A Data Envelopment Analysis Approach to Multicriteria Decision Problems with Incomplete Information

E. TAKEDA AND J. SATOH
Graduate School of Economics, Osaka University
Toyonaka, Osaka 560-0043, Japan

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Abstract—As a decision aid for discrete multicriteria decision problems, this paper proposes a multilevel graph of alternatives to represent the ranking, to the extent that this is possible when incomplete information on weights is available under the assumption of the additive value function. To construct it, the nested decomposition of the set of alternatives is established along the lines of data envelopment analysis (DEA). A numerical example is given to illustrate a multilevel graph based on the nested decomposition and compare it with the hierarchical dominance graph based on dominance relations proposed by Park and Kim. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In discrete alternative multicriteria decision problems, the primary concern for the decision aid is the following:

(1) choosing the most preferred alternative to the decision maker (DM),
(2) ranking alternatives in order of importance for selection problems, or
(3) screening alternatives for the final decision.

The general concepts of domination structures and nondominated solutions play an important role in describing the decision problems and the decision maker’s revealed preferences described above (see [1]). So far, various approaches have been developed as the decision aid (see, for example, [2]). Within the category (1), interactive methods based on the preference cones have been proposed to effectively get the most preferred solution (see, for instance, [3-5]). In these approaches, under the assumption of an implicit quasi-concave increasing value function, preference cones are constructed by pairwise comparisons among alternatives at each iteration. Then, the set of alternatives is gradually reduced to a smaller one by identifying and eliminating inferior alternatives from the set of alternatives by preference cones and finally end up with the most preferred alternative.

On the other hand, it is not uncommon that the DM is only willing or able to provide incomplete information, due to time pressure, lack of knowledge, fear of commitment, etc. Thus, from the
necessity of considering incomplete information, Weber [6] presented a general framework for
decision making with incomplete information. Kirkwood and Sarin [7] derived conditions to
determine whether a pair of alternatives can be ranked and presented a procedure for ranking
alternatives using an additive value function with the incomplete information on the weights.
Kmietowicz and Pearman [8] dealt with decision problems under conditions of linear partial
information (LPI) on probabilities of occurrence for the states of nature and derived conditions
ensuring strict and weak statistical dominance of one strategy over another. Pearman [9] proposed
an ordered metric method for establishing the dominance of alternatives using the linear additive
weighting rule in multiattribute decision making under the LPI on the weights. Park and Kim [10]
proposed a hierarchical dominance graph (HDG) by using pairwise dominance relations in the
multiattribute decision making with the decision maker's incomplete information on both weights
and utilities under the assumption of the additive value function. The HDG can be used to aid
in selecting one or more preferred alternatives.

The purpose of this paper is to propose a multilevel graph which visualizes an incomplete
ranking of alternatives, to the extent that this is possible when incomplete information on the
weights is available under the assumption of the additive value function. To construct it, the
nested decomposition of the set of alternatives is established by sequentially locating alternatives,
each of which is a top ranking for some weight, and then deleting them from the set of alternatives
and locating alternatives, each of which is a top ranking for some weight among the remaining
set, and so on. At the same time, the reference set on an immediate higher level of the alternative
being evaluated is located since, for any weight, at least one alternative in the reference set is a
higher ranking than it. These can be done along the lines of data envelopment analysis (see, for
instance, [11,12]). According to the reference set, a multilevel graph is constructed.

In the following section, we first show how to decompose the set of alternatives and construct
a multilevel graph by the DEA formulation. It is shown that alternatives on the pth level of
the graph based on the nested decomposition have at best pth ranking. Then, using a numerical
example, we compare it with the HDG based on dominance relations proposed by Park and
Kim [10]. Concluding remarks are given in the final section.

