1}\right).$$

Thus $f^*_{T_1}(c^*_{u,u^{-q}}) = f^{(1)}_{T_1}(c^{(1)}_{u,u^{-1}})$ for all $u \in \mathbf{F}_q^*$. $\square$

Lemma 10.2. *Let $b \in \mathcal{B}^{F^*}$ then*

(i) *for $b = c_u^{(2)}$ we have $N_i^* f_{T_i}^{(2)}(c_u^{(2)}) = (u^{-q+1}, u^{-q+1}) N_i^{(2)} f_{T_i}^{(2)}(c_u^{(2)})$ and*
(ii) *for $b = c_{t,t^{-q}}^{(2)}$ we have $N_i^* f_{T_i}^{(2)}(c_{t,t^{-q}}^{(2)}) = (t^{-q+1}, t^{-q+1}) N_i^{(2)} f_{T_i}^{(2)}(c_{t,t^{-q}}^{(2)})$ for $i = 0, 1$.*

Proof. First assume $b = c_u^{(2)}$. Since $b \in \mathcal{B}^{F^*}$ we have $u^{q+1} = 1$. Thus $f_{T_i}(c_u^{(2)}) = (u, u)$ and thus $N_i^{(2)} f_{T_i}(c_u^{(2)}) = (u^{q+1}, u^{q+1}) = (1, 1)$. On the other hand $N_i^{(2)} f_{T_i}(c_u^{(2)}) = (u^{-q+1}, u^{-q+1})$. Thus $N_i^* f_{T_i}^{(2)}(c_u^{(2)}) = (u^{-q+1}, u^{-q+1}) N_i^{(2)} f_{T_i}^{(2)}(c_u^{(2)})$.

Now assume $b = c_{t,t^{-q}}^{(2)}$ for some $t \in \mathbf{F}_{q^2}^*$. We have, using Proposition 8.1,

$$\begin{aligned}
N_0^* f_{T_0}^{(2)}\big(c_{t,t^{-q}}^{(2)}\big) &= N_0^* \sum_{a,b \in \mathbf{F}_{q^2}^*, ab = t^{-q+1}} \chi_2\big(-t^q(a+b)\big)(a,b) \\
&= N_0^* \sum_{a \in \mathbf{F}_{q^2}^*} \chi_2\big(-t^q\big(a + a^{-1}t^{-q+1}\big)\big)\big(a, a^{-1}t^{-q+1}\big) \\
&= \sum_{a \in \mathbf{F}_{q^2}^*} \chi_2\big(-t^q a - ta^{-1}\big)\big(a^{1+q}t^{-q+1}, a^{-1-q}t^{-q+1}\big) \\
&= \big(t^{-q+1}, t^{-q+1}\big) \sum_{a \in \mathbf{F}_{q^2}^*} \chi_2\big(-t^q a - ta^{-1}\big)\big(a^{1+q}, a^{-1-q}\big) \\
&= \big(t^{-q+1}, t^{-q+1}\big) N_0^{(2)} \sum_{a \in \mathbf{F}_{q^2}^*} \chi_2\big(-t^q a - ta^{-1}\big)\big(a, a^{-1}t^{-q+1}\big) \\
&= \big(t^{-q+1}, t^{-q+1}\big) N_0^{(2)} f_{T_0}^{(2)}\big(c_{t,t^{-q}}^{(2)}\big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
N_1^* f_{T_1}^{(2)}\big(c_{t,t^{-q}}^{(2)}\big) &= N_1^* \sum_{a \in \mathbf{F}_{q^4}^*,\, a^{q^2+1} = t^{-q+1}} \chi_2\big(-t^q\big(a + a^{q^2}\big)\big)\big(a, a^{q^2}\big) \\
&= \sum_{a \in \mathbf{F}_{q^4}^*,\, q^{q^2+1} = t^{-q+1}} \chi_2\big(-t^q\big(a + a^{-1}t^{-q+1}\big)\big)\big(a^{1-q}, a^{q^2-q^3}\big).
\end{aligned}$$

