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FORMULATING THE SINGLE MACHINE SEQUENCING PROBLEM WITH RELEASE DATES AS A MIXED INTEGER PROGRAM

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For the problem of minimising the weighted sum of start (or completion) times for the n-jobs, 1-machine problem with release dates, we consider a hierarchy of relaxations obtained by combining enumeration of initial sequences with Smith's rule. It is then shown that each of these relaxations can be formulated as a linear programming problem (i.e., the minimisation of a linear function over a polyhedron) in an enlarged space of variables. By projecting these polyhedra we obtain new valid inequalities for the problem, and in particular complete descriptions for 1-regular problems partially studied by Balas (1985) and Posner (1986).

A second hierarchy of relaxations is obtained by studying various relaxations and alternative formulations from the literature.

1. Introduction

In this paper we study the formulation of the single machine sequencing problem with release dates as a mixed integer program. This is not an end in itself, but it is we believe one of the inherently difficult single machine problems for which it is a challenge to obtain strong lower bounds. We hope that ultimately this approach will allow us to tackle and solve problems including many machines and other types of constraints including deadlines, precedence constraints and order dependent processing times. For earlier work in this vein, see Balas [1], who considers release dates and order dependent processing times, and Peters [4] who considers deadlines.

In Sections 2–4 we consider a hierarchy of relaxations for the release date problem involving partial enumeration. We then derive linear programming formulations of these relaxations involving a large number of variables. In special cases we derive equivalent formulations involving a smaller number of variables. The inequalities from these formulations are then used as strong valid inequalities for the original problem. In Section 5 we analyse the strength of the lower bounds obtained from a variety of different relaxations obtained from mixed integer formulations proposed in the literature.

2. The problem with release dates

The problem to be treated is that of processing n jobs on a single machine subject to release dates so as to minimise the weighted sum of start times.

$$
(R) \qquad \begin{aligned}
& z=\min \sum_{j=1}^{n} w_j t_j, && \\
& t_j \geq r_j, && \text{for } j \in N, \\
& t_j - t_i \geq p_i \text{ or } t_i - t_j \geq p_j, && \text{for } i, j \in N \text{ with } i \neq j,
\end{aligned}
$$

where $N=\{1,\dots,n\}$ is the set of jobs, $r_j \geq 0$ is the release date of job j, $p_j > 0$ is the processing time of job j, $w_j > 0$ is the weight associated with job j, and t_j is a variable representing the start time of job j.

Throughout the paper we assume that the jobs are ordered so that $w_1/p_1 \geq w_2/p_2 \geq \cdots \geq w_n/p_n > 0$ and $\min_{j \in N} r_j = 0$. We will make repeated reference to Smith's rule which is an algorithm for problem (R) when all release dates are zero.

Smith's Rule. Given that $r_j = 0$ for $j \in N$ and the jobs are ordered so that $w_i/p_i \geq w_{i+1}/p_{i+1}$ for $i=1,\dots,n-1$, an optimal solution to (R) is to sequence the jobs in the order $1,2,\dots,n$ giving objective value $\sum_{k=1}^{n} w_k (\sum_{l=1}^{k-1} p_l)$.

Notation. (i) We let $\sigma=(j_1,\dots,j_s)$ denote an ordering of the first s jobs of a sequence where j_i is the ith job in the sequence $i=1,\dots,s$.

(ii) $t^*(\sigma)$ denotes the finish time of job j_s using the initial sequence σ and starting each job as early as possible taking into account processing times and release dates.

(iii) $\tau(\sigma)$ denotes the ordering of all the jobs obtained by starting with σ, and then adding the remaining jobs in increasing order of their indices

$$\tau(\sigma)=(\sigma, i_1, \dots, i_{n-s}) \quad \text{where } i_1 < i_2 < \cdots < i_{n-s},$$

and $\{i_1,\dots,i_{n-s}\} = N \setminus \{j_1,\dots,j_s\}$, i.e., if $N=5$ and $\sigma=(4,2)$, then $\tau(\sigma)=(4,2,1,3,5)$. Now if σ is fixed, no job j not in σ can start before time $t^*(\sigma)$.

(iv) $z(\tau(\sigma), w)$ denotes the weighted sum of start times where σ is the initial sequence taking into account release dates, and the release dates on the remaining jobs are ignored.

Lemma 2.1. *If $\sigma=(j_1,\dots,j_s)$ and $\tau(\sigma)=(\sigma, i_1,\dots,i_{n-s})$, the associated start times and objective values are given by*:

(i) $t_{j_1} = r_{j_1}$, $t_{j_k} = r_{j_1} + p_{j_1} + \cdots + p_{j_{k-1}} + \beta_{j_1 j_2} + \cdots + \beta_{j_1 j_2 \dots j_k}$, *for* $k=2,\dots,s$, *where* $\beta_{j_1,\dots,j_u} = (r_{j_u} - t_{j_{u-1}} - p_{j_{u-1}})^+$ *for* $u=2,\dots,s$, *and* x^+ *denotes* $\max\{x,0\}$.

