



New coherent structures of the Vakhnenko–Parkes equation

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ABSTRACT

A variable separation solution with two arbitrary functions is obtained for the Vakhnenko–Parkes equation. New coherent structures such as the soliton-type, instanton-type and rogue wave-type structures are presented.

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1. Introduction

Nonlinear evolution equations (NLEEs) are encountered in lots of domains such as fluid mechanics, biology, condensed matter physics, optical fibre communication and quantum field theory. One of the most important research fields in the investigation of NLEEs is to seek for some special solutions with concrete physical meaning such as the N -soliton solutions, the algebraic geometry solutions and the rogue wave solutions. It is well known that Fourier transform and variable separation are the two most effective ways to find exact solutions for linear equations. The inverse scattering transformation (IST) serves as a nonlinear Fourier transform for integrable NLEEs. However, it is very difficult to extend the variable separation approach to nonlinear cases effectively and consistently. Fortunately, several kinds of special nonlinear variable separation approaches have been established recently, i.e. the classical symmetry method [1], the geometric method [2], the ansatz-based method [3,4], the conditional symmetry method [5–7], the formal variable separation approach [8] and the variable separation approach based on the corresponding Bäcklund transformation (BT-VSA) [9–17].

The BT-VSA has been established perfectly for many famous NLEEs such as the Davey–Stewartson equation, the Nizhnik–Novikov–Veselov equation, the dispersive long wave equation, the Broer–Kaup–Kupershmidt equation, the general $(N + M)$ -component AKNS equation, the symmetric sine–Gordon equation and the differential-difference special Toda lattice [9–17]. Because the variable separation solution includes some lower-dimensional arbitrary functions, we can construct abundant coherent structures such as the dromions, the lumps, the ring solitons, the breathers and the instantons.

The Vakhnenko equation

$$\frac{\partial}{\partial X} \mathfrak{D}u + u = 0, \quad \mathfrak{D} := \frac{\partial}{\partial t} + u \frac{\partial}{\partial X},$$

is first presented by Vakhnenko [18] to describe high-frequency waves in a relaxing medium [19]. In [20–23] authors have discussed the multi-loopsoliton solution to the Vakhnenko equation with boundary condition $u \rightarrow 0$ as $|x| \rightarrow \infty$. The key step in finding this solution is to introduce the transformation

$$x = \theta(X, T) := T + W(X, T) + x_0, \quad t = X, \quad W = \int_{-\infty}^X U(X', T) dX',$$

where x_0 is a constant, $u(x, t) = U(X, T) = W_X(X, T)$ and it is assumed that, as $|X| \rightarrow \infty$, $U \rightarrow 0$, the derivatives of W vanish, and W tend to be a constant. In terms of the new variables the Vakhnenko equation may be written as

$$W_{XXT} + (1 + W_T)W_X = 0.$$

In Refs. [24–26], authors also have studied two new equations which have loop-soliton, hump-soliton, and cusp-soliton solutions, namely the generalized Vakhnenko equation, and the modified generalized Vakhnenko equation. Following the papers [27–29], hereafter the above equation is referred to as the (1+1)-dimensional Vakhnenko–Parkes (VP) equation.

In Section 2 of this letter, a variable separation solution of the VP equation is obtained. New coherent structures and interactions are constructed and depicted in Section 3. The last section contains the conclusions and discussions.

2. A exact solution of the VP equation

In this section we will consider the VP equation,

$$V_{XXT} + V_X V_T = 0. \quad (1)$$

This equation arises from the equation $W_{XXT} + (1 + W_T)W_X = 0$ through the transformation $W(X, T) = V(X, T) - T$.

We suppose the exact solution of Eq. (1) has form

$$V(X, T) = \frac{V_0(X, T)}{\phi(X, T)} + V_1(X, T). \quad (2)$$

Substituting this expression into Eq. (1), we have

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$$\begin{aligned}
& \phi^{-4} [V_0 \phi_X \phi_T (V_0 - 6\phi_X)] + \phi^{-3} \{2\phi_X (2\phi_T V_{0X} + V_{0T} \phi_X) \\
& - V_0 [\phi_X (V_{0T} - 4\phi_{XT}) + \phi_T (V_{0X} - 2\phi_{XX})]\} \\
& + \phi^{-2} [-2\phi_X V_{0XT} - 2V_{0X} \phi_{XT} - \phi_T V_{0XX} + V_{0T} (V_{0X} - \phi_{XX}) \\
& - V_0 (\phi_T V_{1X} + V_{1T} \phi_X + \phi_{XXT})] + \phi^{-1} (V_{1T} V_{0X} + V_{0T} V_{1X} + V_{0XXT}) \\
& + V_{1XXT} + V_{1X} V_{1T} = 0.
\end{aligned} \quad (3)$$