Since $a^{q^2+1} = t^{-q+1}$ we have $a^{q^2} = a^{-1}t^{-q+1}$ and $a^{q^3} = a^{-q}t^{-1+q}$. Thus $a^{1-q} = a^{1+q^3}t^{-q+1}$ and $a^{q^2-q^3} = a^{q^2+q}t^{-q+1}$. Thus

$$\begin{aligned}
N_1^* f_{T_1}^{(2)}\big(c_{t,t^{-q}}^{(2)}\big) &= \sum_{a \in \mathbf{F}_{q^4}^*,\, q^{q^2+1} = t^{-q+1}} \chi_2\big(-t^q\big(a + a^{-1}t^{-q+1}\big)\big)\big(a^{1+q^3}t^{-q+1}, a^{q^2+q}t^{-q+1}\big) \\
&= \big(t^{-q+1}, t^{-q+1}\big) \sum_{\substack{a \in \mathbf{F}_{q^4}^* \\ q^{q^2+1} = t^{-q+1}}} \chi_2\big(-t^q\big(a + a^{-1}t^{-q+1}\big)\big)\big(a^{1+q^3}, a^{q^2+q}\big) \\
&= \big(t^{-q+1}, t^{-q+1}\big) N_1^{(2)} \sum_{\substack{a \in \mathbf{F}_{q^4}^* \\ q^{q^2+1} = t^{-q+1}}} \chi_2\big(-t^q\big(a + a^{-1}t^{-q+1}\big)\big)\big(a, a^{q^2}\big) \\
&= \big(t^{-q+1}, t^{-q+1}\big) N_1^{(2)} f_{T_1}^{(2)}\big(c_{t,t^{-q}}^{(2)}\big). \qquad \square
\end{aligned}$$

By Proposition 9.4, $\Delta^{(2)}(c^{(2)}_{v,v^{-q}}) = c^{(1)}_{v^{-q-1}} P_2(c^{(1)}_{v^{q+1},1}, qc^{(1)}_{-v^{q+1}})$. Thus

$$\begin{aligned}\Delta^{(2)}\big(c^{(2)}_{v,v^{-q}}\big) &= \big(v^{-q-1}, v^{-q-1}\big)\big(-c^{(1)}_{v^{q+1},1} c^{(1)}_{v^{q+1},1} + 2qc^{(1)}_{-v^{q+1}}\big) \\ &= -c^{(1)}_{1,v^{-q-1}} c^{(1)}_{v^{q+1},1} + 2qc^{(1)}_{-1} \\ &= -\sum_{w\in \mathbf{F}_q^*} \chi\big(-v^{-q-1}w - v^{q+1}v^{q+1}w^{-1} - v^{-q-1}w\big)c^{(1)}_{v^{-q-1}w, v^{q+1}w^{-1}} + qc^{(1)}_{-1}.\end{aligned}$$

But $v^{-q-1}w$ is an element of $\mathbf{F}_q$ thus $\Delta^{(2)}(c^{(2)}_{v,v^{-q}}) \in H^{(1)} \cap H^*$.

Lemma 10.3. $f^*_{T_i}\Delta^{(2)}(c^{(2)}_{t,t^{-q}}) = f^{(1)}_{T_i}\Delta^{(2)}(c^{(2)}_{t,t^{-q}})$.

Proof. This follows immediately from the comments preceding this lemma and Lemma 10.1. □

Theorem 10.4. *Let $b \in \mathcal{B}^{F^*}$ then*

(i) *for $b = c^{(2)}_u$, $\Delta^*(c^{(2)}_u) = (u^{-q+1}, u^{-q+1})\Delta^{(2)}(c^{(2)}_u)$ and*
(ii) *for $b = c^{(2)}_{t,t^{-q}}$, $\Delta^*(c^{(2)}_{t,t^{-q}}) = (t^{-q+1}, t^{-q+1})\Delta^{(2)}(c^{(2)}_{t,t^{-q}})$.*

Proof. Let $b \in \mathcal{B}^{F^*}$. Then $b = c^{(2)}_x$ for some $x \in \mathbf{F}^*_{q^2}$ such that $x^{q+1} = 1$ or $b = c^{(2)}_{x,x^{-q}}$ for some $x \in \mathbf{F}^*_{q^2}$. Thus

$$\begin{aligned}f^*_{T_i}\big(\big(x^{-q+1}, x^{-q+1}\big)\Delta^{(2)}(b)\big) &= \big(x^{-q+1}, x^{-q+1}\big) f^*_{T_i}\Delta^{(2)}(b) \\ &\overset{\text{(by Lemma 10.3)}}{=} \big(x^{-q+1}, x^{-q+1}\big) f^{(1)}_{T_i}\Delta^{(2)}(b) \\ &\overset{\text{(by Theorem 1.2)}}{=} \big(x^{-q+1}, x^{-q+1}\big) N^{(2)}_i f^{(2)}_{T_i}(b) \\ &\overset{\text{(by Lemma 10.2)}}{=} N^*_i f^{(2)}_{T_i}(b).\end{aligned}$$

