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A generalization of the firefighter problem on $\mathbb{Z}\times\mathbb{Z}$

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Abstract

We consider a generalization of the firefighter problem where the number of firefighters available per time step t is not a constant. We show that if the number of firefighters available is periodic in t and the average number per time period exceeds $\frac{3}{2}$, then a fire starting at any finite number of vertices in the two dimensional infinite grid graph can always be contained.

Keywords: Graph; Firefighter problem; Containment certificate

1. Introduction and terminology

The *firefighter problem* is a dynamic problem introduced by Hartnell [6], that can be described as follows: given a connected rooted graph (G, r), r is initially set on fire. At the beginning of each discrete time period $t \geqslant 1$, a number of firefighters are available to be positioned at different vertices in G that are currently not on fire nor already have a firefighter positioned. For this paper, we shall represent the number of firefighters available at each $t \geqslant 1$ by a function $f(t)$. These firefighters remain on their assigned vertices and thus prevent the fire from spreading to that vertex. At the end of each time period, all vertices that are not defended and are adjacent to at least one vertex on fire will catch the fire and become burned. Once the vertex is burned or defended, it remains that way permanently.

If G is a finite graph, the process ends when one of the following occurs:

(i) The fire is *contained*, meaning the fire is unable to spread, and there are still vertices in G that are neither burned nor defended.
(ii) The fire spreads until every vertex in G is either burned or defended.

If G is infinite, then (i) could still happen but (ii) is replaced by

(ii′) The fire cannot be contained, meaning the fire spreads indefinitely.

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The firefighter problem was considered on a variety of graphs, including finite grids [10,12], infinite grids [1,3,12] and trees [6,11]. Other related publications [2,4,5,7–9] are listed in the reference section.

In this paper, we will consider the two dimensional infinite grid graph $G = \mathbb{L}_2$ defined by

$$V(G) = \mathbb{Z} \times \mathbb{Z},$$

$$E(G) = \{\{(m, n), (m', n')\} \mid |m - m'| + |n - n'| = 1\}.$$

Suppose we are given a function $f(t)$ representing the number of firefighters available for deployment at each time period t, our goal is to determine if it is possible to position the firefighters on the vertices of $\mathbb{L}_2$ such that at some finite time t', the fire is unable to spread any further. For our purposes, we shall only consider functions $f(t)$ that are periodic in t. Thus, we can state our problem formally as:

CONTAINMENT. *Instance*: A rooted graph $(\mathbb{L}_2, r)$ and a periodic function $f(t)$.

Question: Is there a finite t' such that by positioning $f(t)$ firefighters at each time period t, the fire can be contained after t' time periods.

Most of the existing literature considers $f(t)$ to be a constant function (usually $f(t)=1$) independent of t. Specifically, Wang and Moeller [12] showed that one firefighter per time period ($f(t) = 1 \forall t$) is insufficient to prevent the fire from spreading indefinitely while $f(t) = 2$ for all t suffices, in which case a minimum of 8 time periods are required to successfully contain the fire. An alternative proof (using a computer program) to the minimum number of time periods required when $f(t) = 2$ for all t was provided by Develin and Hartke [1], who also established that a minimum of 18 vertices in $\mathbb{L}_2$ would be burnt before containment can be achieved. One way to generalize the firefighter problem introduced by Hartnell is to allow the fire to start initially at a finite number of vertices in $\mathbb{L}_2$ rather than a single root r. This was considered by Fogarty [3] when it was shown that $f(t) = 2$ for all t is sufficient to contain a fire that starts at any *finite* number of vertices in $\mathbb{L}_2$. For the remainder of this paper, we shall consider the firefighter problem where the fire could start initially at either a single vertex or a finite collection of vertices in $\mathbb{L}_2$.

The results by Wang and Moeller [12], Develin and Hartke [1] and Fogarty [3] described above provide the motivation for this paper. We would like to know if $f(t)$ is not a constant function, and the *average* (whose notion will be made precise below) number of firefighters available per time period is a number between 1 and 2, is there a finite t' such that by positioning $f(t)$ firefighters at each time period t, the fire can be contained after t' time periods?

To make the notion of the average number of firefighters per time period precise, let $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be a periodic function with period p_f. Define

$$N_f = \sum_{t=1}^{p_f} f(t) \quad \text{and} \quad R_f = \frac{N_f}{p_f}.$$

Thus, if the number of firefighters available for deployment at each time period is given by f, then R_f tells us the *average number* of firefighters available for deployment at each time period. We will frequently identify f with a sequence of its period. For example, we write $f = [2, 1, 2, 2]$ to correspond to the function defined as

$$f(t) = \begin{cases} 2 & \text{if } t \equiv 1 \bmod 4, \\ 1 & \text{if } t \equiv 2 \bmod 4, \\ 2 & \text{if } t \equiv 3 \bmod 4, \\ 2 & \text{if } t \equiv 0 \bmod 4. \end{cases}$$

Observe that $R_f = 1.75$ in this example. For any function $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$, define $f^{-1} : \mathbb{N} \to \mathbb{N}$ as

$$f^{-1}(n) = \min\left\{ j \in \mathbb{N} \,\middle|\, \sum_{t=1}^{j} f(t) \geqslant n \right\}.$$

In other words, $f^{-1}(n)$ can be thought of as the time t when the nth firefighter becomes available for deployment. Note that $f^{-1}(n)$ is a non-decreasing function of n. For a finite set $S \subset \mathbb{Z} \times \mathbb{Z}$ and some $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, define

$$d(S, (x, y)) = \min\{|x' - x| + |y' - y| \,|\, (x', y') \in S\}.$$

For any periodic function f and $S \subset \mathbb{Z} \times \mathbb{Z}$, we say that there is a *containment certificate of f for S* if and only if there exists a set $C_S(f) \subset \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}$ that satisfies the following conditions:

(1) For all $t \in \mathbb{N}$, $f(t) \geqslant |\{(x, y, j) \in C_S(f) | j = t\}|$;
(2) For all $(x, y, t) \in C_S(f)$, $d(S, (x, y)) \geqslant t$;
(3) The number of vertices that have at least one path in $\mathbb{L}_2$ to a vertex in S without passing through any vertex (x, y) where $(x, y, t) \in C_S(f)$ for some $t \in \mathbb{N}$ is finite.

