Compactifications of quasi-uniform hyperspaces

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Abstract

Several results on compactification of quasi-uniform hyperspaces are obtained. For instance, we prove that if \( C_0(X) \) denotes the family of all nonempty closed subsets of a quasi-uniform space \((X, \mathcal{U})\) and \( \mathcal{U}_H \) the Bourbaki quasi-uniformity of \( \mathcal{U} \), then \((C_0(X), \mathcal{U}_H)\) is *-compactifiable if and only if \((X, \mathcal{U})\) is closed symmetric and *-compactifiable and \( \mathcal{U}^{-1} \) is hereditarily precompact. We deduce that for any normal Hausdorff space \( X \), \( 2^\beta X \) is equivalent to the *-compactification of \((C_0(X), \mathcal{P}_N \mathcal{U}_H)\), where \( \mathcal{P}_N \) denotes the Pervin quasi-uniformity of \( X \).

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1. Introduction

As usual we denote by \( \beta X \) the Stone–Čech compactification of a Tychonoff space \( X \) and by \( 2^X \) the hyperspace of nonempty closed subsets of \( X \) with the Vietoris topology.

Compactification of hyperspaces has been investigated by several authors. In particular, Keesling [10] and Ginsburg [9] independently stated that if \( X \) is a normal Hausdorff space such that \( \beta(2^X) = 2^\beta X \), then \( 2^X \) is pseudocompact. Recently, and solving a question posed in [9], Natsheh [17] proved that the converse of this result is true. We are concerned

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here with compactifications of quasi-uniform hyperspaces; more precisely, with the *
-compactification of the Bourbaki quasi-uniformity—a question which seems of interest
in the light of the recent contributions to the fields of quasi-uniform hyperspaces and quasi-
uniform compactifications, respectively (see Section 9 of \[12\], and \[19\]).

Terms and undefined concepts on quasi-uniform spaces may be found in \[7\] and in \[12\].

If \( U \) is a quasi-uniformity on a set \( X \), then \( U^{-1} = \{ U^{-1} : U \in U \} \) is also a quasi-
uniformity on \( X \) called the conjugate of \( U \). The uniformity \( U \vee U^{-1} \) will be denoted by
\( U^* \). If \( U \in \mathcal{U} \), the entourage \( U \cap U^{-1} \) of \( U^* \) will be denoted by \( U^* \).

Each quasi-uniformity \( U \) on \( X \) induces a topology \( \tau(U) \) on \( X \), defined as follows:
\[
\tau(U) = \{ A \subseteq X : \text{ for each } x \in A \text{ there is } U \in \mathcal{U} \text{ such that } U(x) \subseteq A \}.
\]

A quasi-uniform space \((X, U)\) is called precompact if for each \( U \in \mathcal{U} \) there is a finite
subset \( F \) of \( X \) such that \( U(F) = X \). \((X, U)\) is called totally bounded if the uniform
space \((X, U^*)\) is totally bounded. It is well known that, for quasi-uniform spaces, total
boundedness implies hereditary precompactness and hereditary precompactness implies
precompactness. However, the converses do not hold, in general.

A quasi-uniform space \((X, U)\) is called point symmetric if \( \tau(U) \subseteq \tau(U^{-1}) \).

The Pervin quasi-uniformity of any \( T_1 \) topological space provides an interesting
example of a point symmetric totally bounded quasi-uniformity (see \[7\]). Let us recall
that the Pervin quasi-uniformity of a topological space \( X \) is the quasi-uniformity \( \mathcal{P}N \) on
\( X \) which is generated by all sets of the form \( (A \times A) \cup ((X \setminus A) \times X) \), where \( A \) is open
in \( X \).

Following \[8\] (see also \[7\]), a compactification of a \( T_1 \) quasi-uniform space \((X, U)\)
is a compact \( T_1 \) quasi-uniform space \((Y, V)\) that has a \( \tau(V) \)-dense subspace quasi-
isomorphic to \((X, U)\). If \((Y, \tau(V))\) is a Hausdorff space we say that \((Y, V)\) is a Hausdorff
compactification of \((X, U)\) and \((X, U)\) is said to be Hausdorff compactifiable.

It is proved in \[8\] that a totally bounded \( T_1 \) quasi-uniform space has a compactification
if and only if it is point symmetric.

In \[19\] the authors introduce and study the notion of an *
-compactification of a \( T_1 \) quasi-uniform space. While a point symmetric totally bounded \( T_1 \) quasi-uniform space may have
many totally bounded compactifications (see \[8, p. 34\]), an *
-compactifiable quasi-uniform space has an (up to quasi-isomorphism) unique *
-compactification as it is proved in \[19\]. This fact justifies in great part the interest in constructing *
-compactification(s) rather than compactifications of (totally bounded) \( T_1 \) quasi-uniform spaces.

Since the construction of the *
-compactification of a *
-compactifiable quasi-uniform space is based on the theory of bicompletion, we recall some concepts and results in order
to help the reader.

A quasi-uniform space \((X, U)\) is said to be bicomplete if each Cauchy filter on \((X, U^*)\)
converges with respect to the topology \( \tau(U^*) \), i.e., if the uniform space \((X, U^*)\) is
complete \[7,20\].

A bicompletion of a quasi-uniform space \((X, U)\) is a bicomplete quasi-uniform space
\((Y, V)\) that has a \( \tau(V^*) \)-dense subspace quasi-isomorphic to \((X, U)\).

Each \( T_0 \) quasi-uniform space \((X, U)\) has an (up to quasi-isomorphism) unique \( T_0 \)
bicompletion, which will be denoted by \((\tilde{X}, \tilde{U})\).

The construction of \((\tilde{X}, \tilde{U})\) is described in detail in Chapter 3 of \[7\] (see also \[20\]). For
our purposes here it suffices to recall that the family \( \{ \tilde{U} : U \in \mathcal{U} \} \) is a base for \( \mathcal{U} \), where
for each \( U \in \mathcal{U} \), \( \tilde{U} = \{(\mathcal{F}, \mathcal{G}) : \mathcal{F} \text{ and } \mathcal{G} \text{ are minimal Cauchy filters on } (X, \mathcal{U}^*) \) such that \( F \times G \subseteq U \) for some \( F \in \mathcal{F} \) and some \( G \in \mathcal{G} \).

A *-compactification of a \( T_1 \) quasi-uniform space \((X, \mathcal{U})\) is a compact \( T_1 \) quasi-uniform space \((Y, \mathcal{V})\) that has a \( \tau(\mathcal{V}^*) \)-dense subspace quasi-isomorphic to \((X, \mathcal{U})\). We say that a \( T_1 \) quasi-uniform space is *-compactifiable if it has a *-compactification.

Let \((X, \mathcal{U})\) be a \( T_1 \) quasi-uniform space and \((\tilde{X}, \tilde{\mathcal{U}})\) its bicompletion. We will denote by \( G(X) \) the set of closed points of \((\tilde{X}, \tau(\tilde{\mathcal{U}}))\). Clearly \( G(X) = \tilde{X} \) whenever \((\tilde{X}, \tilde{\mathcal{U}})\) is a \( T_1 \) quasi-uniform space.

It is proved in [19] that if a \( T_1 \) quasi-uniform space \((X, \mathcal{U})\) has a *-compactification, then any *-compactification of \((X, \mathcal{U})\) is quasi-isomorphic to \((G(X), \tilde{\mathcal{U}}_{|G(X)})\).

In the sequel the quasi-uniformity \( \tilde{\mathcal{U}}_{|G(X)} \) will be simply denoted by \( \tilde{\mathcal{U}} \) if no confusion arises. Thus, if the \( T_1 \) quasi-uniform space \((X, \mathcal{U})\) is *-compactifiable, \((G(X), \tilde{\mathcal{U}})\) will be called the *-compactification of \((X, \mathcal{U})\). If the *-compactification of a *-compactifiable quasi-uniform space \((X, \mathcal{U})\) is Hausdorff, we say that \((X, \mathcal{U})\) is Hausdorff *-compactifiable.

