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Graphical condensation of plane graphs: A combinatorial approach

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Abstract

The method of graphical vertex-condensation for enumerating perfect matchings of plane bipartite graph was found by Propp [Generalized Domino-shuffling, Theoret. Comput. Sci. 303 (2003) 267–301], and was generalized by Kuo [Applications of graphical condensation for enumerating matchings and tilings, Theoret. Comput. Sci. 319 (2004) 29–57] and Yan and Zhang [Graphical condensation for enumerating perfect matchings, J. Combin. Theory Ser. A 110 (2005) 113–125]. In this paper, by a purely combinatorial method some explicit identities on graphical vertex-condensation for enumerating perfect matchings of plane graphs (which do not need to be bipartite) are obtained. As applications of our results, some results on graphical edge-condensation for enumerating perfect matchings are proved, and we count the sum of weights of perfect matchings of weighted Aztec diamond.

Keywords: Graphical vertex-condensation; Graphical edge-condensation; Perfect matching; Aztec diamond

1. Introduction

Throughout this paper, we suppose that $G = (V(G), E(G))$ is a simple graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, if not specified. A perfect matching of G is a set of independent edges of G covering all vertices of G. Denote the set of perfect matchings of G by $\mathcal{M}(G)$ and the number of perfect matchings of G by $M(G)$. If G is a weighted graph, the weight of a perfect matching P of G is defined to be the product of weights of edges in P. We also denote the sum of weights of perfect matchings of G by $M(G)$. Let $A = \{a_1, a_2, \ldots, a_s\}$ (resp. $E_1 = \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\}$) be a subset of the vertex set $V(G)$ (resp. a subset of the edge set $E(G)$). By $G - A$ or $G - a_1 - a_2 - \cdots - a_s$ (resp. $G - E_1$ or $G - e_{i_1} - e_{i_2} - \cdots - e_{i_t}$) we denote the induced subgraph of G by deleting all vertices in A and the incident edges from G (resp. by deleting all edges in E_1).

By the method of graphical condensation for enumerating perfect matchings of plane bipartite graphs, Propp [13] obtained the following result:

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Proposition 1.1 (*Propp [13]*). *Let $G = (U, V)$ be a plane bipartite graph in which $|U| = |V|$. Let vertices a, b, c and d form a 4-cycle face in G, $a, c \in U$, and $b, d \in V$. Then*

$$M(G)M(G - \{a, b, c, d\}) = M(G - \{a, b\})M(G - \{c, d\}) + M(G - \{a, d\})M(G - \{b, c\}).$$

By a combinatorial method, Kuo [12] generalized Propp's result above as follows.

Proposition 1.2 (*Kuo [12]*). *Let $G = (U, V)$ be a plane bipartite graph in which $|U| = |V|$. Let vertices a, b, c, and d appear in a cyclic order on a face of G.*

(1) *If $a, c \in U$, and $b, d \in V$, then*

$$M(G)M(G-\{a, b, c, d\}) = M(G-\{a, b\})M(G-\{c, d\}) + M(G-\{a, d\})M(G-\{b, c\}).$$

(2) *If $a, b \in U$, and $c, d \in V$, then*

$$M(G-\{a, d\})M(G-\{b, c\}) = M(G)M(G-\{a, b, c, d\}) + M(G-\{a, c\})M(G-\{b, d\}).$$

By Ciucu's Matching Factorization Theorem in [3], Yan and Zhang [16] obtained a more general result than Kuo's as follows.