Here we employ the symbolic computation softwares such as Mathematica, Maple and Matlab to achieve some lengthy calculations. Letting the coefficients of $\phi^n (n = -4, \dots, 0)$ to zero, we can get the following determining equations

$$\begin{aligned}
V_0 - 6\phi_X &= 0, \\
2\phi_X (2\phi_T V_{0X} + V_{0T} \phi_X) - V_0 [\phi_X (V_{0T} - 4\phi_{XT}) + \phi_T (V_{0X} - 2\phi_{XX})] &= 0, \\
2\phi_X V_{0XT} + 2V_{0X} \phi_{XT} + \phi_T V_{0XX} - V_{0T} (V_{0X} - \phi_{XX}) + V_0 (\phi_T V_{1X} + V_{1T} \phi_X + \phi_{XXT}) &= 0, \\
V_{1T} V_{0X} + V_{0T} V_{1X} + V_{0XXT} &= 0, \\
V_{1XXT} + V_{1X} V_{1T} &= 0.
\end{aligned}$$

These nonlinear equations constitute an auto-Bäcklund transformation (BT), if they are consistent, i.e., if this system is solvable with respect to ϕ , V_0 and V_1 .

To find out a variable separation solution, we set $V_0 = 6\phi_X$, seed solution $V_1 \equiv V_1(X)$ and ϕ has prior variable separation ansatz $\phi = F(T) + G(X)$. Then we can reduce the auto-BT to

$$G'V_1' + G''' = 0.$$

This means that the VP equation admits a variable separation solution

$$V(X, T) = 6 \frac{G'(X)}{F(T) + G(X)} - \int^X \frac{G'''(\tilde{X})}{G'(\tilde{X})} d\tilde{X}. \quad (4)$$

The presence of two 1-dimensional arbitrary functions $F(T)$ and $G(X)$ implies the existence of a rich diversity of coherent structures for the physical quantity

$$U = V_T = -6 \frac{F'G'}{(F+G)^2}. \quad (5)$$

Remark 1. Similarly, the generalized VP equation

$$V_{XXT} + \gamma V_X V_T + \alpha V_{XT} + \frac{\alpha\gamma}{3} V_T V + \beta V_T = 0$$

admits a variable separation solution

$$V(X, T) = \frac{6}{\gamma} \frac{G'(X)}{F(T) + G(X)} + e^{\frac{\alpha\gamma}{3} X} \left[c - \int^X \left(\frac{\alpha G''(\tilde{X}) + G'''(\tilde{X})}{\gamma G'(\tilde{X})} + \beta \right) e^{\frac{\alpha\gamma}{3} \tilde{X}} d\tilde{X} \right].$$

Remark 2. For some NLEEs such as the coupled mKdV systems [30], variable separation ansatz may be of more general form $\phi = a_0 + a_1 F + a_2 G + a_3 FG$.

3. New coherent structures of Eq. (5)

In this section, we will construct some new coherent structures for the physical quantity (5). It is known that there are some singularities for (5) for general selections of F and G . However, when the arbitrary functions are selected suitably, there may exist abundant coherent structures without singularities.

Example 1. It is well known that many (1+1)-dimensional integrable NLEEs such as the KdV equation, the Boussinesq equation, the KP equation and the BKP equation possess N -soliton solutions which are constructed by multiple exp-functions. Inspired by this structure, we construct new soliton-type coherent structures by selecting functions F and G appropriately. If we let $G(T) = e^T + e^{-T}$, $F(X) = e^{X^2}$, the two solitons collision with parabolic motion can be obtained and shown in Fig. 1.

Example 2. If some periodic functions in time variables are included in the function $G(T)$, we may obtain some new types of oscillating soliton-type structures. More concretely, if we select the functions $G(T) = e^{T+\sin T} + e^{-T}$, $F(X) = e^{X^2}$ and $G(T) = e^{T+\sin T} + e^{-T-\sin T}$, $F(X) = e^{X^2}$, then we have the structures which are plotted in Figs. 2 and 3.

