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On multiparameter weighted ergodic theorem for noncommutative L_p-spaces

Farrukh Mukhamedov [a,*,1], Maksut Mukhamedov [b], Seyit Temir [c]

[a] *Departamento de Fisica, Universidade de Aveiro, Campus Universitario de Santiago, 3810-193 Aveiro, Portugal*
[b] *Faculty of Economy, Department of High Mathematics, Tashkent State Agrar University, Tashkent, Uzbekistan*
[c] *Department of Mathematics, Art and Science Faculty, Harran University, 63200 Sanliurfa, Turkey*

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Abstract

In the paper we consider $T_1, \ldots, T_d$ absolute contractions of von Neumann algebra $\mathcal{M}$ with normal, semifinite, faithful trace, and prove that for every bounded Besicovitch weight $\{a(\mathbf{k})\}_{\mathbf{k}\in\mathbb{N}^d}$ and every $x \in L_p(\mathcal{M})$ $(p > 1)$ the averages

$$A_{\mathbf{N}}(x) = \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}(x)$$

converge bilaterally almost uniformly in $L_p(\mathcal{M})$.

Keywords: Besicovitch weights; Ergodic theorem; Bilaterally almost uniformly; Noncommutative

1. Introduction

It is known the almost everywhere convergence of sequences of operators were applied to study of the individual ergodic theorem in von Neumann algebras by many authors [7,8,17,28] (see [11,12] for review). Very recently, in [15,16] various maximal ergodic theorems in noncommutative L_p-spaces have been established. As an application of such results the corresponding individual ergodic theorems were obtained. Study of the almost everywhere convergence of weighted averages in von Neumann algebras is relatively new. In [10], the Besicovitch weighted ergodic theorem were firstly proved in a semifinite von Neumann algebra with a faithful normal state, which was a generalization of [24] to a noncommutative setting. In [9] the noncommutative Banach principle firstly obtained, further using it in [19] a particular case of one-dimensional Besicovitch weighted ergodic theorems were proved in the space

* Corresponding author.
E-mail addresses: far75m@yandex.ru, farruh@fis.ua.pt (F. Mukhamedov), temirseyit@harran.edu.tr (S. Temir).
[1] Current address: Department of Comput. & Theor. Sci., Faculty of Sciences, IIUM, PO Box, 141, 25710, Kuantan, Pahang, Malaysia.

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of integrable operators affiliated with a von Neumann algebra. Latter on, in [3] by means of the principle that the Besicovitch weighted ergodic theorem has been proved in noncommutative L_1-spaces.

In [13] the multiparameter Besicovitch weights were introduced and weighted ergodic theorems were obtained in commutative L_p-spaces. Some other related investigations were done in [1,6,18]. The present paper is devoted to the noncommutative extension of that result. Further, we are going to prove the bilateral almost uniform convergence of weighted multiparameter averages with respect to bounded Besicovitch families in noncommutative L_p-spaces. To prove it, we use the maximal ergodic inequality for absolute contractions given in [16].

2. Preliminaries and notations

In what follows, $\mathcal{M}$ would be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Let S_+ denote the set of all $x \in \mathcal{M}_+$ such that $\tau(\operatorname{supp} x) < \infty$, where $\operatorname{supp} x$ denotes the support of x. Let S be the linear span of S_+. Then S is a $w*$-dense $*$-subalgebra of $\mathcal{M}$. Given $1 \leqslant p < \infty$, we define

$$\|x\|_p = \big[\tau\big(|x|^p\big)\big]^{1/p}, \quad x \in S,$$

where $|x| = (x^*x)^{1/2}$ is the modulus of x. Then $(S, \|\cdot\|_p)$ is a normed space, whose completion is the noncommutative L_p-space associated with $(\mathcal{M}, \tau)$, denoted by $L_p(\mathcal{M}, \tau)$ or simply by $L_p(\mathcal{M})$. As usual, we set $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_\infty$. We refer a reader to [23] for more information about noncommutative integration and to [27] for general terminology of von Neumann algebras.