For a topological space \( X \), let \( \mathcal{P}_0(X) \) be the family of nonempty subsets of \( X \), \( \mathcal{C}_0(X) \) the family of nonempty closed subsets of \( X \), \( \mathcal{K}_0(X) \) the family of nonempty compact subsets of \( X \) and \( \mathcal{F}_0(X) \) the family of nonempty finite subsets of \( X \).

The Bourbaki quasi-uniformity (or the Hausdorff quasi-uniformity) of a quasi-uniform space \((X, \mathcal{U})\) is defined as the quasi-uniformity \( \mathcal{U}_H \) on \( \mathcal{P}_0(X) \) which has as a base the family of sets of the form

\[
\mathcal{U}_H = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A) \text{ and } A \subseteq U^{-1}(B)\},
\]

whenever \( U \in \mathcal{U} \) [1,16].

The restriction of \( \mathcal{U}_H \) to \( \mathcal{C}_0(X), \mathcal{K}_0(X) \) and \( \mathcal{F}_0(X) \), respectively, is also denoted by \( \mathcal{U}_H \) if no confusion arises.

Here we prove that if \((X, \mathcal{U})\) is a \( T_1 \) quasi-uniform space, then \((\mathcal{C}_0(X), \mathcal{U}_H)\) is *-compactifiable if and only if \((X, \mathcal{U})\) is closed symmetric and *-compactifiable and \( \mathcal{U}^{-1} \) is hereditarily precompact. The notion of a closed symmetric quasi-uniform space is here used in the sense of [4] (see Section 3). Furthermore, we show that if \((\mathcal{C}_0(X), \mathcal{U}_H)\) is *-compactifiable, then its *-compactification is quasi-isomorphic to \((\mathcal{C}_0(G(X)), \tilde{\mathcal{U}}_{|G(X)})\). The corresponding situation for \((\mathcal{K}_0(X), \mathcal{U}_H)\) and \((\mathcal{F}_0(X), \mathcal{U}_H)\), respectively, is also explored. In particular, we prove that for a quasi-uniform space \((X, \mathcal{U}), (\mathcal{K}_0(X), \mathcal{U}_H)\) is Hausdorff *-compactifiable if and only if \((X, \mathcal{U})\) is Hausdorff *-compactifiable and \( \mathcal{U}^{-1} \) is hereditarily precompact. From our methods and results we deduce the following description of \( 2^B_X \) in terms of *-compactifications: if \( X \) is a Tychonoff space, then \( 2^B_X \) is equivalent to the *-compactification of \((\mathcal{C}_0(X), \mathcal{P}_{\mathcal{N}_H})\) if and only if \( X \) is normal.

2. Preliminary results

In this section we state several facts which will be useful to prove our main results.

**Lemma 2.1** [19]. A \( T_1 \) quasi-uniform space \((X, \mathcal{U})\) is *-compactifiable if and only if it is point symmetric and its bicompletion is compact.
Lemma 2.2. Let \((X, \mathcal{U})\) be a \(T_1\) quasi-uniform space such that \(\mathcal{U}^{-1}\) is hereditarily precompact. Then \((X, \mathcal{U})\) is \(^*\)-compactifiable if and only if it is point symmetric and precompact.

Proof. The statement follows immediately from the preceding lemma and the corollary of Theorem 6 in [19], because every quasi-uniform space \((X, \mathcal{U})\) such that \(\mathcal{U}^{-1}\) is hereditarily precompact, is Smyth completable (see Example 6 of [11]). ✷

As an immediate consequence of the above lemma we obtain the easy but useful fact that every point symmetric totally bounded \(T_1\) quasi-uniform space is \(^*\)-compactifiable [19]. In particular, for each \(T_1\) topological space \(X\), \((X, \mathcal{P}(X))\) is \(^*\)-compactifiable.

The proof of the following result is straightforward, so it is omitted.

Lemma 2.3. Let \((X, \mathcal{U})\) be a quasi-uniform space and let \(M\) be such that \(F_0(X) \subseteq M \subseteq \mathcal{P}_0(X)\). If \((M, \mathcal{U}_H)\) is point symmetric, then \((X, \mathcal{U})\) is point symmetric.

Lemma 2.4. Let \((X, \mathcal{U})\) be a \(T_1\) quasi-uniform space and let \(M\) be such that \(F_0(X) \subseteq M \subseteq \mathcal{P}_0(X)\). If \((M, \mathcal{U}_H)\) is compact, then \((X, \mathcal{U})\) is compact and \(\mathcal{U}^{-1}\) is hereditarily precompact.

Proof. Let \((x_\lambda)_{\lambda \in \Lambda}\) be a net in \(X\). Since \((M, \mathcal{U}_H)\) is compact, the net \((\{x_\lambda\})_{\lambda \in \Lambda}\) has a cluster point \(C \in M\). It is easy to see that for each \(c \in C\), \(c\) is a cluster point of \((x_\lambda)_{\lambda \in \Lambda}\). We conclude that \((X, \mathcal{U})\) is compact. The proof that \(\mathcal{U}^{-1}\) is hereditarily precompact follows similarly to the first part of the proof of Proposition 5 of [15]. ✷

Lemma 2.5. Let \((X, \mathcal{U})\) be a \(T_1\) quasi-uniform space. Then the following statements are equivalent.

(1) \((\mathcal{P}_0(X), \mathcal{U}_H)\) is compact.
(2) \((\mathcal{C}_0(X), \mathcal{U}_H)\) is compact.
(3) \((\mathcal{K}_0(X), \mathcal{U}_H)\) is compact.
(4) \((X, \mathcal{U})\) is compact and \(\mathcal{U}^{-1}\) is hereditarily precompact.

Proof. (1) ⇔ (4). Corollary 2 of [15].

(1) ⇔ (3). This is Remark 1 of [14].

(2) ⇒ (4). Lemma 2.4.

(1) ⇒ (2). The proof follows similarly to the proof of Remark 1 of [14]. ✷

Let \((X, \mathcal{U})\) be a quasi-uniform space. In Proposition 1 of [16] and Remark 2 of [14] it was proved that \((\mathcal{P}_0(X), \mathcal{U}_H)\) is precompact if and only if \((X, \mathcal{U})\) is precompact, and that \((\mathcal{K}_0(X), \mathcal{U}_H)\) is precompact if and only if \((X, \mathcal{U})\) is precompact, respectively. A slight modification of the proofs of these results gives the following.

Lemma 2.6. Let \((X, \mathcal{U})\) be a quasi-uniform space and let \(M\) be such that \(F_0(X) \subseteq M \subseteq \mathcal{P}_0(X)\). Then \((M, \mathcal{U}_H)\) is precompact if and only if \((X, \mathcal{U})\) is precompact.
Proposition 2.7. Let \((X, \mathcal{U})\) be a quasi-uniform space and let \(\mathcal{M}\) be such that \(\mathcal{F}_0(X) \subseteq \mathcal{M} \subseteq \mathcal{P}_0(X)\). If \((\mathcal{M}, \mathcal{U}_{\mathcal{M}})\) is *-compactifiable, then \((X, \mathcal{U})\) is *-compactifiable and \(\mathcal{U}^{-1}\) is hereditarily precompact.