Proposition 1.3 (*Yan and Zhang [16]*). *Let $G = (U, V)$ be a plane weighted bipartite graph in which $|U| = |V| = n$. Let vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ $(2 \leqslant k \leqslant n)$ appear in a cyclic order on a face of G, and let $A_1 = \{a_i \mid a_i \in U,\ 1 \leqslant i \leqslant k\}$, $A_2 = \{a_i \mid a_i \in V,\ 1 \leqslant i \leqslant k\}$, $B_1 = \{b_i \mid b_i \in V,\ 1 \leqslant i \leqslant k\}$ and $B_2 = \{b_i \mid b_i \in U,\ 1 \leqslant i \leqslant k\}$. If $|A_1 \cup B_2| = |A_2 \cup B_1| = k$, then*

$$2^k M(G-A_1-B_1)M(G-A_2-B_2) = \sum_{(X,Y)\subseteq(A_1\cup B_2)\times(A_2\cup B_1),|X|=|Y|} M(G-X-Y)M(G-\overline{X}-\overline{Y}),$$

where the sum ranges over all subsets (X, Y) of $(A_1 \cup B_2) \times (A_2 \cup B_1)$ such that $|X| = |Y|$, and $X \subseteq (A_1 \cup B_2)$, $Y \subseteq (A_2 \cup B_1)$, $\overline{X} = (A_1 \cup B_2)\backslash X$, $\overline{Y} = (A_2 \cup B_1)\backslash Y$.

The results above hold under the condition that the plane graph considered is bipartite. For the case in which the plane graph does not need to be bipartite, in an email sent to "Domino Forum" Propp wrote that Kenyon recently told him about an identity of Pfaff's that, in combination with Kasteleyn's Pfaffian method (see [9,10]), implies the following combinatorial assertion:

Proposition 1.4. *Let G be a plane graph with four vertices a, b, c, d (in the cyclic order) adjacent to a single face. Then*

$$\begin{aligned} &M(G)M(G - \{a, b, c, d\}) + M(G - \{a, c\})M(G - \{b, d\}) \\ &\quad = M(G - \{a, b\})M(G - \{c, d\}) + M(G - \{a, d\})M(G - \{b, c\}). \end{aligned} \tag{1}$$

Propp also hoped to find a combinatorial proof of (1). Kuo told a result similar to Proposition 1.4 in "Domino Forum". But it seems that the explicit results (including the identity (1)) have not been published. Furthermore, it seems that nobody has published a purely combinatorial proof of (1).

In the next section, inspired by an interesting lemma in Ciucu [3] and some Pfaffian identities (see [5,8,11,15]), we find a purely combinatorial method to obtain some explicit identities concerning the enumeration of perfect matchings of plane graphs, which do not need to be bipartite. Our results imply Propositions 1.2 and 1.4. On the other hand, an obvious observation in the identities in Propositions 1.1–1.4 is that the graphs related in these identities are either G or the induced subgraphs of G by deleting some vertices. For the sake of convenience, we call these procedures for enumerating perfect matchings "graphical vertex-condensation" in place of "graphical condensation", the term used by Kuo [12]. In other words, we regard Kuo's "graphical condensation" as "condensing vertices of bipartite graphs". Based on this, it is natural to ask whether we can condense edges of G or both of edges and vertices. The theorems and corollaries in Section 3 answer this question in the affirmative. We call these results "graphical edge-condensation" for enumerating perfect matchings of plane graphs. In Section 4, we obtain a new proof of Stanley's multivariate version of the Aztec diamond theorem.

2. Graphical vertex-condensation

We say a plane graph G is symmetric if it is invariant under the reflection across some straight line ℓ (say symmetry axis). Fig. 1(a) shows an example of a symmetric plane graph. A weighted symmetric graph is a symmetric graph equipped with weight on every edge of G that is constant on the orbits of the reflection. The width of a symmetric graph G, denoted by $\omega(G)$, is defined to be half the number of vertices of G lying on the symmetric axis. Clearly, if $\omega(G)$ is not an integer then $M(G) = 0$. Hence we suppose that there are even number of vertices of G lying on the symmetry axis.