Now recall some notions about the noncommutative $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d))$-spaces. Let $d \geqslant 1$. Given $1 \leqslant p \leqslant \infty$, the space $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d))$ is defined as the space of all families $x = (x_\mathbf{k})_{\mathbf{k}\in\mathbb{N}^d}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there are $a, b \in L_{2p}(\mathcal{M})$ and $y = (y_\mathbf{k}) \subset L_\infty(\mathcal{M})$ such that

$$x_\mathbf{k} = a y_\mathbf{k} b, \quad \forall \mathbf{k} \in \mathbb{N}^d.$$

We then define

$$\|x\|_{L_p(\mathcal{M};\ell_\infty(\mathbb{N}^d))} = \inf\Big\{\|a\|_{2p} \sup_{\mathbf{k}\in\mathbb{N}^d} \|y_\mathbf{k}\|_\infty \|b\|_{2p}\Big\},$$

where the infimum runs over all factorizations as above. Then $(L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d)), \|\cdot\|_{L_p(\mathcal{M};\ell_\infty(\mathbb{N}^d))})$ is a Banach space [5,14]. Note that these spaces were firstly introduced in [22], when $\mathcal{M}$ was a hyperfinite von Neumann algebra. In [14] it was shown that a family of positive elements $x = (x_\mathbf{k})_{\mathbf{k}\in\mathbb{N}^d}$ belongs to $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d))$ iff there is $a \in L_p(\mathcal{M})_+$ such that $x_\mathbf{k} \leqslant a$ for all $\mathbf{k} \in \mathbb{N}^d$, and moreover,

$$\|x\|_{L_p(\mathcal{M};\ell_\infty(\mathbb{N}^d))} = \inf\big\{\|a\|_p\colon a \in L_p(\mathcal{M})_+ \text{ s.t. } x_\mathbf{k} \leqslant a,\ \forall \mathbf{k} \in \mathbb{N}^d\big\}.$$

The norm of x in $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d))$ will be very often denoted by $\|\sup^+_\mathbf{n} x_\mathbf{n}\|_p$.

We should be noted that $\|\sup^+_\mathbf{n} x_\mathbf{n}\|_p$ is just a notation for $\sup_\mathbf{n} x_\mathbf{n}$ does not make any sense in the noncommutative setting. Some elementary properties of these spaces were presented in [5,14,16].

For $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ denote $m(\mathbf{n}) = \min\{n_1, \dots, n_d\}$, $M(\mathbf{n}) = \max\{n_1, \dots, n_d\}$. Let us denote $\Lambda_{[m,n]} = \{\mathbf{k} = (k_1 \cdots k_d) \in \mathbb{N}^d\colon m \leqslant m(\mathbf{k}),\ M(\mathbf{k}) \leqslant n\}$. In the sequel we will deal with the following convergence, namely, a family $(x_\mathbf{n})_{\mathbf{n}\in\mathbb{N}^d}$ in a Banach space X converges to $x \in X$ if $\forall\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\|x_\mathbf{n} - x\|_X < \varepsilon$ for all $\mathbf{n}$: $m(\mathbf{n}) \geqslant N_0$.

We denote by $L_p(\mathcal{M}; \ell_\infty^{\Lambda_{[m,n]}}(\mathbb{N}^d))$ the subspace of $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d))$ consisting of all finite sequences $\{x_\mathbf{k}\}_{\mathbf{k}\in\mathbb{N}^d}$ such that $x_\mathbf{k} = 0$ if $\mathbf{k} \notin \Lambda_{[m,n]}$. In accordance with our preceding convention, the norm of x in $L_p(\mathcal{M}; \ell_\infty^{\Lambda_{[m,n]}}(\mathbb{N}^d))$ will be denoted by $\|\sup^+_{\mathbf{k}\in\Lambda_{[m,n]}} x_\mathbf{k}\|_p$.

Now introduce a subspace $L_p(\mathcal{M}; c_0(\mathbb{N}^d))$ of $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d))$, which is defined as the space of all families $(x_\mathbf{n})_{\mathbf{n}\in\mathbb{N}^d} \subset L_p(\mathcal{M})$ such that there are $a, b \in L_{2p}(\mathcal{M})$ and $(y_\mathbf{n}) \subset \mathcal{M}$ verifying

$$x_\mathbf{n} = a y_\mathbf{n} b \quad \text{and} \quad \lim_{\mathbf{n}\to\infty} \|y_\mathbf{n}\|_\infty = 0.$$

One can check that $L_p(\mathcal{M}; c_0(\mathbb{N}^d))$ is a closed subspace of $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d))$ (see [5,14], for more details) and

$$\Big\|\sup_\mathbf{n}{}^+ x_\mathbf{n}\Big\|_p = \inf\Big\{\|a\|_{2p} \sup_{\mathbf{n}\in\mathbb{N}^d} \|y_\mathbf{n}\|_\infty \|b\|_{2p}\Big\},$$

where the infimum runs over all factorizations of $(x_\mathbf{n})$ as above.