Proof. By Lemma 2.1, \((\mathcal{M}, \mathcal{U}_{\mathcal{M}})\) is a point symmetric \(T_1\) quasi-uniform space and \((\mathcal{M}, \mathcal{U}_{\mathcal{M}})\) is compact. Hence \((X, \mathcal{U})\) is a \(T_1\) quasi-uniform space and by Lemma 2.3, it is point symmetric. Moreover \(\mathcal{U}_{\mathcal{M}}\) is precompact, so \(\mathcal{U}_{\mathcal{M}}\) is precompact by Proposition 4(c) of [13]. Thus \(\mathcal{U}\) is precompact by Lemma 2.6.

Now suppose that \(\mathcal{U}^{-1}\) is not hereditarily precompact. Then there exist \(A \subseteq X\), \(U_0 \in \mathcal{U}\) and a sequence \((a_n)_{n \in \mathbb{N}}\) in \(A\) such that \(a_{n+1} \not\in \bigcup_{i=1}^{n} U_0^{-1}(a_i)\) for all \(n \in \mathbb{N}\). For each \(n \in \mathbb{N}\), put \(A_n = \{a_i : i \leq n\}\).

By assumption, the sequence \((A_n)_{n \in \mathbb{N}}\) clusters to some point \(\hat{x}\) in \((G(\mathcal{M}), \mathcal{U}_{\mathcal{M}})\). (Note that \(x\) is a (minimal) \((\mathcal{U}_{\mathcal{M}})^*\)-Cauchy filter on \(\mathcal{M}\).)

Choose \(U \in \mathcal{U}\) with \(U^2 \subseteq U_0\) and let \(k \in \mathbb{N}\) be such that \(A_k \subseteq \mathcal{U}_{\mathcal{M}}(\hat{x})\). Thus, there is \(F \in \mathcal{U}\) with \(F \times \{A_k\} \subseteq \mathcal{U}_{\mathcal{M}}\). Hence \((f, A_k) \in \mathcal{U}_{\mathcal{M}}\) for all \(f \in F\), so in particular \(f \subseteq U^{-1}(A_k)\) for all \(f \in F\). (Observe that \(F \subseteq \mathcal{M}\), and thus each \(f \in F\) is a nonempty subset of \(X\).)

Now consider the point \(a_{k+1}\). Since \(\mathcal{U}\) is point symmetric, \(V^{-1}(a_{k+1}) \subseteq U(a_{k+1})\) for some \(V \in \mathcal{U}\). Moreover, there is \(n \geq k + 1\) such that \(A_n \subseteq \mathcal{U}_{\mathcal{M}}(\hat{x})\). So there is \(G \in \mathcal{U}\), with \(G \subseteq F\), such that \(G \times \{A_n\} \subseteq \mathcal{U}_{\mathcal{M}}(\hat{x})\). Fix \(g \in G\). Then \(A_n \subseteq V(g)\). Since \(a_{k+1} \in A_n\), \(a_{k+1} \in V(g)\), so \(a_{k+1} \in U^{-1}(g)\). Since \(g \in F\), \(a_{k+1} \in U^{-1}(A_k)\). Hence \(a_{k+1} \in U_0^{-1}(A_k) = \bigcup_{i=1}^{k} U_0^{-1}(A_i)\), a contradiction. Therefore \(\mathcal{U}^{-1}\) is hereditarily precompact.

We have shown that \((X, \mathcal{U})\) is a point symmetric precompact \(T_1\) quasi-uniform space such that \(\mathcal{U}^{-1}\) is hereditarily precompact. By Lemma 2.2, \((X, \mathcal{U})\) is *-compactifiable. □

Lemma 2.8. Let \((X, \mathcal{U})\) be a point symmetric quasi-uniform space. Then \((C_0(X), \mathcal{U}_{\mathcal{M}})\) is a \(T_1\) quasi-uniform space.

Proof. Let \(A, B \in C_0(X)\) with \(A \neq B\). Suppose that there exists \(x \in B \setminus A\). By assumption there exists \(U \in \mathcal{U}\) such that \(U(x) \cap A = \emptyset\) and \(U^{-1}(x) \cap A = \emptyset\). It follows that \(x \not\in U^{-1}(A)\) and \(x \not\in U(A)\) and hence \(A \not\in \mathcal{U}(A)\) and \(B \not\in \mathcal{U}(B)\). We conclude that \((C_0(X), \mathcal{U}_{\mathcal{M}})\) is a \(T_1\) quasi-uniform space. □

Lemma 2.9. Let \((X, \mathcal{U})\) be a \(T_1\) quasi-uniform space such that \(\mathcal{U}^{-1}\) is hereditarily precompact and each closed subset of \((X, \mathcal{U})\) is precompact. If \(A\) is dense in \((X, \mathcal{U}^*)\), then \(\mathcal{F}_0(A)\) is dense in \((C_0(X), (\mathcal{U}_{\mathcal{M}})^*)\).

Proof. We first note that \(\mathcal{F}_0(A)\) is a subset of \(C_0(X)\) because \((X, \mathcal{U})\) is assumed to be a \(T_1\) quasi-uniform space.

Let \(C \in C_0(X)\) and \(U \in \mathcal{U}\). Choose \(V \in \mathcal{U}\) such that \(V^2 \subseteq U\). By assumption there are \(x_1, \ldots, x_n \in C\) such that \(C \subseteq \bigcup_{i=1}^{n} V^{-1}(x_i)\). Moreover, by precompactness of \(\mathcal{U}(C)\), there are \(y_1, \ldots, y_m \in C\) such that \(C \subseteq \bigcup_{i=1}^{m} V(y_i)\).

Let \(a_1, \ldots, a_n\), and \(b_1, \ldots, b_m\), be points of \(A\) such that \(a_i \in V^*(x_i), i = 1, \ldots, n\), and \(b_i \in V^*(y_i), i = 1, \ldots, m\). Put \(F = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_m\}\). Then \(F \in \mathcal{F}_0(A)\).
We wish to show that $F \in (U_H)^*(C)$. Indeed, the inclusions $F \subseteq U(C)$ and $F \subseteq U^{-1}(C)$ are clear because $F \subseteq V^*(C)$. Furthermore, given $x \in C$ we have $x \in V^{-1}(x_i)$ for some $i \in \{1, \ldots, n\}$, and $x \in V(y_j)$, for some $j \in \{1, \ldots, m\}$, so $x \in V^{-2}(a_i)$ and $x \in V^2(b_j)$. Therefore $C \subseteq U^{-1}(F)$ and $C \subseteq U(F)$. Hence $F \in (U_H)^*(C)$. We conclude that $F_0(A)$ is dense in $(C_0(X), (U_H)^*)$. □

**Proposition 2.10.** Let $(X, \mathcal{U})$ be a *-compactifiable quasi-uniform space such that $\mathcal{U}^{-1}$ is hereditarily precompact. Then $(C_0(G(X)), \tilde{U}_H)$ is a compact $T_1$ quasi-uniform space that contains $F_0(X)$ as a $\tau((\tilde{U}_H)^*)$-dense subset.

**Proof.** By assumption $(G(X), \tilde{U})$ is a compact $T_1$ quasi-uniform space. Moreover, by Proposition 4(b) of [13], $\tilde{U}^{-1}$ is hereditarily precompact. Hence $(C_0(G(X)), \tilde{U}_H)$ is compact by Lemma 2.5. Furthermore $(C_0(G(X)), \tilde{U}_H)$ is a $T_1$ quasi-uniform space by Lemma 2.8.

Finally, since each closed subset of $(G(X), \tilde{U})$ is compact, it follows from Lemma 2.9 that $F_0(X)$ is dense in $(C_0(G(X)), (\tilde{U}_H)^*)$. □

3. *-compactification of $(C_0(X), U_H)$

Following Deák [4], a quasi-uniform space $(X, \mathcal{U})$ is closed symmetric provided that whenever $A$ and $B$ are closed subsets of $(X, \mathcal{U})$ and there is $U \in \mathcal{U}$ such that $U(A) \cap B = \emptyset$, then there is $V \in \mathcal{U}$ such that $V(B) \cap A = \emptyset$. In this case, we say that $\mathcal{U}$ is closed symmetric.