(ii) $t^*(\sigma) = t_{j_s} + p_{j_s}$,

(iii) $t_{i_k} = t^*(\sigma) + \sum_{l=1}^{k-1} p_{i_l}$,

(iv) $z(\tau(\varphi), w) = \sum_{k=1}^{n} w_k \sum_{l=1}^{k-1} p_l$,

(v) $z(\tau(j_1, \ldots, j_r), w) = z(\tau(\phi), w) + \sum_{k<j_1}(w_k p_{j_1} - w_{j_1} p_k) + \cdots + \sum_{k<j_r}(w_k p_{j_r} - w_{j_r} p_k) + (\sum_k w_k) r_{j_1} + (\sum_{k \neq j_1} w_k)\beta_{j_1 j_2} + \cdots + (\sum_{k \neq j_1 \ldots j_{r-1}} w_k)\beta_{j_1 \ldots j_r}$.

Definition 2.2. An initial subsequence σ is *S*(*Smith*)-*feasible* for (R) (with respect to w) if

$$t_{i_k} \geq r_{i_k} \quad \text{for } k = 1, \ldots, n-s.$$

Based on Smith's rule we obtain a simple hierarchy of lower bounds on the optimal value z of (R). Consider the problem:

$$(R_s) \qquad \underline{z}_s = \min_{\sigma} \{z(\tau(\sigma), w) \colon \sigma = (j_1, \ldots, j_s) \text{ is of length } s\}.$$

Proposition 2.3. $z(\tau(\varphi), w) = \underline{z}_0 \leq \underline{z}_1 \leq \cdots \leq \underline{z}_{n-1} \leq z$.

Proof. Let $(j_1^*, \ldots, j_n^*)$ be an optimal sequence for (R). Then $z \geq z(\tau(j_1^*, \ldots, j_s^*), w) \geq \underline{z}_s$, and $z(\tau(j_1, \ldots, j_s), w) \geq z(\tau(j_1, \ldots, j_{s-1}), w)$. □

This leads to an immediate optimality test:

Proposition 2.4. *Suppose* $\underline{z}_s = z(\tau(\sigma'), w)$ *with* $\tau(\sigma') = (\sigma', i_1, \ldots, i_{n-s})$. *Then if* σ' *is S-feasible in* (R), $\tau(\sigma')$ *is an optimal solution for* (R).

It is natural to ask when it can be guaranteed that the condition of Proposition 2.4 will hold.

Definition 2.5. A problem (R) is *s-regular* if for every initial subsequence $\sigma = (j_1, \ldots, j_s)$ of length s, σ is S-feasible for (R).

Clearly, (R) is s-regular is a sufficient but not necessary condition that $\underline{z}_s = z$. In certain cases we can say a priori that an instance (R) is s-regular.

Proposition 2.6. *A sufficient condition for an instance* (R) *to be s-regular is*

$$\min_{j \in K} \{r_j\} + \sum_{j \in K} p_j \geq r_i,$$

for all i, K *where* $K \subseteq N \setminus \{i\}$ *and* $|K| = s$.

Proof. Let $\sigma = (j_1, \ldots, j_s)$ be any sequence of length s. Using Lemma 2.1,

$$t_{i_k} \geq t^*(\sigma) \geq r_{j_1} + \sum_{k=1}^{s} p_{j_k} \geq r_{i_k}. \qquad \square$$

Note that if the conditions of Proposition 2.6 hold, (R) is s-regular for all weight vectors $w \in R_+^n$.

We now introduce integer programming formulations of (R) and (R_s) respectively. Problem (R) can be "badly" formulated as:

$$z=\min \sum_{j=1}^{n} w_j t_j,$$

$$t_j - t_i \geq p_i - M\delta_{ji}, \quad \text{for all } i,j \in N,\ i \neq j, \tag{1a}$$

$$t_j \geq r_j, \quad \text{for } j \in N, \tag{1b}$$

$$\delta_{ij} + \delta_{ji} = 1, \quad \text{for all } i<j, \tag{1c}$$

$$\delta \in \Delta, \tag{1d}$$

$$t \in R_+^n, \quad \delta \in R_+^{n(n-1)}, \tag{1e}$$

where

$$\Delta = \{\delta \in R_+^{n(n-1)}\colon \delta_{ij} \in \{0,1\},\ \delta_{ij}+\delta_{jk}+\delta_{ki} \leq 2 \text{ for all } i,\ j,\ k,\ i \neq j \neq k\}.$$

Note that δ satisfies (1c) and (1d) if and only if δ corresponds to a sequence $(j_1,\ldots,j_n)$ with $\delta_{j_p j_q}=1$ if and only if j_p precedes j_q. The problem is badly formulated because of the big "M" which means that the constraints (1a) are inactive in the linear programming relaxation of (1), so that $t_j = r_j$ for $j \in N$ is an optimal solution of the linear program. (We ignore the effects of reducing M as much as possible. Though this is advisable in practice, we do not consider that this essentially changes the formulation.)

