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A Data Envelopment Analysis Approach to Multicriteria Decision Problems with Incomplete Information

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Abstract—As a decision aid for discrete multicriteria decision problems, this paper proposes a multilevel graph of alternatives to represent the ranking, to the extent that this is possible when incomplete information on weights is available under the assumption of the additive value function. To construct it, the nested decomposition of the set of alternatives is established along the lines of data envelopment analysis (DEA). A numerical example is given to illustrate a multilevel graph based on the nested decomposition and compare it with the hierarchical dominance graph based on dominance relations proposed by Park and Kim. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In discrete alternative multicriteria decision problems, the primary concern for the decision aid is the following:

(1) choosing the most preferred alternative to the decision maker (DM),
(2) ranking alternatives in order of importance for selection problems, or
(3) screening alternatives for the final decision.

The general concepts of domination structures and nondominated solutions play an important role in describing the decision problems and the decision maker's revealed preferences described above (see [1]). So far, various approaches have been developed as the decision aid (see, for example, [2]). Within the category (1), interactive methods based on the preference cones have been proposed to effectively get the most preferred solution (see, for instance, [3–5]). In these approaches, under the assumption of an implicit quasi-concave increasing value function, preference cones are constructed by pairwise comparisons among alternatives at each iteration. Then, the set of alternatives is gradually reduced to a smaller one by identifying and eliminating inferior alternatives from the set of alternatives by preference cones and finally end up with the most preferred alternative.

On the other hand, it is not uncommon that the DM is only willing or able to provide incomplete information, due to time pressure, lack of knowledge, fear of commitment, etc. Thus, from the

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necessity of considering incomplete information, Weber [6] presented a general framework for decision making with incomplete information. Kirkwood and Sarin [7] derived conditions to determine whether a pair of alternatives can be ranked and presented a procedure for ranking alternatives using an additive value function with the incomplete information on the weights. Kmietowicz and Pearman [8] dealt with decision problems under conditions of linear partial information (LPI) on probabilities of occurrence for the states of nature and derived conditions ensuring strict and weak statistical dominance of one strategy over another. Pearman [9] proposed an ordered metric method for establishing the dominance of alternatives using the linear additive weighting rule in multiattribute decision making under the LPI on the weights. Park and Kim [10] proposed a hierarchical dominance graph (HDG) by using pairwise dominance relations in the multiattribute decision making with the decision maker's incomplete information on both weights and utilities under the assumption of the additive value function. The HDG can be used to aid in selecting one or more preferred alternatives.

The purpose of this paper is to propose a multilevel graph which visualizes an incomplete ranking of alternatives, to the extent that this is possible when incomplete information on the weights is available under the assumption of the additive value function. To construct it, the nested decomposition of the set of alternatives is established by sequentially locating alternatives, each of which is a top ranking for some weight, and then deleting them from the set of alternatives and locating alternatives, each of which is a top ranking for some weight among the remaining set, and so on. At the same time, the reference set on an immediate higher level of the alternative being evaluated is located since, for any weight, at least one alternative in the reference set is a higher ranking than it. These can be done along the lines of data envelopment analysis (see, for instance, [11,12]). According to the reference set, a multilevel graph is constructed.

In the following section, we first show how to decompose the set of alternatives and construct a multilevel graph by the DEA formulation. It is shown that alternatives on the p^{th} level of the graph based on the nested decomposition have at best p^{th} ranking. Then, using a numerical example, we compare it with the HDG based on dominance relations proposed by Park and Kim [10]. Concluding remarks are given in the final section.

2. MULTILEVEL GRAPHS BASED ON THE NESTED DECOMPOSITION

Let us consider n alternatives $\mathbf{a}_i$, $i = 1, 2, \ldots, n$. And let

$$A = \{\mathbf{a}_i\}.$$

Suppose that each alternative $\mathbf{a}_i$ has a multiattribute outcome denoted by a vector

$$\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{mi}),$$

where x_{ji} is the measurement on attribute j, $j = 1, 2, \ldots, m$. We assume that the attributes are additively independent. Thus, the value function $v(\mathbf{x}_i)$ is written by a weighted additive value function

$$v(\mathbf{x}_i) = \sum w_j v_j (x_{ji}),$$

where $w_j > 0$ is a relative weight of attribute j (it is not necessary to sum up to unity) and v_j is a single-attribute value function satisfying

$$0 \leqq v_j (x_{ji}) \leqq 1, \qquad j = 1, 2, \ldots, m.$$

This paper is concerned with multicriteria decision problems with incomplete information on the weights, when v_j is given. In what follows, $\mathbf{a}_i$ and $\mathbf{v}_i$ are used interchangeably whenever no confusion arises. Thus, let

$$v(\mathbf{a}_i) = \mathbf{w}^{\top} \mathbf{v}_i,$$

where $v_{ji} = v_j(x_{ji})$, $\mathbf{v}_i = (v_{ji})$, and $\mathbf{w} = (w_j) > 0$.