Let G be a plane weighted symmetric graph with symmetry axis ℓ, which we consider to be horizontal. Let $s_1, t_1, s_2, t_2, \dots, s_k, t_k$ be the vertices lying on ℓ as they occur from left to right. A reduced subgraph of G is a graph obtained from G by deleting at each vertex s_i either all incident edges above ℓ or all incident edges below ℓ. Fig. 1(b) shows a reduced subgraph of the graph presented in Fig. 1(a) (the deleted edges of the original graph are represented by dotted lines). Obviously, there exist exactly 2^k reduced subgraphs of G. Now, we can introduce a lemma found by Ciucu [3] and proved by a purely combinatorial method, which plays a key role in the proof of one of our main theorems.

Lemma 2.1 (*Ciucu [3]*). *Let G be a plane weighted symmetric graph and there exist $2k$ vertices lying on the symmetry axis. Then all 2^k reduced subgraphs of G have the same sum of weights of perfect matchings.*

Now we are in the position to prove one of our main results.

Theorem 2.2. *Let G be a plane weighted graph with $2n$ vertices. Let vertices $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ $(2 \leqslant k \leqslant n)$ appear in a cyclic order on a face of G, and let $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$. Then, for any $j = 1, 2, \dots, k$, we have*

$$\begin{aligned} &\sum_{Y \subseteq B,\ |Y| \text{ is odd}} M(G - a_j - Y) M(G - A \backslash \{a_j\} - \overline{Y}) \\ &\quad = \sum_{W \subseteq B,\ |W| \text{ is even}} M(G - W) M(G - A - \overline{W}). \end{aligned} \tag{2}$$

where the first sum ranges over all odd subsets Y of B and the second sum ranges over all even subsets W of B, $\overline{Y} = B \backslash Y$ and $\overline{W} = B \backslash W$.

Proof. Since G is a plane graph, for an arbitrary face F of G there exists a planar embedding of G such that the face F is the unbounded one. Hence we may assume that vertices $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ appear in a cyclic order on the unbounded face of G. Take two copies of the weighted graph G, denoted by $G_1 = (V(G_1), E(G_1))$ with the vertex set $V(G_1) = \{v_i^{(1)} \mid 1 \leqslant i \leqslant 2n\}$, and $G_2 = (V(G_2), E(G_2))$ with the vertex set $V(G_2) = \{v_i^{(2)} \mid 1 \leqslant i \leqslant 2n\}$, respectively, and leave weights of all edges unchanged. Hence $a_1^{(1)}, b_1^{(1)}, a_2^{(1)}, b_2^{(1)}, \dots, a_k^{(1)}, b_k^{(1)}$ appear in a cyclic order on the unbounded face of G_1 and $a_1^{(2)}, b_1^{(2)}, a_2^{(2)}, b_2^{(2)}, \dots, a_k^{(2)}, b_k^{(2)}$ appear in a cyclic order on the unbounded face of G_2. Construct

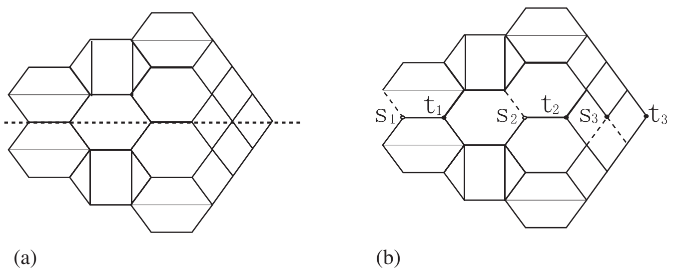
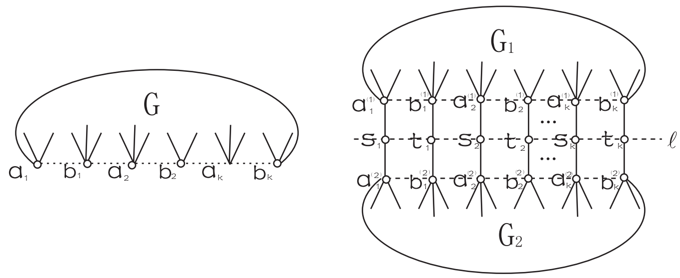


Fig. 1. (a) A symmetric graph G. (b) A reduced subgraph of symmetric graph G.



