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## Multiplicativity properties of entrywise positive maps

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### Abstract

Multiplicativity of certain maximal  $p \rightarrow q$  norms of a tensor product of linear maps on matrix algebras is proved in situations in which the condition of complete positivity (CP) is either augmented by, or replaced by, the requirement that the entries of a matrix representative of the map are non-negative (EP). In particular, for integer  $t$ , multiplicativity holds for the maximal  $2 \rightarrow 2t$  norm of a product of two maps, whenever one of the pair is EP; for the maximal  $1 \rightarrow t$  norm for pairs of CP maps when one of them is also EP; and for the maximal  $1 \rightarrow 2t$  norm for the product of an EP and a 2-positive map. Similar results are shown in the infinite-dimensional setting of convolution operators on  $L^2(\mathbb{R})$ , with the pointwise positivity of an integral kernel replacing entrywise positivity of a matrix. These results apply in particular to Gaussian bosonic channels.

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## 1. Introduction

The additivity conjecture for minimal output entropy of product channels remains a challenging open problem in quantum information theory [21]. In this paper we study a class of completely positive (CP) maps for which the related question of multiplicativity of maximal output purity [2] can be demonstrated for *integer values* of the parameter  $p$ . The multiplicativity property follows from the existence of a basis in which the map satisfies a condition we call entrywise positive (EP), so that Hölder's inequality can be applied in a useful way. Several classes of maps satisfying the EP property are presented.

## 2. Statement of results

Throughout we will denote by  $M_n$  the vector space of complex-valued  $n \times n$  matrices. The Schatten norm of  $A \in M_n$  is defined for  $p \geq 1$  by

$$\|A\|_p = (\text{Tr}|A|^p)^{1/p}. \quad (1)$$

For a linear map  $K : M_n \rightarrow M_m$  and  $p, q \geq 1$  we define the family of norms

$$\|K\|_{p \rightarrow q} = \sup \left\{ \frac{\|K(A)\|_q}{\|A\|_p} : A \neq 0 \right\}. \quad (2)$$

One can also consider such norms when  $A$  is restricted to the real vector space of self-adjoint matrices.<sup>1</sup> We will not do this here, except for the case  $p = 1$  which we denote by  $v_q(K)$ . As noted in [1,16], this is equivalent to

$$v_q(K) = \sup \left\{ \frac{\|K(A)\|_q}{\text{Tr } A} : A \geq 0, A \neq 0 \right\}. \quad (3)$$

**Definition 1.** A linear map  $\Phi : M_n \mapsto M_m$  is called *entrywise positive* (EP) if all entries of  $\Phi$  are nonnegative with respect to some pair of orthonormal bases  $\{|e_j\rangle\}$  and  $\{|f_k\rangle\}$  for  $\mathbb{C}_n$  and  $\mathbb{C}_m$ , respectively. That is,

$$\text{Tr}|f_k\rangle\langle f_\ell|\Phi(|e_i\rangle\langle e_j|) = \langle f_\ell|\Phi(|e_i\rangle\langle e_j|)|f_k\rangle \geq 0 \quad (4)$$

for all  $i, j, k, \ell$ .

Recall that the matrix representative of a linear operator using orthonormal bases  $\{|e_j\rangle\}$  and  $\{|f_k\rangle\}$  for its domain and range is  $a_{jk} = \langle f_k, Ae_j \rangle$ . Thus, for each fixed  $i, j$  the expression (4) gives the  $\ell, k$  entry of the  $m \times m$  matrix representative for

<sup>1</sup> In a recent paper [23], Watrous showed that in the case of CP maps, the norm in (2) is unchanged if the supremum is restricted to  $A > 0$ .

the operator  $\Phi(|e_i\rangle\langle e_j|)$ , i.e, the  $\ell, k$  entry of the  $i, j$  block in the  $mn \times mn$  Choi-Jamiolkowski matrix (or state representative of  $\Phi$ ). Alternatively, one can regard (4) as describing the  $m^2 \times n^2$  matrix representative of  $\Phi$  using input and output bases with the form  $E_{ij} = |e_i\rangle\langle e_j|$  and  $F_{k\ell} = |f_k\rangle\langle f_\ell|$ , respectively. The condition that the  $m^2n^2$  numbers given by (4) are non-negative is independent of whether or not they are arranged in any particular matrix form.