Let $\mathcal{M}$ be, as before, a von Neumann algebra equipped with a semifinite normal faithful trace τ. Let $x, (x_{\mathbf{n}}) \subset L_p(\mathcal{M})$. A family $(x_{\mathbf{n}})$ is said to converge *bilaterally almost uniformly* (*b.a.u.* in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau\bigl(e^{\perp}\bigr) < \varepsilon \quad \text{and} \quad \lim_{m(\mathbf{n})\to\infty} \bigl\| e(x_{\mathbf{n}} - x)e \bigr\|_{\infty} = 0.$$

In the commutative case, the convergence in the definition above is equivalent to the usual almost everywhere convergence by virtue of Egorov's theorem [20,25].

The following lemma gives a relation between the b.a.u. convergence and $L_p(\mathcal{M}; c_0(\mathbb{N}^d))$.

Lemma 2.1. *If for every $\mathbf{p} \in \mathbb{N}^d$, $\{x_{\mathbf{n}+\mathbf{p}} - x_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}; c_0(\mathbb{N}^d))$ with $1 \leqslant p < \infty$, then $x_{\mathbf{n}}$ converges b.a.u. to some x from $L_p(\mathcal{M})$.*

The proof immediately follows from [16, Lemma 6.2] and [3, Theorems 1.2 and 2.3].

Let $(Z, \mathcal{F}, \mu)$ be a measurable space with a probability measure μ. Let $\widetilde{\mathcal{M}}$ be the von Neumann algebra of all essentially bounded ultra-weakly measurable functions $h : (Z, \mu) \to \mathcal{M}$ equipped with the trace

$$\tilde{\tau}(h) = \int_Z \tau\bigl(h(z)\bigr)\, d\mu(z),$$

and let $\widetilde{L}_p = L_p(\widetilde{\mathcal{M}}, \tilde{\tau})$. It is known [2] that the space $\widetilde{L}_p$ is isomorphic to $L_p(Z, \mu; L_p(\mathcal{M}))$.

For the sake of completeness, we provide the proof for the next lemma, which is an analog of Lemma 2 in [4] (see also [3]).

Lemma 2.2. *Let $\{x_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\widetilde{\mathcal{M}}, c_0(\mathbb{N}^d))$. Then $\{x_{\mathbf{n}}(z)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}, c_0(\mathbb{N}^d))$ for almost all $z \in Z$.*

Proof. From the definition of $L_p(\widetilde{\mathcal{M}}, c_0(\mathbb{N}^d))$ we have $x_{\mathbf{n}} = a y_{\mathbf{n}} b$, where $a, b \in L_{2p}(Z, \mu; L^p(\mathcal{M}))$, $y_{\mathbf{n}} \in \widetilde{\mathcal{M}}$, and $\|y_{\mathbf{n}}\|_{\widetilde{\mathcal{M}}} \to 0$ as $\mathbf{n} \to \infty$.

For any $z \in Z$ consider $x_{\mathbf{n}}(z) = a(z) y_{\mathbf{n}}(z) b(z)$. We have

$$\|y_{\mathbf{n}}\|_{\widetilde{M}} = \operatorname*{ess\,sup}_{z\in Z} \bigl\| y_{\mathbf{n}}(z) \bigr\|_{\mathcal{M}} \to 0,$$

therefore $\|y_{\mathbf{n}}(z)\|_{\mathcal{M}} \to 0$, $\forall z \in Z \setminus D$, where $\mu(D) = 0$.

