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Nonlinear free vibrations of planar elastic beams: A unified treatment of geometrical and mechanical effects

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Abstract

The exact equations of motion of a planar, initially straight, beam are determined within the large displacement framework, by considering geometrical nonlinearities and linear elastic behaviour of the material. With the aim of investigating the behaviour also for low slenderness, shear deformations and rotational inertia are taken into account, together with axial inertia. An axial linear spring is added to one end of the beam, permitting us to investigate the effect of varying boundary conditions, from the hinged-supported (stiffness=0) to the hinged-hinged (stiffness=∞) limit cases. The Poincaré-Lindstedt method is applied to obtain an approximate analytical solution. The nonlinear frequency correction ω_2, responsible for the hardening vs softening nonlinear behaviour, is determined. Preliminary results on its dependence on the system parameters are illustrated.

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1. Introduction

The nonlinear dynamic behaviour of a beam strongly depends on whether one constrained boundary is axially immovable or movable, being hardening (softening) in the former (latter) case[1,2]. For axially restrained - e.g., hinged - beams, the axial inertia and the nonlinearity due to the curvature are negligible, and the dominant nonlinearity is due to the axial stretching, seemingly introduced for the first time by Mettler[3]; in contrast, for axially unrestrained - e.g., simply supported - beams, the axial inertia is likely to provide the most important nonlinear contribution, and the beam is assumed inextensible in the absence of axial loads.

These outcomes are confirmed by the direct perturbation analysis of approximate models of extensible and inextensible beams derived from the geometrically exact theory of rods, as well as by experimental results[4].

Shearable beam models are presented in a number of specific and general[5,6] works, but they are rarely used to specifically investigate the effects of shear deformation on nonlinear vibrations also because slender beams (with a minimum slenderness of about 20) are commonly considered, by consistently neglecting rotatory inertia and shear deformation.

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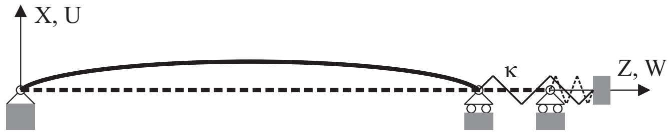

Fig. 1. Current configuration (continuous line) of the initially straight (dashed line) beam with end spring of stiffness κ. U and W are the transversal and axial displacements, respectively.

The present work aims at comprehensively revisiting the matter, by investigating the nonlinear behaviour of axially restrained or unrestrained beams of whatever slenderness. This is made in a unified framework where rotatory inertia and shear deformation are taken into account in addition to the other mechanical features (axial inertia, axial stretching, etc.). This permits to determine the limit of low slenderness for which the common simplifying hypotheses hold, and to have general results also for non-slender beams. Furthermore, we also consider the effects of the end spring stiffness κ, which provides a means of transition from the axially restrained ($\kappa \to \infty$) to the axially unrestrained case ($\kappa = 0$).

We attack directly the exact governing partial differential equations of motion, without introducing any approximation or condensation. Attention is focused on the nonlinear correction of the linear frequency, which is positive (negative) for hardening (softening) behavior, thus corresponding to qualitatively regimes of major difference also from a practical viewpoint. The paper is focused on the analytical developments necessary to obtain the nonlinear correction. Preliminary results are reported, while a detailed investigation will be the object of a forthcoming paper[7].

2. The beam model

Let us consider an initially straight planar beam (Fig. 1), and let us denote by $W(Z,T)$ and $U(Z,T)$ the axial and the transversal displacements of the beam axis, respectively. Z is the spatial coordinate in the *rest* rectilinear configuration, which ranges from 0 to the length L. T is the time. κ is the stiffness of the spring at the right-end of the beam.

2.1. Kinematics

By referring to Fig. 2a we have

$$S' = \sqrt{(1+W')^2 + U'^2}, \quad \cos\varphi = \frac{1+W'}{S'}, \quad \sin\varphi = \frac{U'}{S'}, \quad \tan\varphi = \frac{U'}{1+W'}, \tag{1}$$

where the prime denotes derivative with respect to Z, φ is the slope angle of the beam axis, and θ is the rotation of the beam cross-section. The measures of strain are (see Fig. 2a for γ)

$$e = S' - 1, \quad k = \frac{d\theta}{dS} = \frac{\theta'}{S'}, \quad \gamma = \theta - \varphi, \tag{2}$$

where e is the elongation of the beam axis, k the curvature and γ the shear strain. Note that, according to the Timoshenko beam model[8] (see also the works of Huang[9] and Cao and Tucker[10]), k is not the curvature of the axis of the beam, i.e. $k \neq \frac{d\varphi}{dS}$ (unless the beam is unshearable, $\gamma = 0$); moreover, owed to the considered axial deformability, $k = \frac{d\theta}{dS}$ instead of the usual definition $k = \frac{d\theta}{dZ}$, which only holds for inextensible beams.