The condition (4) can be restated as follows. Let  $\Gamma_U$  denote the map which acts by conjugation with  $U$ , that is  $\Gamma_U(Q) = UQU^*$ . Then  $\Phi$  is EP if there are unitary matrices  $U \in M_n$  and  $V \in M_m$  such that the operator  $\Gamma_V \circ \Phi \circ \Gamma_U$  satisfies (4) in the standard bases for  $\mathbb{C}_n$  and  $\mathbb{C}_m$ .

The entrywise positivity property also arises for integral operators on function spaces, where it is expressed as pointwise positivity of the integral kernel. In this context, multiplicativity of the  $p \rightarrow q$  norm defined in (2) was proved for integral operators for all  $1 \leq p \leq q$  [3,17]. Using the additional assumption of pointwise positivity, Lieb extended this result to all  $1 \leq p, q$ . (See Theorem 3.2 in [17]. Although the proofs in [3,17] are given for the tensor product of a kernel with itself, the argument extends to different kernels.) Restricting to finite dimensions yields multiplicativity for linear maps acting on the subalgebra of diagonal matrices, where again the result holds in general for  $1 \leq p \leq q$ , and under the EP assumption for all  $1 \leq p, q$ .

Less is known about multiplicativity for maps on the full matrix algebra. In the following theorems we use the EP property to demonstrate this property in several cases. The first result applies to linear maps on matrix algebras without the additional assumption of complete positivity. It asserts multiplicativity of the maximal  $2 \rightarrow 2t$  norm for integer  $t$  whenever one of the maps is EP, and provides an upper bound on the  $p \rightarrow 2t$  norm for  $1 \leq p \leq 2$ .

**Theorem 2.** *Let  $K$  and  $L$  be linear maps on  $M_n$  and  $M_m$  respectively, and suppose that  $K$  is EP. Then for all  $1 \leq p \leq 2$ , and all integers  $t$ ,*

$$\|K \otimes L\|_{p \rightarrow 2t} \leq \|K\|_{2 \rightarrow 2t} \|L\|_{p \rightarrow 2t} \tag{5}$$

with equality when  $p = 2$ .

The next result uses the assumption of complete positivity to deduce a multiplicativity result for (3), which is the case of interest in quantum information theory.

**Theorem 3.** *Let  $\Phi$  and  $\Omega$  be CP maps on  $M_n$  and  $M_m$  respectively, and assume that  $\Phi$  is also EP. Then for all integers  $t$ ,*

$$v_t(\Phi \otimes \Omega) = v_t(\Phi) v_t(\Omega). \tag{6}$$

When  $t$  is an even integer the hypothesis of Theorem 3 can be weakened; the requirement that  $\Phi$  be CP is not necessary and  $\Omega$  need only be 2-positive.

**Theorem 4.** Let  $\Phi$  be an EP linear map and let  $\Omega$  be a 2-positive map on  $M_n$  and  $M_m$  respectively. Then for all integers  $t$ ,

$$v_{2t}(\Phi \otimes \Omega) = v_{2t}(\Phi) v_{2t}(\Omega). \quad (7)$$

It remains an open question whether the equality (6) holds for other values of  $t$ , in particular for the range  $1 \leq t \leq 2$ . It can be shown that (6) is true at  $t = 2$  under the weaker condition that  $\widehat{\Phi} \circ \Phi$  is EP [16], where  $\widehat{\Phi}$  is the adjoint with respect to the Hilbert-Schmidt inner product (note however that (6) does *not* hold for general integer  $t$  under this weaker condition, as demonstrated by the well-known example of Holevo–Werner maps [22]).