Since $a, b \in L_{2p}(Z, L_p(\mathcal{M}))$ we conclude that

$$\int_Z \bigl\|a(z)\bigr\|_{L_p(\mathcal{M})}^{2p}\, d\mu < \infty; \qquad \int_Z \bigl\|b(z)\bigr\|_{L_p(\mathcal{M})}^{2p}\, d\mu < \infty.$$

Denote

$$Z_k^{(\phi)} = \bigl\{ z \in Z\colon\ \bigl\|\phi(z)\bigr\|_{L_p(\mathcal{M})}^{2p} > k^2 \bigr\},$$

and put $Z^{(\phi)} = \bigcap_{n=1}^{\infty} \bigcup_{k\geqslant n} Z_k^{(\phi)}$, where $\phi = a, b$. Then

$$\mu\bigl(Z^{(\phi)}\bigr) \leqslant \mu\biggl(\bigcup_{k\geqslant n} Z_k^{(\phi)}\biggr) \leqslant \sum_{k=n}^{\infty} \mu\bigl(Z_k^{(\phi)}\bigr) \leqslant \|\phi\|_{L_{2p}(Z, L_p(\mathcal{M}))}^{1/2p} \sum_{k=n}^{\infty} \frac{1}{k^2} \underset{n\to\infty}{\longrightarrow} 0.$$

Here, we have used that

$$\mu\bigl(Z_k^{(\phi)}\bigr) \leqslant \frac{1}{k^2} \int \bigl\|\phi(z)\bigr\|_{L_p(\mathcal{M})}^{2p}\, d\mu.$$

So, $\mu(Z^{(\phi)}) = 0$. Putting $N = Z^{(a)} \cup Z^{(b)} \cup D$, we have $\mu(N) = 0$, and $a(z), b(z) \in L_{2p}(\mathcal{M})$, $y_{\mathbf{n}}(z) \in \mathcal{M}$ with $\|y_{\mathbf{n}}(z)\|_{\mathcal{M}} \to 0$ as $\mathbf{n} \to \infty$ for every $z \in Z \setminus N$. Hence $\{x_{\mathbf{n}}(z)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}, c_0(\mathbb{N}^d))$ a.e. $z \in Z$. $\square$

Let $T : \mathcal{M} \to \mathcal{M}$ be a linear map. We say that T is an *absolute contraction* if it satisfies the following conditions:

(i) T is a contraction on $\mathcal{M}$: $\|Tx\|_\infty \leqslant \|x\|_\infty$ for all $x \in \mathcal{M}$.
(ii) T is positive: $Tx \geqslant 0$ if $x \geqslant 0$.
(iii) $\tau \circ T \leqslant \tau$: $\tau(T(x)) \leqslant \tau(x)$ for all $x \in L_1(\mathcal{M}) \cap \mathcal{M}_+$.

It is well known [16,28] that if T satisfies these properties, then T naturally extends to a contraction on $L_p(\mathcal{M})$ for all $1 \leqslant p < \infty$.

In the sequel, unless explicitly specified otherwise, T will always denote an absolute contraction of $\mathcal{M}$. The same symbol T will also stand for the extensions of T on $L_p(\mathcal{M})$. Now let $T_1, \dots, T_d$ be such kind of mappings.

We form their ergodic averages:

$$M_{\mathbf{N}}(\mathbf{T}) = \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}} \mathbf{T}^{\mathbf{k}}, \tag{2.1}$$

where $\mathbf{T}^{\mathbf{k}} = T_d^{k_d} \cdots T_1^{k_1}$ with $\mathbf{N} = (N_1, \dots, N_d)$, $\mathbf{k} = (k_1, \dots, k_d)$ and $|\mathbf{N}| = N_1 \cdots N_d$.

In [16] the following maximal inequality has been proved

Theorem 2.3. *Let $p > 1$ and $T_1, \dots, T_d$ be absolute contractions of $\mathcal{M}$. For any $x \in L_p(M)_+$ there is $a \in L_p(M)_+$ such that*

$$M_{\mathbf{N}}(\mathbf{T})(x) \leqslant a, \quad \forall \mathbf{N} \in \mathbb{N}^d \quad \text{and} \quad \|a\|_p \leqslant C_p^d \|x\|_p,$$

where C_p is a positive constant depending only on p. Moreover, $C_p \leqslant C\, p^2(p-1)^{-2}$ and this is the optimal order of C_p as $p \to 1$.

In the present paper we are going to consider the Besicovitch weights in $\mathbb{N}^d$. It is said [13] that a family of complex numbers $\{a(\mathbf{k})\}$ to be *Besicovitch weight* if for every $\varepsilon > 0$ there is a family of trigonometric polynomials $\{P_\varepsilon(\mathbf{k})\}$ in d variables such that

$$\limsup_{m(\mathbf{N})\to\infty} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}} \bigl|a(\mathbf{k}) - P_\varepsilon(\mathbf{k})\bigr| < \varepsilon. \tag{2.2}$$

A Besicovitch weight $\{a(\mathbf{k})\}$ is said to be *bounded* if $\{a(\mathbf{k})\} \in \ell_\infty(\mathbb{N}^d)$. Note that certain properties of Besicovitch weights were studied in [13,18,24].