Problem (R_s) can be formulated as:

(P_s^I)

$$z_s = \min \sum_{j=1}^{n} w_j t_j,$$

$$t_k \geq \sum_{\sigma=(j_1,\ldots,j_s)} (r_{j_1} + \beta_{j_1 j_2}^k + \cdots + \beta_{j_1 \ldots j_s}^k) y_\sigma + \sum_{\{i:\, i \neq k\}} p_i \delta_{ik},$$

$$\text{for all } k, \tag{2a}$$

$$\sum_{\{\sigma:\, |\sigma|=s\}} y_\sigma = 1, \tag{2b}$$

$$\delta_{ik} \geq \sum_{\sigma \in Q_s^{ik}} y_\sigma,$$

$$\text{for all } i,k, \tag{2c}$$

$$\delta_{ij} + \delta_{ji} = 1,$$

$$\text{for all } 1 \leq i \leq j \leq n, \tag{2d}$$

$$\delta \in \Delta, \tag{2e}$$

$$t \in R_+^n, \quad \delta \in R_+^{n(n-1)}, \quad y_\sigma \in Z_+^1,$$

$$\text{for all } \sigma \text{ with } |\sigma| = s, \tag{2f}$$

where

$$\beta_\pi^k = \begin{cases} \beta_\pi, & \text{if } k \notin \{j_1, \dots, j_s\} \text{ and } \pi = (j_1, \dots, j_q), \text{ for } q \le s, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$Q_s^{ik} = \{\sigma\colon |\sigma| = s,\ i \in \{j_1, \dots, j_s\},\ k \notin \{j_1, \dots, j_s\}\}$$
$$\cup \{\sigma\colon |\sigma| = s,\ i, k \in \{j_1, \dots, j_s\} \text{ and } i \text{ precedes } k\}.$$

Interpreting $y_\sigma = 1$ to mean that σ is the initial sequence, and $y_\sigma = 0$ otherwise, we obtain:

Proposition 2.7. *(P_s^I) is a reformulation of (R_s).*

Now let (P_s) be the linear programming problem obtained from (P_s^I) by dropping the constraints $\delta \in \Delta$, and the integrality of y_σ.

Theorem 2.8. *The linear programming problem (P_s) solves (R_s).*

Proof. For simplicity of notation we give the proof for $s = 2$. For any feasible solution of (P_2) using (2a)

$$\begin{aligned} \sum_k w_k t_k &\ge \sum_k w_k \Big(\sum_{j_1, j_2} (r_{j_1} + \beta_{j_1 j_2}^k) y_{j_1 j_2} \Big) + \sum_k w_k \sum_{l \ne k} p_l \delta_{lk} \\ &= \sum_{j_1, j_2} \Big[\Big(\sum_k w_k \Big) r_{j_1} + \Big(\sum_{k \ne j_1} w_k \Big) \beta_{j_1 j_2} \Big] y_{j_1 j_2} \\ &\quad + \sum_{1 \le k < l \le n} (w_k p_l \delta_{lk} + w_l p_k \delta_{kl}). \end{aligned}$$

Now as $w_i/p_i \ge w_{i+1}/p_{i+1}$ for $i = 1, \dots, n-1$ and $\delta_{lk} + \delta_{kl} = 1$,

$$\begin{aligned} &\sum_{1 \le k < l \le n} (w_k p_l \delta_{lk} + w_l p_k \delta_{kl}) \\ &\quad = \sum_{l=1}^{n} w_l \sum_{k=1}^{l-1} p_k + \sum_{l=1}^{n} \sum_{k=1}^{l-1} (w_k p_l - w_l p_k) \delta_{lk}, \end{aligned}$$

where $w_k p_l - w_l p_k \ge 0$ for $k < l$.

Using (2b),

$$\sum_{l=1}^{n} w_l \sum_{k=1}^{l-1} p_k = \sum_{j_1, j_2} \Big(\sum_{l=1}^{n} w_l \sum_{k=1}^{l-1} p_k \Big) y_{j_1 j_2},$$

and using (2c)

$$\begin{aligned} \sum_{l=1}^{n} \sum_{k=1}^{l-1} (w_k p_l - w_l p_k) \delta_{lk} &\ge \sum_{l=1}^{n} \sum_{k=1}^{l-1} (w_k p_l - w_l p_k) \Big(\sum_{\{q:\, q \ne l\}} y_{lq} + \sum_{\{q:\, q \ne l, k\}} y_{ql} \Big) \\ &= \sum_{j_1, j_2} \Big[\sum_{k=1}^{j_1 - 1} (w_k p_{j_1} - w_{j_1} p_k) \Big] y_{j_1 j_2} \end{aligned}$$

$$+\sum_{j_1,j_2}\left[\sum_{\{k:\,k<j_2,\,k\neq j_1\}}(w_kp_{j_2}-w_{j_2}p_k)\right]y_{j_1j_2}.$$

Therefore

$$\sum_k w_kt_k\ge\sum_{j_1,j_2}\left[\left(\sum_k w_k\right)r_{j_1}+\left(\sum_{\{k:\,k\neq j_1\}}w_k\right)\beta_{j_1j_2}+\sum_{l=1}^{n}w_l\sum_{k=1}^{l-1}p_k\right.$$
$$\left.+\sum_{k=1}^{j_1-1}(w_kp_{j_1}-w_{j_1}p_k)+\sum_{\{k:\,k<j_2,\,k\neq j_1\}}(w_kp_{j_2}-w_{j_2}p_k)\right]y_{j_1j_2}$$
$$=\sum_{j_1,j_2}[z(\tau(j_1,j_2),w)]y_{j_1,j_2}$$

using Lemma 2.1(v). Therefore for any feasible solution of (P_2),

$$\sum_k w_kt_k\ge\min\left\{\sum_{\{\sigma:\,|\sigma|=2\}}z(\tau(\sigma),w)y_\sigma:\ \sum_{\{\sigma:\,|\sigma|=2\}}y_\sigma=1,\ y_\sigma\ge 0\right\}=\underline{z}_2.$$

As (P_2) is a relaxation of (P_2^I) which is equivalent to (R_2), the claim follows. □

3. Alternative representations of 1-regular problems

Let Q_s be the feasible region associated with problem (P_s). Q_s is a polyhedron in a space involving n variables t_j, $O(n^2)$ variables δ_{ij} and $O(n^s)$ variables y_σ. For practical purposes $O(n^2)$ is probably the maximum number of variables that can be handled. Here we investigate certain projections of Q_s for $s=1$.