Our last results concern the one-particle Hilbert space  $L^2(\mathbb{R})$ , where states are represented by kernels  $K(x, y)$  satisfying  $K(x, y) = \overline{K(y, x)}$ ,

$$\int \int \overline{\psi(x)} K(x, y) \psi(y) dx dy \geq 0 \quad (8)$$

for all  $\psi \in L^2(\mathbb{R})$ , and

$$\int K(x, x) dx = 1. \quad (9)$$

In this setting, a linear map  $\Phi$  is a convolution operator:

$$\Phi : K(x, y) \rightarrow \int \int G(x, y; u, v) K(u, v) du dv. \quad (10)$$

The analog of the entrywise positive (EP) property in the finite-dimensional case is pointwise positivity of the kernel  $G$ , that is

$$G(x, y; u, v) \geq 0 \quad (11)$$

for all  $u, v, x, y \in \mathbf{R}$ .

For integer  $t$ , the Schatten norm of a state is defined by

$$\|\rho\|_t = (\text{Tr } \rho^t)^{1/t} = \left( \int \cdots \int K(x_1, x_2) K(x_2, x_3) \cdots K(x_t, x_1) dx_1 \cdots dx_t \right)^{1/t}. \quad (12)$$

When  $\rho$  is positive and trace class, (12) is well-defined for all integer  $t$ , and hence (3) extends to this case also.

**Theorem 5.** Let  $\Phi$  be a completely positive map defined as in (10), with kernel  $G$  satisfying the positivity condition (11). Let  $\Omega$  be any other CP map on  $L^2(\mathbb{R})$ . Then for integer  $t$ ,

$$v_t(\Phi \otimes \Omega) = v_t(\Phi) v_t(\Omega). \quad (13)$$

As an application of the previous result, recall the definition of a bosonic channel [6,10]:

$$N(\rho) = \int P(z)D(z)\rho D(z)^\dagger dz. \tag{14}$$

Here  $D(z)$  is the unitary displacement operator for a coherent state, which acts on  $L^2(\mathbb{R})$  according to

$$(D(z)\psi)(x) = (D(\alpha, \beta)\psi)(x) = e^{i\alpha x} \psi(x - \beta). \tag{15}$$

The function  $P(z) = P(\alpha, \beta)$  is a probability density function on  $\mathbb{R}^2$ , so (14) defines a unital trace-preserving CP channel on states over  $L^2(\mathbb{R})$ . In the main case of interest for applications  $P(z)$  is a Gaussian density [6,10], and some multiplicativity results have been proved under this assumption [7,8,20]. As our next result shows, Theorem 5 can be applied to bosonic channels of the form (14) in the case where  $P$  satisfies two positivity conditions:

$$P(\alpha, \beta) \geq 0, \quad \int e^{i\alpha x} P(\alpha, \beta) d\alpha \geq 0 \quad \text{for all } x, \beta. \tag{16}$$

In particular note that (16) holds for any Gaussian density.

**Theorem 6.** *Let  $N$  be a map of the form (14) where  $P(z)$  satisfies (16), and let  $\Omega$  be any CP map on  $L^2(\mathbb{R})$ . Then for integer  $t$ ,*

$$v_t(\Phi \otimes \Omega) = v_t(\Phi) v_t(\Omega). \tag{17}$$

### 3. Examples of CP–EP maps

#### 3.1. Quantum-classical maps

A map  $\Phi : M_n \mapsto M_m$  takes a quantum system to a classical one if its range is contained in the subset of diagonal matrices. In this case, the map is EP if and only if it is CP.