Example 3. If we let $G(T) = 1 + \tanh T$, $F(X) = 1 - \tanh X$, the physical quantity U shows the soliton vanish phenomenon. From Fig. 4, we can see that when time $T < 0$, there is a soliton, and as T increases over 0, it vanishes very soon.

Example 4. Recently, instantons have attracted the attention of scientists [9]. Here we construct a compacton-type instanton. When choosing the following piecewise continuous functions

$$G(T) = 20 + \begin{cases} 0, & T \leq -\frac{\pi}{2}, \\ -2 \sin T - 2, & -\frac{\pi}{2} < T \leq \frac{\pi}{2}, \\ -4, & T > \frac{\pi}{2}, \end{cases}$$

and

$$F(X) = 20 + \begin{cases} 0, & X \leq -\frac{\pi}{2}, \\ \sin X + 1, & -\frac{\pi}{2} < X \leq \frac{\pi}{2}, \\ 2, & X > \frac{\pi}{2}, \end{cases}$$

then we can derive a compacton-type instanton which is plotted in Fig. 5. In fact, this coherent structure is not stable because the amplitude changes as t changes till decaying.

Example 5. Rogue waves (or freak waves) are single waves appearing in the ocean with amplitudes much higher than the average wave crests around them. If we let

$$\begin{aligned}
F'(X) &= 0.6 \operatorname{sech}^2(\xi + 7.5) + \operatorname{sech}^2(\xi + 2.5) + 1.6 \operatorname{sech}^2(\xi - 2.5), \\
X &= \xi - 0.5 \tanh(\xi + 7.5) - \tanh(\xi + 2.5) - 1.5 \tanh^2(\xi - 2.5), \\
G'(T) &= 20 \exp\left(\frac{\eta}{10} - 6\right), \\
T &= \eta,
\end{aligned}$$

the physical quantity U shows three rogue waves phenomenon including the bell-type, peakon-type and loop-type waves (see Fig. 6).

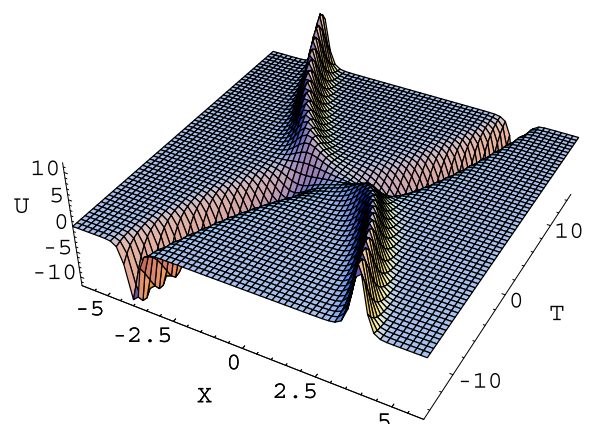


Fig. 1. Two solitons collision with parabolic motion.

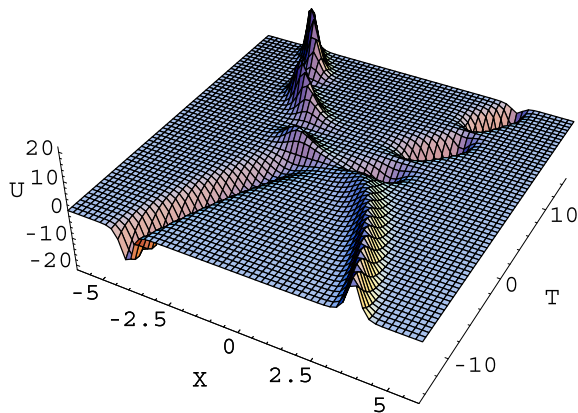


Fig. 2. Oscillating soliton-type structure (I).

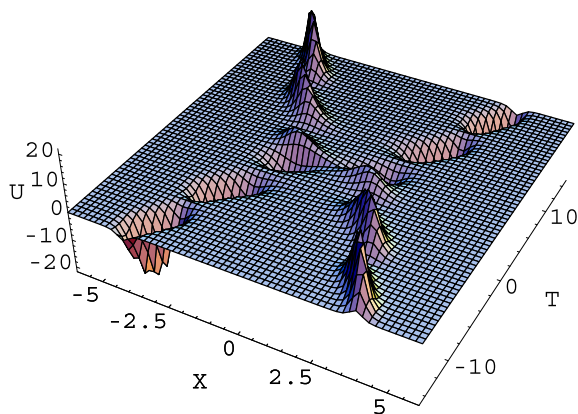


Fig. 3. Oscillating soliton-type structure (II).