First we consider the projection $\mathrm{proj}_{t,\delta}(Q_1)$ in which the n variables $y_{\{j\}}$ for $j\in N$ are eliminated. Let $D_S=\{\alpha:S\to S:\ \alpha(i)\neq i \text{ for all } i\in S\}$.

Theorem 3.1. *The polyhedron* $\mathrm{proj}_{t,\delta}(Q_1)$ *is described by the inequalities*:

$$t_j-\sum_{\{i:\,i\neq j\}}p_i\delta_{ij}\ge r_k-\sum_i(r_k-r_i)^+\delta_{i,\alpha(i)},\quad \textit{for all } j,k\in N, \textit{ and all } \alpha\in D_N,$$
$$\sum_i\delta_{i,\gamma(i)}\ge 1,\quad \textit{for all } \gamma\in D_N,$$
$$\delta_{ij}+\delta_{ji}=1,\quad \textit{for } 1\le i<j\le n,$$
$$t\in R_+^n,\quad \delta\in R_+^{n(n-1)}.$$

Proof. Q_1 is the polyhedron

$$\begin{aligned}&t_k\ge\sum_{j=1}^n r_jy_j+\sum_{\{i:\,i\neq k\}}p_i\delta_{ik}, && \text{for } k=1,\dots,n,\\ &\sum_{j=1}^n y_j=1,\\ &-y_i+\delta_{ij}\ge 0, && \text{for all } i,j\in N \text{ with } i\neq j,\\ &\delta_{ij}+\delta_{ji}=1, && \text{for } 1\le i<j\le n,\\ &t\in R_+^n,\quad \delta\in R_+^{n(n-1)},\quad y\in R_+^n.\end{aligned}\qquad (3$$

To find $\text{proj}_{t,\delta}(Q_1)$, we consider the constraints involving y_j for $j \in N$ with t, δ fixed, namely the polyhedron

$$\begin{aligned}
&\sum_{j=1}^{n} r_j y_j \le t_k - \sum_{\{i:\, i \ne k\}} p_i \delta_{ik}, && \text{for } k \in N,\\
&-\sum_{j=1}^{n} y_j = -1, &&\\
&y_i \le \delta_{ij}, && \text{for all } i, j \in N \text{ with } i \ne j,\\
&y \in R^n_+. &&
\end{aligned}$$

Now by Farkas' lemma, or Benders' algorithm, it suffices to find the extreme rays of the dual cone U, where U is of the form:

$$U = \left\{(u, v, w) \in R^n_+ \times R^1 \times R^{n(n-1)}_+ :\ r_l\left(\sum_{i=1}^{n} u_i\right) - v + \sum_{\{i:\, i \ne l\}} w_{li} \ge 0 \text{ for } l \in N\right\}.$$

The extreme rays of U are of two forms:

(a) $u = 0,\ v = 1,\ w_{l,\gamma(l)} = 1$ for $l \in N,\ \gamma \in D_N$;
(b) $u_j = 1,\ u_i = 0$ for $i \ne j,\ v = r_k,\ w_{i,\alpha(i)} = (r_k - r_i)^+$ for $i \in N,\ \alpha \in D_N$, for all $j, k \in N$.

The claim follows. □

The following corollary is a consequence of Theorem 2.8.

Corollary 3.2. *The linear program*

$$\min\left\{\sum_{j=1}^{n} w_j t_j :\ (t, \delta) \in \text{proj}_{t,\delta}(Q_1)\right\}$$

solves (R_1).

The separation problem for $\text{proj}_{t,\delta}(Q_1)$ is also easily solved.

Corollary 3.3. *Given* $(t^*, \delta^*) \in R^n_+ \times R^{n(n-1)}_+$ *satisfying* $\delta^*_{ij} + \delta^*_{ji} = 1$ *for* $i < j$, *let*

$$\xi_1 = \min_{j \in N}\left\{t^*_j - \sum_{\{i:\, i \ne j\}} p_i \delta^*_{ij}\right\} - \max_{k}\left\{r_k - \sum_{i \in N} (r_k - r_i)^+ \min_{\alpha_i \in N \setminus \{i\}} \delta^*_{i,\alpha_i}\right\},$$

and

$$\xi_2 = \sum_{i}\left(\min_{\gamma_i \in N \setminus \{i\}} \delta^*_{i,\gamma_i}\right) - 1.$$

Then $(t^*, \delta^*) \in \text{proj}_{t,\delta}(Q_1)$ *if and only if* $\xi_1 \ge 0$ *and* $\xi_2 \ge 0$.