Let the set of weights be

$$W^{(0)} = \{\mathbf{w} : \mathbf{w} > 0\}.$$

Now let us consider a situation where the set of weights is further restricted by

$$W^{(1)} = \left\{\mathbf{w} : \mathbf{w}^\top \mathbf{b}_j^{(1)} > 0,\ j = 1, \dots, t^{(1)}\right\},$$
$$W^{(2)} = \left\{\mathbf{w} : \mathbf{w}^\top \mathbf{b}_j^{(2)} \geqq 0,\ j = 1, \dots, t^{(2)}\right\}, \qquad \text{and}$$
$$W^{(3)} = \left\{\mathbf{w} : \mathbf{w}^\top \mathbf{b}_j^{(3)} = 0,\ j = 1, \dots, t^{(3)}\right\},$$

where $\top$ represents the transpose and $\mathbf{b}_j^{(i)}$, $i = 1, 2, 3$, are m-dimensional vectors.

REMARK 1. $W^{(i)}$, $i = 1, 2, 3$, may be constructed as follows:

(a) the DMs respond to some preliminary questions for any pairs of alternatives, say $\mathbf{a}_1$ and $\mathbf{a}_2$,
  (1) if the DM prefers $\mathbf{a}_1$ to $\mathbf{a}_2$, then one can deduce from this $\mathbf{w}\mathbf{a}_1 > \mathbf{w}\mathbf{a}_2$ and $\mathbf{b}_j^{(1)} = \mathbf{a}_1 - \mathbf{a}_2$,
  (2) if $\mathbf{a}_1$ is at least as good as $\mathbf{a}_2$, this is interpreted as $\mathbf{w}\mathbf{a}_1 \geqq \mathbf{w}\mathbf{a}_2$ and $\mathbf{b}_j^{(2)} = \mathbf{a}_1 - \mathbf{a}_2$,
  (3) if $\mathbf{a}_1$ is indifferent to $\mathbf{a}_2$, then construct an equality of the form $\mathbf{w}\mathbf{a}_1 = \mathbf{w}\mathbf{a}_2$ and $\mathbf{b}_j^{(3)} = \mathbf{a}_1 - \mathbf{a}_2$,

  and/or

(b) the information on the criteria, for instance, the order of importance, $w_1 \geqq w_2 \geqq \cdots \geqq w_m$.

In what follows, let

$$W = \bigcap W^{(i)}, \quad \text{for all nonempty sets } W^{(i)}, \qquad i = 0, 1, 2, 3.$$

To rank alternatives, to the extent that this is possible, according to W consider the following linear programming problem.

For each $\mathbf{a}_k$,

$$\begin{aligned} \text{maximize} \quad & v_k = \mathbf{w}^\top \mathbf{v}_k, \\ \text{subject to} \quad & v_j = \mathbf{w}^\top \mathbf{v}_j \leqq 1, \qquad j = 1, 2, \dots, n, \\ & \mathbf{w} \in W, \end{aligned}$$

or equivalently,

$$\begin{aligned} \text{maximize} \quad & v_k = \mathbf{w}^\top \mathbf{v}_k, \\ \text{subject to} \quad & v_j = \mathbf{w}^\top \mathbf{v}_j \leqq 1, \qquad j = 1, 2, \dots, n, \\ & \mathbf{w} \geqq \varepsilon \mathbf{e}_m, \\ & \mathbf{w}^\top B^{(1)} \geqq \varepsilon \mathbf{e}_{t^{(1)}}^\top, \\ & \mathbf{w}^\top B^{(2)} \geqq \mathbf{0}^\top, \\ & \mathbf{w}^\top B^{(3)} = \mathbf{0}^\top, \end{aligned} \tag{Pk}$$

where ε is a positive non-Archimedean infinitesimal which is used to replace $>$ with $\geqq$ as is used in DEA, $\mathbf{e}_m = (1, 1, \dots, 1)^\top$ is an m-dimensional vector, $B^{(i)}, i = 1, 2, 3$, is an $m \times t^{(i)}$ matrix whose column vectors are $\mathbf{b}_j^{(i)}$, $j = 1, 2, \dots, t^{(i)}$.

REMARK 2.

(i) Note that the last three constraints in (Pk) correspond to $W^{(i)}$, $i = 1, 2, 3$, respectively, and therefore, whenever $W^{(i)} = \emptyset$, $i = 1, 2, 3$, the corresponding constraint is removed from the constraints.

(ii) In the formulation of (Pk), observe that we can assume $v_j = \mathbf{w}^\top \mathbf{v}_j \leqq c$, $j = 1, 2, \dots, n$, for any $c > 0$, instead of $v_j = \mathbf{w}^\top \mathbf{v}_j \leqq 1$, $j = 1, 2, \dots, n$, since $\mathbf{w}$ is not normalized to one.