2. MULTILEVEL GRAPHS BASED
ON THE NESTED DECOMPOSITION

Let us consider n alternatives a_i, i = 1,2,...,n. And let
\[ A = \{a_i\} \]
Suppose that each alternative a_i has a multiattribute outcome denoted by a vector
\[ x_i = (x_{i1}, x_{i2}, \ldots, x_{im}) \]
where x_{ji} is the measurement on attribute j, j = 1,2,...,m. We assume that the attributes are
additively independent. Thus, the value function \( v(x_i) \) is written by a weighted additive value
function
\[ v(x_i) = \sum w_j v_j (x_{ji}) , \]
where \( w_j > 0 \) is a relative weight of attribute j (it is not necessary to sum up to unity) and \( v_j \)
is a single-attribute value function satisfying
\[ 0 \leq v_j (x_{ji}) \leq 1, \quad j = 1,2,...,m. \]

This paper is concerned with multicriteria decision problems with incomplete information on
the weights, when \( v_j \) is given. In what follows, a_i and v_i are used interchangeably whenever no
confusion arises. Thus, let
\[ v(a_i) = w^\top v_i, \]
where \( v_{ji} = v_j (x_{ji}) \), \( v_i = (v_{ji}) \), and \( w = (w_j) > 0 \).
Let the set of weights be

\[ W^{(0)} = \{ w : w > 0 \} \]

Now let us consider a situation where the set of weights is further restricted by

\[ W^{(1)} = \{ w : w^T b_j^{(1)} > 0, \quad j = 1, \ldots, t(1) \} \]
\[ W^{(2)} = \{ w : w^T b_j^{(2)} \geq 0, \quad j = 1, \ldots, t(2) \} \]
\[ W^{(3)} = \{ w : w^T b_j^{(3)} = 0, \quad j = 1, \ldots, t(3) \} \]

where \( T \) represents the transpose and \( b_j^{(i)} \), \( i = 1, 2, 3 \), are \( m \)-dimensional vectors.

**Remark 1.** \( W^{(i)}, i = 1, 2, 3 \), may be constructed as follows:

(a) the DMs respond to some preliminary questions for any pairs of alternatives, say \( a_1 \) and \( a_2 \),
   (1) if the DM prefers \( a_1 \) to \( a_2 \), then one can deduce from this \( w_{a_1} > w_{a_2} \) and \( b_j^{(1)} = a_1 - a_2 \),
   (2) if \( a_1 \) is at least as good as \( a_2 \), this is interpreted as \( w_{a_1} \geq w_{a_2} \) and \( b_j^{(2)} = a_1 - a_2 \),
   (3) if \( a_1 \) is indifferent to \( a_2 \), then construct an equality of the form \( w_{a_1} = w_{a_2} \) and \( b_j^{(3)} = a_1 - a_2 \),
   and/or
(b) the information on the criteria, for instance, the order of importance, \( w_1 \geq w_2 \geq \ldots \geq w_m \).

In what follows, let

\[ W = \bigcap_i W^{(i)}, \quad \text{for all nonempty sets } W^{(i)}, \quad i = 0, 1, 2, 3. \]

To rank alternatives, to the extent that this is possible, according to \( W \) consider the following linear programming problem.

For each \( a_k \),

\[
\begin{align*}
\text{maximize} & \quad v_k = w^T v_k, \\
\text{subject to} & \quad v_j = w^T v_j \leq 1, \quad j = 1, 2, \ldots, n, \\
& \quad w \in W,
\end{align*}
\]

or equivalently,

\[
\begin{align*}
\text{maximize} & \quad v_k = w^T v_k, \\
\text{subject to} & \quad v_j = w^T v_j \leq 1, \quad j = 1, 2, \ldots, n, \\
& \quad w \succeq e_m, \\
& \quad w^T B^{(1)} \succeq e_{m+1}, \\
& \quad w^T B^{(2)} \succeq 0^T, \\
& \quad w^T B^{(3)} = 0^T,
\end{align*}
\]

where \( e \) is a positive non-Archimedean infinitesimal which is used to replace > with \( \succeq \) as is used in DEA, \( e_m = (1, 1, \ldots, 1)^T \) is an \( m \)-dimensional vector, \( B^{(i)}, i = 1, 2, 3, \) is an \( m \times t^{(i)} \) matrix whose column vectors are \( b_j^{(i)} \), \( j = 1, 2, \ldots, t^{(i)} \).