#### 3.2. Qubit maps

We use the diagonal representation of qubit maps introduced in [15] and used, e.g., in [19]. In this representation a qubit map  $\Phi$  acts as follows:

$$\Phi \left( I + \sum w_k \sigma_k \right) = I + \sum (\lambda_k w_k + t_k) \sigma_k, \tag{18}$$

where  $\sigma_k$  are the Pauli matrices. The Choi matrix of  $\Phi$  in this representation is

$$\begin{pmatrix} \Phi(E_{11}) & \Phi(E_{12}) \\ \Phi(E_{21}) & \Phi(E_{22}) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 + \lambda_3 + t_3 & t_1 - it_2 & 0 & \lambda_1 + \lambda_2 \\ t_1 + it_2 & 1 - \lambda_3 - t_3 & \lambda_1 - \lambda_2 & 0 \\ 0 & \lambda_1 - \lambda_2 & 1 - \lambda_3 + t_3 & t_1 - it_2 \\ \lambda_1 + \lambda_2 & 0 & t_1 + it_2 & 1 + \lambda_3 - t_3 \end{pmatrix}.$$

The CP condition puts some constraints on the six parameters  $\{\lambda_k, t_k\}$ , and these are fully explored in [19]. By changing bases if necessary in the domain and range of  $\Phi$ , (i.e., using  $\Gamma_U \circ \Phi \circ \Gamma_V$  as discussed in [11,15]) it can be assumed that the following conditions are satisfied:

$$\lambda_1 \geq |\lambda_2|, \quad t_1 \geq 0. \quad (19)$$

The EP condition is satisfied if

$$\lambda_1 \geq |\lambda_2|, \quad t_1 \geq 0, \quad t_2 = 0. \quad (20)$$

Hence the only additional restriction coming from the EP condition is  $t_2 = 0$ . The index ‘2’ here has a geometric meaning as it labels one of the two smaller axes of the image ellipsoid. Theorem 3 then implies the following.

**Corollary 7.** *Let  $\Phi$  be a qubit channel, and suppose that  $t_2 = 0$  in the diagonal representation, where  $\lambda_2$  describes one of the two smaller axes of the image ellipsoid. (If any two axes have equal length, there is no restriction.) Then  $v_t(\Phi \otimes \Omega) = v_t(\Phi)v_t(\Omega)$  for any CP map  $\Omega$ , for all integer  $t$ .*

The methods of [14] can be used to extend this multiplicativity result to all non-integer values  $p \geq 2$  for the same class of qubit channels. Unfortunately these methods do not apply for values  $1 < p < 2$ , and for this interval the multiplicativity question for qubit maps is open except for unital channels [12].

### 3.3. Depolarizing channels and generalizations

The  $d$ -dimensional depolarizing channel is the map

$$\rho \rightarrow \lambda\rho + (1 - \lambda)(\text{Tr } \rho) \frac{1}{d}I, \quad (21)$$

where  $I$  is the  $d \times d$  identity matrix. It is well-known (see e.g., [13]) that this map is CPT (CP and trace-preserving) for values of  $\lambda$  in the range  $-\frac{1}{d^2-1} \leq \lambda \leq 1$ . The map (21) is clearly EP for  $0 \leq \lambda \leq 1$ , and hence Theorem 3 can be used to show that  $v_t(\Phi^{\otimes m}) = [v_t(\Phi)]^m$  for integer  $t$ . This result for products of depolarizing channels was first established in [1]; subsequently, it was extended to all  $t \geq 1$  in [5] and [13].

In [9] the map (21) was generalized by replacing  $\frac{1}{d}I$  by a fixed arbitrary density matrix  $\gamma$ :

$$\rho \rightarrow \lambda\rho + (1 - \lambda)\text{Tr } \rho \gamma. \quad (22)$$

Using a basis in which  $\gamma$  is diagonal, it is easy to verify that (22) is CPT and EP for  $0 \leq \lambda \leq 1$ . Thus, Theorem 3 can again be used to show that  $v_t(\Phi^{\otimes m}) = [v_t(\Phi)]^m$  for all integer  $t$ . This result was established for  $t = 2$  in [9].

### 3.4. Positive Kraus operators

If a channel has a Kraus representation  $\Phi(\rho) = \sum A_k \rho A_k^*$  where each matrix  $A_k$  is EP, then the map  $\Phi$  is EP, and Theorem 3 can be applied. In particular, this holds when  $A_k = \sqrt{p_k} P_k$  where  $P_k$  is a permutation matrix, and  $\sum p_k = 1$ . This is a particular case of the class of so-called “random unitary” channels.