Note that if $v_k^* = 1$, $\mathbf{a}_k$ is a top ranking alternative for $\mathbf{w}^* \in W$,

$$v_k \geqq v_j, \qquad j = 1, 2, \ldots, n, \quad \text{for } \mathbf{w}^* \in W,$$

where v_k^* and w^* is an optimal solution to (Pk).

On the other hand, if $v_k^* < 1$, $\mathbf{a}_k$ does not become a top ranking alternative for any $\mathbf{w} \in W$, since for any $\mathbf{w} \in W$ there exists at least one $\mathbf{v}_i$ such that

$$v_k < v_i.$$

Thus, there is a possibility that $\mathbf{a}_k$ is top ranking alternative, if and only if $v_k^* = 1$.

DEFINITION 1. *Alternative* $\mathbf{a}_i$ *dominates* $\mathbf{a}_j$ *with respect to* W *if and only if*

$$v_i = \mathbf{w}^\top \mathbf{v}_i > v_j = \mathbf{w}^\top \mathbf{v}_j, \qquad \textit{for all } \mathbf{w} \in W.$$

REMARK 3. From the context of the value function, $\mathbf{a}_i$ dominates $\mathbf{a}_j$ with respect to W if and only if $\mathbf{a}_i$ is preferred to $\mathbf{a}_i$ for each $w \in W$ (see [6]).

Let us now consider the dual problem of (Pk):

$$\begin{aligned}
\text{minimize} \quad & z_k = \mathbf{e}_n^\top \boldsymbol{\lambda} - \varepsilon \mathbf{e}_m^\top \boldsymbol{\mu} - \varepsilon \mathbf{e}_{t(1)}^\top \boldsymbol{\mu}^{(1)}, \\
\text{subject to} \quad & \mathbf{v}_k = X\boldsymbol{\lambda} - \boldsymbol{\mu} - B^{(1)}\boldsymbol{\mu}^{(1)} - B^{(2)}\boldsymbol{\mu}^{(2)} - B^{(3)}\boldsymbol{\mu}^{(3)}, \\
& \boldsymbol{\lambda} \geqq \mathbf{0}, \\
& \boldsymbol{\mu} \geqq \mathbf{0}, \\
& \boldsymbol{\mu}^{(1)} \geqq \mathbf{0}, \\
& \boldsymbol{\mu}^{(2)} \geqq \mathbf{0}, \\
& \boldsymbol{\mu}^{(3)} : \text{free},
\end{aligned} \tag{Dk}$$

where $\boldsymbol{\lambda} = (\lambda_i)$, $i = 1, 2, \ldots, n$, $\boldsymbol{\mu} = (\mu_j)$, $j = 1, 2, \ldots, m$, $\boldsymbol{\mu}^{(i)} = (\mu_j^{(i)})$, $i = 1, 2, 3$; $j = 1, 2, \ldots, t^{(i)}$, and X is the $m \times n$ matrix whose column vectors are $\mathbf{v}^i$, $i = 1, 2, \ldots, n$.

REMARK 4.

(i) If $W^{(i)} = \phi$, $i = 1, 2, 3$, in (Pk), then the corresponding $B^{(i)}$ and $\boldsymbol{\mu}^{(i)}$ are deleted in (Dk).

(ii) The non-Archimedean infinitesimal $\varepsilon > 0$ allows the minimization over $\mathbf{e}_n^\top \lambda$ to preempt the maximization of the sum of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^{(1)}$. In this way, (Dk) is a computational form without any need to specify it explicitly (see, for instance, [12, p. 9]).

(iii) It follows from the dual theorem that $v_k^* = z_k^* \leqq 1$, where v_k^* and z_k^* are optimal values of (Pk) and (Dk), respectively. Thus, there is a possibility that $\mathbf{a}_k$ is a top ranking alternative if and only if $z_k^*{=}1$.

(iv) If $\mathbf{e}_n^\top \lambda^* > 1$ in (Dk), then $\mathbf{e}_n^\top \lambda^* - c\varepsilon > 1$, for any $c > 0$, which contradicts with $z_k^* \leqq 1$. Therefore, $\mathbf{e}_n^\top \lambda^* \leqq 1$.

(v) It follows that $z_k^* = 1$, if and only if $\mathbf{e}_n^\top \lambda^* = 1, \boldsymbol{\mu}^* = \mathbf{0}$, and $\boldsymbol{\mu}^{(1)*} = \mathbf{0}$, where $z_k^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*$, and $\boldsymbol{\mu}^{(1)*}$ is an optimal solution to (Dk).

The procedure for constructing a multilevel graph is as follows.