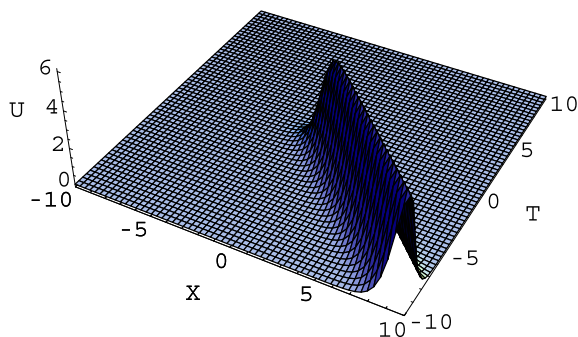


Fig. 4. Soliton vanish phenomenon.

Remark 3. For some concrete rogue waves, they may only exist in limited time, so instantons in Example 4 can be considered as a kind of rogue waves. Thus, rogue waves in Example 5 are called standing waves.

Remark 4. In (2+1)-dimensional NLEEs, the similar formula is $U = a \frac{F_X G_Y}{(F+G)^2}$. $F \equiv F(X, T)$ and $G \equiv G(Y, T)$ are arbitrary functions for some NLEEs like the DS and NNV equations. If we set

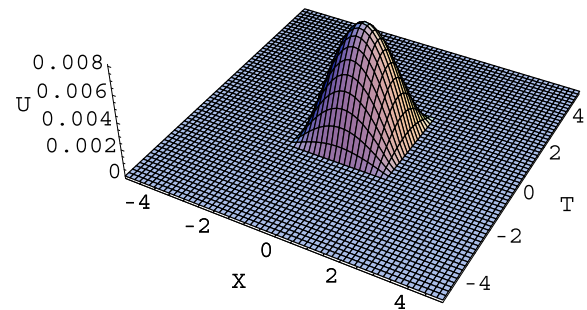


Fig. 5. Compacton-type instanton.

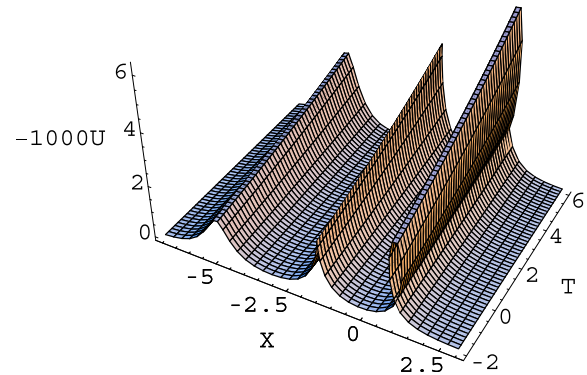
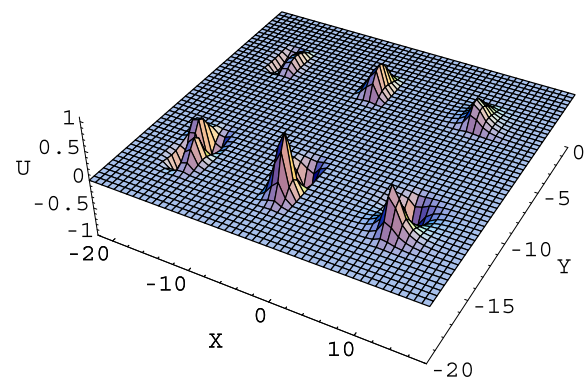


Fig. 6. Three rogue waves.

Fig. 7. Six coherent structures with $a = -10$ at $T = -5$.