3. Weighted vector ergodic theorem for noncommutative L_p-spaces

In this section we are going to prove the following a weighted ergodic theorem for absolute contractions.

Theorem 3.1. *Let $T_1, \dots, T_d$ be absolute contractions of von Neumann algebra $\mathcal{M}$. For every bounded Besicovitch weight $\{a(\mathbf{k})\}$ and every $x \in L_p(\mathcal{M})$ $(p > 1)$ the averages*

$$A_{\mathbf{N}}(x) = \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}} a(\mathbf{k})\mathbf{T}^{\mathbf{k}}(x) \tag{3.1}$$

converge b.a.u. in $L_p(\mathcal{M})$.

To prove the main result now we need some auxiliary lemmas. Let us first recall the following well-known principle:

Theorem 3.2. *Let X, Y be Banach spaces and Z be a subspace of Y. Assume that $T : X \to Y$ is a linear continuous mapping such that $T(X_0) \subset Z$ for a dense subset $X_0 \subset X$. Then $T(X) \subset Z$.*

From this principle, by taking $Z = L_p(\mathcal{M}, c_0(\mathbb{N}^d))$, $Y = L_p(\mathcal{M}, \ell_\infty(\mathbb{N}^d))$ we immediately get the following:

Lemma 3.3. *Let X be a Banach space and $a_{\mathbf{n}} : X \to L_p(\mathcal{M}, \ell_\infty(\mathbb{N}^d))$ ($\mathbf{n} \in \mathbb{N}^d$) be linear mappings such that*

(i) $\|a_{\mathbf{n}}(x)\|_{L_p(\mathcal{M},\ell_\infty(\mathbb{N}^d))} \leqslant C\|x\|_X$, $\forall x \in X$;
(ii) $\{a_{\mathbf{n}+\mathbf{p}}(x) - a_{\mathbf{n}}(x)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}, c_0(\mathbb{N}^d))$ *for every* $\mathbf{p} \in \mathbb{N}^d$ *and* $x \in X_0$*, where* X_0 *is a dense subspace of* X.

Then $\{a_{\mathbf{n}+\mathbf{p}}(x) - a_{\mathbf{n}}(x)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}; c_0(\mathbb{N}^d))$ *for all* $x \in X$.

Lemma 3.4. *Let $T_1, \ldots, T_d$ be as in Theorem* 3.1. *Then for every trigonometric polynomial $P(\mathbf{k})$ on $\mathbb{N}^d$ and every $x \in L_p(\mathcal{M})$ the averages*

$$\widetilde{A}_{\mathbf{N}}(x) = \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}} P(\mathbf{k})\mathbf{T}^{\mathbf{k}}(x) \tag{3.2}$$

satisfy the following relation $\{\widetilde{A}_{\mathbf{n}+\mathbf{p}}(x) - \widetilde{A}_{\mathbf{n}}(x)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(M, c_0(\mathbb{N}^d))$ *for every* $\mathbf{p} \in \mathbb{N}^d$.

Proof. Let $\mathbb{B} = \{z \in \mathbb{C}\colon |z| = 1\}$ be the unite circle in $\mathbb{C}$ with the normalized Lebesgue measure σ. Let d be a fixed positive integer, then denote

$$\mathbb{K} = \underbrace{\mathbb{B} \times \cdots \times \mathbb{B}}_{d}, \qquad \mu = \underbrace{\sigma \otimes \cdots \otimes \sigma}_{d}.$$

Now consider $\widetilde{L}_p = L_p(\widetilde{\mathcal{M}})$, where $\widetilde{\mathcal{M}} = \mathcal{M} \otimes L_\infty(\mathbb{K}, \mu)$ and $\tilde{\tau} = \tau \otimes \mu$.