Clearly, every equinormal quasi-uniformity is closed symmetric, so the Pervin quasi-uniformity of any topological space is closed symmetric.

Closed symmetric quasi-uniform spaces were originally called semi-symmetric. This property was introduced in [3] (see also [6]), to study the equivalence between several notions of quasi-uniform completeness.

**Proposition 3.1.** A quasi-uniform space $(X, \mathcal{U})$ is closed symmetric if and only if for each $A \in C_0(X)$ and each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V^{-1}(A) \subseteq U(A)$.

**Proof.** Suppose that for each $A \in C_0(X)$ and each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V^{-1}(A) \subseteq U(A)$.

Let $A, B \in C_0(X)$ and $U \in \mathcal{U}$ such that $U(A) \cap B = \emptyset$. Let $V \in \mathcal{U}$ such that $V^{-1}(A) \subseteq U(A)$. Then $B \cap V^{-1}(A) = \emptyset$, and hence $V(B) \cap A = \emptyset$.

Conversely, suppose that $(X, \mathcal{U})$ is closed symmetric and let $A \in C_0(X)$ and $U \in \mathcal{U}$ (we can suppose, without loss of generality, that $U(x)$ is open for each $x \in X$). Since $(X \setminus U(A)) \cap U(A) = \emptyset$, there exists $V \in \mathcal{U}$ with $V(X \setminus U(A)) \cap A = \emptyset$. It follows that $V^{-1}(A) \subseteq U(A)$. ■

**Proposition 3.2.** Let $(X, \mathcal{U})$ be a quasi-uniform space such that $(C_0(X), (U_H)^*)$ is point symmetric. Then $(X, \mathcal{U})$ is closed symmetric.
Proof. Let $A \in C_0(X)$ and $U \in \mathcal{U}$. Then there is $V \in \mathcal{U}$ such that $V^{-1}(A) \subseteq U_H(A)$. Let $W \in \mathcal{U}$ with $W^2 \subseteq V$. Then $\text{Cl}_{\tau(U)}(W^{-1}(A)) \subseteq W^{-2}(A) \subseteq V^{-1}(A)$, and hence $\text{Cl}_{\tau(U)}(W^{-1}(A)) \subseteq V^{-1}(A) \subseteq U_H(A)$. In particular $W^{-1}(A) \subseteq \text{Cl}_{\tau(U)}(W^{-1}(A)) \subseteq U(A)$, so $(X, \mathcal{U})$ is closed symmetric by Proposition 3.1. □

**Proposition 3.3.** Let $(X, \mathcal{U})$ be a closed symmetric and *-compactifiable quasi-uniform space. Then for each $A \in C_0(X)$, it holds $\text{Cl}_{\tau(\tilde{\mathcal{U}})} A = \text{Cl}_{\tau(\tilde{\mathcal{U}}^{-1})} A$, where closures are taken in $G(X)$.

**Proof.** Since $(G(X), \tilde{\mathcal{U}})$ is point symmetric, then $\text{Cl}_{\tau(\tilde{\mathcal{U}}^{-1})} A \subseteq \text{Cl}_{\tau(\tilde{\mathcal{U}})} A$. Let $\tilde{x} \in \text{Cl}_{\tau(\tilde{\mathcal{U}})} A$ and $U_0 \in \mathcal{U}$. By Proposition 3.1, there is $V \in \mathcal{U}$ with $V \subseteq U$ and $V^{-1}(A) \subseteq U(A)$. Let $W \in \mathcal{U}$ with $W^2 \subseteq V$. There is $x \in X$ such that $x \in \tilde{W}^*(\tilde{x})$. Since $\tilde{x} \in \text{Cl}_{\tau(\tilde{\mathcal{U}})} A$, $\tilde{W}(\tilde{x}) \cap A \neq \emptyset$, and hence $\tilde{V}(x) \cap A \neq \emptyset$. Then $x \in V^{-1}(A) \subseteq U(A)$ and hence $U^{-1}(x) \cap A \neq \emptyset$. It follows that $\tilde{U}_0^{-1}(\tilde{x}) \cap A \neq \emptyset$. Therefore $\tilde{x} \in \text{Cl}_{\tau(\tilde{\mathcal{U}}^{-1})} A$, so $\text{Cl}_{\tau(\tilde{\mathcal{U}})} A = \text{Cl}_{\tau(\tilde{\mathcal{U}}^{-1})} A$. □

**Proposition 3.4.** Let $(X, \mathcal{U})$ be a closed symmetric and *-compactifiable quasi-uniform space. Then the map $\phi : (C_0(X), \mathcal{U}_H) \to (\hat{C_0}(G(X)), \hat{\mathcal{U}}_H)$, defined by $\phi(A) = \text{Cl}_{\tau(\tilde{\mathcal{U}})} A$, is a quasi-isomorphism between $(C_0(X), \mathcal{U}_H)$ and $(\hat{C_0}(G(X)), \hat{\mathcal{U}}_H)$.

**Proof.** Let $A, B \in C_0(X)$ with $\phi(A) = \phi(B)$. Then $X \cap \phi(A) = A$ and $X \cap \phi(B) = B$, and hence $A = B$. Thus $\phi$ is injective.

Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ with $V^2 \subseteq U$. Let $A, B \in C_0(X)$ such that $B \in V_H(A)$. Then $B \subseteq V(A) \subseteq \tilde{V}(\phi(A))$ and $A \subseteq V^{-1}(B) \subseteq \tilde{V}^{-1}(\phi(B))$. By Proposition 3.3, $\phi(B)$ is closed in $(G(X), \tilde{\mathcal{U}}^{-1})$, so $\phi(B) \subseteq \tilde{V}(\phi(A))$ and $\phi(A) \subseteq \tilde{V}^{-1}(A) \subseteq \tilde{U}^{-1}(\phi(B))$. Therefore $\phi(B) \subseteq \tilde{U}_H(\phi(A))$.

Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ with $V^2 \subseteq U$. Let $A, B \in C_0(X)$ such that $\phi(B) \subseteq \tilde{V}_H(\phi(A))$. Then $B \subseteq \phi(B) \subseteq \tilde{V}(\phi(A))$ and $\phi(A) \subseteq \tilde{V}^{-1}(\phi(B)) \subseteq \tilde{U}^{-1}(B) \subseteq \tilde{U}(B)$. Since $\phi(A)$ is closed in $(G(X), \tilde{\mathcal{U}}^{-1})$, $B \subseteq \tilde{V}(\phi(A)) \subseteq \tilde{V}^2(A) \subseteq \tilde{U}(A)$. Therefore $B \in U_H(A)$.

We conclude that $\phi$ is a quasi-isomorphism. □

**Theorem 3.5.** Let $(X, \mathcal{U})$ be a $T_1$ quasi-uniform space. Then the following statements are equivalent.

1. $(C_0(X), \mathcal{U}_H)$ is *-compactifiable and the *-compactification $(G(C_0(X)), \hat{\mathcal{U}}_H)$ is quasi-isomorphic to $(C_0(G(X)), \hat{\mathcal{U}}_H)$.
2. $(C_0(X), \mathcal{U}_H)$ is *-compactifiable.
3. $(X, \mathcal{U})$ is closed symmetric and *-compactifiable and $\mathcal{U}^{-1}$ is hereditarily precompact.

**Proof.** (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Suppose that $(C_0(X), \mathcal{U}_H)$ is *-compactifiable. Since by assumption $(X, \mathcal{U})$ is a $T_1$ quasi-uniform space, $\mathcal{F}_0(X) \subseteq C_0(X)$, so by Proposition 2.7, $(X, \mathcal{U})$ is
*-compactifiable and $U^{-1}$ is hereditarily precompact. Moreover, $(X, U)$ is closed symmetric by Proposition 3.2.