### 3.5. Maps which are not EP

To give an example of a map which is not EP, it suffices to recall that Werner and Holevo [22] found maps for which (6) is false for  $t$  sufficiently large; therefore, these maps cannot be EP.

We now show that there are also qubit maps which are not EP by observing that (4) implies that  $\text{Tr } E_{ij} \Phi(E_{k\ell})$  is real for all  $i, j, k, \ell$ . Let  $\{a_{jk}\}_{j,k=0,1,2,3}$  be the matrix representing the qubit map  $\Phi$  in the basis consisting of  $\{I, \sigma_1, \sigma_2, \sigma_3\}$ , i.e., the identity and the three Pauli matrices with the implicit convention  $\sigma_0 = I$ , and let  $E_{ij} = |i\rangle\langle j|$  in the standard basis for  $\mathbb{C}_m$ . Then, e.g.,

$$\begin{aligned} 4\text{Tr } E_{12} \Phi(E_{11}) &= \text{Tr}(\sigma_1 + i\sigma_2) \Phi(I + \sigma_3) = (a_{10} + a_{13}) + i(a_{20} + a_{23}), \\ 4\text{Tr } E_{12} \Phi(E_{22}) &= \text{Tr}(\sigma_1 + i\sigma_2) \Phi(I - \sigma_3) = (a_{10} - a_{13}) + i(a_{20} - a_{23}). \end{aligned}$$

Therefore, the requirement that  $\text{Im Tr } E_{12} \Phi(E_{11}) = \text{Im Tr } E_{12} \Phi(E_{22}) = 0$  implies that  $(a_{20} \pm a_{23}) = 0$  which implies  $a_{20} = a_{23} = 0$ . Proceeding in this way, one can show that a necessary condition for  $\text{Tr } E_{ij} \Phi(E_{k\ell})$  to be real is that

$$a_{j2} = a_{2k} = 0 \quad \text{for } j, k = 0, 1, 3, \tag{23}$$

i.e., all  $a_{jk}$  with  $j = 2$  or  $k = 2$  vanish unless  $j = k$ . The map (18) corresponds to the choice  $a_{00} = 1, a_{0k} = 0, a_{j0} = t_k$  and  $a_{jk} = \lambda_j \delta_{jk}, j, k = 1, 2, 3$ . This map does not satisfy the condition (23) when  $t_2 \neq 0$ . Now recall that a change of basis on  $\mathbb{C}_m$  corresponds to a rotation on  $\mathbb{R}_3$ . As explained in Appendix B of [15], making a change of basis on the domain and range of  $\mathbb{C}_m$ , corresponds to changing

$$\mathbf{v} \rightarrow O_1 \mathbf{v}, \quad T \rightarrow O_1 T O_2, \tag{24}$$

where  $O_1, O_2$  are  $3 \times 3$  orthogonal matrices,  $T$  is the  $3 \times 3$  matrix  $a_{jk}$  with  $j, k = 1, 2, 3$  and  $\mathbf{v} = (a_{01}, a_{02}, a_{03})^T$ . Thus, for the map (18),  $T$  has elements  $\lambda_j \delta_{jk}, \mathbf{v} = (t_1, t_2, t_3)^T$ . When all  $\lambda_j \neq 0$  are distinct and all  $t_j \neq 0$ , any  $O_1$  which makes  $t_2 = 0$  will make either  $a_{21} \neq 0$  or  $a_{23} \neq 0$ , violating (23). In general there is no choice of  $O_1, O_2$  for which (23) holds.