Suppose that the set of vertices in S are initially set on fire and $f(t)$ represents the number of firefighters available for deployment at time t. A containment certificate of f for S, if it exists, contains all the information on where and when each available firefighter is deployed such that the spread of the fire can eventually be contained at some finite time t'. For example, if $(8, 9, 4) \in C_S(f)$, then we would place a firefighter on $(8, 9)$ at time $t = 4$. Condition 1 of the containment certificate ensures that there are at most $f(t)$ firefighters deployed at time t. Condition 2 ensures that (x, y) is not already on fire when a firefighter is deployed there at time t. Condition 3 guarantees that there exists some $t' \geqslant \max\{t | (x, y, t) \in C_S(f)\}$ such that the number of vertices on fire at times $t \geqslant t'$ is a constant, meaning that the fire is indeed contained.

Suppose $C_S(f)$ is a containment certificate of f for S. For each $n \in \mathbb{N}$, define

$$C_S^{>n}(f) = \{(x, y, t) \in C_S(f) | t > n\};$$
$$C_S^{=n}(f) = \{(x, y, t) \in C_S(f) | t = n\};$$
$$C_S^{<n}(f) = \{(x, y, t) \in C_S(f) | t < n\}.$$

We will consider two partial orders associated with periodic functions, defined by

$$f \preceq g \iff \sum_{t=1}^{k} f(t) \leqslant \sum_{t=1}^{k} g(t) \quad \forall k \in \mathbb{N}$$

and

$$f \preceq^* g \iff \exists n \in \mathbb{N} \quad \text{such that} \sum_{t=1}^{k} f(t) \leqslant \sum_{t=1}^{k} g(t) \quad \forall k \geqslant n.$$

We say that g *dominates* f if $f \preceq g$ and g *eventually dominates* f if $f \preceq^* g$. Observe the fact that g dominates f implies g eventually dominates f. It is useful to note that to establish $f \preceq g$ for periodic f and g, it suffices to show that

$$\sum_{t=1}^{k} f(t) \leqslant \sum_{t=1}^{k} g(t) \quad \text{for all } 1 \leqslant k \leqslant lcm(p_f, p_g).$$

Several specific periodic functions will be used frequently in this paper. Their definitions and notations are introduced below.

For any $n, k \in \mathbb{Z}^+$, define $g_{n,k}$ to be the periodic function with period n by

$$g_{n,k}(t) = \begin{cases} 0 & \text{if } t \not\equiv 0 \bmod n, \\ k & \text{if } t \equiv 0 \bmod n. \end{cases}$$

In other words, $g_{n,k} = [\overbrace{0, 0, \ldots, 0}^{n-1}, k]$. For any integer $n \geqslant 2$, let $Z_n = g_{n,z_n}$ where

$$z_n = \begin{cases} \frac{3n}{2} + 1 & \text{if } n \text{is even,} \\ \frac{1}{2}(3n+1) & \text{if } n \text{is odd.} \end{cases}$$

Note that for each n, z_n is defined to be the smallest positive integer such that $R_{Z_n} > 1.5$. For any integer $n \geqslant 1$, define F_n by

$$F_n(t) = \begin{cases} 1 & \text{if } t \equiv k \bmod 2n+1, \text{ where } (k \in \{1, 2, \ldots, n\} \\ 2 & \text{if } t \equiv k \bmod 2n+1, \text{ where } k \in \{0, n+1, n+2, \ldots, 2n\}. \end{cases}$$

In other words, $F_n = [\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{2, 2, \ldots, 2}^{n+1}]$. Note that $p_{F_n} = 2n + 1$ and $R_{F_n} > 1.5$ for all $n \geqslant 1$. If f is a periodic function and i is any non-negative integer, let f_{+i} be the *i-translate* of f defined by

$$f_{+i}(t) = f(t+i) \quad \text{for all } t \geqslant 1.$$

Note that $f_{+0} = f$. We are now ready to state the main result of this paper.

Theorem 1. *Suppose a finite set $S \subset \mathbb{Z} \times \mathbb{Z}$ of vertices are initially set on fire. If the number of firefighters available for deployment per time period is given by a periodic function f such that $R_f > 1.5$, then there exists a containment certificate of f for S.*

Remark. The above theorem gives no conclusion about containment of the fire if the function f is such that $R_f \leqslant 1.5$. We will discuss this briefly at the end of the paper.

In Section 2, we will prove several lemmas regarding some of the periodic functions defined above. The main result is proven in Section 3 and the paper concludes in Section 4 with a brief discussion on possible future work.

2. Several lemmas

We first show that the relation $\preceq^*$ is transitive.

Lemma 2.1. *If f, g and h are periodic functions such that $f \preceq^* g$ and $g \preceq^* h$, then $f \preceq^* h$.*

Proof. Let $n_1, n_2 \in \mathbb{N}$ be such that

$$\sum_{t=1}^{k} f(t) \leqslant \sum_{t=1}^{k} g(t) \quad \forall k \geqslant n_1 \quad \text{and} \quad \sum_{t=1}^{k} g(t) \leqslant \sum_{t=1}^{k} h(t) \quad \forall k \geqslant n_2.$$

Let $n = \max\{n_1, n_2\}$. We have

$$\sum_{t=1}^{k} f(t) \leqslant \sum_{t=1}^{k} h(t) \quad \forall k \geqslant n$$

and thus $f \preceq^* h$. $\square$

Lemma 2.2. *For any periodic function f, we have $g_{p_f,N_f} \preceq f$.*

Proof. Note that g_{p_f,N_f} and f have the same period. If $k < p_f$ then we have

$$0 = \sum_{t=1}^{k} g_{p_f,N_f}(t) \leqslant \sum_{t=1}^{k} f(t)$$

since f must take on non-negative values. If $k = p_f$ then

$$\sum_{t=1}^{p_f} g_{p_f,N_f}(t) = \sum_{t=1}^{p_f} f(t)$$

and so by definition we have $g_{p_f,N_f} \preceq f$. □

Lemma 2.3. *If f is a periodic function that is non-decreasing on its period, then $f \preceq f_{+i}$ for all $i \in \mathbb{Z}^+$.*