2.2. Balance

The balance equations are (see Fig. 2b)

$$H_o' = \rho B \ddot{W}, \quad V_e' = \rho A \ddot{U}, \quad M' - VS' = \rho J \ddot{\theta}, \tag{3}$$

where the dot denotes derivative with respect to the time T, where we have not considered external loads and damping (since we are interested in free and undamped oscillations), and where:

- $H_o = N\cos\varphi + V\sin\varphi$ and $V_e = N\sin\varphi - V\cos\varphi$ are the horizontal (in the Z-direction) and vertical (in the X-direction) internal forces, respectively;

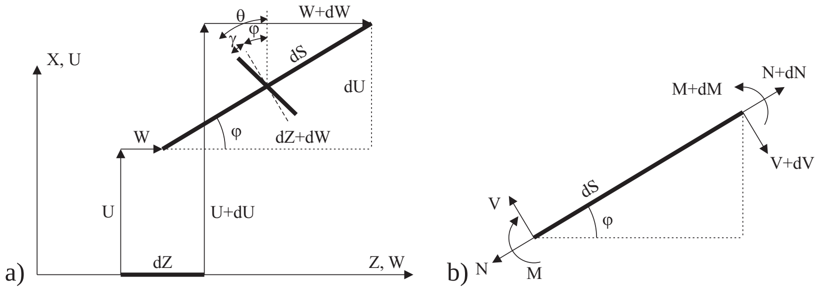

Fig. 2. (a) Undeformed (dZ) and deformed (dS) beam element. φ is the slope angle of the beam axis, and γ the shear strain. (b) Internal forces: N=axial force, V=shear force and M=bending moment.

- N, V and M are the axial force, shear force and bending moment, respectively;
- ρB is the mass per unit length in the *reference* configuration in the horizontal Z-direction;
- ρA is the mass per unit length in the *reference* configuration in the vertical X-direction;
- ρJ is the second moment of inertia of the beam cross-section in the *reference* configuration.

Note that N and V are tangent and perpendicular to the axis of the beam, respectively, so that they are *not* perpendicular and tangent to the beam cross-section (unless the beam is unshearable, $\gamma = 0$, see Fig. 2a). This is correct for the axial force, while it may appear a rough approximation for the shear. The other option, i.e. to assume the internal forces perpendicular and tangential to the cross section, would have the opposite properties, namely the shear would be correct and the axial force approximated, but it is not adopted here.

In general $\rho B = \rho A$, but we prefer to keep them disjoint because often one neglects the axial inertia, i.e. assumes $\rho B = 0$, while the transverse inertia ρA is never negligible in transverse oscillations.

2.3. Constitutive behaviour

We consider the following linear elastic behaviour

$$N = EAe, \quad V = GA\gamma, \quad M = EJk, \tag{4}$$

because we want to focus on geometric nonlinearities only. EA, GA and EJ are the axial, shear and bending stiffnesses, respectively, and are assumed to be constant. Note that, according to the truly 1D approach used in this paper, each of them (as well as ρA, ρB and ρJ) is considered as a *unique* parameter, and *not* as the product of a material modulus times a geometric property of the cross-section.

2.4. Boundary conditions

The following boundary conditions for the transversal displacement are considered:

$$U(0,T) = 0, \quad U(L,T) = 0, \quad M(0,T) = 0, \quad M(L,T) = 0, \tag{5}$$

where L is the length of the beam.

For the horizontal displacement we assume:

$$W(0,T) = 0. \tag{6}$$

Furthermore, three different cases are considered:

$$W(L,T) = 0 \rightarrow \text{hinged-hinged beam}; \tag{7}$$

$$H_o(L,T) = 0 \rightarrow \text{hinged-supported beam}; \tag{8}$$

$$H_o(L,T) + \kappa W(L,T) = 0 \rightarrow \text{hinged-spring beam}. \tag{9}$$

Note that (7) and (8) are obtained by assuming $\kappa \rightarrow \infty$ and $\kappa = 0$ in (9), respectively.