For any $\mathbf{g} = (g_i)_{i=1}^d, \mathbf{z} = (z_i)_{i=1}^d \in \mathbb{K}$ we put

$$\mathbf{g} \circ \mathbf{z} = (g_1 z_1, \ldots, g_d z_d).$$

Now fix $\mathbf{g} \in \mathbb{K}$ and define a linear mapping $\widetilde{\mathbf{T}}_{\mathbf{g}} : \widetilde{L}_p \to \widetilde{L}_p$ by

$$\widetilde{\mathbf{T}}_{\mathbf{g}}(f)(\mathbf{z}) = \mathbf{T}\big(f(\mathbf{g} \circ \mathbf{z})\big), \tag{3.3}$$

where $f = f(\mathbf{z}) \in \widetilde{L}_p$, $\mathbf{z} \in \mathbb{K}$. One can see that the mapping $\widetilde{\mathbf{T}}_{\mathbf{g}}$ is an absolute contraction. Therefore, according to [16, Theorem 6.6] for the averages $M_{\mathbf{N}}(\widetilde{\mathbf{T}}_{\mathbf{g}})$ we have $\{M_{\mathbf{n}+\mathbf{p}}(\widetilde{\mathbf{T}}_{\mathbf{g}})(f) - M_{\mathbf{n}}(\widetilde{\mathbf{T}}_{\mathbf{g}})(f)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\widetilde{\mathcal{M}}, c_0(\mathbb{N}^d))$ for every $\mathbf{p} \in \mathbb{N}^d$ and $f \in \widetilde{L}_p$. Hence, Lemma 2.2 implies that $\{M_{\mathbf{n}+\mathbf{p}}(\widetilde{\mathbf{T}}_{\mathbf{g}})(f(\mathbf{z})) - M_{\mathbf{n}}(\widetilde{\mathbf{T}}_{\mathbf{g}})(f(\mathbf{z}))\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}, c_0(\mathbb{N}^d))$ a.e. $\mathbf{z} \in \mathbb{K}$. Now applying the letter one to the function $f_x(\mathbf{z}) = \Pi(\mathbf{z})x$, where $\Pi(\mathbf{z}) = z_1 \cdots z_d$, $x \in L_p(\mathcal{M})$ is fixed, which clearly belongs to $\widetilde{L}_p$, we obtain

$$\left\{\Pi(\mathbf{z})\left(\frac{1}{|\mathbf{n}+\mathbf{p}|}\sum_{\mathbf{k}=1}^{\mathbf{n}+\mathbf{p}} \Pi\big(\mathbf{g}^{\mathbf{k}}\big)\mathbf{T}^{\mathbf{k}}(x) - \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} \Pi\big(\mathbf{g}^{\mathbf{k}}\big)\mathbf{T}^{\mathbf{k}}(x)\right)\right\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p\big(\mathcal{M}, c_0\big(\mathbb{N}^d\big)\big)$$

for almost all $\mathbf{z} \in \mathbb{K}$. Here we have used (3.3) to get $(\widetilde{\mathbf{T}}_{\mathbf{g}})^{\mathbf{k}}(f_x)(\mathbf{z}) = \Pi(\mathbf{g}^{\mathbf{k}})\Pi(\mathbf{z})\mathbf{T}^{\mathbf{k}}x$ for every $\mathbf{k} \in \mathbb{N}^d$, where $\mathbf{g}^{\mathbf{k}} = (g_1^{k_1}, \ldots, g_d^{k_d})$.

Consequently, due to $\Pi(\mathbf{z}) \neq 0$, we conclude that

$$\left\{\frac{1}{|\mathbf{n}+\mathbf{p}|}\sum_{\mathbf{k}=1}^{\mathbf{n}+\mathbf{p}} \Pi\big(\mathbf{g}^{\mathbf{k}}\big)\mathbf{T}^{\mathbf{k}}(x) - \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} \Pi\big(\mathbf{g}^{\mathbf{k}}\big)\mathbf{T}^{\mathbf{k}}(x)\right\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p\big(\mathcal{M}, c_0\big(\mathbb{N}^d\big)\big).$$

One can see that the theorem holds for finite linear combinations of $\Pi(\mathbf{g}^{\mathbf{k}})$, hence holds for trigonometric polynomials in d variables. □

Now let us turn to the proof of the main Theorem 3.1.