**Remark 2.**

(i) Note that the last three constraints in \( \text{(Pk)} \) correspond to \( W^{(i)} \), \( i = 1, 2, 3 \), respectively, and therefore, whenever \( W^{(i)} = \emptyset, \quad i = 1, 2, 3 \), the corresponding constraint is removed from the constraints.

(ii) In the formulation of \( \text{(Pk)} \), observe that we can assume \( v_j = w^T v_j \leq c, \quad j = 1, 2, \ldots, n \), for any \( c > 0 \), instead of \( v_j = w^T v_j \leq 1, \quad j = 1, 2, \ldots, n \), since \( w \) is not normalized to one.
Note that if \( v_k^* = 1 \), \( a_k \) is a top ranking alternative for \( w^* \in W \),

\[
v_k \geq v_j, \quad j = 1, 2, \ldots, n, \quad \text{for } w^* \in W,
\]

where \( v_k^* \) and \( w^* \) is an optimal solution to \( (P_k) \).

On the other hand, if \( v_k^* < 1 \), \( a_k \) does not become a top ranking alternative for any \( w \in W \), since for any \( w \in W \) there exists at least one \( v_i \) such that

\[
v_k < v_i.
\]

Thus, there is a possibility that \( a_k \) is top ranking alternative, if and only if \( v_k^* = 1 \).

**Definition 1.** Alternative \( a_i \) dominates \( a_j \) with respect to \( W \) if and only if

\[
v_i = w^T v_i > v_j = w^T v_j, \quad \text{for all } w \in W.
\]

**Remark 3.** From the context of the value function, \( a_i \) dominates \( a_j \) with respect to \( W \) if and only if \( a_i \) is preferred to \( a_j \) for each \( w \in W \) (see [6]).

Let us now consider the dual problem of \( (P_k) \):

\[
\begin{align*}
\text{minimize} & \quad z_k = e_n^T \lambda - \varepsilon e_m^T \mu - \varepsilon e_{t(i)}^T \mu^{(1)}, \\
\text{subject to} & \quad v_k = X \lambda - \mu - B^{(1)} \mu^{(1)} - B^{(2)} \mu^{(2)} - B^{(3)} \mu^{(3)}, \\
& \quad \lambda \geq 0, \\
& \quad \mu \geq 0, \\
& \quad \mu^{(1)} \geq 0, \\
& \quad \mu^{(2)} \geq 0, \\
& \quad \mu^{(3)} : \text{free}, \\
& \quad \lambda = (\lambda_i), \quad i = 1, 2, \ldots, n, \\
& \quad \mu = (\mu_j), \quad j = 1, 2, \ldots, m, \\
& \quad \mu^{(i)} = (\mu_j^{(i)}), \quad i = 1, 2, 3; \\
& \quad X = \text{the } m \times n \text{ matrix whose column vectors are } v^i, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

**Remark 4.**

(i) If \( W^{(i)} = \phi, \) \( i = 1, 2, 3, \) in \( (P_k) \), then the corresponding \( B^{(i)} \) and \( \mu^{(i)} \) are deleted in \( (D_k) \).

(ii) The non-Archimedean infinitesimal \( \varepsilon > 0 \) allows the minimization over \( e_n^T \lambda \) to preempt the maximization of the sum of \( \mu \) and \( \mu^{(i)} \). In this way, \( (D_k) \) is a computational form without any need to specify it explicitly (see, for instance, [12, p. 9]).

(iii) It follows from the dual theorem that \( v_k^* = z_k^* \leq 1 \), where \( v_k^* \) and \( z_k^* \) are optimal values of \( (P_k) \) and \( (D_k) \), respectively. Thus, there is a possibility that \( a_k \) is a top ranking alternative if and only if \( z_k^* \leq 1 \).