3. Asymptotic solution

Based on kinematics, balance and constitutive behaviour, the following exact PDEs of motion are obtained

$$\begin{aligned}
&\left\{EA[\sqrt{(1+W')^2+U'^2}-1]\frac{1+W'}{\sqrt{(1+W')^2+U'^2}}+GA\left[\theta-\arctan\left(\frac{U'}{1+W'}\right)\right]\frac{U'}{\sqrt{(1+W')^2+U'^2}}\right\}'=\omega^2\rho B\ddot{W},\\
&\left\{EA[\sqrt{(1+W')^2+U'^2}-1]\frac{U'}{\sqrt{(1+W')^2+U'^2}}-GA\left[\theta-\arctan\left(\frac{U'}{1+W'}\right)\right]\frac{1+W'}{\sqrt{(1+W')^2+U'^2}}\right\}'=\omega^2\rho A\ddot{U},\\
&\left[EJ\frac{\theta'}{\sqrt{(1+W')^2+U'^2}}\right]'-GA\left[\theta-\arctan\left(\frac{U'}{1+W'}\right)\right]\sqrt{(1+W')^2+U'^2}=\omega^2\rho J\ddot{\theta},
\end{aligned}\tag{10}$$

where the time is rescaled as $t=\omega T$, and the dot means derivative with respect to the dimensionless time t. In view of pursuing an asymptotic solution, it is convenient to develop (10) up to the third order:

$$\begin{aligned}
&EA(W'+\tfrac{1}{2}U'^2-U'^2W')'+GA(U'\theta-U'^2+2U'^2W'-U'W'\theta)'=\omega^2\rho A\ddot{W},\\
&EA(U'W'+\tfrac{1}{2}U'^3-U'W'^2)'+GA(U'-\theta-U'W'+\tfrac{1}{2}U'^2\theta-\tfrac{5}{6}U'^3+U'W'^2)'=\omega^2\rho A\ddot{U},\\
&EJ(\theta'-W'\theta'+W'^2\theta'-\tfrac{1}{2}U'^2\theta')'+GA(U'-\theta-W'\theta-\tfrac{1}{2}U'^2\theta+\tfrac{1}{6}U'^3)=\omega^2\rho J\ddot{\theta}.
\end{aligned}\tag{11}$$

An asymptotic solution by means of the Poincaré-Lindstedt method [11] is considered, by expanding the configuration variables and the frequency in the form (ε is a small book-keeping parameter)

$$\begin{aligned}
&W(Z,t)=\varepsilon W_1(Z,t)+\varepsilon^2W_2(Z,t)+\varepsilon^3W_3(Z,t)+...,\\
&U(Z,t)=\varepsilon U_1(Z,t)+\varepsilon^2U_2(Z,t)+\varepsilon^3U_3(Z,t)+...,\quad \theta(Z,t)=\varepsilon\theta_1(Z,t)+\varepsilon^2\theta_2(Z,t)+\varepsilon^3\theta_3(Z,t)+...,\\
&\omega=\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+...,
\end{aligned}\tag{12}$$

Inserting the expressions (12) in the governing equations, and equating to zero the coefficients of ε^n, we get the following sequence of linear problems.

First order

$$EAW_1''-\rho B\omega_0^2\ddot{W}_1=0,\quad GA(\theta_1-U_1')'+\rho A\omega_0^2\ddot{U}_1=0,\quad EJ\theta_1''-GA(\theta_1-U_1')-\rho J\omega_0^2\ddot{\theta}_1=0,\tag{13}$$

with the boundary conditions (here and at following orders we consider directly the most general case (9))

$$W_1(0,t)=0,\quad EAW_1'(L,t)+\kappa W_1(L,t)=0,\quad U_1(0,t)=U_1(L,t)=0,\quad \theta_1'(0,t)=\theta_1'(L,t)=0.\tag{14}$$

Second order

$$\begin{aligned}
&EAW_2''-\rho B\omega_0^2\ddot{W}_2=-GA[U_1'(\theta_1-U_1')]'-EA(\tfrac{U_1'^2}{2})'+2\omega_0\omega_1\rho B\ddot{W}_1,\\
&GA(\theta_2-U_2')'+\rho A\omega_0^2\ddot{U}_2=(EA-GA)(U_1'W_1')'-2\omega_0\omega_1\rho A\ddot{U}_1,\\
&EJ\theta_2''-GA(\theta_2-U_2')-\rho J\omega_0^2\ddot{\theta}_2=GA\theta_1W_1'+EJ(W_1'\theta_1')'+2\omega_0\omega_1\rho J\ddot{\theta}_1,
\end{aligned}\tag{15}$$

with the boundary conditions (use is made of the boundary conditions at the previous order)

$$\begin{aligned}
&W_2(0,t)=0,\quad EAW_2'(L,t)+\kappa W_2(L,t)+(\tfrac{EA}{2}-GA)U_1'^2(L,t)+GA\theta_1(L,t)U_1'(L,t)=0,\\
&U_2(0,t)=U_2(L,t)=0,\quad \theta_2'(0,t)=\theta_2'(L,t)=0.
\end{aligned}\tag{16}$$