One can similarly show that qubit maps of the form (18) with the additional restrictions above do not satisfy the weaker condition that  $\widehat{\Phi} \circ \Phi$  is EP. Note that

$\widehat{\Phi} \circ \Phi$  is represented by the  $4 \times 4$  matrix  $B \equiv A^*A$  (indexed by 0, 1, 2, 3). When  $t_2 = 0$ , all elements of  $B$  are explicitly non-negative except for  $b_{12} = b_{21} = \lambda_1^2 - \lambda_2^2$ . This will be negative when  $|\lambda_1| < |\lambda_2|$ , which suggests that maps which do not satisfy (20) do not satisfy the condition that  $\widehat{\Phi} \circ \Phi$  is EP.

#### 4. Proofs of Theorems

Our proofs will use the following consequence of Hölder's inequality: for any matrices  $B_1, B_2, \dots$  and integer  $n$ ,

$$|\text{Tr}(B_1 B_2 \dots B_n)| \leq \|B_1\|_n \|B_2\|_n \dots \|B_n\|_n. \quad (25)$$

Furthermore the definition of the  $(p \rightarrow q)$  norm implies that for any matrix  $B$  and linear operator  $L$ ,

$$\|L(B)\|_q \leq \|L\|_{p \rightarrow q} \|B\|_p. \quad (26)$$

##### 4.1. Proof of Theorem 2

Let  $A$  be any  $nm \times nm$  matrix, then

$$A = \sum_{ij} E_{ij} \otimes A_{ij}, \quad (27)$$

where  $\{A_{ij}\}$  are the  $m \times m$  blocks. Hence

$$(K \otimes L)A = \sum_{ij} K(E_{ij}) \otimes L(A_{ij}). \quad (28)$$

For any integer  $t$ ,

$$\text{Tr} |(K \otimes L)(A)|^{2t} = \text{Tr} ((K \otimes L)(A) [(K \otimes L)(A)]^*)^t. \quad (29)$$

Using the representation (28) in (29) we get

$$\text{Tr} |(K \otimes L)A|^{2t} = \sum \text{Tr}(K(E_{i_1 j_1}) K(E_{i_2 j_2})^* \dots) \text{Tr}(L(A_{i_1 j_1}) L(A_{i_2 j_2})^* \dots). \quad (30)$$

Now we apply (25) with  $n = 2t$  and (26) with  $q = 2t$ : this gives

$$\text{Tr} |(K \otimes L)A|^{2t} \leq (\|L\|_{p \rightarrow 2t})^{2t} \sum \text{Tr}(K(E_{i_1 j_1}) K(E_{i_2 j_2})^* \dots) \|A_{i_1 j_1}\|_p \dots \quad (31)$$

The assumption that  $K$  is EP implies that

$$\text{Tr}(K(E_{i_1 j_1}) K(E_{i_2 j_2})^* \dots) \geq 0 \quad (32)$$

for all indices  $i_1, j_1, \dots$ . It follows that



$$\text{Tr}|(K \otimes L)A|^{2t} \leq (\|L\|_{p \rightarrow 2t})^{2t} \text{Tr}|K(\alpha)|^{2t}, \tag{33}$$

where  $\alpha$  is the  $n \times n$  matrix with entries

$$\alpha_{ij} = \|A_{ij}\|_p = \|A_{ij}^*\|_p. \tag{34}$$

Finally we use the following result of Bhatia and Kittaneh [4]: for  $1 \leq p \leq 2$

$$\text{Tr} \alpha^2 = \sum_{ij} \|A_{ij}\|_p^2 \leq \|A\|_p^2. \tag{35}$$

This implies

$$\|(K \otimes L)A\|_{2t} \leq \|L\|_{p \rightarrow 2t} \|K\|_{2 \rightarrow 2t} \|\alpha\|_2 \leq \|L\|_{p \rightarrow 2t} \|K\|_{2 \rightarrow 2t} \|A\|_p, \tag{36}$$

which completes the proof that

$$\|K \otimes L\|_{p \rightarrow 2t} \leq \|K\|_{2 \rightarrow 2t} \|L\|_{p \rightarrow 2t}. \tag{37}$$

At  $p = 2$ , equality can be achieved using the product of states which maximize  $\|K\|_{p \rightarrow 2t}$  and  $\|L\|_{p \rightarrow 2t}$ .  $\square$