(iv) If \( e_n^T \lambda^* > 1 \) in \( (D_k) \), then \( e_n^T \lambda^* - \varepsilon > 1 \), for any \( c > 0 \), which contradicts with \( z_k^* \leq 1 \). Therefore, \( e_n^T \lambda^* \leq 1 \).

(v) It follows that \( z_k^* = 1 \), if and only if \( e_n^T \lambda^* = 1, \mu^* = 0, \) and \( \mu^{(1)*} = 0 \), where \( z_k^*, \lambda^*, \mu^*, \) and \( \mu^{(1)*} \) is an optimal solution to \( (D_k) \).

The procedure for constructing a multilevel graph is as follows.

**Nested Decomposition**

**Step 1.** To begin with, let \( p = 1, A_p = A \).

**Step 2.** For each \( a_k \) in \( A_p \), solve the following linear programming.
minimize $z_k = e_n^T \lambda - e_m^T \mu - e_{(l+1)}^T \mu^{(1)}$
subject to $v_k = X_p \lambda - \mu - B^{(1)} \mu^{(1)} - B^{(2)} \mu^{(2)} - B^{(3)} \mu^{(3)}$

where $\lambda = (\lambda_i), i = 1, 2, \ldots, n_p, X_p$ is an $m \times n_p$ matrix whose column vectors are $v_i, i = 1, 2, \ldots, n_p$ and $n_p$ is the number of alternatives in $A_p$ (renumbering, if necessary).

When $z_k < 1$ in problem (Dk)p for $a_k$, let the reference set of $a_k$ be

$E_k = \{ j \mid \lambda_j^* > 0 \}$

where $\lambda_j^*$ is the optimum solution to (Dk)p for $a_k$.

STEP 3. Let $C_1$ be the set of all alternatives with $z_k^* = 1$ in $A_1 = A$. Removing $C_1$ from the set of alternatives, let $p = 2$ and the remaining set be $A_2$, i.e., $A_2 = A_1 \setminus C_1$. For each alternative $a_k$ in $A_2$, again solve problem (Dk)p. Let $C_2$ be a set consisting of all alternatives with $z_k^* = 1$ in $A_2$. Continue the above process until the remaining set $A_p$ is empty.

Constructing a Multilevel Graph

STEP 4. A multilevel graph $G(X, A)$ is constructed as follows: put $C_1$ in the top level of the graph and $C_2$ in the second level of the graph and so on. Finally, an arc $(i, j)$ from $a_i$ in $C_p$ to $a_j$ in $C_{p+1}$ ($p = 1, 2, \ldots$) is placed if and only if $a_i$ belongs to the reference set $E_j^p$ of $a_j$. Thus,

$A = \{(a_i, a_j) : a_i \in C_p; a_i \in E_j^p \text{ to } a_j \in C_{p+1} (p = 1, 2, \ldots)\}$

REMARK 5. In Step 3, alternatives within $C_p$ are incomparable without any further information on the weights.

THEOREM 1. Let $a_i$ be in $C_{p+1}$. Then, for any $w \in W$, there exists at least one $a_i$ in $E_j^p \subset C_p$, such that

$w^T v_i < w^T v_j$

PROOF. Since $a_k \in C_{p+1}$, we have, $z_k^* < 1$ in problem (Dk)p for $a_k$, i.e.,

minimize $z_k = e_n^T \lambda - e_m^T \mu^* - e_{(l+1)}^T \mu^{(1)*} < 1$
subject to $v_k = X_p \lambda^* - \mu^* - B^{(1)} \mu^{(1)*} - B^{(2)} \mu^{(2)*} - B^{(3)} \mu^{(3)*}$

where $\lambda^* = (\lambda_i^*), \mu^* = (\mu_j^*), \mu^{(1)*} = (\mu_j^{(1)*}), i = 1, 2, 3$, are an optimum solution to problem (Dk)p.