Nested Decomposition

STEP 1. To begin with, let $p = 1, A_p = A$.

STEP 2. For each $\mathbf{a}_k$ in A_p, solve the following linear programming.

$$\begin{aligned} \text{minimize} \quad & z_k = \mathbf{e}_{n_p}^{\mathsf{T}}\boldsymbol{\lambda} - \varepsilon \mathbf{e}_m^{\mathsf{T}}\boldsymbol{\mu} - \varepsilon \mathbf{e}_{t(1)}^{\mathsf{T}}\boldsymbol{\mu}^{(1)}, \\ \text{subject to} \quad & \mathbf{v}_k = X_p\boldsymbol{\lambda} - \boldsymbol{\mu} - B^{(1)}\boldsymbol{\mu}^{(1)} - B^{(2)}\boldsymbol{\mu}^{(2)} - B^{(3)}\boldsymbol{\mu}^{(3)}, \\ & \boldsymbol{\lambda} \geqq \mathbf{0}, \\ & \boldsymbol{\mu} \geqq \mathbf{0}, \\ & \boldsymbol{\mu}^{(1)} \geqq \mathbf{0}, \\ & \boldsymbol{\mu}^{(2)} \geqq \mathbf{0}, \\ & \boldsymbol{\mu}^{(3)} : \text{free}, \end{aligned} \qquad \text{(Dk)p}$$

where $\boldsymbol{\lambda} = (\lambda_i)$, $i = 1, 2, \ldots, n_p$, X_p is an $m \times n_p$ matrix whose column vectors are $\mathbf{v}^i$, $i = 1, 2, \ldots, n_p$ and n_p is the number of alternatives in A_p (renumbering, if necessary).

When $z_k^* < 1$ in problem (Dk)p for $\mathbf{a}_k$, let the reference set of $\mathbf{a}_k$ be

$$E_k^p = \{j \mid \lambda_j^* > 0\},$$

where λ_j^* is the optimum solution to (Dk)p for $\mathbf{a}_k$.

STEP 3. Let C_1 be the set of all alternatives with $z_k^* = 1$ in $A_1 = A$. Removing C_1 from the set of alternatives, let $p = 2$ and the remaining set be A_p, i.e., $A_p = A_1 \setminus C_1$. For each alternative $\mathbf{a}_k$ in A_p, again solve problem (Dk)p. Let C_2 be a set consisting of all alternatives with $z_k^* = 1$ in A_p. Continue the above process until the remaining set A_p is empty.

Constructing a Multilevel Graph

STEP 4. A multilevel graph $G(X, A)$ is constructed as follows: put C_1 in the top level of the graph and C_2 in the second level of the graph and so on. Finally, an arc (i, j) from $\mathbf{a}_i$ in C_p to $\mathbf{a}_j$ in C_{p+1} $(p = 1, 2, \ldots)$ is placed if and only if $\mathbf{a}_i$ belongs to the reference set E_j^p of $\mathbf{a}_j$. Thus,

$$A = \left\{(\mathbf{a}_i, \mathbf{a}_j) : \mathbf{a}_i \in C_p;\ \mathbf{a}_i \in E_j^p \text{ to } \mathbf{a}_j \in C_{p+1}\, (p = 1, 2, \ldots)\right\}.$$

REMARK 5. In Step 3, alternatives within C_p are incomparable without any further information on the weights.

THEOREM 1. *Let $\mathbf{a}_i$ be in C_{p+1}. Then, for any $\mathbf{w} \in W$, there exists at least one $\mathbf{a}_i$ in $E_k^p \subset C_p$, such that*

$$\mathbf{w}^{\mathsf{T}}\mathbf{v}_k < \mathbf{w}^{\mathsf{T}}\mathbf{v}_i.$$

PROOF. Since $\mathbf{a}_k \in C_{p+1}$, we have, $z_k^* < 1$ in problem (Dk)p for $\mathbf{a}_k$, i.e.,

$$\begin{aligned} \text{minimize} \quad & z_k = \mathbf{e}_{n_p}^{\mathsf{T}}\boldsymbol{\lambda}^* - \varepsilon \mathbf{e}_m^{\mathsf{T}}\boldsymbol{\mu}^* - \varepsilon \mathbf{e}_{t(1)}^{\mathsf{T}}\boldsymbol{\mu}^{(1)*} < 1, \\ \text{subject to} \quad & \mathbf{v}_k = X_p\boldsymbol{\lambda}^* - \boldsymbol{\mu}^* - B^{(1)}\boldsymbol{\mu}^{(1)*} - B^{(2)}\boldsymbol{\mu}^{(2*)} - B^{(3)}\boldsymbol{\mu}^{(3)*}, \\ & \lambda_i^* > 0, \qquad \text{for all } \mathbf{a}_i \in E_k^p, \\ & \lambda_j^* = 0, \qquad \text{otherwise}, \\ & \boldsymbol{\mu}^* \geqq \mathbf{0}, \\ & \boldsymbol{\mu}^{(1)*} \geqq \mathbf{0}, \\ & \boldsymbol{\mu}^{(2)*} \geqq \mathbf{0}, \\ & \boldsymbol{\mu}^{(3)*} : \text{free}, \end{aligned}$$