#### 4.2. Proof of Theorem 3

Let  $A \geq 0$  and write

$$A = \sum_{ij} E_{ij} \otimes A_{ij}. \tag{38}$$

Since  $\Phi \otimes \Omega$  is positivity preserving it follows that

$$\text{Tr}(\Phi \otimes \Omega)(A)^t = \sum \text{Tr}(\Phi(E_{i_1 j_1}) \Phi(E_{i_2 j_2}) \cdots) \text{Tr}(\Omega(A_{i_1 j_1}) \Omega(A_{i_2 j_2}) \cdots). \tag{39}$$

Using Hölder’s inequality again as in (25), and using the fact that  $\Phi$  is EP, we deduce

$$\text{Tr}(\Phi \otimes \Omega)(A)^t \leq \sum \text{Tr}(\Phi(E_{i_1 j_1}) \Phi(E_{i_2 j_2}) \cdots) \|\Omega(A_{i_1 j_1})\|_t \|\Omega(A_{i_2 j_2})\|_t \cdots \tag{40}$$

Since  $(I \otimes \Omega)(A) \geq 0$  it follows that for all  $i, j$

$$\Omega(A_{ij}) = \Omega(A_{ii})^{1/2} R_{ij} \Omega(A_{jj})^{1/2}, \tag{41}$$

where  $R_{ij}$  is a contraction, that is  $\|R_{ij}\|_\infty \leq 1$ . Hence

$$\|\Omega(A_{ij})\|_t \leq \|\Omega(A_{ii})\|_t^{1/2} \|\Omega(A_{jj})\|_t^{1/2}. \tag{42}$$

Substituting into (40) we deduce

$$\text{Tr}(\Phi \otimes \Omega)(A)^t \leq \text{Tr} \Phi(\beta)^t, \tag{43}$$

where now  $\beta$  is the  $n \times n$  matrix with entries

$$\beta_{ij} = \|\Omega(A_{ii})\|_t^{1/2} \|\Omega(A_{jj})\|_t^{1/2}. \tag{44}$$

Since  $\beta \geq 0$  we deduce

$$\begin{aligned} \text{Tr}(\Phi \otimes \Omega)(A)^t &\leq v_t(\Phi)^t (\text{Tr}(\beta))^t \\ &= v_t(\Phi)^t \left( \sum_{i=1}^n \|\Omega(A_{ii})\|_t \right)^t \\ &\leq v_t(\Phi)^t v_t(\Omega)^t (\text{Tr}A)^t. \end{aligned} \tag{45}$$

Taking the  $t$ th root of both sides and taking the sup over  $A$  shows that

$$v_t(\Phi \otimes \Omega) \leq v_t(\Phi)v_t(\Omega) \tag{46}$$

as required. Equality can be achieved using a product of states which maximize  $v_t(\Phi)$  and  $v_t(\Omega)$ .  $\square$

4.3. Proof of Theorem 4

First, observe that (41) only involves the  $2 \times 2$  submatrix  $\begin{pmatrix} \Omega(A_{ii}) & \Omega(A_{ij}) \\ \Omega(A_{ij}^*) & \Omega(A_{jj}) \end{pmatrix}$ ; therefore, the inequality (42) holds whenever  $\Omega$  is 2-positive. Next, proceed as in the proof of Theorem 2 up to (30). Since  $\Phi$  is EP, one can then conclude that a variant of (40) holds with  $\Phi(E_{i_1 j_1})\Phi(E_{i_2 j_2})$  replaced by  $\Phi(E_{i_1 j_1})[\Phi(E_{i_2 j_2})]^*$  and  $t$  replaced by the even integer  $2t$ . When  $\Omega$  is 2-positive, the remainder of the proof of Theorem 3 goes through to yield  $v_{2t}(\Phi \otimes \Omega) \leq v_{2t}(\Phi)v_{2t}(\Omega)$ .  $\square$