(3) $\Rightarrow$ (1). Suppose that $(X, U)$ is closed symmetric and *-compactifiable and $U^{-1}$ is hereditarily precompact. By Proposition 2.10, $(C_0(G(X)), \tilde{U}_H)$ is a compact $T_1$ quasi-uniform space such that $F_0(X)$ is dense in $(C_0(G(X)), \tilde{U}_H^*)$. Now, if $\phi$ is the map of Proposition 3.4, we have $F_0(X) \subseteq \phi(C_0(X)) \subseteq C_0(G(X))$, and so $\phi(C_0(X))$ is dense in $(C_0(G(X)), \tilde{U}_H^*)$. By Proposition 3.4, $(C_0(G(X)), \tilde{U}_H)$ is *-compactifiable and $(C_0(G(X)), \tilde{U}_H)$ is quasi-isomorphic to the *-compactification of $(C_0(X), U_H)$. □

Next we give a characterization of those quasi-uniform spaces $(X, U)$ for which $(C_0(X), U_H)$ is Hausdorff *-compactifiable. The following observation will be useful.

**Remark 3.6.** If $(X, U)$ is a compact Hausdorff quasi-uniform space, then $2^X$ is a compact Hausdorff space, so $(K_0(X), U_H)$ is Hausdorff by Proposition 2.1 of [2]. Of course, $K_0(X) = C_0(X)$ in this case.

**Lemma 3.7.** Let $(X, U)$ be a Hausdorff *-compactifiable quasi-uniform space. Then $(K_0(G(X)), \tilde{U}_H)$ is Hausdorff and $K_0(G(X)) = C_0(G(X))$ by Remark 3.6. □

**Theorem 3.8.** Let $(X, U)$ be a $T_1$ quasi-uniform space. Then the following statements are equivalent.

1. $(C_0(X), U_H)$ is Hausdorff *-compactifiable and the *-compactification $(G(C_0(X)), \tilde{U}_H)$ is quasi-isomorphic to $(C_0(G(X)), \tilde{U}_H)$.
2. $(C_0(X), U_H)$ is Hausdorff *-compactifiable.
3. $(X, U)$ is closed symmetric and Hausdorff *-compactifiable and $U^{-1}$ is hereditarily precompact.

**Proof.** (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Suppose that $(C_0(X), U_H)$ is Hausdorff *-compactifiable. By Theorem 3.5, $(X, U)$ is closed symmetric and *-compactifiable and $U^{-1}$ is hereditarily precompact. So, by applying Theorem 3.5 again, $(G(C_0(X)), \tilde{U}_H)$ is quasi-isomorphic to $(C_0(G(X)), \tilde{U}_H)$. Therefore $(C_0(G(X)), \tilde{U}_H)$ is Hausdorff and thus $(G(X), U)$ is Hausdorff. Consequently $(X, U)$ is Hausdorff *-compactifiable.

(3) $\Rightarrow$ (1). By Theorem 3.5, $(C_0(X), U_H)$ is *-compactifiable and its *-compactification is quasi-isomorphic to $(C_0(G(X)), \tilde{U}_H)$. Since, by assumption, $(G(X), U_H)$ is Hausdorff, Lemma 3.7 shows that $(C_0(G(X)), \tilde{U}_H)$ is Hausdorff. This completes the proof. □

Next we give an example of a totally bounded Hausdorff *-compactifiable quasi-uniform space $(X, U)$ such that $(C_0(X), U_H)$ is not *-compactifiable.
Recall (Proposition 7 of [19]), that a point symmetric totally bounded $T_1$ quasi-uniform space $(X, \mathcal{U})$, is Hausdorff *-compactifiable if and only if it satisfies the following condition (introduced in [8]):

$$(\ast) \text{ if } A \text{ and } B \text{ are subsets of } X \text{ such that } U^{-1}(A) \cap U^{-1}(B) = \emptyset \text{ for some } U \in \mathcal{U}, \text{ then there is } V \in \mathcal{U} \text{ such that } V(A) \cap V(B) = \emptyset.$$ 

**Example 3.9.** Let $X$ be the set of natural numbers and let $d$ be the quasi-metric on $X$ given by $d(n, m) = 1/n + 1/m$ if $n$ is even and $n \neq m$, $d(n, m) = 1/n + 1/m$ if $n$ and $m$ are odd and $n \neq m$, $d(n, n) = 0$ for all natural $n$, and $d(n, m) = 1$ otherwise. Clearly both $\tau(\mathcal{U}_d)$ and $\tau(\mathcal{U}_{d-1})$ are the discrete topology on $X$, and $(X, \mathcal{U}_d)$ is totally bounded. Moreover, it is routine to check that $(X, \mathcal{U}_d)$ satisfies condition $(\ast)$, so it is Hausdorff *-compactifiable. In fact, the bicompletion of $(X, \mathcal{U}_d)$ is the pair $(\tilde{X}, \mathcal{U}_{d-1})$, where $\tilde{X} = X \cup \{a, b\}$, with $a, b \notin X$, $a \neq b$, and $\tilde{d}$ is the quasi-pseudo-metric on $\tilde{X}$ such that $d_{\tilde{X} \times \tilde{X}} = d$, $\tilde{d}(a, a) = \tilde{d}(b, b) = 0$, $\tilde{d}(a, n) = \tilde{d}(n, a) = 1/n$ if $n$ is even, $\tilde{d}(b, n) = \tilde{d}(n, b) = 1/n$ if $n$ is odd, $\tilde{d}(a, n) = 1/n$ if $n$ is odd, $\tilde{d}(a, b) = 0$, and $\tilde{d}(x, y) = 1$ otherwise. Thus $G(X) = X \cup \{a\}$ and hence $(G(X), \mathcal{U}_{d-1})$ is a compact Hausdorff quasi-uniform space.

However, $\mathcal{U}_d$ is not closed symmetric because for $A = \{2n - 1 : n \in X\}$ and $B = \{2n : n \in X\}$, we have $d(A, B) = 1$ but $d(B, A) = 0$. Therefore $(C_0(X), (\mathcal{U}_d)_H)$ is not *-compactifiable by Theorem 3.5.

At the end of this section we shall describe the hyperspace $2^{\beta X}$ of a normal Hausdorff space $X$ in terms of the *-compactification of $(C_0(X), \mathcal{P}N_H)$.

**Proposition 3.10.** Let $X$ be a topological space. Then $(X, \mathcal{P}N)$ satisfies condition $(\ast)$ if and only if $X$ is normal.

**Proof.** Suppose that $(X, \mathcal{P}N)$ satisfies condition $(\ast)$. Let $A$ and $B$ be two disjoint nonempty closed subsets of $X$. Let $U = ((X \setminus A) \times (X \setminus A)) \cup (A \times X)$ and $V = ((X \setminus B) \times (X \setminus B)) \cup (B \times X)$. Then $U$ and $V$ are entourages of $\mathcal{P}N$, and it immediately follows that $(U \cap V)^{-1}(A) \cap (U \cap V)^{-1}(B) = \emptyset$. By assumption, there is $W \in \mathcal{P}N$ such that $W(A) \cap W(B) = \emptyset$. Hence $X$ is a normal topological space.