(a) (b)

Fig. 2. (a) The graph G. (b) The graph $\widetilde{G}$.

a new plane weighted graph with $4n + 2k$ vertices, denoted by $\widetilde{G} = (V(\widetilde{G}), E(\widetilde{G}))$, such that $V(\widetilde{G}) = V(G_1) \cup V(G_2) \cup W$, $E(\widetilde{G}) = E(G_1) \cup E(G_2) \cup \{a_i^{(1)}s_i, a_i^{(2)}s_i, b_i^{(1)}t_i, b_i^{(2)}t_i \mid 1 \leqslant i \leqslant k\}$, where $W = \{s_1, t_1, s_2, t_2, \ldots, s_k, t_k\}$. Let the weight of every edge in $\{a_i^{(1)}s_i, a_i^{(2)}s_i, b_i^{(1)}t_i, b_i^{(2)}t_i \mid 1 \leqslant i \leqslant k\}$ in $\widetilde{G}$ be 1 and leave all other weights unchanged. The resulting weighted graph is $\widetilde{G}$. Fig. 2(a) and (b) show this procedure constructing the new weighted graph $\widetilde{G}$ from the weighted graph G. Obviously, $\widetilde{G}$ is a plane weighted graph. Furthermore, by the definition of the symmetric graph, $\widetilde{G}$ can be regarded as a symmetric weighted plane graph with symmetry axis ℓ, which contains $2k$ vertices lying on ℓ.

Now, we consider the following $k + 1$ reduced subgraphs of $\widetilde{G}$, denoted by $G^{(0)}, G^{(1)}, \ldots, G^{(k)}$, respectively, where $G^{(i)} = \widetilde{G} - E_i$, $E_0 = \{s_p a_p^{(1)} \mid p = 1, 2, \ldots, k\}$, $E_i = \{s_p a_p^{(1)} \mid p = 1, 2, \ldots, i-1, i+1, \ldots, k\} \cup \{s_i a_i^{(2)}\}$ for $i = 1, 2, \ldots, k$. Hence, by Lemma 2.1, we have

$$M(G^{(0)}) = M(G^{(1)}) = \cdots = M(G^{(k)}). \tag{3}$$

We partition the set $\mathcal{M}(G^{(0)})$ of perfect matchings of $G^{(0)}$ such that

$$\mathcal{M}(G^{(0)}) = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{[k/2]},$$

where $\mathcal{M}_i$ denotes the set of perfect matchings of $G^{(0)}$ containing exactly $2i$ edges in subset $\{t_j b_j^{(1)} \mid 1 \leqslant j \leqslant k\}$ of $E(G^{(0)})$. It is obvious that, for any i $(0 \leqslant i \leqslant [k/2])$, after removing the forced edges we have

$$|\mathcal{M}_i| = \sum_{Y \subseteq B,\ |Y|=2i} M(G - Y)M(G - A - \overline{Y}),$$

where the sum ranges over all subsets Y of B such that $|Y| = 2i$. Hence we have

$$M(G^{(0)}) = |\mathcal{M}(G^{(0)})| = \sum_{i=0}^{[k/2]} |\mathcal{M}_i| = \sum_{Y \subseteq B,\ |Y| \text{ is even}} M(G - Y)M(G - A - \overline{Y}), \tag{4}$$

where the second sum ranges over all even subsets of B.

Similarly, for any $j = 1, 2, \ldots, k$, we can prove that

$$M(G^{(j)}) = \sum_{Y \subseteq B,\ |Y| \text{ is odd}} M(G - a_j - Y)M(G - A \backslash \{a_j\} - \overline{Y}), \tag{5}$$

where the sum ranges over all odd subsets of B.

The theorem thus follows from (3) to (5). □

Remark 1. Note that Ciucu [3] used a purely combinatorial method to prove Lemma 2.1. Hence, by the procedure proving Theorem 2.2, our method to prove Theorem 2.2 is also combinatorial.