where $\boldsymbol{\lambda}^* = (\lambda_i^*)$, $\boldsymbol{\mu}^* = (\mu_j^*)$, and $\boldsymbol{\mu}^{(i)*} = (\mu_j^{(i)*})$, $i = 1, 2, 3$, are an optimum solution to problem (Dk)p.

When $\mathbf{v}_k = \mathbf{0}$, it is obvious. So, let $\mathbf{v}_k \neq \mathbf{0}$.

Suppose, to the contrary, that there is a weight $\mathbf{w} \in W$ such that

$$\mathbf{w}^\top \mathbf{v}_k \geqq \mathbf{w}^\top \mathbf{v}_i, \qquad \text{for all } \mathbf{a}_i \in E_k^p \subset C_p. \tag{a}$$

Note that $z_k^* < 1$ implies either

(i) $\mathbf{e}_{n_p}^\top \boldsymbol{\lambda}^* < 1$, or
(ii) $\mathbf{e}_{n_p}^\top \boldsymbol{\lambda}^* = 1$.

In the case of (i), since $\mathbf{w}^\top \mathbf{v}_k > 0$, it follows from (a) that

$$\mathbf{w}^\top \mathbf{v}_k > \sum \lambda_i^* \mathbf{w}^\top \mathbf{v}_k \geqq \sum \lambda_i^* \mathbf{w}^\top \mathbf{v}_i = \mathbf{w}^\top X_p \boldsymbol{\lambda}^*.$$

On the other hand,

$$\mathbf{w}^\top \mathbf{v}_k = \mathbf{w}^\top X_p \boldsymbol{\lambda}^* - \mathbf{w}^\top \boldsymbol{\mu}^* - \mathbf{w}^\top B^{(1)} \boldsymbol{\mu}^{(1)*} - \mathbf{w}^\top B^{(2)} \boldsymbol{\mu}^{(2)*} - \mathbf{w}^\top B^{(3)} \boldsymbol{\mu}^{(3)*}.$$

Since $\mathbf{w} \in W$, it follows that $\mathbf{w} > \mathbf{0}$, $\mathbf{w}^\top B^{(1)} > \mathbf{0}$, $\mathbf{w}^\top B^{(2)} \geqq \mathbf{0}$, and $\mathbf{w}^\top B^{(3)} = \mathbf{0}$. Therefore, we obtain

$$\mathbf{w}^\top \mathbf{v}_k \leqq \mathbf{w}^\top X_p \boldsymbol{\lambda}^*,$$

which is a contradiction. In the case of (ii), it follows from (a) that

$$\mathbf{w}^\top \mathbf{v}_k = \sum \lambda_i^* \mathbf{w}^\top \mathbf{v}_k \geqq \sum \lambda_i^* \mathbf{w}^\top \mathbf{v}_i = \mathbf{w}^\top X_p \lambda^*.$$

Note that either

(1) $\mu_j^* > 0$ for at least one j, and/or
(2) $\mu_j^{(1)*} > 0$ for at least one j

holds. Since $\mathbf{w} > \mathbf{0}$ and $\mathbf{w}^\top B^{(1)} > \mathbf{0}$, in either case, we have

$$\mathbf{w}^\top \mathbf{v}_k < \mathbf{w}^\top X_p \boldsymbol{\lambda}^*,$$

which leads to a contradiction. ∎

REMARK 6.

(i) From Theorem 1, we can see that any $\mathbf{a}_k$ in C_p has, at best, a p^{th} ranking, that is, a ranking of p or less.
(ii) The reference set $E_k^p (\subset C_p)$ is not necessarily unique. One can, however, say that $\mathbf{a}_k \in C_{p+1}$ is dominated by E_k^p.
(iii) We can conclude that, if only one alternative in E_k^p of $\mathbf{a}_k \in C_{p+1}$ exists, that is, only one arc $(\mathbf{a}_i, \mathbf{a}_k)$ from C_p to $\mathbf{a}_k \in C_{p+1}$ exists in the multilevel graph, then $\mathbf{a}_k$ is dominated by $\mathbf{a}_i$.