Proof. Let $i \in \mathbb{Z}^+$. Since f and f_{+i} have the same period, it suffices to show

$$\sum_{t=1}^{n} f(t) \leqslant \sum_{t=1}^{n} f_{+i}(t) \quad \text{for all } n \leqslant p_f.$$

Case 1. Suppose $n + i \leqslant p_f$. In this case, as f is non-decreasing, we have $f(t) \leqslant f(t+i)$ for all $t = 1, 2, \ldots, n$, implying

$$\sum_{t=1}^{n} f(t) \leqslant \sum_{t=1}^{n} f(t+i)$$

and thus $f \preceq f_{+i}$.

Case 2. Suppose $n + i > p_f$. Note that

$$\sum_{t=1}^{n} f(t+i) = \sum_{t=i+1}^{n+i} f(t) = \sum_{t=i+1}^{p_f} f(t) + \sum_{t=p_f+1}^{n+i} f(t)$$

$$= \sum_{t=i+1}^{p_f} f(t) + \sum_{t=1}^{n+i-p_f} f(t).$$

Thus,

$$\sum_{t=1}^{n} f(t) = \sum_{t=1}^{n+i-p_f} f(t) + \sum_{t=n+i-p_f+1}^{n} f(t)$$

$$\leqslant \sum_{t=1}^{n+i-p_f} f(t) + \sum_{t=i+1}^{p_f} f(t) \quad (\text{since } f \text{ is non-decreasing})$$

$$= \sum_{t=i+1}^{n+i} f(t) = \sum_{t=1}^{n} f_{+i}(t)$$

and we are done. □

Lemma 2.4. *If f is a periodic function such that $R_f > 1.5$, then $Z_n \preceq f$ for some $n \geqslant 2$.*

Proof. Recall that for each $n \in \mathbb{N}$, z_n was defined to be the smallest positive integer such that $R_{Z_n} > 1.5$. Since f is a periodic with $R_f > 1.5$ and $Z_n(t) = 0$ for all $t \not\equiv n \pmod{n}$, it is clear that $Z_{p_f} \preceq f$. □

If we want to compare two periodic functions f and g, then as stated before we would have to compare f and g up to $lcm(p_f, p_g)$, which could be as large as $p_f p_g$. The following lemma adds a hypothesis but the end result allows us to simply compare the two functions up to the larger of the two periods.

Lemma 2.5. *Let g be a periodic function that is non-decreasing on its period and f be a periodic function such that $p_f \geqslant p_g$ and*

$$\sum_{t=1}^{p_f} f(t) < \sum_{t=1}^{p_f} g(t).$$

Then $f \preceq^* g$, meaning there exists $n \in \mathbb{N}$ such that

$$\sum_{t=1}^{k} f(t) \leqslant \sum_{t=1}^{k} g(t) \quad \text{for all } k \geqslant n.$$

Proof. We first prove the following claim.

Claim. *For each $k = 1, 2, 3, \ldots,$*

$$\sum_{t=kp_f+1}^{(k+1)p_f} f(t) < \sum_{t=kp_f+1}^{(k+1)p_f} g(t).$$

Proof of Claim. Let $kp_f + 1 = k' p_g + r$, with $0 < r \leqslant p_g$. Then we have

$$\begin{aligned}\sum_{t=kp_f+1}^{(k+1)p_f} g(t) &= \sum_{t=r}^{r+p_f-1} g(t) \\ &= \sum_{t=1}^{p_f} g_{+(r-1)}(t) \\ &\geqslant \sum_{t=1}^{p_f} g(t) \quad \text{by Lemma 2.3} \\ &> \sum_{t=1}^{p_f} f(t) = \sum_{t=kp_f+1}^{(k+1)p_f} f(t).\end{aligned}$$

So from the above claim, the following function

$$h(k) = \sum_{t=1}^{kp_f} g(t) - \sum_{t=1}^{kp_f} f(t)$$

is a strictly increasing function in k. Define k^* by

$$k^* = \min\{k \in \mathbb{N} | h(k) > N_f\}.$$

Now let $n=k^* p_f$. This is the n that we require in order to prove the lemma. To see this, suppose $k\geqslant n$ and $k=a_k p_f+b_k$, where $0\leqslant b_k < p_f$. Then

$$\begin{aligned}
\sum_{t=1}^{k} f(t) &= \sum_{t=1}^{a_k p_f+b_k} f(t) \\
&\leqslant \sum_{t=1}^{a_k p_f+p_f} f(t) \\
&= \sum_{t=1}^{(a_k+1)p_f} f(t) \\
&= \sum_{t=1}^{a_k p_f} f(t) + \sum_{t=a_k p_f+1}^{(a_k+1)p_f} f(t) \\
&= \sum_{t=1}^{a_k p_f} f(t) + \sum_{t=1}^{p_f} f(t) \\
&= \left(\sum_{t=1}^{a_k p_f} g(t) - h(a_k)\right) + \sum_{t=1}^{p_f} f(t) \\
&\leqslant \left(\sum_{t=1}^{a_k p_f} g(t) - h(k^*)\right) + \sum_{t=1}^{p_f} f(t) \quad (\text{since } a_k\geqslant k^*) \\
&< \left(\sum_{t=1}^{a_k p_f} g(t) - \sum_{t=1}^{p_f} f(t)\right) + \sum_{t=1}^{p_f} f(t) \\
&= \sum_{t=1}^{a_k p_f} g(t) \leqslant \sum_{t=1}^{a_k p_f+b_k} g(t) = \sum_{t=1}^{k} g(t).
\end{aligned}$$

The proof of the lemma is thus complete. □

Using Lemma 2.5 we can prove the next lemma easily.