4.4. Proof of Theorem 5

The proof is a transcription of the proof of Theorem 3, with matrices replaced by integral kernels. First note that a bipartite state  $R$  on  $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$  is described by a kernel  $K(x_1, y_1; x_2, y_2)$ . For fixed  $x_1$  and  $x_2$  we define the function

$$T_{x_1, x_2}(y_1, y_2) = K(x_1, y_1; x_2, y_2). \tag{47}$$

Then by analogy with (40) we have

$$\begin{aligned} \text{Tr}(\Phi \otimes \Omega)(R)^t & \\ &= \int \cdots \int \prod_{i=1}^t G(x_i, x_{i+1}; u_i, v_i) \text{Tr}(\Omega(T_{u_1, v_1}) \cdots \Omega(T_{u_t, v_t})) \prod_{i=1}^t dx_i du_i dv_i, \end{aligned} \tag{48}$$

$$\tag{49}$$

where we use the labelling convention  $t + 1 \equiv 1$ . Applying Hölder’s inequality (see, for example, Appendix B of [18]) gives

$$|\text{Tr}(\Omega(T_{u_1, v_1}) \cdots \Omega(T_{u_t, v_t}))| \leq \|\Omega(T_{u_1, v_1})\|_t \cdots \|\Omega(T_{u_t, v_t})\|_t. \tag{50}$$

The analog of (42) is

$$\|\Omega(T_{u,v})\|_t \leq \|\Omega(T_{u,u})\|_t^{1/2} \|\Omega(T_{v,v})\|_t^{1/2}. \tag{51}$$

Using the pointwise positivity of  $G$ , and substituting (50) and (51) back into (49) gives

$$\text{Tr}(\Phi \otimes \Omega)(R)^t \leq \text{Tr} \Phi(S)^t, \tag{52}$$

where  $S$  is the operator with kernel

$$S(u, v) = \|\Omega(T_{u,u})\|_t^{1/2} \|\Omega(T_{v,v})\|_t^{1/2}. \tag{53}$$

Using

$$\text{Tr}(S) = \int \|\Omega(T_{u,u})\|_t \, du \tag{54}$$

$$\leq v_t(\Omega) \int \text{Tr}(T_{u,u}) \, du \tag{55}$$

$$= v_t(\Omega) \int \int K(u, v; u, v) \, du \, dv, \tag{56}$$

the rest of the argument follows as before.  $\square$

#### 4.5. Proof of Theorem 6

In terms of integral kernels, the channel (14) acts by

$$\Phi : K(x, y) \rightarrow \int h(x - y, \beta) K(x - \beta, y - \beta) \, d\beta, \tag{57}$$

where

$$h(x - y, \beta) = \int P(\alpha, \beta) e^{i\alpha(x-y)} \, d\alpha. \tag{58}$$

This can be re-written in the form (10) with

$$G(x, y; u, v) = \delta(u - x + y - v) h(u - v, y - v). \tag{59}$$

The condition (16) implies that  $h(a, b) \geq 0$  for all  $a, b$ . By using a sequence of positive approximations  $\delta_n$  for the  $\delta$ -function in (59), we obtain a sequence of positive kernels  $G_n$  for which Theorem 5 can be applied. The result is then obtained in the limit  $n \rightarrow \infty$ .  $\square$

### 5. Conclusion

The additivity conjecture arose in quantum information theory in the context of entropy-related properties of completely positive trace-preserving (CPT) maps. In the course of seeking a proof of the conjecture, Amosov and Holevo [1] proposed a

more general multiplicativity result involving Schatten  $p$ -norms. In this larger context it is natural to drop the trace-preserving condition, and consider just completely positive maps, and most of the known results hold for this more general class. A further natural generalization of the question is to consider multiplicativity properties involving  $p$  to  $q$  norms of CP maps for general values  $p, q \geq 1$ . In this paper we have demonstrated some multiplicativity results in this case for large classes of maps characterized by conditions which are not equivalent to the CP property in the case of non-commutative systems. This may be an indication that the multiplicativity property has its roots in a different setting.

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