Observe that if $\mathbf{a}_i$ dominates $\mathbf{a}_k$, then $\mathbf{a}_k$ is at a lower level than $\mathbf{a}_i$. To show this, let $\mathbf{a}_k$ be in C_p. Note that $\mathbf{a}_k$ is in X_p. Now, let us suppose that $\mathbf{a}_i$ is also in X_p. The dual of problem (Dk)p is

$$\begin{aligned} &\text{maximize} && v_k = \mathbf{w}^\top \mathbf{v}_k, \\ &\text{subject to} && \mathbf{w}^\top \mathbf{v}_i \leqq 1, \qquad \text{for } i = 1, 2, \ldots, n_p, \\ &&& \mathbf{w} \in W. \end{aligned} \tag{Pk)p}$$

Since $\mathbf{a}_i$ dominates $\mathbf{a}_k$,

$$v_i = \mathbf{w}^\top \mathbf{v}_i > v_k = \mathbf{w}^\top \mathbf{v}_k, \qquad \text{for all } \leq \mathbf{w} \in W,$$

which yields

$$\text{maximize} \quad v_k = \mathbf{w}^\top \mathbf{v}_k < 1.$$

Therefore, $\mathbf{a}_k$ cannot belong to C_p, which is a contradiction. Thus, we have the following theorem.

THEOREM 2. *If* $\mathbf{a}_i$ *dominates* $\mathbf{a}_k$*, then* $\mathbf{a}_k$ *is at a lower level than* $\mathbf{a}_i$ *in the multilevel graph.*

Though the dominance relations can be defined in the decision problems with incomplete information about both weights and utilities (see [10]), in a special case where the value of utilities is known precisely, it is easy to establish the dominance relations.

The set of collecting dominance relations between the alternatives $\Omega \subseteq A \times A$ is defined so as to include the indifference relations as $(\mathbf{a}_i, \mathbf{a}_j) \in \Omega$ if and only if $\mathbf{a}_i$ is at least as preferred as $\mathbf{a}_j$, where $\mathbf{a}_j \neq \mathbf{a}_j$. After Ω is identified, a hierarchical dominance graph $G_H(H(A), E)$ with $H(A) = [H_1, \dots, H_L]$, where a set of arcs $E \subseteq A \times A$ is the set Ω, $H_k \subseteq A$ is a set of alternatives in the k^{th} level, L the number of levels of G_H, and $H_k \neq \emptyset \; \forall\, k$, is constructed as follows.

Construction of a Hierarchical Dominance Graph Based on the Dominance Relation

STEP 1. Construct the adjacent matrix M by using the information of Ω.

STEP 2. Compute the reachability matrix R of M.

STEP 3. Perform the following iterative procedure with $H_0 = \emptyset$ and $k = 1$.

a. Construct $B_k = A - \bigcup_{i=1}^{k-1} H_i$. If $B_k = \emptyset$, the set $L = k - 1$ and go to Step 4.
b. Find $H_k = \{\mathbf{a}_i \in B_k \mid P_k(\mathbf{a}_i) = P_k(\mathbf{a}_i) \cap S_k(\mathbf{a}_i)\}$, where $P_k(\mathbf{a}_i)$ and $S_k(\mathbf{a}_i)$, respectively, are sets of predecessors and successors denoted by the subgraph consisting of the elements in B_k.
c. Set $k = k + 1$ and go to Step 3.a.

STEP 4. Display $G_H(H(A), E)$ with $H(A) = [H_1, \dots, H_L]$.

NUMERICAL EXAMPLE. Let us consider the following example, with three criteria, O_1, O_2, and O_3, as shown in Table 1.

Table 1.

	O_1	O_2	O_3
$\mathbf{a}_1$	.65	.72	.38
$\mathbf{a}_2$	.40	.88	.19
$\mathbf{a}_3$	.77	.30	.64
$\mathbf{a}_4$	.68	.39	.26
$\mathbf{a}_5$	.58	.56	.67
$\mathbf{a}_6$	.26	.65	.35
$\mathbf{a}_7$	.63	.43	.48
$\mathbf{a}_8$	.47	.72	.12
$\mathbf{a}_9$	.34	.78	.23
$\mathbf{a}_{10}$	.78	.67	.45
$\mathbf{a}_{11}$	.73	.24	.91
$\mathbf{a}_{12}$	.27	.68	.56
$\mathbf{a}_{13}$	.89	.21	.52
$\mathbf{a}_{14}$	.44	.57	.62
$\mathbf{a}_{15}$	.33	.76	.39

Supposing that

(i) $\mathbf{a}_{11}$ is preferred to $\mathbf{a}_5$, and
(ii) $\mathbf{a}_{10}$ is preferred to $\mathbf{a}_{11}$,

let the set of weights W be

$$W^{(0)} = \{\mathbf{w} : \mathbf{w} > \mathbf{0}\}$$
$$W^{(1)} = \left\{\mathbf{w} : \mathbf{w}^\top \mathbf{b}_j^{(1)} > 0,\ j = 1, 2\right\}$$
$$W^{(2)} = W^{(3)} = \phi,$$