Lemma 2.6. *For each* $n\geqslant 2$, $F_{n^2}\preceq^* Z_n$.

Proof. Note that F_{n^2} is periodic, $p_{F_{n^2}} = 2n^2+1\geqslant n = p_{Z_n}$ and

$$\sum_{t=1}^{2n^2+1} F_{n^2}(t) = n^2 + 2(n^2+1) = 3n^2+2.$$

If n is even, then

$$\sum_{t=1}^{2n^2+1} Z_n(t) = 2n\left(\frac{3n}{2}+1\right) = 3n^2+2n.$$

On the other hand, if n is odd, then

$$\sum_{t=1}^{2n^2+1} Z_n(t) = 2n\left(\frac{3n+1}{2}\right) = 3n^2 + n.$$

In either case, we have

$$\sum_{t=1}^{2n^2+1} F_{n^2}(t) < \sum_{t=1}^{2n^2+1} Z_n(t)$$

and thus by Lemma 2.5, $F_{n^2} \preceq^* Z_n$. $\square$

Lemma 2.7. *Given any periodic function f such that $p_f \geqslant 2$ and $R_f > 1.5$, there exists some $n \geqslant 2$ such that $F_{n^2} \preceq^* f$.*

Proof. Suppose *f* is periodic, $p_f \geqslant 2$ and $R_f > 1.5$. By Lemma 2.2, $g_{p_f,N_f} \preceq^* f$. Note that $R_{g_{p_f,N_f}} = R_f > 1.5$, so by Lemmas 2.4 and 2.6, for some $n \geqslant 2$,

$$F_{n^2} \preceq^* Z_n \preceq^* g_{p_f,N_f}.$$

Applying Lemma 2.1 to

$$F_{n^2} \preceq^* Z_n \preceq^* g_{p_f,N_f} \preceq^* f$$

completes the proof. $\square$

3. Proof of main result

We first state two lemmas without proof.

Lemma 3.1. *Suppose S_1 and S_2 are both finite subsets of $\mathbb{Z} \times \mathbb{Z}$ such that $S_1 \subseteq S_2$. For any function f, if $C_{S_2}(f)$ is a containment certificate of f for S_2, then $C_{S_2}(f)$ is also a containment certificate of f for S_1.*

Lemma 3.2. *For any $d \in \mathbb{N} \cup \{0\}$, let*

$$S_d = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \,|\, |x| + |y| \leqslant d\}.$$

For any $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $(x, y) \notin S_d$,

$$d(S_d, (x, y)) = |x| + |y| - d.$$

Now for any $n \in \mathbb{N}$, recall that $F_n = [\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{2, 2, \ldots, 2}^{n+1}]$ is a periodic function with period $2n + 1$. Let

$$F_n^2 = [\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{2, 2, \ldots, 2}^{n+1}, \overbrace{1, 1, \ldots, 1}^{n}, \overbrace{2, 2, \ldots, 2}^{n+1}].$$

Note that F_n^2 is periodic with period $2(2n + 1)$ and $F_n^2 \preceq F_n$. Let $p = 2(2n + 1)$ and define the function G_p of period p by

$$G_p = [\overbrace{1, 0, 1, 0, \ldots, 1}^{p-1}, p + 1].$$

It is easy to see that $G_p \preceq F_n^2$.

Lemma 3.3. *For any $n, d \in \mathbb{N}$, let $p = 2(2n + 1)$. There exists a containment certificate of G_p for S_d.*

Proof. Consider the following eight sets:

$$A_0 = \bigcup_{i=1,\, i \text{ odd}}^{2(p+1)^3(d+p)-1} \left\{ \left(\frac{i-1}{2}, -\left(d+p+\frac{i-1}{2} \right), i \right) \right\},$$

$$A_1 = \bigcup_{i=1}^{d+p} \bigcup_{k=1}^{p} \{(-(i-1)p-k, -(d+p), ip)\},$$

$$A_2 = \bigcup_{i=1}^{d+p} \{(-p(d+p)-i, -(d+p)+i, ip)\},$$

$$A_3 = \bigcup_{i=1}^{(p+1)(d+p)} \bigcup_{k=1}^{p} \{(-(p+1)(d+p), (i-1)p+k, (d+p+i)p)\},$$

$$A_4 = \bigcup_{i=1}^{(p+1)(d+p)} \{(-(p+1)(d+p)+i, p(p+1)(d+p)+i, (d+p+i)p)\},$$

$$A_5 = \bigcup_{i=1}^{(p+1)^2(d+p)} \bigcup_{k=1}^{p} \{((i-1)p+k, (p+1)^2(d+p), ((p+2)(d+p)+i)p)\},$$

$$A_6 = \bigcup_{i=1}^{(p+1)^2(d+p)} \{(p(p+1)^2(d+p)+i, (p+1)^2(d+p)-i, ((p+2)(d+p)+i)p)\},$$

$$A_7 = \bigcup_{i=1}^{N} \left\{ \begin{array}{l} ((p+1)^3(d+p), -(i-1)p-k, ((d+p)(p+2+(p+1)^2)+i)p) \mid \\ 1 \leqslant k \leqslant p \text{ and } (i-1)p+k \leqslant ((p+1)^3+1)(d+p)+2 \end{array} \right\},$$

where

$$N = \left\lceil \frac{((p+1)^3+1)(d+p)+2}{p} \right\rceil.$$

We claim that $A = \bigcup_{i=0}^{7} A_i$ is a containment certificate of G_p for S_d.

Fig. 1illustrates the positions corresponding to the set $A = \bigcup_{i=0}^{7} A_i$. Recall that an element (x, y, t) in a containment certificate can be thought of as the time t where a firefighter is positioned at (x, y). To show that the first condition in the definition of a containment certificate is satisfied, it is easier to describe the elements of the eight sets in terms on their positions on $\mathbb{Z} \times \mathbb{Z}$ and when these positions are taken up by the firefighters. Note that $G_p(t) = 1$ for all odd t, $G_p(t) = p + 1$ if $t = kp$ for some $k \in \mathbb{N}$ and $G_p(t) = 0$ otherwise.