Conversely, let $A$ and $B$ be two nonempty subsets of $X$ such that $U^{-1}(A) \cap U^{-1}(B) = \emptyset$. By Proposition 1.7 of [7], $\overline{A} \cap \overline{B} = \emptyset$, so there exist two disjoint open subsets $G$ and $H$ of $X$, such that $\overline{A} \subseteq G$ and $\overline{B} \subseteq H$. Since $(X, \mathcal{P}N)$ is equinormal, there is $V \in \mathcal{U}$ such that $V(\overline{A}) \subseteq G$ and $V(\overline{B}) \subseteq H$. Therefore $V(A) \cap V(B) = \emptyset$. We conclude that $(X, \mathcal{P}N)$ satisfies condition $(\ast)$. $\square$

The following auxiliary result was proved in [18].

**Lemma 3.11.** Let $(X, \mathcal{U})$ be a $T_1$ quasi-uniform space. Then $\mathcal{U}_H$ is compatible with the Vietoris topology of $(X, \tau(\mathcal{U}))$ on $C_0(X)$ if and only if $\mathcal{U}$ is equinormal and $U^{-1}$ is hereditarily precompact.
Theorem 3.12. Let \( X \) be a Tychonoff space. Then \( 2^\beta X \) is equivalent to the *-compactification of \((\mathcal{C}_0(X), \mathcal{P}^N_H)\) if and only if \( X \) is normal.

Proof. Suppose that \( X \) is normal. By Proposition 3.10 and Proposition 7 of [19] cited above, \((X, \mathcal{P}^N)\) is Hausdorff *-compactifiable, so \((\mathcal{C}_0(X), \mathcal{P}^N_H)\) is Hausdorff *-compactifiable and its *-compactification is quasi-isomorphic to \((\mathcal{C}_0(G(X)), \overline{\mathcal{P}}^N_H)\) by Theorem 3.8. Moreover, \((G(X), \tau(\overline{\mathcal{P}}^N))\) is equivalent to the Stone–Čech compactification \( \beta X \) of \( X \) (Theorem 3.8 of [8]). Since by Proposition 4(a) of [13], \((G(X), \mathcal{P}^N)\) is totally bounded, it follows from Lemma 3.11 that \( \overline{\mathcal{P}}^N_H \) is compatible with the Vietoris topology on \( \mathcal{C}_0(G(X)) \). We conclude that \( 2^\beta X \) is equivalent to \((\mathcal{C}_0(G(X)), \tau(\overline{\mathcal{P}}^N_H))\), and thus it is equivalent to the *-compactification of \((\mathcal{C}_0(X), \mathcal{P}^N_H)\).

Conversely, if \( 2^\beta X \) is equivalent to the *-compactification of \((\mathcal{C}_0(X), \mathcal{P}^N_H)\), it follows that \((\mathcal{C}_0(X), \mathcal{P}^N_H)\) is Hausdorff *-compactifiable. So, by Theorem 3.8, \((X, \mathcal{P}^N)\) is Hausdorff *-compactifiable, and thus, it satisfies condition (⋆). Therefore, \( X \) is normal by Proposition 3.10.

From Theorem 3.12 and the results of [9,10,17], cited in Section 1, we deduce the following.

Corollary 3.13. Let \( X \) be a normal Hausdorff space. Then the Stone–Čech compactification of \( 2^X \) is equivalent to the *-compactification of \((\mathcal{C}_0(X), \mathcal{P}^N_H)\) if and only if \( 2^X \) is pseudo-compact.

It seems interesting to note that one can construct examples of point symmetric totally bounded Hausdorff quasi-uniform spaces \((X, \mathcal{U})\) that are not Hausdorff *-compactifiable but \((\mathcal{C}_0(X), \mathcal{U}_H)\) is *-compactifiable.

Example 3.14. Let \( X \) be the set of natural numbers and let \( d \) be the quasi-metric on \( X \) given by \( d(n,m) = 1/m \) if \( n \neq m \), and \( d(n,n) = 0 \) for all natural \( n \). Denote by \( \mathcal{U}_d \) the quasi-uniformity induced by \( d \). Then \( \tau(\mathcal{U}_d) \) is the cofinite topology on \( X \). Clearly \((X, \mathcal{U}_d)\) is compact and totally bounded. Now let \( A = X \) and \( B = X \setminus \{1\} \). Then \( A, B \in \mathcal{K}_0(X) \), \( A \neq B \) but \( B \in \bigcap_{U \in \mathcal{U}_H} U_H(A) \). So \((\mathcal{K}_0(X), \mathcal{U}_d(H))\) is not a \( T_1 \) quasi-uniform space.

4. *-compactification of \((\mathcal{K}_0(X), \mathcal{U}_H)\)

We start this section with an example of a compact totally bounded \( T_1 \) quasi-uniform space \((X, \mathcal{U})\) such that \((\mathcal{K}_0(X), \mathcal{U}_H)\) is not a \( T_1 \) quasi-uniform space, and hence it is not *-compactifiable.

Example 4.1. Let \( X \) be the set of natural numbers and let \( d \) be the quasi-metric on \( X \) given by \( d(n,m) = 1/m \) if \( n \neq m \), and \( d(n,n) = 0 \) for all natural \( n \). Denote by \( \mathcal{U}_d \) the quasi-uniformity induced by \( d \). Then \( \tau(\mathcal{U}_d) \) is the cofinite topology on \( X \). Clearly \((X, \mathcal{U}_d)\) is compact and totally bounded. Now let \( A = X \) and \( B = X \setminus \{1\} \). Then \( A, B \in \mathcal{K}_0(X) \), \( A \neq B \) but \( B \in \bigcap_{U \in \mathcal{U}_H} U_H(A) \). So \((\mathcal{K}_0(X), \mathcal{U}_d(H))\) is not a \( T_1 \) quasi-uniform space.
However, it is possible to obtain a satisfactory characterization of those Hausdorff quasi-uniform spaces \((X, \mathcal{U})\) for which \((\mathcal{K}_0(X), \mathcal{U}_H)\) is \(*\)-compactifiable. We will need the following concept.

**Definition 4.2.** A quasi-uniform space \((X, \mathcal{U})\) is said to be **compact symmetric** if for each \(A \in \mathcal{K}_0(X)\) and \(B \in \mathcal{C}_0(X)\) such that there is \(U \in \mathcal{U}\) with \(U(A) \cap B = \emptyset\), then there is \(V \in \mathcal{U}\) with \(V(B) \cap A = \emptyset\).

Clearly, each closed symmetric Hausdorff quasi-uniform space is compact symmetric and each compact symmetric quasi-uniform space is point symmetric.

**Proposition 4.3.** A quasi-uniform space \((X, \mathcal{U})\) is compact symmetric if and only if for each \(A \in \mathcal{K}_0(X)\) and each \(U \in \mathcal{U}\) there exists \(V \in \mathcal{U}\) with \(V^{-1}(A) \subseteq U(A)\).

**Proof.** Suppose that for each \(A \in \mathcal{K}_0(X)\) and each \(U \in \mathcal{U}\) there exists \(V \in \mathcal{U}\) with \(V^{-1}(A) \subseteq U(A)\).

Let \(A \in \mathcal{K}_0(X)\) and \(U \in \mathcal{U}\) such that \(U(A) \cap B = \emptyset\). Let \(V \in \mathcal{U}\) such that \(V^{-1}(A) \subseteq U(A)\). Then \(B \cap V^{-1}(A) = \emptyset\) and hence \(V(B) \cap A = \emptyset\).

Conversely, suppose that \((X, \mathcal{U})\) is compact symmetric and let \(A \in \mathcal{K}_0(X)\) and \(U \in \mathcal{U}\) (we can suppose, without loss of generality, that \(U(x)\) is open for each \(x \in X\)). Since \((X \setminus U(A)) \cap U(A) = \emptyset\), there exists \(V \in \mathcal{U}\) with \(V(X \setminus U(A)) \cap A = \emptyset\). It follows that \(V^{-1}(A) \subseteq U(A)\). \(\Box\)

**Proposition 4.4.** Let \((X, \mathcal{U})\) be a quasi-uniform space such that \((\mathcal{K}_0(X), \mathcal{U}_H)\) is point symmetric. Then \((X, \mathcal{U})\) is compact symmetric.