Now we eliminate the δ variables so as to obtain $\text{proj}_{t,y}(Q_1)$. We let

$$p(S) = \sum_{\{i \in S,\, j \in S:\, i < j\}} p_i p_j, \qquad q(S) = \sum_{i \in S} p_i.$$

Theorem 3.4. $\mathrm{proj}_{t,y}(Q_1)$ *is described by the inequalities*

$$\begin{aligned} &\sum_{i\in S} p_i t_i \geq \sum_{i,j\in S} p_i p_j + \sum_{k\in S} p_k \left[\sum_{i\in S} r_i y_i + \sum_{i\notin S} (r_i+p_i) y_i \right], \\ &\quad \text{for all } S\subseteq N, \\ &\sum_{i\in N} y_i = 1, \\ &t\in R^n_+, \qquad y\in R^n_+. \end{aligned} \tag{4}$$

Proof. Rather then projecting, we give a proof by showing that the optimal value of the linear program $\min\{\sum_{j=1}^n w_j t_j$: (t,y) is feasible in (4)$\}$ is $\underline{z}_1$. More precisely we will show that the dual of this linear program

$$\max \sum_S p(S) u_S + v,$$

$$\sum_{S\ni i} p_i u_S \leq w_i, \quad \text{for } i\in N, \tag{5a}$$

$$-\sum_{S\ni i} q(S) r_i u_S - \sum_{S\not\ni i} q(S)(r_i+p_i) u_S + v \leq 0, \quad \text{for } i\in N, \tag{5b}$$

$$u_S\geq 0, \quad \text{for } S\subseteq N,\ v\in R^1$$

has optimal value greater than or equal to $\underline{z}_1$. We assume that $\underline{z}_1 = z(\tau(\{j^*\}), w)$, i.e., the optimal solution of (R_1) is to set j^* first.

We claim that $u^*_{S_j} = w_j/p_j - w_{j+1}/p_{j+1}$ and $u^*_{S_n} = w_n/p_n$ where $S_j = \{1,\dots,j\}$, $u^*_S = 0$ for $S\neq S_j$ for $j=1,\dots,n$, and $v^* = (\sum_k w_k) r_{j^*} + \sum_{k<j^*} (w_k p_{j^*} - w_{j^*} p_k)$ is dual feasible and optimal.

To see that (5a) is satisfied, observe that

$$\sum_{S\ni i} p_i u^*_S = p_i \sum_{j\geq i} \left(\frac{w_j}{p_j} - \frac{w_{j+1}}{p_{j+1}} \right) = w_i, \quad \text{for } i\in N.$$

Now to check (5b), observe that

$$\begin{aligned} &\sum_{S\ni i} q(S) r_i u^*_S + \sum_{S\not\ni i} q(S)(r_i+p_i) u^*_S \\ &= \sum_S q(S) r_i u^*_S + \sum_{S\not\ni i} q(S) p_i u^*_S \\ &= r_i \sum_j q(S_j) \left(\frac{w_j}{p_j} - \frac{w_{j+1}}{p_{j+1}} \right) + p_i \sum_{j<i} q(S_j) \left(\frac{w_j}{p_j} - \frac{w_{j+1}}{p_{j+1}} \right) \end{aligned}$$

$$= r_i\left[\sum_j \frac{w_j}{p_j}(q(S_j) - q(S_{j-1}))\right] + p_i\left[\sum_{j<i}(q(S_j) - q(S_{j-1}))\frac{w_j}{p_j} - q(S_{i-1})\frac{w_i}{p_i}\right]$$

$$= r_i\left(\sum_j w_j\right) + p_i\left(\sum_{j<i} w_j\right) - w_i\left(\sum_{j<i} p_j\right).$$

Hence (u^*, v^*) is dual feasible. Now the corresponding objective value is $\sum_S p(S)u_S^* + v^* = \sum_i w_i(\sum_{k<i} p_k) + v^* = \underline{z}_1$, and the claim follows. □

The separation problem for the inequalities (4) can be solved as a max flow problem. Given (t^*, y^*) it suffices to solve

$$\min_{z \in B^n}\left\{\sum_{i \in N}\left[t_i^* - \sum_k y_k^*(r_k + p_k)\right]p_i z_i - \sum_{\{i,j:i<j\}}(1 - y_i^* - y_j^*)p_i p_j z_i z_j\right\},$$

where $y_i^* + y_j^* \le 1$ so that the quadratic terms all have nonpositive coefficients.

One surprising consequence of the proof of Theorem 3.4 is that a linear system with $2n$ variables, and $n+1$ constraints apart from nonnegativity, suffices to give the bound $\underline{z}_1$.

Example 3.5. We consider 1-regular problems with $r_1 \le r_2 \le r_3$.

(a) For $n = 2$, $\text{proj}_{t,\delta}(Q_1)$ is described in the (t, δ) space by

$$t_1 - p_2\delta_{21} \ge r_2 - (r_2 - r_1)\delta_{12},$$
$$t_2 - p_1\delta_{12} \ge r_2 - (r_2 - r_1)\delta_{12},$$
$$\delta_{12} + \delta_{21} = 1,$$
$$t, \delta \ge 0.$$

It is easily seen that in this case $\text{proj}_t(Q_1)$ is given by:

$$t_1 \ge r_1, \qquad t_2 \ge r_2,$$
$$(r_1 + p_1 - r_2)(t_1 - r_1) + (r_2 + p_2 - r_1)(t_2 - r_2) \ge (r_1 + p_1 - r_2)(r_2 + p_2 - r_1),$$

see Balas (1985).