Third order

$$\begin{aligned}
&EAW_3''-\rho B\omega_0^2\ddot{W}_3=-GA[-U_1'W_1'\theta_1+2U_1'^2W_1'-2U_1'U_2'+U_1'\theta_2+U_2'\theta_1]'\\
&\quad -EA[-U_1'^2W_1'+U_1'U_2']'+2\omega_0\omega_2\rho B\ddot{W}_1+2\omega_0\omega_1\rho B\ddot{W}_2+\omega_1^2\rho B\ddot{W}_1,\\
&GA(\theta_3-U_3')'+\rho A\omega_0^2\ddot{U}_3=GA[-\tfrac{U_1'^3}{3}+\theta_1\tfrac{U_1'^2}{2}]'+(EA-GA)[U_2'W_1'-W_1'^2U_1'+\tfrac{U_1'^3}{2}+U_1'W_2']'-\\
&\quad -2\omega_0\omega_2\rho A\ddot{U}_1-2\omega_0\omega_1\rho A\ddot{U}_2-\omega_1^2\rho A\ddot{U}_1,\\
&EJ\theta_3''-GA(\theta_3-U_3')-\rho J\omega_0^2\ddot{\theta}_3=GA[-\tfrac{U_1'^3}{6}+\theta_1(W_2'+\tfrac{U_1'^2}{2})+\theta_2W_1']+EJ[\theta_2'W_1'+\theta_1'W_2'-\theta_1'(W_1'^2-\tfrac{U_1'^2}{2})]'\\
&\quad +2\omega_0\omega_2\rho J\ddot{\theta}_1+2\omega_0\omega_1\rho J\ddot{\theta}_2+\omega_1^2\rho J\ddot{\theta}_1,
\end{aligned}\tag{17}$$

with the boundary conditions (use is made of the boundary conditions at the previous orders)

$$\begin{aligned}
&W_3(0,t)=0,\quad EAW_3'(L,t)+\kappa W_3(L,t)+(EA-2GA)[U_2'(L,t)U_1'(L,t)-W_1'(L,t)U_1'^2(L,t)]+\\
&\quad +GA[-\theta_1(L,t)W_1'(L,t)U_1'(L,t)+\theta_2(L,t)U_1'(L,t)+U_2'(L,t)\theta_1(L,t)]=0,\\
&U_3(0,t)=U_3(L,t)=0,\quad \theta_3'(0,t)=\theta_3'(L,t)=0.
\end{aligned}\tag{18}$$

3.1. First order solution

The first order terms $U_1(Z,t)$, $W_1(Z,t)$ and $\theta_1(Z,t)$ satisfy the equations (13) and the boundary conditions (14), which are the same equations reported in[8,9] for a shearable beam. In these equations the transversal (U_1 and θ_1) and axial (W_1) displacements are decoupled. We assume as dominant the transversal behaviour, i.e. we consider $W_1 = 0$.

The general solution of $(13)_2$ and $(13)_3$ is given by

$$\begin{aligned}
&U_1(Z,t) = U_{1a}(Z)\sin(t), \quad \theta_1(Z,t) = \theta_{1a}(Z)\sin(t),\\
&U_{1a}(Z) = U_a\sin(\lambda_{U1}Z) + U_b\cos(\lambda_{U1}Z) + U_c\sinh(\lambda_{U2}Z) + U_d\cosh(\lambda_{U2}Z),\\
&\theta_{1a}(Z) = \alpha_1\lambda_{U1}[U_a\cos(\lambda_{U1}Z) - U_b\sin(\lambda_{U1}Z)] + \alpha_2\lambda_{U2}[U_c\cosh(\lambda_{U2}Z) + U_d\sinh(\lambda_{U2}Z)],
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
&\alpha_1 = \frac{GA}{GA-\rho J\omega_0^2+EJ\lambda_{U1}^2}, \quad \lambda_{U1} = \sqrt{\frac{\omega_0}{2}\frac{(EJ\rho A+GA\rho J)\omega_0+\sqrt{(EJ\rho A-GA\rho J)^2\omega_0^2+4GA^2EJ\rho A}}{GAEJ}},\\
&\alpha_2 = \frac{GA}{GA-\rho J\omega_0^2-EJ\lambda_{U2}^2}, \quad \lambda_{U2} = \sqrt{\frac{\omega_0}{2}\frac{-(EJ\rho A+GA\rho J)\omega_0+\sqrt{(EJ\rho A-GA\rho J)^2\omega_0^2+4GA^2EJ\rho A}}{GAEJ}}.
\end{aligned} \tag{20}$$

Note that the denominators of α_1 and α_2 never vanish for the values of ω_0 and λ_{U1} determined in the following.