**Proof.** Let \(A \in \mathcal{K}_0(X)\) and \(U \in \mathcal{U}\). Then there is \(V \in \mathcal{U}\) such that \(V^{-1}(A) \subseteq U_H(A)\). Let \(x \in V^{-1}(A)\) and \(B = A \cup \{x\}\). Then \(B \in \mathcal{K}_0(X)\) and \(B \in V^{-1}(A) \subseteq U_H(A)\). In particular \(B \subseteq U(A)\) and hence \(x \in U(A)\). It follows that \(V^{-1}(A) \subseteq U(A)\), so \((X, \mathcal{U})\) is compact symmetric by Proposition 4.3. \(\Box\)

**Proposition 4.5.** Let \((X, \mathcal{U})\) be a compact symmetric and \(*\)-compactifiable quasi-uniform space. If \((X, \mathcal{U})\) is Hausdorff, then for each \(A \in \mathcal{K}_0(X)\), it holds \(\overline{\text{Cl}_t(\mathcal{U}))} A = \overline{\text{Cl}_t(\mathcal{U}^{-1})} A\), where closures are taken in \(G(X)\).

**Proof.** The proof is analogous to the proof of Proposition 3.3, but using Proposition 4.3 instead of Proposition 3.1. \(\Box\)

**Proposition 4.6.** Let \((X, \mathcal{U})\) be a compact symmetric and \(*\)-compactifiable quasi-uniform space. If \((X, \mathcal{U})\) is Hausdorff, then the map \(\phi: (\mathcal{K}_0(X), \mathcal{U}_H) \rightarrow (\mathcal{C}_0(G(X)), \overline{\mathcal{U}_H})\) defined by \(\phi(A) = \overline{\text{Cl}_t(\mathcal{U}))} A\) is a quasi-isomorphism between \((\mathcal{K}_0(X), \mathcal{U}_H)\) and \((\mathcal{C}_0(G(X)), \overline{\mathcal{U}_H})\).

**Proof.** Let \(A, B \in \mathcal{K}_0(X)\) with \(\phi(A) = \phi(B)\). By Hausdorffness of \((X, \mathcal{U})\), \(X \cap \phi(A) = A\) and \(X \cap \phi(B) = B\). Hence \(A = B\), and thus \(\phi\) is injective. The rest of the proof is analogous to the proof of Proposition 3.4. \(\Box\)
Theorem 4.7. Let \((X, \mathcal{U})\) be a Hausdorff quasi-uniform space. Then the following statements are equivalent.

1. \((K_0(X), \mathcal{U}_H)\) is *-compactifiable and the *-compactification \((G(K_0(X)), \tilde{\mathcal{U}}_H)\) is quasi-isomorphic to \((C_0(G(X)), \tilde{\mathcal{U}}_H)\).
2. \((K_0(X), \mathcal{U}_H)\) is *-compactifiable.
3. \((X, \mathcal{U})\) is compact symmetric and *-compactifiable and \(\mathcal{U}^{-1}\) is hereditarily precompact.

Proof. (1) \(\Rightarrow\) (2). Obvious.

(2) \(\Rightarrow\) (3). Suppose that \((K_0(X), \mathcal{U}_H)\) is *-compactifiable. Since \(F_0(X) \subseteq K_0(X)\), it follows from Proposition 2.7 that \((X, \mathcal{U})\) is *-compactifiable and \(\mathcal{U}^{-1}\) is hereditarily precompact. Moreover, \((X, \mathcal{U})\) is compact symmetric by Proposition 4.4.

(3) \(\Rightarrow\) (1). Suppose that \((X, \mathcal{U})\) is compact symmetric and *-compactifiable and \(\mathcal{U}^{-1}\) is hereditarily precompact. By Proposition 2.10, \((C_0(G(X)), \tilde{\mathcal{U}}_H)\) is a compact \(T_1\) quasi-uniform space such that \(F_0(X)\) is dense in \((C_0(G(X)), \tilde{\mathcal{U}}_H)^*\). Now, if \(\phi\) is the map of Proposition 4.6, we have \(F_0(X) \subseteq \phi(K_0(X)) \subseteq C_0(G(X))\). So \(\phi(K_0(X))\) is dense in \((C_0(G(X)), (\tilde{\mathcal{U}}_H)^*\), and hence \((K_0(X), \mathcal{U}_H)\) is *-compactifiable and its *-compactification is quasi-isomorphic to \((C_0(G(X)), \tilde{\mathcal{U}}_H)\). □

Our next result characterizes those quasi-uniform spaces \((X, \mathcal{U})\) for which \((K_0(X), \mathcal{U}_H)\) is Hausdorff *-compactifiable.

Theorem 4.8. Let \((X, \mathcal{U})\) be a quasi-uniform space. Then the following statements are equivalent.

1. \((K_0(X), \mathcal{U}_H)\) is Hausdorff *-compactifiable and the *-compactification \((G(K_0(X)), \tilde{\mathcal{U}}_H)\) is quasi-isomorphic to \((K_0(G(X)), \tilde{\mathcal{U}}_H)\).
2. \((K_0(X), \mathcal{U}_H)\) is Hausdorff *-compactifiable.
3. \((X, \mathcal{U})\) is Hausdorff *-compactifiable and \(\mathcal{U}^{-1}\) is hereditarily precompact.

Proof. (1) \(\Rightarrow\) (2). Obvious.

(2) \(\Rightarrow\) (3). Suppose that \((K_0(X), \mathcal{U}_H)\) is Hausdorff *-compactifiable. Clearly \((X, \mathcal{U})\) is Hausdorff. So, by Theorem 4.7, \((X, \mathcal{U})\) is *-compactifiable, \(\mathcal{U}^{-1}\) is hereditarily precompact and \((G(K_0(X)), \tilde{\mathcal{U}}_H)\) is quasi-isomorphic to \((K_0(G(X)), \tilde{\mathcal{U}}_H)\). Therefore \((C_0(G(X)), \tilde{\mathcal{U}}_H)\) is Hausdorff and thus \((G(X), \tilde{\mathcal{U}})\) is Hausdorff. We conclude that \((X, \mathcal{U})\) is Hausdorff *-compactifiable.

(3) \(\Rightarrow\) (1). Suppose that \((X, \mathcal{U})\) is Hausdorff *-compactifiable and \(\mathcal{U}^{-1}\) is hereditarily precompact. By Proposition 2.10 and Lemma 3.7, \((K_0(G(X)), \tilde{\mathcal{U}}_H)\) is a compact Hausdorff quasi-uniform space such that \(F_0(X)\), and hence \(K_0(X)\), is dense in \((K_0(G(X)), (\tilde{\mathcal{U}}_H)^*)\). Therefore \((K_0(X), \mathcal{U}_H)\) is *-compactifiable and its *-compactification is quasi-isomorphic to \((K_0(G(X)), \tilde{\mathcal{U}}_H)\). □

By Theorems 3.5, 4.7 and 4.8 it follows that if \((X, \mathcal{U})\) is a Hausdorff quasi-uniform space such that \((C_0(X), \mathcal{U}_H)\) is (Hausdorff) *-compactifiable, then \((K_0(X), \mathcal{U}_H)\) is Hausdorff *-compactifiable.
is (Hausdorff) *-compactifiable. The converse does not hold in general, even for totally bounded quasi-uniform spaces, as the non closed symmetric Hausdorff *-compactifiable quasi-uniform space \((X, \mathcal{U}_d)\) of Example 3.9 shows. Indeed \((K_0(X), (\mathcal{U}_d)_H)\) is Hausdorff *-compactifiable by Theorem 4.8.