Proof. Take $x \in L_p(\mathcal{M})_+$. Without lost of generality we may assume that $|a(\mathbf{k})| \leqslant 1$ for all $\mathbf{k} \in \mathbb{N}^d$. We are going to show that $A_{\mathbf{N}}(x)$ belongs to $L_p(\mathcal{M}, \ell_\infty(\mathbb{N}^d))$. To do it let us consider[2]

[2] Note that $\{a(\mathbf{k})\}$ are complex numbers, therefore we need to consider their real and imaginary parts separately.

$$A_{\mathbf{N}}^{(R)}(x)=\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}}\Re\big(a(\mathbf{k})\big)\mathbf{T}^{\mathbf{k}}(x), \tag{3.4}$$

$$A_{\mathbf{N}}^{(I)}(x)=\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}}\Im\big(a(\mathbf{k})\big)\mathbf{T}^{\mathbf{k}}(x). \tag{3.5}$$

Due to our assumption (i.e. $|a(\mathbf{k})|\leqslant 1$) and Theorem 2.3 one has

$$-a\leqslant -M_{\mathbf{N}}(\mathbf{T})(x)\leqslant A_{\mathbf{N}}^{(R)}(x)\leqslant M_{\mathbf{N}}(\mathbf{T})(x)\leqslant a, \tag{3.6}$$

where $\|a\|_p\leqslant C_p^d\|x\|_p$. From (3.6) we have $A_{\mathbf{N}}^{(R)}(x)\in L_p(\mathcal{M},\ell_\infty(\mathbb{N}^d))$, moreover

$$\Big\|\sup_{\mathbf{N}}{}^{+}A_{\mathbf{N}}^{(R)}(x)\Big\|_{L_p(\mathcal{M},\ell_\infty(\mathbb{N}^d))}\leqslant C\|x\|_p \tag{3.7}$$

for some constant C (more exactly, $C=3C_p^d$). Similarly, one gets

$$\Big\|\sup_{\mathbf{N}}{}^{+}A_{\mathbf{N}}^{(I)}(x)\Big\|_{L_p(\mathcal{M},\ell_\infty(\mathbb{N}^d))}\leqslant C\|x\|_p. \tag{3.8}$$

Consequently, (3.7), (3.8) imply

$$\Big\|\sup_{\mathbf{N}}{}^{+}A_{\mathbf{N}}(x)\Big\|_{L_p(\mathcal{M},\ell_\infty(\mathbb{N}^d))}\leqslant 2C\|x\|_p,\quad \forall x\in L_p(\mathcal{M})_+. \tag{3.9}$$

Take any $x\in L_p(\mathcal{M})$, then one can be represented as $x=\sum_{k=0}^{3}i^kx_k$, where $x_k\in L_p(\mathcal{M})_+$, $k=0,1,2,3$. Therefore, the inequality (3.9) implies that

$$\Big\|\sup_{\mathbf{N}}{}^{+}A_{\mathbf{N}}(x)\Big\|_{L_p(\mathcal{M},\ell_\infty(\mathbb{N}^d))}\leqslant 8C\|x\|_p,\quad \forall x\in L_p(\mathcal{M}). \tag{3.10}$$

Now let us assume that $x\in L_1(\mathcal{M})\cap\mathcal{M}$, then $x\in L_q(\mathcal{M})$ for any $1<q<\infty$. For an arbitrary $\varepsilon>0$ due to the definition of the Besicovitch weight there is a trigonometric polynomial $\{P_\varepsilon(\mathbf{k})\}$ for which (2.2) holds. Let $\widetilde{A}_{\mathbf{N}}(x)$ be the corresponding averages (see (3.2)). Then from (3.10) we conclude that $\{A_{\mathbf{N}}(x)\}\in L_q(\mathcal{M},\ell_\infty(\mathbb{N}^d))$, $\{\widetilde{A}_{\mathbf{N}}(x)\}\in L_q(\mathcal{M},\ell_\infty(\mathbb{N}^d))$, so $\{A_{\mathbf{N}}(x)-\widetilde{A}_{\mathbf{N}}(x)\}\in L_q(\mathcal{M},\ell_\infty(\mathbb{N}^d))$. Moreover,

$$\big\|A_{\mathbf{k}}(x)-\widetilde{A}_{\mathbf{k}}(x)\big\|_\infty\leqslant\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}}\big|a(\mathbf{k})-P_\varepsilon(\mathbf{k})\big|\big\|\mathbf{T}^{\mathbf{k}}(x)\big\|_\infty\leqslant\|x\|_\infty\frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=1}^{\mathbf{N}}\big|a(\mathbf{k})-P_\varepsilon(\mathbf{k})\big|<\varepsilon\|x\|_\infty.$$