Using the boundary conditions (14) we get $U_b = U_c = U_d = 0$, namely $U_{1a}(Z) = U_a\sin(\lambda_{U1}Z)$ and $\theta_{1a}(Z) = \alpha_1\lambda_{U1}U_a\cos(\lambda_{U1}Z)$, and

$$\sin(\lambda_{U1}L) = 0 \rightarrow \lambda_{U1} = \frac{n\pi}{L}. \tag{21}$$

Inserting (21) in the λ_{U1} formula of (20) and inverting provides ω_0. To simplify its expression we assume

$$EA = \frac{EJ}{L^2}l^2, \quad \rho B = \rho A x, \quad \rho J = \frac{\rho A L^2}{l^2}y, \quad GA = \frac{EJ}{L^2}l^2 z, \quad \kappa = \frac{EJ}{L^3}k_h, \tag{22}$$

where

- $l = L\sqrt{(A/J)}$ is the slenderness of the beam;
- $x = 0$ if we neglect the axial inertia and $x = 1$ if we consider it;
- $y = 0$ if we neglect the rotational inertia and $y = 1$ if we consider it;
- z is a parameter that measures the shear stiffness, which ranges from $[2(1+\nu)\chi]^{-1}$ (ν is the Poisson coefficient and χ is the shear correction factor, equal to 6/5 for rectangular cross-section) to ∞ (for unshearable beams);
- κ_h is the dimensionless stiffness of the spring at $Z = L$, to be used later on.

Using (22) we get the following expression:

$$\omega_0 = \frac{1}{L^2}\sqrt{\frac{EJ}{\rho A}}\bar{\omega}_0, \quad \bar{\omega}_0 = l\sqrt{\frac{zl^2+n^2\pi^2(1+zy)-\sqrt{z^2l^4+2zn^2\pi^2(1+zy)l^2+n^4\pi^4(1-zy)^2}}{2y}}. \tag{23}$$

The previous one is the *linear* natural (circular) frequency of the problem, which takes into account all the mechanical characteristics that we have considered, apart from the longitudinal inertia ρB and the end spring stiffness (i.e. x and κ_h, see (22)) that do not appear at this order.

3.2. Second order solution

The second order terms $U_2(Z,t)$, $W_2(Z,t)$ and $\theta_2(Z,t)$ satisfy the following equations, which are obtained by inserting the first order solution in (15):

$$\begin{aligned}
&EAW_2'' - \rho B\omega_0^2\ddot{W}_2 = (\tfrac{EA}{2} + GA\alpha_1 - GA)U_a^2\lambda_{U1}^3\sin(2\lambda_{U1}Z)\sin^2(t),\\
&GA(\theta_2' - U_2'') + \rho A\omega_0^2\ddot{U}_2 = 2U_a\rho A\omega_0\omega_1\sin(\lambda_{U1}Z)\sin(t),\\
&EJ\theta_2'' - GA(\theta_2 - U_2') - \rho J\omega_0^2\ddot{\theta}_2 = -2U_a\rho J\omega_0\omega_1\alpha_1\lambda_{U1}\cos(\lambda_{U1}Z)\sin(t).
\end{aligned} \tag{24}$$

The solution of the last two equations is given by

$$U_2(Z,t) = U_{2a}(Z)\sin(t), \quad \theta_2(Z,t) = \theta_{2a}(Z)\sin(t), \tag{25}$$

where here and in the following order we do not consider the solutions of the homogeneous equations since they are yet considered at the first order. $U_{2a}(Z)$ and $\theta_{2a}(Z)$ satisfy the equations

$$-GA(\theta_{2a} - U'_{2a})' + \rho A\omega_0^2 U_{2a} + f_{2U} = 0, \quad EJ\theta''_{2a} - GA(\theta_{2a} - U'_{2a}) + \rho J\omega_0^2\theta_{2a} + f_{2\theta} = 0, \tag{26}$$

where

$$f_{2U} = 2U_a\rho A\omega_0\omega_1 \sin(\lambda_{U1}Z), \quad f_{2\theta} = 2U_a\rho J\omega_0\omega_1\alpha_1\lambda_{U1}\cos(\lambda_{U1}Z). \tag{27}$$

The solvability condition of (26) is

$$\int_0^L [f_{2U}(Z)U_{1a}(Z) + f_{2\theta}(Z)\theta_{1a}(Z)]\, dZ = 0 \to U_a^2\omega_0\omega_1(\rho AL + \rho J\alpha_1^2 n^2\pi^2/L) = 0, \tag{28}$$

which provides $\omega_1 = 0$ and $f_{2U} = f_{2\theta} = 0$, thus also entailing $U_2(Z,t) = 0$ and $\theta_2(Z,t) = 0$. The condition $\omega_1 = 0$ is not surprising, since it is well known that the nonlinear frequency depends quadratically, and not linearly, on the excitation amplitude, see equation $(12)_4$.