When $v_k = 0$, it is obvious. So, let $v_k \neq 0$. 

weird
Suppose, to the contrary, that there is a weight \( w \in W \) such that
\[
 w^T v_k \geq w^T v_i, \quad \text{for all } a_i \in E^p_k \subset C_p. \tag{a}
\]
Note that \( z^*_k < 1 \) implies either
(i) \( e_{np}^*, \lambda^* < 1 \), or
(ii) \( \sigma_{np}^*, \lambda^* - 1 \).
In the case of (i), since \( w^T v_k > 0 \), it follows from (a) that
\[
 w^T v_k > \sum \lambda_i^* w^T v_i \geq \sum \lambda_i^* w^T v_i = w^T X_p \lambda^*.
\]
On the other hand,
\[
 w^T v_k = w^T X_p \lambda^* - w^T \mu^* - w^T B^{(1)}(\mu^{(1)*}) - w^T B^{(2)}(\mu^{(2)*}) - w^T B^{(3)}(\mu^{(3)*}).
\]
Since \( w \in W \), it follows that \( w > 0 \), \( w^T B^{(1)} > 0 \), \( w^T B^{(2)} \geq 0 \), and \( w^T B^{(3)} = 0 \). Therefore, we obtain
\[
 w^T v_k \leq w^T X_p \lambda^*,
\]
which is a contradiction. In the case of (ii), it follows from (a) that
\[
 w^T v_k = \sum \lambda_i^* w^T v_k \geq \sum \lambda_i^* w^T v_i = w^T X_p \lambda^*.
\]
Note that either
(1) \( \mu_j^* > 0 \) for at least one \( j \), and/or
(2) \( \mu_j^{(1)*} > 0 \) for at least one \( j \)
holds. Since \( w > 0 \) and \( w^T B^{(1)} > 0 \), in either case, we have
\[
 w^T v_k < w^T X_p \lambda^*.
\]
which leads to a contradiction.

REMARK 6.
(i) From Theorem 1, we can see that any \( a_k \) in \( C_p \) has, at best, a \( p \)th ranking, that is, a ranking of \( p \) or less.
(ii) The reference set \( E^p_k (\subset C_p) \) is not necessarily unique. One can, however, say that \( a_k \in C_{p+1} \) is dominated by \( E^p_k \).
(iii) We can conclude that, if only one alternative in \( E^p_k \) of \( a_k \in C_{p+1} \) exists, that is, only one arc \( (a_i, a_k) \) from \( C_p \) to \( a_k \in C_{p+1} \) exists in the multilevel graph, then \( a_k \) is dominated by \( a_i \).

Observe that if \( a_i \) dominates \( a_k \), then \( a_k \) is at a lower level than \( a_i \). To show this, let \( a_k \) be in \( C_p \). Note that \( a_k \) is in \( X_p \). Now, let us suppose that \( a_i \) is also in \( X_p \). The dual of problem (Dk)p is
\[
\begin{align*}
\text{maximize} & \quad v_k = w^T v_k, \\
\text{subject to} & \quad w^T v_i \leq 1, \quad \text{for } i = 1, 2, \ldots, n_p, \\
& \quad w \in W. \quad \text{(Pk)p}
\end{align*}
\]
Since \( a_i \) dominates \( a_k \),
\[
v_i = w^T v_i > v_k = w^T v_k, \quad \text{for all } \leq w \in W,
\]
which yields
\[
\text{maximize} \quad v_k = w^T v_k < 1.
\]
Therefore, \( a_k \) cannot belong to \( C_p \), which is a contradiction. Thus, we have the following theorem.
THEOREM 2. If \( a_1 \) dominates \( a_k \), then \( a_k \) is at a lower level than \( a_1 \) in the multilevel graph.

Though the dominance relations can be defined in the decision problems with incomplete information about both weights and utilities (see [10]), in a special case where the value of utilities is known precisely, it is easy to establish the dominance relations.