(1) At each odd $t = 1, 3, \ldots, 2(p+1)^3(d+p) - 1$, a firefighter is positioned at $((t-1)/2, -(d+p+(t-1)/2))$. This corresponds to the set A_0.
(2) At each $t = ip$, $i = 1, 2, \ldots, d+p$, we have $p+1$ firefighters available, p of which have positions given by A_1 (forming a horizontal line) and the remaining one has position given by A_2 (forming a diagonal line).
(3) At each $t = (d+p+i)p$, $i = 1, \ldots, (p+1)(d+p)p$, we have $p+1$ firefighters available, p of which have positions given by A_3 (forming a vertical line) and the remaining one has position given by A_4 (forming a diagonal line).
(4) At each $t = ((p+2)(d+p)+i)p$, $i = 1, \ldots, (p+1)^2(d+p)$, we have $p+1$ firefighters available, p of which have positions given by A_5 (forming a horizontal line) and the remaining one has position given by A_6 (forming a diagonal line).

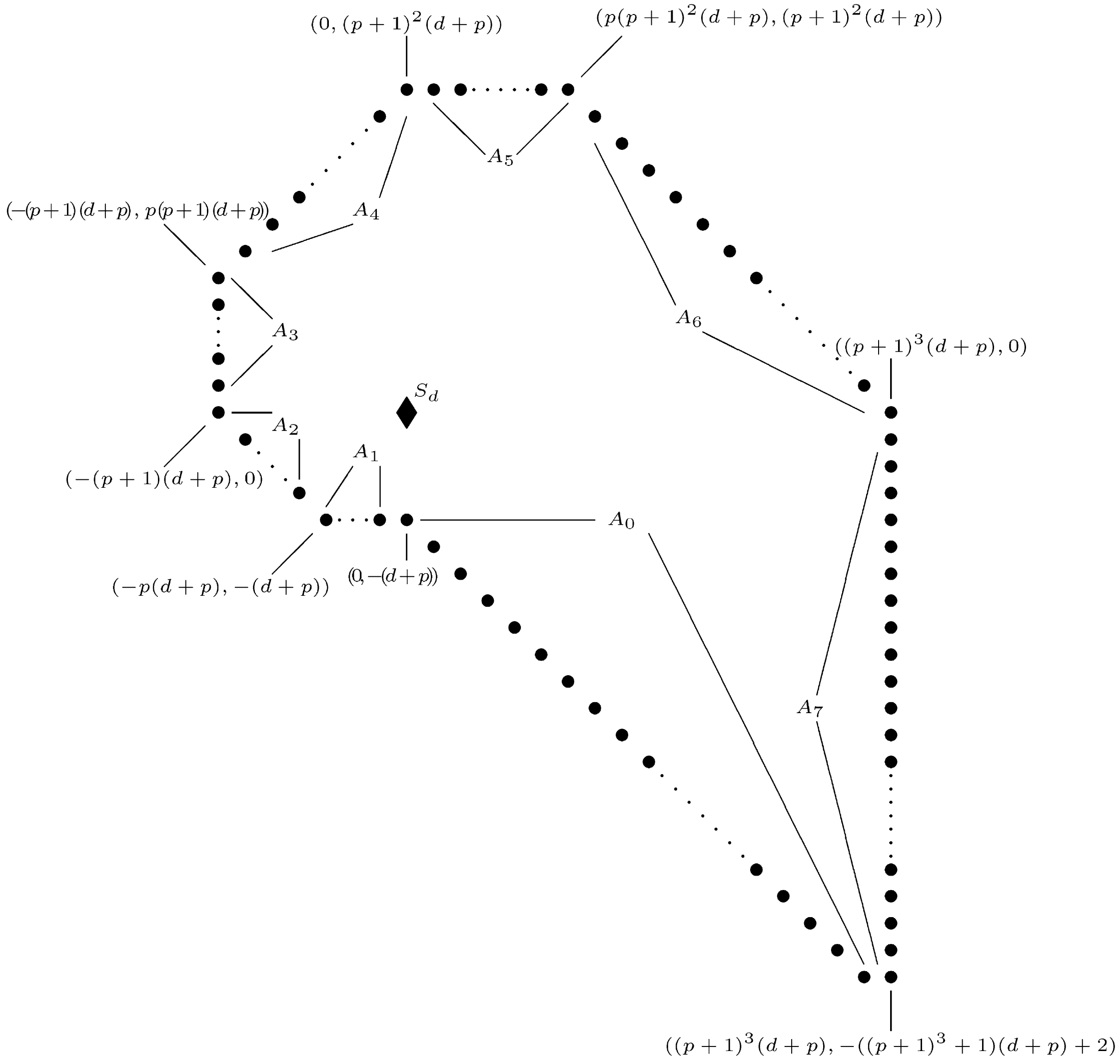

Fig. 1. A global view of the containment certificate.

(5) At each $t = ((d + p)(p + 2 + (p + 1)^2) + i)p$, $i = 1, \ldots, N$, we place p firefighters at positions given by A_7. This forms a vertical line and the positioning ends when this vertical line meets with the diagonal line formed by firefighters whose positions corresponds to the set A_0.

We next check the second condition in the definition of a containment certificate.