(b) For $n = 3$, it can be checked that $\text{proj}_{t,\delta}(Q_1)$ is described by the 12 inequalities:

$$t_k - \sum_{\{j:j \ne k\}} p_j\delta_{jk} \ge r_2 - (r_2 - r_1)\delta_{12},$$
$$t_k - \sum_{\{j:j \ne k\}} p_j\delta_{jk} \ge r_3 - (r_3 - r_1)\delta_{13} - (r_3 - r_2)\delta_{23},$$
$$t_k - \sum_{\{j:j \ne k\}} p_j\delta_{jk} \ge r_3 - (r_3 - r_1)\delta_{13} - (r_3 - r_2)\delta_{21},$$
$$t_k - \sum_{\{j:j \ne k\}} p_j\delta_{jk} \ge r_3 - (r_3 - r_1)\delta_{12} - (r_3 - r_2)\delta_{23},$$

for $k = 1, 2, 3$, and $\delta_{ij} + \delta_{ij} = 1$ for $i < j$, $t \in R_+^3$, $\delta \in R_+^6$.

4. Valid inequalities and a formulation for the general problem with release dates

Using the s-regular inequalities, we now obtain a larger family of valid inequalities, and also a valid reformulation of (R). We derive inequalities for $s=1$.

Proposition 4.1. *Given $S\subseteq N$, and $j,k\in S$, the inequality*

$$t_j-\sum_{i\in S\setminus\{j\}} p_i\delta_{ij}\geq r_k-\sum_{i\in S}(r_k-r_i)^+\delta_{i,\alpha(i)} \tag{6}$$

is valid for formulation (1) *of problem* (R), *for all $\alpha\in D_S$.*

We now derive certain special cases of these inequalities. For the following proposition we suppose the jobs to be ordered so that $r_1\leq r_2\leq\cdots\leq r_n$.

Proposition 4.2. *For all j,k,*

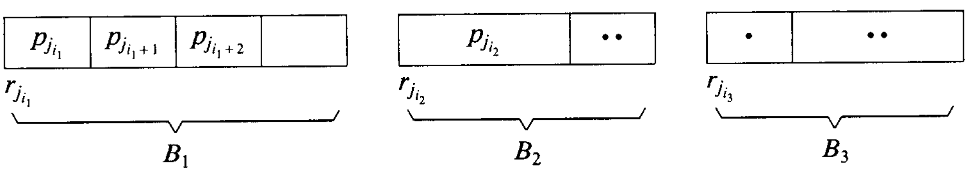
$$t_j\geq r_k+\sum_{\substack{i<k\\ i\neq j}}(p_i+r_i-r_k)^+\delta_{ij}+\sum_{\substack{i\geq k\\ i\neq j}}p_i\delta_{ij}-(r_k-r_j)^+\delta_{jk}$$

is a valid inequality for (R).

Proof. Take $S=\{k,\ldots,n\}\cup\{i\colon\ i<k,\ p_i+r_i-r_k>0\}\cup\{j\}$, $\alpha(i)=j$ for all $i\neq j$ and $\alpha(j)=k$ in Proposition 4.1. □

We now observe that by taking all possible sets S we necessarily obtain a valid formulation of (R).

In particular suppose (t,δ) is a feasible sequence corresponding to $\sigma=(j_1,\ldots,j_n)$. Let $1=i_1<\cdots<i_p\leq n$ be the subset of $(1,\ldots,n)$ for which $t_{j_i}=r_{j_i}$. σ breaks up into blocks as shown below, where $B_1=\{j_{i_1},\ldots,j_{i_2-1}\}$, $B_2=\{j_{i_2},\ldots,j_{i_3-1}\}$, etc.



Observation. Taking $S=\bigcup_{i=q}^{p}B_i$ or $S=B_q$ and $k=j_{i_q}$, the (j,k,S) inequality of Proposition 4.1 forces the correct value of t_j for each $j\in B_q$.

Hence we obtain:

Theorem 4.3. *The (S,j,k) inequalities* (6) *plus the constraints* (1b), (1c) *and* (1d) *give a valid formulation of* (R).

Posner [5] has considered a model in which blocks $B_1, \dots, B_p$ are imposed with $\delta_{ij}=1$ if $i \in B_r$, $j \in B_t$ and $r<t$. He gives a polynomial algorithm when the jobs in each block are 1-regular. An apparently natural formulation for this model is:

$$
\begin{aligned}
& t_k \geq \sum_{j \in B_u} r_j y_j + \sum_{i=u}^{q} \sum_{j \in B_i} p_j \delta_{jk}, && \text{for } k \in B_q,\ u \leq q \text{ and } 1 \leq q \leq p, \\
& \sum_{j \in B_q} y_j = 1, && \text{for } q = 1, \dots, p, \\
& y_i \leq \delta_{ij}, && \text{for } (i,j) \in B_q,\ q = 1, \dots, p, \\
& \delta_{ij} = 1, && \text{if } i \in B_r,\ j \in B_s \text{ with } r<s, \\
& \delta_{ij} + \delta_{ji} = 1, && \text{for } i<j,\ i,j \in B_q \text{ and } q = 1, \dots, p.
\end{aligned}
$$

Both the inequalities of Proposition 4.2 and this model indicate that a judicious choice of sets S is important in solving (R).