The set of collecting dominance relations between the alternatives \( \Omega \subseteq A \times A \) is defined so as to include the indifference relations as \( (a_i, a_j) \in \Omega \) if and only if \( a_i \) is at least as preferred as \( a_j \), where \( a_j \neq a_j \). After \( \Omega \) is identified, a hierarchical dominance graph \( G_H(H(A), E) \) with \( H(A) = [H_1, \ldots, H_L] \), where a set of arcs \( E \subseteq A \times A \) is the set \( \Omega \), \( H_k \subseteq A \) is a set of alternatives in the \( k^{th} \) level, \( L \) the number of levels of \( G_H \), and \( H_k \neq \emptyset \ \forall k \), is constructed as follows.

Construction of a Hierarchical Dominance Graph Based on the Dominance Relation

**Step 1.** Construct the adjacent matrix \( M \) by using the information of \( \Omega \).

**Step 2.** Compute the reachability matrix \( R \) of \( M \).

**Step 3.** Perform the following iterative procedure with \( H_0 = \emptyset \) and \( k = 1 \).

a. Construct \( B_k = A - \bigcup_{i=1}^{k-1} H_i \). If \( B_k = \emptyset \), the set \( L = k - 1 \) and go to Step 4.

b. Find \( H_k = \{ a_i \in B_k \mid P_k(a_i) = P_k(a_i) \cap S_k(a_i) \} \), where \( P_k(a_i) \) and \( S_k(a_i) \), respectively, are sets of predecessors and successors denoted by the subgraph consisting of the elements in \( B_k \).

c. Set \( k = k + 1 \) and go to Step 3.a.

**Step 4.** Display \( G_H(H(A), E) \) with \( H(A) = [H_1, \ldots, H_L] \).

**Numerical Example.** Let us consider the following example, with three criteria, \( O_1 \), \( O_2 \), and \( O_3 \), as shown in Table 1.

<table>
<thead>
<tr>
<th>( a_i )</th>
<th>( O_1 )</th>
<th>( O_2 )</th>
<th>( O_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>.65</td>
<td>.72</td>
<td>.38</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>.40</td>
<td>.88</td>
<td>.19</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>.77</td>
<td>.30</td>
<td>.04</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>.68</td>
<td>.39</td>
<td>.26</td>
</tr>
<tr>
<td>( a_5 )</td>
<td>.58</td>
<td>.56</td>
<td>.67</td>
</tr>
<tr>
<td>( a_6 )</td>
<td>.26</td>
<td>.65</td>
<td>.35</td>
</tr>
<tr>
<td>( a_7 )</td>
<td>.63</td>
<td>.43</td>
<td>.48</td>
</tr>
<tr>
<td>( a_8 )</td>
<td>.47</td>
<td>.72</td>
<td>.12</td>
</tr>
<tr>
<td>( a_9 )</td>
<td>.34</td>
<td>.78</td>
<td>.23</td>
</tr>
<tr>
<td>( a_{10} )</td>
<td>.78</td>
<td>.67</td>
<td>.45</td>
</tr>
<tr>
<td>( a_{11} )</td>
<td>.73</td>
<td>.24</td>
<td>.91</td>
</tr>
<tr>
<td>( a_{12} )</td>
<td>.37</td>
<td>.68</td>
<td>.56</td>
</tr>
<tr>
<td>( a_{13} )</td>
<td>.89</td>
<td>.21</td>
<td>.52</td>
</tr>
<tr>
<td>( a_{14} )</td>
<td>.44</td>
<td>.57</td>
<td>.62</td>
</tr>
<tr>
<td>( a_{15} )</td>
<td>.33</td>
<td>.76</td>
<td>.39</td>
</tr>
</tbody>
</table>

Supposing that

(i) \( a_{11} \) is preferred to \( a_5 \), and
(ii) \( a_{10} \) is preferred to \( a_{11} \),

let the set of weights \( W \) be

\[
W^{(0)} = \{ w : w > 0 \} \\
W^{(1)} = \{ w : w^T b_j^{(1)} > 0, \ j = 1, 2 \} \\
W^{(2)} = W^{(3)} = \phi
\]
and therefore,

\[ W = W^{(0)} \cap W^{(1)}, \]

where \( b_1^{(1)} = a_{11} - a_5 = (0.15, -0.32, 0.24) \) and \( b_2^{(1)} = a_{10} - a_{11} = (0.05, 0.43, -0.46) \).