Case 1. Suppose $((i-1)/2, -(d+p+(i-1)/2), i) \in A_0$ for some $i \in \{1, 3, \ldots, 2(p+1)^3(d+p)-1\}$. By Lemma 3.2,

$$
\begin{aligned}
d\left(S_d, \left(\frac{i-1}{2}, -\left(d+p+\frac{i-1}{2}\right)\right)\right) &= \left|\frac{i-1}{2}\right| + \left|-\left(d+p+\frac{i-1}{2}\right)\right| - d \\
&= \frac{i-1}{2} + \left(d+p+\frac{i-1}{2}\right) - d \\
&= p + i - 1 \geqslant i \quad (\text{since } p \geqslant 6).
\end{aligned}
$$

Case 2. Suppose $(-(i-1)p-k, -(d+p), ip) \in A_1$ for some $i \in \{1, 2, \ldots, d+p\}$ and some $k \in \{1, \ldots, p\}$. By Lemma 3.2,

$$\begin{aligned} d(S_d, (-(i-1)p-k, -(d+p))) &= |-(i-1)p-k| + |-(d+p)| - d \\ &= (i-1)p + k + (d+p) - d \\ &= ip + k \geqslant ip. \end{aligned}$$

Case 3. Suppose $(-p(d+p)-i, -(d+p)+i, ip) \in A_2$ for some $i \in \{1, 2, \ldots, d+p\}$. By Lemma 3.2,

$$\begin{aligned} d(S_d, (-p(d+p)-i, -(d+p)+i)) &= |-p(d+p)-i| + |-(d+p)+i| - d \\ &= p(d+p) + i + (d+p) - i - d \\ &= p(d+p+1) \geqslant ip. \end{aligned}$$

Case 4. Suppose $(-(p+1)(d+p), (i-1)p+k, (d+p+i)p) \in A_3$ for some $i \in \{1, 2, \ldots, (p+1)(d+p)\}$ and $k \in \{1, \ldots, p\}$. By Lemma 3.2,

$$\begin{aligned} d(S_d, (-(p+1)(d+p), (i-1)p+k)) &= |-(p+1)(d+p)| + |(i-1)p+k| - d \\ &= (p+1)(d+p) + (i-1)p + k - d \\ &= pd + p^2 + d + p + ip - p + k - d \\ &= pd + p^2 + ip + k \geqslant (d+p+i)p. \end{aligned}$$

Case 5. Suppose $(-(p+1)(d+p)+i, p(p+1)(d+p)+i, (d+p+i)p) \in A_4$ for some $i \in \{1, 2, \ldots, (p+1)(d+p)\}$. By Lemma 3.2,

$$\begin{aligned} & d(S_d, (-(p+1)(d+p)+i, p(p+1)(d+p)+i)) \\ &\quad = |-(p+1)(d+p)+i| + |p(p+1)(d+p)+i| - d \\ &\quad = (p+1)(d+p) - i + p(p+1)(d+p) + i - d \\ &\quad = (p+1)^2(d+p) - d \\ &\quad = p^2d + p^3 + 2pd + 2p^2 + p \\ &\quad \geqslant p^2d + p^3 + 2pd + 2p^2 \\ &\quad = (p+2)(d+p)p \\ &\quad = (d+p+(p+1)(d+p))p \\ &\quad \geqslant (d+p+i)p. \end{aligned}$$

Case 6. Suppose $((i-1)p+k, (p+1)^2(d+p), ((p+2)(d+p)+i)p) \in A_5$ for some $i \in \{1, \ldots, (p+1)^2(d+p)\}$ and $k \in \{1, \ldots, p\}$. By Lemma 3.2,

$$\begin{aligned} d(S_d, ((i-1)p+k, (p+1)^2(d+p)) &= |(i-1)p+k| + |(p+1)^2(d+p)| - d \\ &= ip - p + k + (p^2+2p+1)(d+p) - d \\ &= ip + k + p^2d + p^3 + 2pd + 2p^2 \\ &\geqslant ip + p(pd + p^2 + 2d + 2p) \\ &= ((p+2)(d+p)+i)p. \end{aligned}$$

Case 7. Suppose $(p(p+1)^2(d+p)+i, (p+1)^2(d+p)-i, ((p+2)(d+p)+i)p) \in A_6$ for some $i \in \{1, \ldots, (p+1)^2(d+p)\}$. By Lemma 3.2,

$$\begin{aligned}
&d(S_d, (p(p+1)^2(d+p)+i, (p+1)^2(d+p)-i)) \\
&\quad = |p(p+1)^2(d+p)+i| + |(p+1)^2(d+p)-i| - d \\
&\quad = (p+1)^3(d+p) - d \\
&\quad = p^3d + p^4 + 3p^2d + 3p^3 + 3pd + 3p^2 + p \\
&\quad \geqslant p^3d + p^4 + 3p^2d + 3p^3 + 3pd + 3p^2 \\
&\quad = (p^2d + p^3 + 3pd + 3p^2 + 3d + 3p)p \\
&\quad = ((p+1)^2 + p + 2)(d+p)p \\
&\quad = ((p+2)(d+p) + (p+1)^2(d+p))p \\
&\quad \geqslant ((p+2)(d+p)+i)p.
\end{aligned}$$

Case 8. Suppose $((p+1)^3(d+p), -((i-1)p+k), ((d+p)(p+2+(p+1)^2)+i)p) \in A_7$ for some $i \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, p\}$. By Lemma 3.2,

$$\begin{aligned}
&d(S_d, ((p+1)^3(d+p), -((i-1)p+k))) \\
&\quad = |(p+1)^3(d+p)| + |-((i-1)p+k)| - d \\
&\quad = (p+1)^3(d+p) + (i-1)p + k - d \\
&\quad \geqslant (p^3 + 3p^2 + 3p + 1)(d+p) - p - d + ip \\
&\quad = (p+d)(p^3 + 3p^2 + 3p) + ip \\
&\quad = (p+d)p(p+2+(p+1)^2) + ip \\
&\quad = ((d+p)(p+2+(p+1)^2)+i)p.
\end{aligned}$$

Thus, the second condition in the definition of a containment certificate is satisfied. To see that A satisfies the third condition, let us consider the closed curve (in $\mathbb{R}^2$) determined by A by "connecting the dots", meaning we draw a line segment between two adjacent points (x, y, t) and $(x', y', t') \in A$ that satisfy

$$\max\{|x-x'|, |y-y'|\} = 1.$$

Note that this produces a polygon P with nine sides. P separates $\mathbb{R}^2$ into an interior and an exterior. Since the interior has finite area as a subset of $\mathbb{R}^2$, there are only a finite number of lattice points in the interior. Also, note that S_d is a subset of the interior, thus any point on the exterior must cross P in order to reach any point is S_d. This implies that the only vertices that have at least one path to a vertex in S_d without passing through any vertex in A are precisely the lattice points in the interior of P, which is finite. □

Lemma 3.4. *Suppose f and g are two periodic functions such that* $f \preceq^* g$. *If there is a containment certificate of f for* S_d *for all* $d \geqslant 0$, *then there is a containment certificate of g for* S_d *for all* $d \geqslant 0$.