The solution of $(24)_1$ is given by

$$W_2(Z,t) = W_{2a}(Z) + W_{2b}(Z)\cos(2t),$$

$$\frac{W_{2a}(Z)}{U_a^2} = -\frac{\lambda_{U1}}{16}\frac{EA+2GA(\alpha_1-1)}{EA}\sin(2\lambda_{U1}Z) + \frac{c_1}{L}\frac{Z}{L}, \quad \frac{W_{2b}(Z)}{U_a^2} = \frac{\lambda_{U1}^3}{16}\frac{EA+2GA(\alpha_1-1)}{EA\lambda_{U1}^2-\rho B\omega_0^2}\sin(2\lambda_{U1}Z) + \frac{c_2}{L}\sin\left(\frac{2\omega_0\sqrt{\rho B}}{\sqrt{EA}}Z\right). \tag{29}$$

Note that:

- the axial displacement (and the axial force) oscillates with a frequency double of the frequency of the transversal displacements. The oscillations are not around the rest position, since $W_{2a}(Z) \neq 0$;
- the c_2 term is present only when axial inertia ρB is considered;
- when $\omega_0 = \lambda_{U1}\sqrt{(EA/\rho B)}$ the function $W_{2b}(Z)$ is not defined. This corresponds to the (linear) natural frequencies of axial vibrations. However, it is well-known that transversal vibrations (those considered here) have principal frequencies that are much lower than the frequencies of the axial vibrations (not considered here, as $W_1 = 0$), thus we can assume that $EA\lambda_{U1}^2 \neq \rho B\omega_0^2$. More precisely, we are assuming that n is sufficiently small or, if it is large, that no internal resonance occurs between transversal and longitudinal modes.

With the expressions (29) we have $W_2(0,t) = 0$, i.e. (6) is satisfied to the second order. We also have

$$\frac{W_2(L,t)}{U_a^2} = \frac{c_1}{L} + \frac{c_2}{L}\sin\left(\frac{2\omega_0\sqrt{\rho B}}{\sqrt{EA}}L\right)\cos(2t), \quad \frac{H_{o2}(L,t)}{U_a^2} = \left[\frac{EA}{L}\frac{c_1}{L} + \lambda_{U1}^2\left(\frac{EA}{8} + \frac{GA}{4}(\alpha_1-1)\right)\right] + \\ + \left[-\frac{\lambda_{U1}^2}{8}\frac{(EA+2GA(\alpha_1-1))(2\rho B\omega_0^2 - EA\lambda_{U1}^2)}{\rho B\omega_0^2 - EA\lambda_{U1}^2} + 2\frac{c_2}{L}\omega_0\sqrt{\rho B}\sqrt{EA}\cos\left(\frac{2\omega_0\sqrt{\rho B}}{\sqrt{EA}}L\right)\right]\cos(2t). \tag{30}$$

From the previous relations we see that:

- assuming $c_1 = c_2 = 0$ we have $W_2(L,t) = 0$ and $H_{o2}(L,t) \neq 0$, namely (7) is satisfied to the second order and we have a hinged-hinged beam;
- assuming

$$\frac{c_1}{L^2\lambda_{U1}^2} = -\left[\frac{1}{8} + \frac{GA}{4EA}(\alpha_1-1)\right], \quad c_2 = \frac{[EA+2GA(\alpha_1-1)](2\rho B\omega_0^2 - EA\lambda_{U1}^2)}{2\omega_0\sqrt{\rho B}\sqrt{EA}\cos\left(\frac{2\omega_0\sqrt{\rho B}}{\sqrt{EA}}L\right)}\frac{L}{8\left(\frac{\rho B\omega_0^2}{\lambda_{U1}^2} - EA\right)}, \tag{31}$$

  we have $H_{o2}(L,t) = 0$ and $W_2(L,t) \neq 0$, namely (8) is satisfied to the second order and we have a hinged-supported beam.
- assuming

$$\frac{c_1}{L^2\lambda_{U1}^2} = -\frac{\frac{1}{8} + \frac{GA}{4EA}(\alpha_1-1)}{1 + \frac{\kappa L}{EA}}, \quad c_2 = \frac{[EA+2GA(\alpha_1-1)](2\rho B\omega_0^2 - EA\lambda_{U1}^2)}{2\omega_0\sqrt{\rho B}\sqrt{EA}\cos\left(\frac{2\omega_0\sqrt{\rho B}}{\sqrt{EA}}L\right) + \kappa L\sin\left(\frac{2\omega_0\sqrt{\rho B}}{\sqrt{EA}}L\right)}\frac{L}{8\left(\frac{\rho B\omega_0^2}{\lambda_{U1}^2} - EA\right)}, \tag{32}$$

we have $H_{o2}(L,t) + \kappa W_2(L,t) = 0$ (Fig. 1), namely (9) is satisfied to the second order and we have a hinged-spring beam.