Since this is true for all maximal tori T_i, we have $\Delta^*(b) = (x^{-q+1}, x^{-q+1})\Delta^{(2)}(b)$ by uniqueness of Δ^* in Theorem 1.2. □

Corollary 10.5. *Let $t \in \mathbf{F}^*_q$ then $\Delta^*(c^{(2)}_{t,t^{-1}}) = \Delta^{(2)}(c^{(2)}_{t,t^{-1}})$ and $\Delta^*(c^{(2)}_{\pm 1}) = \Delta^{(2)}(c^{(2)}_{\pm 1})$.*

Proof. This follows immediately from Theorem 10.4 since $t^{-q+1} = 1$ when $t \in \mathbf{F}^*_q$ and $\pm 1^{-q+1} = 1$. □

Lemma 10.6. *The structure constants of elements in $\mathcal{C}$ are the same whether the elements are viewed as in H^* or as in $H^{(1)}$.*

Proof. A comparison of Propositions 4.2 and 6.1 immediately shows this lemma is true when one (or both) of the two elements multiplied together is central. Thus we only need to compare $c^{(1)}_{t,t^{-1}} c^{(1)}_{u,u^{-1}}$ and $c^*_{t,t^{-1}} c^*_{u,u^{-1}}$ for $t, u \in \mathbf{F}^*_q$. According to Proposition 4.2

$$c^{(1)}_{t,t^{-1}} c^{(1)}_{u,u^{-1}} = qc^{(1)}_{-tu^{-1}}\delta_{tu^{-1},t^{-1}u} + \sum_{w\in \mathbf{F}_q^*} \chi\big(-u^{-2}w - t^2u^2w^{-1} - t^{-2}w\big)c^{(1)}_{t^{-1}u^{-1}w, tuw}.$$

Whereas, according to Proposition 6.1

$$c^*_{t,t^{-1}} c^*_{u,u^{-1}} = qc^*_{-ut^{-1}} \delta_{t^2,u^2} + \sum_{w \in \mathbf{F}_q^*} \chi\left(-u^{-2}w - t^2u^2w^{-1} - t^{-2}w\right) c^*_{t^{-1}u^{-1}w,tuw}.$$

This lemma then follows from the fact that $\delta_{tu^{-1},t^{-1}u} = \delta_{t^2,u^2}$. □

Theorem 10.7. *For all $t \in \mathbf{F}^*_{q^2}$, $\Delta^*(c^{(2)}_{t,t^{-q}}) = P_2(c^*_{t,t^{-q}}, qc^*_{t^{-q+1}})$.*

Proof.

$$\begin{aligned}
\Delta^*\left(c^{(2)}_{t,t^{-q}}\right) &\overset{\text{(by Theorem 10.4)}}{=} \left(t^{-q+1}, t^{-q+1}\right)\Delta^{(2)}\left(c^{(2)}_{t,t^{-q}}\right) \\
&\overset{\text{(by Proposition 9.4)}}{=} \left(t^{-q+1}, t^{-q+1}\right)c^{(1)}_{t^{-q-1}} P_2\left(c^{(1)}_{t^{q+1},1}, qc^{(1)}_{-t^{q+1}}\right) \\
&\overset{\text{(by definition of } P_2(x,y))}{=} \left(t^{-q+1}, t^{-q+1}\right)c^{(1)}_{t^{-q-1}}\left(-\left(c^{(1)}_{t^{q+1},1}\right)^2 + 2qc^{(1)}_{t^{q+1}}\right) \\
&\overset{\text{(by Proposition 4.2 and Lemma 10.6)}}{=} -\left(c^*_{t,t^{-q}}\right)^2 + 2qc^*_{t^{-q+1}} \\
&= P_2\left(c^*_{t,t^{-q}}, qc^*_{t^{-q+1}}\right). \qquad \square
\end{aligned}$$

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