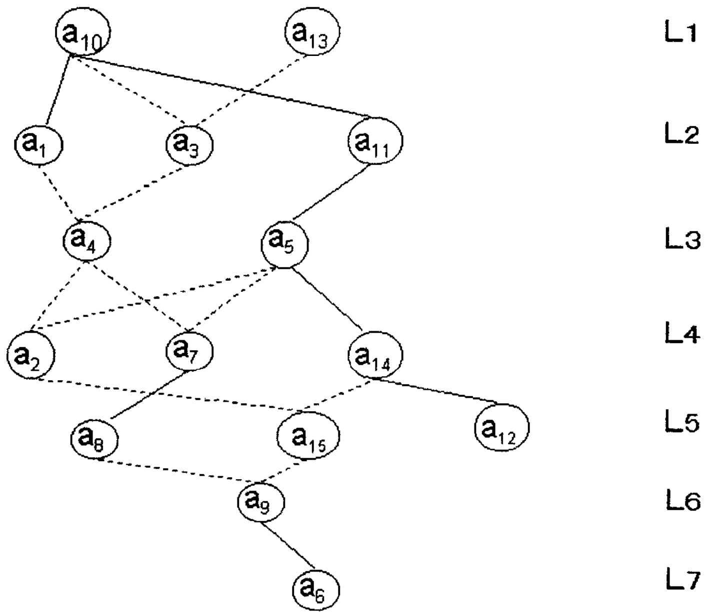

Figure 1. Multilevel graph based on the nested decomposition.

and therefore,

$$W = W^{(0)} \cap W^{(1)},$$

where $\mathbf{b}_1^{(1)} = \mathbf{a}_{11} - \mathbf{a}_5 = (.15, -.32, .24)$ and $\mathbf{b}_2^{(1)} = \mathbf{a}_{10} - \mathbf{a}_{11} = (.05, .43, -.46)$.

A multilevel graph based on the nested decomposition is shown in Figure 1. In this figure, while dotted lines designate the "group dominance" by a reference set, solid lines simply represent the dominance by a single alternative.

On the other hand, a hierarchical dominance graph is constructed in this example. The set Ω is as follows:

$$\{(\mathbf{a}_1,\mathbf{a}_2),(\mathbf{a}_1,\mathbf{a}_6),(\mathbf{a}_1,\mathbf{a}_7),(\mathbf{a}_1,\mathbf{a}_8),(\mathbf{a}_1,\mathbf{a}_9),(\mathbf{a}_1,\mathbf{a}_{12}),(\mathbf{a}_1,\mathbf{a}_{14}),(\mathbf{a}_1,\mathbf{a}_{15}),$$
$$(\mathbf{a}_2,\mathbf{a}_6),(\mathbf{a}_2,\mathbf{a}_9),(\mathbf{a}_3,\mathbf{a}_2),(\mathbf{a}_3,\mathbf{a}_4),(\mathbf{a}_3,\mathbf{a}_6),(\mathbf{a}_3,\mathbf{a}_7),(\mathbf{a}_3,\mathbf{a}_8),(\mathbf{a}_3,\mathbf{a}_9),(\mathbf{a}_3,\mathbf{a}_{15}),(\mathbf{a}_5,\mathbf{a}_2),$$
$$(\mathbf{a}_5,\mathbf{a}_6),(\mathbf{a}_5,\mathbf{a}_8),(\mathbf{a}_5,\mathbf{a}_9),(\mathbf{a}_5,\mathbf{a}_{12}),(\mathbf{a}_5,\mathbf{a}_{14}),(\mathbf{a}_5,\mathbf{a}_{15}),(\mathbf{a}_7,\mathbf{a}_6),(\mathbf{a}_7,\mathbf{a}_8),(\mathbf{a}_7,\mathbf{a}_9),$$
$$(\mathbf{a}_9,\mathbf{a}_6),(\mathbf{a}_{10},\mathbf{a}_1),(\mathbf{a}_{10},\mathbf{a}_2),(\mathbf{a}_{10},\mathbf{a}_4),(\mathbf{a}_{10},\mathbf{a}_5),(\mathbf{a}_{10},\mathbf{a}_6),(\mathbf{a}_{10},\mathbf{a}_7),(\mathbf{a}_{10},\mathbf{a}_8),$$
$$(\mathbf{a}_{10},\mathbf{a}_9),(\mathbf{a}_{10},\mathbf{a}_{11}),(\mathbf{a}_{10},\mathbf{a}_{12}),(\mathbf{a}_{10},\mathbf{a}_{14}),(\mathbf{a}_{10},\mathbf{a}_{15}),(\mathbf{a}_{11},\mathbf{a}_2),(\mathbf{a}_{11},\mathbf{a}_5),(\mathbf{a}_{11}\mathbf{a}_6),$$
$$(\mathbf{a}_{11},\mathbf{a}_7),(\mathbf{a}_{11},\mathbf{a}_8),(\mathbf{a}_{11},\mathbf{a}_9),(\mathbf{a}_{11},\mathbf{a}_{12}),(\mathbf{a}_{11},\mathbf{a}_{14}),(\mathbf{a}_{11},\mathbf{a}_{15}),(\mathbf{a}_{12},\mathbf{a}_6),(\mathbf{a}_{13},\mathbf{a}_6)$$
$$(\mathbf{a}_{13},\mathbf{a}_6),(\mathbf{a}_{13},\mathbf{a}_8),(\mathbf{a}_{13},\mathbf{a}_9),(\mathbf{a}_{14},\mathbf{a}_6),(\mathbf{a}_{14},\mathbf{a}_9),(\mathbf{a}_{14},\mathbf{a}_{12}),(\mathbf{a}_{14},\mathbf{a}_{15}),(\mathbf{a}_{15},\mathbf{a}_6)\}.$$