Proof. Since $f \preceq^* g$, there exists $n \in \mathbb{N}$ such that

$$\sum_{t=1}^{k} f(t) \leqslant \sum_{t=1}^{k} g(t)$$

for all $k \geqslant n$. Since there is a containment certificate of f for S_d for all $d \geqslant 0$, let $C_{S_{n+d+1}}(f)$ be a containment certificate of f for S_{n+d+1}. We will use $C_{S_{n+d+1}}(f)$ to construct a containment certificate of g for S_d. We order the elements in $C_{S_{n+d+1}}(f)$ on the third coordinate such that

$$C_{S_{n+d+1}}(f) = \{(x_1, y_1, t_1), (x_2, y_2, t_2), \ldots, (x_r, y_r, t_r)\},$$

where $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_r$. It is now easy to see that for all $j \geqslant 1$,

$$\sum_{t=1}^{t_j} f(t) \geqslant j.$$

Now define $C_{S_d}(g)$ to be

$$C_{S_d}(g) = \{(x_j, y_j, g^{-1}(j)) | 1 \leqslant j \leqslant r\}.$$

Note that elements in $C_{S_{n+d+1}}(f)$ and $C_{S_d}(g)$ differ only the third coordinate. To prove that $C_{S_d}(g)$ is indeed a containment certificate of g for S_d, we check the three conditions in the definition of a containment certificate.

Condition 1. Note that

$$|\{j \in \mathbb{N} | g^{-1}(j) = i\}| = \text{number of } j \text{ such that} \min\left\{k | \sum_{t=1}^{k} g(t) \geqslant j\right\} = i$$
$$= g(i).$$

Thus $C_{S_d}(g)$ satisfies the first condition since there are exactly $g(i)$ elements in $C_{S_d}(g)$ where that the third coordinate is i.

Condition 2. For the second condition, first consider the case where $(x_j, y_j, t_j) \in C^{\leqslant n}_{S_{n+d+1}}(f)$. This implies $t_j \leqslant n$. We want to show that $d(S_d, (x_j, y_j)) \geqslant g^{-1}(j)$. We claim that $g^{-1}(j) \leqslant n$. Suppose, for a contradiction that $g^{-1}(j) > n$. By the definition of g^{-1}, this implies that

$$\sum_{t=1}^{n} g(t) < j.$$

However,

$$\sum_{t=1}^{n} f(t) \geqslant \sum_{t=1}^{t_j} f(t) \geqslant j \Rightarrow \sum_{t=1}^{n} g(t) < j \leqslant \sum_{t=1}^{n} f(t),$$

which contradicts $f \preceq^* g$. So $g^{-1}(j) \leqslant n$. Since $(x_j, y_j, t_j) \in C_{S_{n+d+1}}(f)$,

$$d(S_{n+d+1}, (x_j, y_j)) \geqslant 1 \Rightarrow d(S_d, (x_j, y_j)) > n \geqslant g^{-1}(j)$$

and we are done. Next consider the case where $(x_j, y_j, t_j) \in C^{>n}_{S_{n+d+1}}(f)$. We claim that $g^{-1}(j) \leqslant t_j$. Suppose, for a contradiction that $g^{-1}(j) > t_j$. By the definition of g^{-1}, this implies that

$$\sum_{t=1}^{t_j} g(t) < j.$$

However,

$$\sum_{t=1}^{t_j} f(t) \geqslant j \Rightarrow \sum_{t=1}^{t_j} g(t) < j \leqslant \sum_{t=1}^{t_j} f(t),$$

which contradicts $f \preceq^* g$ since $t_j > n$. So $g^{-1}(j) \leqslant t_j$. Since $(x_j, y_j, t_j) \in C_{S_{n+d+1}}(f)$,

$$d(S_d, (x_j, y_j)) > d(S_{n+d+1}, (x_j, y_j)) \geqslant t_j \Rightarrow d(S_d, (x_j, y_j)) > g^{-1}(j)$$

and we are done. Thus $C_{S_d}(g)$ satisfies the second condition in the definition of a containment certificate.

Condition 3. The third condition follows naturally because $C_{S_{n+d-1}}(f)$ is a containment certificate and the positions (x_j, y_j) determined by $C_{S_d}(g)$ and those determined by $C_{S_{n+d+1}}(f)$ are exactly identical. □

We are now ready to prove our main result.

Theorem 2. *Suppose a finite set $S \subset \mathbb{Z} \times \mathbb{Z}$ of vertices are initially set on fire. If the number of firefighters available for deployment per time period is given by a periodic function f such that $R_f > 1.5$, then there exists a containment certificate of f for S.*

Proof. Suppose *f* is a periodic function such that $R_f > 1.5$. If $p_f = 1$, this means that $f(t) \geqslant 2$ for all *t*. Fogarty [3] has shown that this is sufficient to contain the fire that starts at any finite set *S*. Suppose $p_f \geqslant 2$. By Lemma 2.7, there exists some $n \geqslant 2$ such that $F_{n^2} \preceq^* f$. Since $F^2_{n^2} \preceq^* F_{n^2}$ and $G_p \preceq^* F^2_{n^2}$ where $p = 2(2n^2 + 1)$, we have $G_p \preceq^* f$.