On the other hand, note that if \((X, \mathcal{U})\) is a Hausdorff *-compactifiable quasi-uniform space such that \(\mathcal{U}^{-1}\) is hereditarily precompact, then \((X, \mathcal{U})\) is compact symmetric by Theorem 4.8 and Proposition 4.4. We shall show that actually each Hausdorff *-compactifiable quasi-uniform space is compact symmetric.

Let us recall [7] that a quasi-uniform space \((X, \mathcal{U}, d)\) is locally symmetric provided that for each \(x \in X\), \([U^{-1}(U(x)): U \in \mathcal{U}]\) is a base for the \(\tau(\mathcal{U})\)-neighborhood filter of \(x\).

It was shown in [3] that each closed symmetric regular quasi-uniform space is locally symmetric, and hence it is metrizable by Theorem 2.32 of [7].

**Proposition 4.9.** Each locally symmetric quasi-uniform space is compact symmetric.

**Proof.** Let \((X, \mathcal{U})\) be a locally symmetric quasi-uniform space. Let \(A \in K_0(X)\) and \(U \in \mathcal{U}\) such that \(V^{-1}(A) \subsetneq U(A)\) for each \(V \in \mathcal{U}\). Given \(V \in \mathcal{U}\), let \(x_V \in V^{-1}(A) \setminus U(A)\) and \(a_V \in V(x_V)\). Let \(a \in A\) be a cluster point of the net \((a_V)_{V \in \mathcal{U}}\). Since \(U\) is locally symmetric, there exists \(V \in \mathcal{U}\) with \(V^{-1}(V(a)) \subseteq U(a)\). Let \(aw\) with \(W \subseteq V\) such that \(aw \in V(a)\). Then \(x_W \in W^{-1}(aw) \subseteq V^{-1}(V(a)) \subseteq U(a)\), a contradiction. Therefore there exists \(V \in \mathcal{U}\) with \(V^{-1}(A) \subseteq U(A)\), and hence \((X, \mathcal{U})\) is compact symmetric by Proposition 4.3. \(\square\)

Since each compact Hausdorff quasi-uniform space is locally symmetric and local symmetry is a hereditary property we have the following.

**Corollary 4.10.** Each Hausdorff *-compactifiable quasi-uniform space is compact symmetric.

**Remark 4.11.** It is not difficult to show that each compact symmetric regular quasi-metric space is locally symmetric, and hence it is metrizable by Theorem 2.32 of [7].

In [2] it was introduced the notion of a compactly symmetric quasi-uniform space in order to study completeness properties of the Bourbaki quasi-uniformity. It is easy to see that each compactly symmetric quasi-uniform space is compact symmetric. However, Example 2.3 and Proposition 2.2 of [2] show that there exists a compact Hausdorff quasi-metric space, hence compact symmetric, that is not compactly symmetric.

We conclude this section with an example of a totally bounded compact symmetric (perfectly normal) Hausdorff quasi-uniform space that is not locally symmetric.

**Example 4.12.** Let \(X = \{0, 0\} \cup (\bigcup_{i \in \mathbb{N}} X_i\), where \(X_i = \{\frac{1}{2^i}, \frac{1}{2^i}; k \in \mathbb{N}\}\).

Let \(P\) be all points \(p\) of the form \(p = ((j_1, \ldots, j_n), ((F_i)_{i \in \mathbb{N}}))\) where \(n \in \mathbb{N}\) and \(F_i\) is a finite subset of \(\mathbb{N}\) for all \(i \in \mathbb{N}\).

For each \(p \in P\) define:

- \(U_p((0, 0)) = X \setminus ((X_{j_1} \cup \cdots \cup X_{j_n}) \cup \{(\frac{1}{2^i}, \frac{1}{2^i}; i \neq j_1, \ldots, j_n; k \in F_i\})\),
- \(U_p((\frac{1}{2^i}, \frac{1}{2^i})) = U_p((0, 0)) \setminus \{(0, 0)\}\) if \(k \neq j_1, \ldots, j_n\), and \(i \neq F_k\).
For each $k \in F_k$, let $U_p^i = \{ (x, y) \in X \times X : d(x, y) < 1/2k \}$. Then $U_p^i$ is a quasi-uniform base for $X$, and $X$ is a compact subset of $X$. Therefore, $X$ is hereditarily precompact.

Next we show that each compact subset of $X$ is finite. Let $A$ be a compact subset of $X$. If there exists an infinite number of $k$'s such that $x_k \in A \cap X_k$, then the sequence $(x_k)_{k \in \mathbb{N}}$ converges to an adherent point, and by compactness of $A$, $A \cap X_k \neq \emptyset$ only for a finite number of $k$'s. Similarly, $A \cap X_k$ must be finite, and hence $A$ is finite. Since $(X, U)$ is point symmetric, we conclude that $(X, U)$ is compact symmetric.

Finally, we note that $(X, U)$ is totally bounded and thus $(K_0(X), U_H)$ is $\ast$-compactifiable by Theorem 4.7.

Hence, this example shows that "compact symmetric" cannot be replaced by "locally symmetric" in the statement of Theorem 4.7.

5. $\ast$-compactification of $(F_0(X), U_H)$

We conclude the paper by studying the $\ast$-compactification of $(F_0(X), U_H)$.

Theorem 5.1. Let $(X, U)$ be a quasi-uniform space. Then the following statements are equivalent.

1. $(F_0(X), U_H)$ is (Hausdorff) $\ast$-compactifiable and the $\ast$-compactification $(G(F_0(X)), U_H)$ is quasi-isomorphic to $(C_0(G(X)), U_H)$.
2. $(F_0(X), U_H)$ is (Hausdorff) $\ast$-compactifiable.
3. $(X, U)$ is (Hausdorff) $\ast$-compactifiable and $U^{-1}$ is hereditarily precompact.

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Suppose that $(F_0(X), U_H)$ is $\ast$-compactifiable. By Proposition 2.7, $(X, U)$ is $\ast$-compactifiable and $U^{-1}$ is hereditarily precompact.

Now suppose that, in addition, $(G(F_0(X)), U_H)$ is Hausdorff. Since by Proposition 2.10, $(C_0(G(X)), U_H)$ is a compact $T_1$ quasi-uniform space such that $F_0(X)$ is dense in $(C_0(G(X)), U_H)$, it follows that $(G(F_0(X)), U_H)$ is quasi-isomorphic to $(C_0(G(X)), U_H)$. Hence $(C_0(G(X)), U_H)$, and thus $(G(X), U_H)$, is Hausdorff.
(3) $\Rightarrow$ (1). Suppose that $(X, \mathcal{U})$ is $^*$-compactifiable and $\mathcal{U}^{-1}$ is hereditarily precompact. From Proposition 2.10 it follows that $(F_0(X), \mathcal{U}_H)$ is $^*$-compactifiable and $(C_0(G(X)), \tilde{\mathcal{U}}_H)$ is quasi-isomorphic to the $^*$-compactification of $(F_0(X), \mathcal{U}_H)$.

Finally, suppose that, in addition, $(G(X), \tilde{\mathcal{U}})$ is Hausdorff. Thus $(C_0(G(X)), \tilde{\mathcal{U}}_H)$ is Hausdorff by Lemma 3.7. We conclude that then $(F_0(X), \mathcal{U}_H)$ is Hausdorff $^*$-compactifiable. $\blacksquare$

**Corollary 5.2.** Let $(X, \mathcal{U})$ be a compact $T_1$ quasi-uniform space with $\mathcal{U}^{-1}$ hereditarily precompact. Then $(F_0(X), \mathcal{U}_H)$ is $^*$-compactifiable and its $^*$-compactification is quasi-isomorphic to $(C_0(X), \mathcal{U}_H)$.

**References**