5. A comparison of lower bounds

Here we compare several different relaxations of (R). We suppose first that $s=0$, so no enumeration is assumed. Let

$$L(u) = z(\tau(\phi), w-u) + \sum_{j=1}^{n} r_j u_j .$$

Remember that $z(\tau(\phi), w-u)$ is the value obtained when applying Smith's rule with weights $w-u$. We examine five relaxations.

Relaxation A (Smith's Rule).

$$
\begin{aligned}
z^A = z_0 = L(0) = \min \sum_{j=1}^{n} w_j t_j, \\
\text{for all } t_j \geq \sum_i p_i \delta_{ij}, \text{ for } j \in N, \\
\delta_{ij} + \delta_{ji} = 1, \text{ for } 1 \leq i < j \leq n, \\
t \in R_+^n,\ \delta \in R_+^{n(n-1)}.
\end{aligned}
$$

Relaxation B (Multiplier Adjustment).

$$
\begin{aligned}
z^B &= \max_{u \geq 0} \left\{ L(u) \colon \frac{w_1 - u_1}{p_1} \geq \cdots \geq \frac{w_n - u_n}{p_n} \geq 0 \right\} \\
&= L(O) + \max_{u \geq 0} \left\{ \sum_{j=1}^{n} \left(r_j - \sum_{i<j} p_i \right) u_j \colon \frac{w_1 - u_1}{p_1} \geq \cdots \geq \frac{w_n - u_n}{p_n} \geq 0 \right\}.
\end{aligned}
$$

z^B can be calculated very rapidly. A similar multiplier adjustment technique has been used several times by Potts and van Wassenhove [7], Hariri and Potts [2], and hopefully gives a good approximation to the next bound.

Relaxation C (Optimal Multiplier Adjustment).

$$z^C = \max_{u \geq 0} L(u).$$

Adding the constraints $t_j \geq r_j$ for $j \in N$ to Relaxation A, and then dualising gives Relaxation C. As the linear program A has the integrality property, Relaxation C is equivalent to a linear program.

Proposition 5.1.

$$z^C = \min \sum_{j \in N} w_j t_j,$$

$$t_j \geq \sum_i p_i \delta_{ij}, \quad \textit{for } j \in N,$$

$$\delta_{ij} + \delta_{ji} = 1, \quad \textit{for } 1 \leq i < j \leq n,$$

$$t_j \geq r_j, \quad \textit{for } j \in N,$$

$$\delta \in R_+^{n(n-1)}.$$

It is in addition obvious that $z^A \leq z^B \leq z^C$.

Relaxation D. We consider

$$(D') \quad \begin{aligned} &\min \sum_{j=1}^{n} w_j t_j, \\ &\sum_j y_{j\tau} \leq 1, \quad \text{for } \tau = 0, 1, \ldots, T, && \text{(7a)} \\ &\sum_\tau y_{j\tau} = p_j, \quad \text{for } j = 1, \ldots, n, && \text{(7b)} \\ &t_j = \frac{1}{p_j}\left[\sum_\tau \tau y_{j\tau} - \frac{(p_j - 1)p_j}{2}\right], && \text{(7c)} \\ &y \in Y^*, && \text{(7d)} \\ &y_{j\tau} \in \{0,1\}, \quad \text{for } r_j \leq \tau \leq T - p_j + 1 \text{ and } j = 1, \ldots, n, \end{aligned}$$

where T is an upper bound on the completion time of the last job, $y_{j\tau} = 1$ if job j is being processed in the time period $[\tau, \tau + 1]$ and Y^* are constraints imposing that each job j is processed during p_j consecutive periods (i.e., $y \in Y^*$ and $y_{js_1} = y_{js_2} = \cdots = y_{js_p} = 1$ implies $s_1, \ldots, s_p$ are consecutive periods).

Proposition 5.2. *Formulation (D′) is a reformulation of (R).*

Proof. If y is feasible in (D'), we have $y_{j\tau}=y_{j,\tau+1}=\cdots=y_{j,\tau+p_j-1}$ for some τ, and $y_{js}=0$ otherwise, and hence

$$t_j=\frac{1}{p_j}\left[\sum_s sy_{js}-\frac{(p_j-1)p_j}{2}\right]$$
$$=\frac{1}{p_j}\left[\tau+(\tau+1)+\cdots+(\tau+p_j-1)-\left(\frac{(p_j-1)p_j}{2}\right)\right]=\tau.$$

In addition the first set of constraints ensures that only one job is processed at a time. □

Now let (D) be the problem obtained from (D') by dropping the constraints $y\in Y^*$, and the integrality constraints, and let $M(u)+\sum_{j=1}^{n} r_j u_j$ be the optimal value of the resulting problem with objective weights $w-u$. It is readily observed that after elimination of the variables t_j, (D) is a transportation problem. In addition because of the particular cost structure there is an $O(n \log n)$ algorithm for (D). See Posner [6] who gives such an algorithm for the problem with deadlines.

We now establish that (D) is at least as strong a relaxation as relaxation (C). Let $z^D=M(O)$.

Proposition 5.3. $z^C\le z^D$.