A multilevel graph based on the nested decomposition is shown in Figure 1. In this figure, while dotted lines designate the "group dominance" by a reference set, solid lines simply represent the dominance by a single alternative.

On the other hand, a hierarchical dominance graph is constructed in this example. The set \( \Omega \) is as follows:

\[
\{(a_1, a_2), (a_1, a_6), (a_1, a_7), (a_1, a_8), (a_1, a_9), (a_1, a_{12}), (a_1, a_{14}), (a_1, a_{15}), \\
(a_2, a_6), (a_2, a_9), (a_3, a_2), (a_3, a_4), (a_3, a_5), (a_3, a_7), (a_3, a_8), (a_3, a_9), (a_3, a_{15}), (a_5, a_2), \\
(a_5, a_6), (a_5, a_8), (a_5, a_{15}), (a_5, a_{17}), (a_6, a_12), (a_6, a_{14}), (a_7, a_8), (a_7, a_9), \\
(a_9, a_{10}), (a_{10}, a_2), (a_{10}, a_4), (a_{10}, a_5), (a_{10}, a_6), (a_{10}, a_{17}), (a_{10}, a_{18}), \\
(a_{10}, a_{19}), (a_{10}, a_{11}), (a_{10}, a_{12}), (a_{10}, a_{14}), (a_{10}, a_{15}), (a_{10}, a_{16}), (a_{12}, a_6), (a_{13}, a_6), \\
(a_{13}, a_6), (a_{13}, a_8), (a_{13}, a_{14}), (a_{14}, a_6), (a_{14}, a_{11}), (a_{14}, a_{12}), (a_{14}, a_{13}), (a_{14}, a_{15}), (a_{15}, a_6)\}.
\]

Since \( \Omega \) is transitive, \( R = (r_{ij}) = I + M \) where \( I \) is an \( n \times n \) identity matrix. From the set \( \Omega \), a hierarchical dominance graph \( G_H(H(A), E) \) is constructed as shown in Figure 2. In this figure, arcs which are derived from transitivity are omitted and therefore, \( E \subseteq \Omega \).

It is clear that

(i) if \( a_k \) dominates \( a_j \), then \( a_k \) is placed in a higher level than \( a_j \) in both graphs,

(ii) if \( a_k \) is indifferent to \( a_j \), then \( a_k \) and \( a_j \) are placed in the same level in both graphs, and

(iii) each \( a_k \) in the multilevel graph is placed in the same or lower level than in the hierarchical dominance graph since the former captures not only the dominance by an alternative but also the "group dominance" by the reference set.

For instance, neither \( a_{10} \) nor \( a_{13} \) dominates \( a_5 \). Therefore, \( a_5 \) is placed in the top level of the hierarchical dominance graph as shown in Figure 2. Since \( a_3 \) is, however, dominated by
Figure 2. Hierarchical dominance graph based on dominance relations.

$E_2^1 = \{a_{10}, a_{13}\}$, it cannot be the top of the ranking and is placed in the second level as shown in Figure 1. And, $a_7$ in the second level in Figure 2 is dominated by $E_2^3 = \{a_1, a_3\}$ and is placed in the third level in Figure 1. Similarly, $a_7, a_8, a_9, a_{10}$ are, respectively, placed in lower levels than those in Figure 2.

3. CONCLUDING REMARKS

We have presented a multilevel graph of alternatives to represent the incomplete ranking, to the extent that this is possible when incomplete information on the weights is available under the assumption of the additive value function. The nested decomposition of the set of alternatives is established by sequentially locating efficient frontiers using the DEA formulation. A numerical example is given to illustrate a multilevel graph based on the nested decomposition and compare it with the hierarchical dominance graph based on dominance relations proposed by Park and Kim. It is shown that our procedure provides at least as much information regarding the ranking as does the hierarchical dominance graph based on dominance relations, since the former captures not only the dominance by an alternative but also the group dominance by the reference set.

REFERENCES