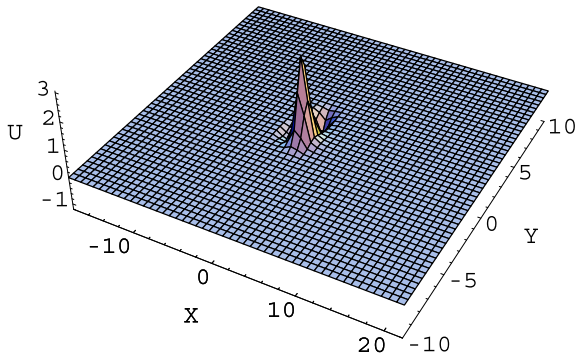
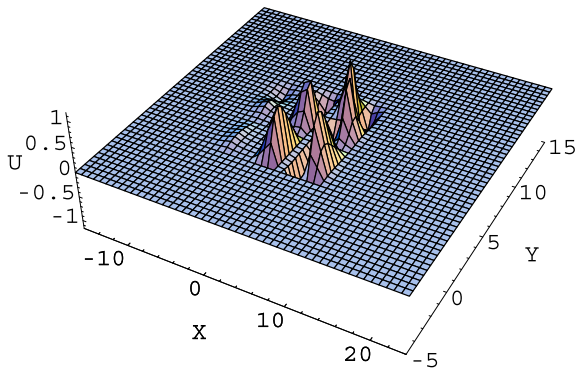
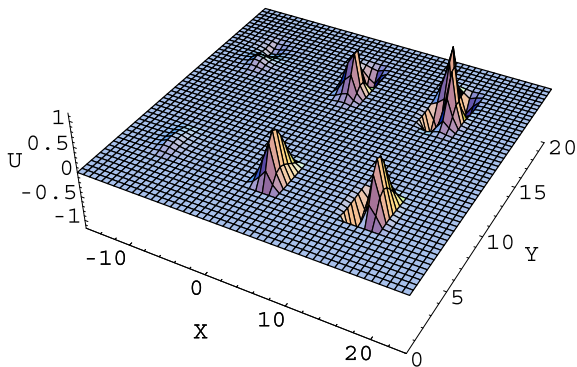
and

$$G = 20 + \frac{1}{10} e^{5-(Y-3T)^2} + \begin{cases} 0, & Y - T \leq -\frac{\pi}{2}, \\ \sin(Y - T) + 1, & -\frac{\pi}{2} < Y - T \leq \frac{\pi}{2}, \\ 2, & Y - T > \frac{\pi}{2}, \end{cases}$$

then the new six coherent structures including a (2+1)-dimensional ring soliton (RR), two semi-ring-compacton structures (RC, CR), a (2+1)-dimensional compacton (CC), a semi-ring-peakon structure (RP) and a peakon-compacton structure (PC) are plotted in Fig. 7. We use Figs. 8–11 to show their interactions with T.

Through the asymptotic analysis [9], we find that the interac-

$$F = 20 + \frac{1}{10} e^{5-(X-3T)^2} + 5 \begin{cases} 0, & X - \frac{3}{5}T \leq -\frac{\pi}{2}, \\ -2 \sin(X - \frac{3}{5}T) - 2, & -\frac{\pi}{2} < X - \frac{3}{5}T \leq \frac{\pi}{2}, \\ -4, & X - \frac{3}{5}T > \frac{\pi}{2}, \end{cases} - 5 \begin{cases} -\ln[\tanh \frac{1-X-2T}{2}], & X + 2T \leq 0, \\ \ln[\tanh \frac{1+X+2T}{2}] - 2 \ln[\tanh \frac{1}{2}], & X + 2T > 0, \end{cases}$$

Fig. 9. Interaction at $T = 0$.Fig. 10. Interaction at $T = 2$.Fig. 11. Six coherent structures after interaction at $T = 5$.

tion among these coherent structures is not elastic and the phase position is changed after the interaction. In (1+1)-dimensional case, these coherent structures are considered as instantons.

4. Summary

In summary, by means of the BT-VSA, a variable separation solution with two arbitrary functions is obtained for the VP equation. New coherent structures such as the soliton-type, instanton-type and rogue wave-type structures are constructed by selecting functions F and G suitably. Some useful remarks on a new solution of the generalized VP equation and (2+1)-dimensional coherent structures are presented. We also believe that more coherent structures of the VP equation can be found and it is worthwhile studying further.

In addition, besides the BT-VSA, there are many other approaches for solving NLEEs such as the linear superposition principle method, the multiple exp-function algorithm and the linear subspaces method [31–33]. Thus it is very meaningful to see whether these methods can be used to solve more NLEEs and obtain more new coherent structures.

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