Since Ω is transitive, $R = (r_{ij}) = I + M$ where I is an $n \times n$ identity matrix. From the set Ω, a hierarchical dominance graph $G_H(H(A), E)$ is constructed as shown in Figure 2. In this figure, arcs which are derived from transitivity are omitted and therefore, $E \subseteq \Omega$.

It is clear that

(i) if $\mathbf{a}_k$ dominates $\mathbf{a}_j$, then $\mathbf{a}_k$ is placed in a higher level than $\mathbf{a}_j$ in both graphs,
(ii) $\mathbf{a}_k$ is indifferent to $\mathbf{a}_j$, then $\mathbf{a}_k$ and $\mathbf{a}_j$ are placed in the same level in both graphs, and
(iii) each $\mathbf{a}_k$ in the multilevel graph is placed in the same or lower level than in the hierarchical domination graph since the former captures not only the dominance by an alternative but also the "group dominance" by the reference set.

For instance, neither $\mathbf{a}_{10}$ nor $\mathbf{a}_{13}$ dominates $\mathbf{a}_3$. Therefore, $\mathbf{a}_3$ is placed in the top level of the hierarchical dominance graph as shown in Figure 2. Since $\mathbf{a}_3$ is, however, dominated by

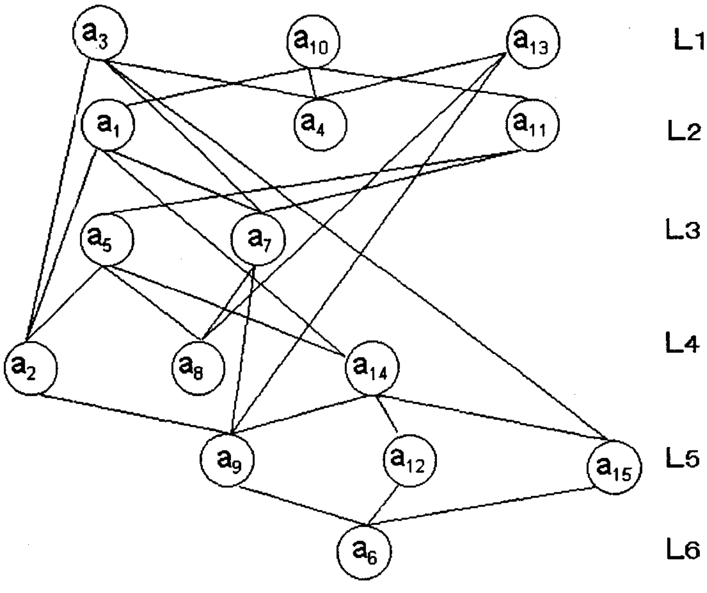

Figure 2. Hierarchical dominance graph based on dominance relations.

$E_3^1 = \{a_{10}, a_{13}\}$, it cannot be the top of the ranking and is placed in the second level as shown in Figure 1. And, a_4 in the second level in Figure 2 is dominated by $E_4^2 = \{a_1, a_3\}$ and is placed in the third level in Figure 1. Similarly, a_7, a_8, a_9, a_6 are, respectively, placed in lower levels than those in Figure 2.

3. CONCLUDING REMARKS

We have presented a multilevel graph of alternatives to represent the incomplete ranking, to the extent that this is possible when incomplete information on the weights is available under the assumption of the additive value function. The nested decomposition of the set of alternatives is established by sequentially locating efficient frontiers using the DEA formulation. A numerical example is given to illustrate a multilevel graph based on the nested decomposition and compare it with the hierarchical dominance graph based on dominance relations proposed by Park and Kim. It is shown that our procedure provides at least as much information regarding the ranking as does the hierarchical dominance graph based on dominance relations, since the former captures not only the dominance by an alternative but also the group dominance by the reference set.

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