3.3. Third order solution

The third order is needed to compute the nonlinear frequency correction ω_2. $W_3(Z,t)$ is not requested for our purposes and so it will not be considered. The other two unknowns in $(17)_2$ and $(17)_3$, where the lower order solutions are inserted, are given by

$$U_3(Z,t) = U_{3a}(Z)\sin(t) + U_{3b}(Z)\sin(3t), \quad \theta_3(Z,t) = \theta_{3a}(Z)\sin(t) + \theta_{3b}(Z)\sin(3t). \tag{33}$$

$U_{3b}(Z)$ and $\theta_{3b}(Z)$ do not provide secular terms in the equations, and so are not interesting for the present work. $U_{3a}(Z)$ and $\theta_{3a}(Z)$, on the other hand, satisfy the equations

$$-GA(\theta_{3a} - U'_{3a})' + \rho A\omega_0^2 U_{3a} + f_{3U} = 0, \quad EJ\theta''_{3a} - GA(\theta_{3a} - U'_{3a}) + \rho J\omega_0^2\theta_{3a} + f_{3\theta} = 0, \tag{34}$$

where

$$\begin{aligned} f_{3U} &= 2\rho A\omega_0\omega_2 U_{1a} + EA[U'_{1a}(W'_{2a} - \tfrac{1}{2}W'_{2b}) + \tfrac{3}{8}U'^3_{1a}]' - GA[U'_{1a}(W'_{2a} - \tfrac{1}{2}W'_{2b}) + \tfrac{5}{8}U'^3_{1a} - \tfrac{3}{8}\theta_{1a}U'^2_{1a}]', \\ f_{3\theta} &= 2\rho J\omega_0\omega_2\theta_{1a} - EJ[\theta'_{1a}(W'_{2a} - \tfrac{1}{2}W'_{2b}) + \tfrac{3}{8}\theta'_{1a}U'^2_{1a}]' + GA[-\theta_{1a}(W'_{2a} - \tfrac{1}{2}W'_{2b} + \tfrac{3}{8}U'^2_{1a}) + \tfrac{1}{8}U'^3_{1a}]. \end{aligned} \tag{35}$$

Similarly to (28), the solvability condition for the system (34) is

$$\int_0^L [f_{3U}(Z)U_{1a}(Z) + f_{3\theta}(Z)\theta_{1a}(Z)]\,dZ = 0 \to -\omega_2\omega_{2d} + U_a^2\left[c_1\omega_{2a} + c_2\sin\left(\frac{2L\omega_0\sqrt{\rho B}}{\sqrt{EA}}\right)\omega_{2b} + \omega_{2c}\right] = 0, \tag{36}$$

where the expressions of ω_{2a}, ω_{2b}, ω_{2c} and ω_{2d} are reported in the Appendix A. Note that ω_{2d} does not vanish for the considered values of ω_0. Solving this equation finally yields ω_2, which can be rewritten in the form

$$\omega_2 = \left(\frac{U_a}{L}\right)^2 \frac{1}{L^2}\sqrt{\frac{EJ}{\rho A}}\bar{\omega}_2, \tag{37}$$

where $\bar{\omega}_2$ is a dimensionless quantity that depends on l (slenderness), x (axial inertia), y (rotational inertia), z (shear stiffness) and κ_h (spring stiffness).

We have that

$$\omega = \frac{1}{L^2}\sqrt{\frac{EJ}{\rho A}}\left[\bar{\omega}_0 + \left(\frac{\varepsilon U_a}{L}\right)^2\bar{\omega}_2 + ...\right]. \tag{38}$$

Since εU_a is the amplitude of the (first order) oscillations (see (12) and (19)), the previous equation provides the so-called “backbone” curve, which shows how the (nonlinear) frequency depends on the square of the oscillation amplitude.

4. Preliminary results and forthcoming work

The nonlinear correction frequency $\bar{\omega}_2$ for $x = 1$ (i.e. considering the axial inertia), $y = 1$ (i.e. considering the rotational inertia), $z = 0.3205$ (i.e. for $\nu = 0.3$ and $\chi = 1.2$) and $n = 1$ (first mode) is reported in Fig. 3 for varying slenderness l and for different values of the end spring stiffness k_h.