Now let

$$d = \max\{|x| + |y| | (x, y) \in S\}.$$

By Lemma 3.3, there exists a containment certificate of G_p for S_d. By Lemma 3.4, since $G_p \preceq^* f$, there also exists a containment certificate of *f* for S_d, $C_{S_d}(f)$. Since $S \subseteq S_d$, by Lemma 3.1, $C_{S_d}(f)$ is also a containment certificate of *f* for *S*. □

4. Discussion and conclusion

For a given periodic function *f* and set $S \subset \mathbb{Z} \times \mathbb{Z}$, if a containment certificate of *f* for *S* exists, it is not necessarily unique. In fact, our initial efforts to prove Theorem 1 resulted in the construction of a containment certificate of the function $F_n = [\overbrace{1, 1, \ldots, 1}^{n}, \overbrace{2, 2, \ldots, 2}^{n+1}]$ for the set S_d, for every $n \geqslant 1$ and $d \geqslant 0$. Of course, with Lemmas 2.7 and 3.4, we are still able to arrive at Theorem 1. The containment certificate of F_n differs significantly from the containment certificate of G_p for S_d presented in Lemma 3.3. Our decision to present the containment certificate of G_p for S_d in this paper is based on its relative simpler form and ease of checking the three conditions of a containment certificate.

In this paper, we have established that if *f* is a periodic function with $R_f > 1.5$, then for any $d \geqslant 0$, there always exists a containment certificate of *f* for S_d. But what about periodic functions *f* with $R_f \leqslant 1.5$? Attempts have been made, for example, with the function $f = [2, 1]$ but with no success. Even in the simplest case when the fire breaks out at just a single vertex of $\mathbb{L}_2$, we were unable to determine if there is a containment certificate of $f = [2, 1]$ for S_0. Through our many attempts, however, we believe that such a containment certificate does not exist.

Conjecture 1. *There is no containment certificate of $f = [2, 1]$ for S_0.*

In this light, if we define the following number:

$$R := \inf\{k \in \mathbb{R} | \forall f \text{ with } R_f = k \text{ there exists a } C_S(f) \text{ for any finite} S\}$$

then the research mentioned in Section 1 showed that $1 \leqslant R \leqslant 2$, and this paper has shown that $1 \leqslant R \leqslant 1.5$. So, it leads to the following question:

Question 1. What is *R*, exactly?

Note that if Conjecture 1 holds, then it would answer Question 1, and the answer would be 1.5. It is clear, however, that new machinery beyond what is covered in this paper will be necessary to answer this question.

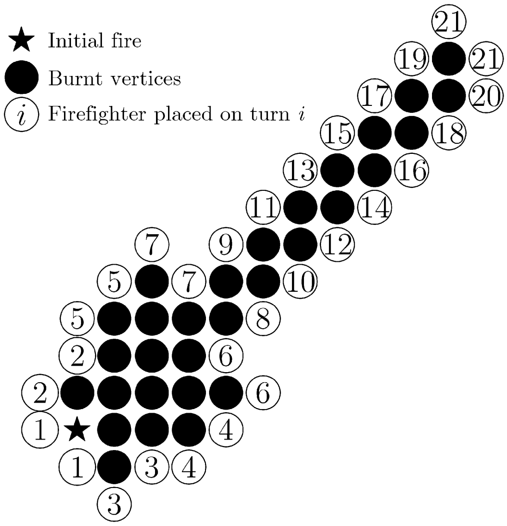

Fig. 2. A not-so-nice containment certificate.

We wish to note, however, that containment certificates exist for "periodic" functions with ratios less than 1.5. The reason for the quotation marks will become clear soon. Consider first the function

$$g = [4, 0, 0, 0, 0, 0, 0, 0].$$

Clearly there is a containment certificate of g for S_0. However, by the way we defined g we would have $R_g = 0.5$, which is much less than 1.5. We can extend this example further to obtain ratios as close to 0 as possible where containment certificates still exist.

For a more subtle second example, consider the function

$$f = [2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2].$$

This function has a containment certificate for S_0, as shown in Fig. 2. With the above example, we have reached a point at turn 8 where we swere able to just hold off the fire indefinitely. Hence we could place one fighter per turn at this stage indefinitely without increasing the number of "exposed" vertices that could catch on fire the next turn. Although the two examples above are valid examples in the context of the paper, they do not contain the spirit of our paper. Rather than finding functions with a certain ratio where containment certificates exist, we are interested in the question of whether *all* functions with a given ratio admit containment certificates.

One final thing to notice is that the restriction on the periodicity of the function can probably be relaxed. For any arbitrary function $f : \mathbb{N} \to \mathbb{N}$, it will still be true that there exists a containment certificate of f for any finite S if f eventually dominates a F_n for some n. Given $f : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$, we define the *running ratio* of f at n to be

$$R_f(n) := \frac{\sum_{t=1}^{n} f(t)}{n}.$$

The authors believe that the following conjecture is true.

Conjecture 2. *If* $R_f(n) > 1.5$ *for all* n *and*

$$\liminf_{n\to\infty} R_f(n) > 1.5,$$

then there is a containment certificate of f *for any finite* S.

Finally, the authors wish to note that this paper stemmed from questions arising from epidemiology and that many extensions to this problem can be thought of by thinking of the problem in this manner. In this simplified model of disease spread, the nodes of the graph represent individuals in the population, and the edges represent relations that may allow for disease spread. Therefore, the results in this paper could be translated into disease control for a population whose social structure is a grid and for a disease that strikes neighbors the next time period after a person is infected. While this is a very simplistic and unlikely setting for population structure and disease spread, we invite readers to extend these results to more general types of graphs and more interesting fire/disease behaviors that more accurate. For example, the first modification that could be made to this problem is to add a probability parameter p to the scenario, which would be the probability that a unprotected node would catch fire given that a neighbor is on fire. Another possible modification would be to modify the graph as t increases, presumably to represent the changes in inter-person behavior as a day goes by: one is rarely likely to catch a disease from a co-worker at four in the morning!

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