Proof. It is easily verified that

$$L(O)=\min\left\{\sum_{j=1}^{T} w_j t_j\text{: (7a), (7b), } y_{j\tau}\in\{0,1\}\right.$$
$$\left.\text{for } 0\le\tau\le T-p_j+1 \text{ and } j=1,\ldots,n\right\}$$

so that $L(u)$ is a relaxation of $M(u)$. What is more, in any feasible solution of (D), $\sum_\tau \tau y_{j\tau}\ge r_j+(r_j+1)+\cdots+(r_j+p_j-1)$, and hence $t_j\ge r_j$ and $\max_{u\ge 0} M(u)=M(O)$. Therefore $z^C=L(u^*)\le M(u^*)\le M(O)=z^D$. □

Finally we consider what is in practice the strongest formulation but, as in the case of (D), it has a very large number nT of variables.

Formulation *E*. Here $x_{j\tau}=1$ if the processing of job j starts at time τ. The formulation is:

$$\min\sum_{j=1}^{n} w_j t_j,$$

$$\sum_\tau x_{j\tau}=1,\qquad \text{for } j=1,\ldots,n, \tag{8a}$$

(E')

$$\sum_j \sum_{\tau-p_j<s\le\tau} x_{js}\le 1,\qquad \text{for } \tau=0,\ldots,T, \tag{8b}$$

$$t_j = \sum_{\tau=0}^{T-p_j+1} \tau x_{j\tau}, \qquad \text{for } j=1,\dots,n, \tag{8c}$$

$$x_{j\tau} \in \{0,1\}, \qquad \text{for } r_j \le \tau \le T-p_j+1 \text{ and } j=1,\dots,n.$$

We let (E) denote the linear programming relaxation of (E'), with optimal value z^E.

Proposition 5.4. $z^D \le z^E$.

Proof. We show that every feasible solution to (E) gives a feasible solution to the linear programming relaxation of (D) with the same objective value using the substitution $y_{j\tau} = \sum_{\tau-p_j<s\le\tau} x_{js}$. If x is feasible in (E), then from (8b), $\sum_j y_{j\tau} \le 1$. In addition multiplying (8a) by p_j, we see that

$$p_j = p_j \sum_\tau x_{j\tau} = \sum_\tau p_j x_{j\tau} = \sum_\tau \sum_{\tau-p_j<s\le\tau} x_{js} = \sum_\tau y_{j\tau}.$$

Finally we consider the objective value. Starting from (8c), $t_j = \sum_\tau \tau x_{j\tau}$. Using (8a), this can be rewritten as

$$t_j = \frac{1}{p_j}\left[p_j \sum_\tau \tau x_{j\tau} + \sum_\tau x_{j\tau}\{1+\cdots+(p_j-1)\} - \tfrac{1}{2}(p_j-1)p_j\right].$$

Rewriting the summations gives

$$t_j = \frac{1}{p_j}\left[\sum_\tau \{\tau+(\tau+1)+\cdots+(\tau+p_j-1)\}x_{j\tau} - \tfrac{1}{2}(p_j-1)p_j\right]$$

$$= \frac{1}{p_j}\left[\sum_\tau \tau \sum_{\tau-p_j<s\le\tau} x_{js} - \tfrac{1}{2}(p_j-1)p_j\right].$$

Finally substituting for $y_{j\tau}$ we obtain

$$t_j = \frac{1}{p_j}\left[\sum_\tau \tau y_{j\tau} - \tfrac{1}{2}(p_j-1)p_j\right]$$

and we see by (7c) that the objective values are equal. □

The above framework of relaxations is useful for the analysis of other relaxations. For instance it is now easily seen that the first heuristic in Hariri and Potts [2] gives a lower bound z_1^{HP1} not exceeding z^C and the second a value z^{HP2} not exceeding z^D.

Example 5.5. The data for a 10-job problem is given in Table 1.

To calculate the lower bound z_B, it suffices to solve

$$\max_{u\ge 0} \quad 11u_1 + 9u_2 + 12u_3 - u_4 + 15u_5 + 2u_6 - 27u_7 - 33u_8 - 35u_9 - 35u_{10},$$

$$\text{s.t.} \quad \frac{w_i - u_i}{p_i} \ge \frac{w_{i+1} - u_{i+1}}{p_{i+1}} \quad \text{for } i=1,\dots,9.$$

Table 1

j	1	2	3	4	5	6	7	8	9	10
p_j	3	1	5	3	9	6	7	4	5	8
w_j	20	4	12	6	10	5	5	2	2	1
r_j	11	12	16	8	27	23	0	1	3	8

The optimal solution is easily verified to be $(w_i - u_i)^*/p_i = \frac{5}{7}$ for $i = 1, \ldots, 7$ $u_i^* = 0$ for $i = 8, 9, 10$.

The resulting bound is $z^B = L(O) + 376\frac{6}{7} = 1037\frac{6}{7}$.

Different bounds obtained for this example are shown in Table 2.

Table 2

s	0	1	2	$\cdots$	n
A	661	908	1041		1118
B	1037.9				1118
C	1051.5	1051.5	1077		1118
D	1064.5				1118
E	1107				1118

6. Concluding remarks

As the bounds in Table 2 show there is little doubt that the formulations based on time discretisation, models D and E, give stronger lower bounds than those presently obtainable using the (t, δ) variables. These models have the additional advantage that the feasible solution sets are essentially independent of the data, which is manifestly not the case in the (t, δ) space. However as the (t, δ) model appears a natural one in which to include precedence constraints, we are pursuing both theoretical and computational research on the range of different formulations discussed above.

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