The main observation is that it is confirmed that for slender beams the hinged-supported boundary conditions ($k_h = 0$) provide softening behaviour ($\bar{\omega}_2 < 0$), while hinged-hinged boundary conditions ($k_h \to \infty$) provide much stronger hardening ($\bar{\omega}_2 > 0$), see e.g.[2]. In the present case, the transition occurs for a value of k_h in the range [1 – 50].

For low values of l, on the other hand, the behaviour is more involved, and even for $k_h = 1$ we can have hardening. Here the transition from hardening to softening is due to both the conditions $\bar{\omega}_2 = 0$ and $\bar{\omega}_2 \to \infty$.

A systematic investigation of the dependence of $\bar{\omega}_2$ on x, y, z, l, k_h and n, including the possible negligibility of the underlying mechanical effects, is outside the scope of this paper, and is the object of a forthcoming paper[7].

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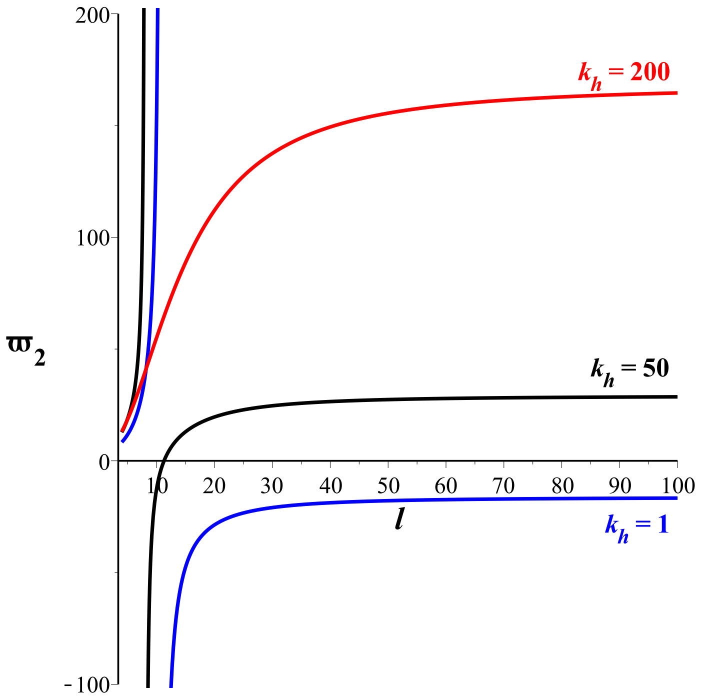

Fig. 3. The nonlinear correction frequency $\bar{\omega}_2(l)$ for different values of k_h and for $x = 1$, $y = 1$, $z = 0.3205$ and $n = 1$.

Appendix A. Appendix

In this appendix the mathematical expressions used in (36) are reported.

$$
\begin{aligned}
\omega_{2a} &= 32EA\pi^2n^2(EA\pi^2n^2 - \omega_0^2L^2\rho B)[EAL^2 - EJ\pi^2n^2\alpha_1^2 + GAL^2(\alpha_1^2 - 1)],\\
\omega_{2b} &= 16EA\pi^2n^2GAL^2(\alpha_1^2 - 1)(2\rho B\omega_0^2L^2 - EA\pi^2n^2) + 16(EA)^2\pi^2n^2(EJ\alpha_1^2\pi^4n^4 - EA\pi^2n^2L^2 + 2\rho B\omega_0^2L^4),\\
\omega_{2c} &= 6\pi^6n^6L^2(EA)^3 - \pi^4n^4(EA)^2[-6\pi^2n^2L^2(\alpha_1^2 - 1)GA + 6\pi^4n^4\alpha_1^2EJ + 7\rho B\omega_0^2L^4]+\\
&\quad +EA\{\pi^6n^6\alpha_1^2[-6\pi^2n^2(\alpha_1 - 1)GA + 5L^2\rho B\omega_0^2]EJ - \pi^4n^4L^2GA(\alpha_1 - 1)[6n^2\pi^2(\alpha_1^2 - 1)GA+\\
&\quad +\omega_0^2L^2\rho B(7\alpha_1 + 9)]\} + 4\rho B(\alpha_1 - 1)L^2GA\pi^4n^4\omega_0^2[(\alpha_1^2 - 1)L^2GA + n^2\pi^2\alpha_1^2EJ],\\
\omega_{2d} &= 64EAL^4\omega_0(EA\pi^2n^2 - \omega_0^2L^2\rho B)(\rho AL^2 + \pi^2n^2\alpha_1^2\rho J).
\end{aligned}
\tag{A.1}
$$

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