From the last relation and Proposition 2.5 in [16] one finds

$$\begin{aligned}&\Big\|\sup_{\mathbf{k}\in\Lambda_{[k,n]}}{}^{+}\big(A_{\mathbf{k}}(x)-\widetilde{A}_{\mathbf{k}}(x)\big)\Big\|_{L_p(\mathcal{M},\ell_\infty(\mathbb{N}^d))}\\&\quad\leqslant\sup_{\mathbf{k}\in\Lambda_{[k,n]}}\big\|A_{\mathbf{k}}(x)-\widetilde{A}_{\mathbf{k}}(x)\big\|_\infty^{1-\frac{q}{p}}\Big\|\sup_{\mathbf{k}\in\Lambda_{[k,n]}}{}^{+}\big(A_{\mathbf{k}}(x)-\widetilde{A}_{\mathbf{k}}(x)\big)\Big\|_{L_q(\mathcal{M},\ell_\infty(\mathbb{N}^d))}^{\frac{q}{p}}\\&\quad\leqslant\varepsilon^{1-\frac{q}{p}}\|x\|_\infty^{1-\frac{q}{p}}\Big\|\sup_{\mathbf{k}\in\mathbb{N}^d}{}^{+}\big(A_{\mathbf{k}}(x)-\widetilde{A}_{\mathbf{k}}(x)\big)\Big\|_{L_q(\mathcal{M},\ell_\infty(\mathbb{N}^d))}^{\frac{q}{p}}.\end{aligned} \tag{3.11}$$

Define a sequence $b^{(k)}=(b_{\mathbf{k}}^{(k)})_{\mathbf{k}\in\mathbb{N}^d}\in L_p(\mathcal{M},c_0(\mathbb{N}^d))$ as follows:

$$b_{\mathbf{k}}^{(k)}=\begin{cases}A_{\mathbf{k}}(x)-\widetilde{A}_{\mathbf{k}}(x), & \text{if } \mathbf{k}\in\Lambda_{[0,k]},\\ 0, & \text{if } \mathbf{k}\notin\Lambda_{[0,k]}.\end{cases}$$

From (3.11) one gets that $b^{(k)}\to\{A_{\mathbf{k}}(x)-\widetilde{A}_{\mathbf{k}}(x)\}$ in $L_p(\mathcal{M},\ell_\infty(\mathbb{N}^d))$ as $k\to\infty$. Since $\{b^{(k)}\}\subset L_p(\mathcal{M},c_0(\mathbb{N}^d))$ we obtain $\{A_{\mathbf{k}}(x)-\widetilde{A}_{\mathbf{k}}(x)\}\in L_p(\mathcal{M},c_0(\mathbb{N}^d))$.

According to Lemma 3.4 we have already known that $\{\widetilde{A}_{\mathbf{n}+\mathbf{p}}(x) - \widetilde{A}_{\mathbf{n}}(x)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}, c_0(\mathbb{N}^d))$. Consequently, the equality

$$A_{\mathbf{n}+\mathbf{p}}(x) - A_{\mathbf{n}}(x) = \big(A_{\mathbf{n}+\mathbf{p}}(x) - \widetilde{A}_{\mathbf{n}+\mathbf{p}}(x)\big) + \big(\widetilde{A}_{\mathbf{n}}(x) - A_{\mathbf{n}}(x)\big) + \big(\widetilde{A}_{\mathbf{n}+\mathbf{p}}(x) - \widetilde{A}_{\mathbf{n}}(x)\big)$$

implies that $\{A_{\mathbf{n}+\mathbf{p}}(x) - A_{\mathbf{n}}(x)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}, c_0(\mathbb{N}^d))$.

Now by means of the density of $L_1(\mathcal{M}) \cap \mathcal{M}$ in $L_p(\mathcal{M})$ and (3.10) with Lemma 3.3, we have $\{A_{\mathbf{n}+\mathbf{p}}(x) - A_{\mathbf{n}}(x)\}_{\mathbf{n}\in\mathbb{N}^d} \in L_p(\mathcal{M}, c_0(\mathbb{N}^d))$ for any $x \in L_p(\mathcal{M})$. Hence, Lemma 2.1 implies that the required assertion. □

Remark. 1. The proved theorem extends the results of the paper [3] to multiparameter. When $a(\mathbf{k}) \equiv 1$, then we get an extension of [21,26] to L_p-spaces.

2. Similar results for multiparameter Besicovitch weights in a commutative setting were obtained in [1,13].

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