Remark 2. Proposition 1.4 is the special case of Theorem 2.2 in which $k = 2$.

The following corollary, which has a simpler form than that in Corollary 2.3 in Yan and Zhang [16], is the special instance of Theorem 2.2.

Corollary 2.3. *Let $G = (U, V)$ be a plane weighted bipartite graph in which $U = \{u_i \mid 1 \leqslant i \leqslant n\}$ and $V = \{v_i \mid 1 \leqslant i \leqslant n\}$. Let vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order on a face of G. If $A = \{a_i \mid 1 \leqslant i \leqslant k\} \subseteq U$, and $B = \{b_i \mid 1 \leqslant i \leqslant k\} \subseteq V$, then*

$$M(G)M(G - A - B) = \sum_{i=1}^{n} M(G - a_j - b_i)M[G - (A \cup B)\backslash\{a_j, b_i\}] \tag{6}$$

for any $j = 1, 2, \ldots, k$.

Proof. Note that $G = (U, V)$ is a bipartite graph, and $A=\{a_i \mid 1 \leqslant i \leqslant k\} \subseteq U$ and $B=\{b_i \mid 1 \leqslant i \leqslant k\} \subseteq V$. Hence, in formula (2) in Theorem 2.2 if $|Y|$ is an odd integer more than 1 we have $M(G-a_j-Y)=0$. Similarly, in formula (2) in Theorem 2.2 if $|W| \neq 0$ we have $M(G-W)=0$. Thus it is not difficult to see that (6) is immediate from (2). □

If we set $k = 3$ in Corollary 2.3, then we have the following formula:

$$\begin{aligned} M(G)M(G-a_1-a_2-a_3-b_1-b_2-b_3) &= M(G-a_1-b_1)M(G-a_2-a_3-b_2-b_3) \\ &\quad + M(G-a_1-b_2)M(G-a_2-a_3-b_1-b_3) \\ &\quad + M(G - a_1 - b_3)M(G - a_2 - a_3 - b_1 - b_2), \end{aligned} \tag{7}$$

Remark 3. Similarly, we can obtain the identities in Corollaries 2.5 and 2.6 in Yan and Zhang [16] from Theorem 2.2.

3. Graphical edge-condensation

Let $G = (V(G), E(G))$ be a weighted graph and $e = ab$ an edge of G. Define a new weighted graph $G' = (V(G'), E(G'))$ from G as follows. Delete the edge $e = ab$ from G and add three edges $aa', a'b', b'b$ with the weights $\sqrt{\omega_e}$, 1 and $\sqrt{\omega_e}$, where ω_e denotes the weight of edge e. The resulting weighted graph is G'. Hence $V(G') = \{a', b'\} \cup V(G)$ and $E(G') = \{aa', a'b', b'b\} \cup E(G)\backslash\{e\}$. Fig. 3(a) and (b) illustrate this procedure.

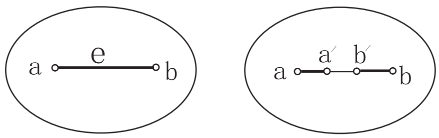
Lemma 3.1 (*Ciucu [2]*). *Let G be a weighted graph and $e = ab$ an edge of G, and let G' be the weighted graph defined above. Then*

$$M(G) = M(G').$$

In order to state our main results, we need to introduce some notation. We use $[\mathbf{k}]$ to denote the set $\{1, 2, \ldots, k\}$. Let G be a graph, and let $e_1 = a_1b_1, e_2 = a_2b_2, \ldots, e_k = a_kb_k$ $(2 \leqslant k \leqslant n)$ be k independent edges (a matching of G with k edges) in G, and $X \subseteq A = \{a_i \mid 1 \leqslant i \leqslant k\}$, $Y \subseteq B = \{b_i \mid 1 \leqslant i \leqslant k\}$. Define: $I_X = \{i \mid a_i \in X\}$, $I_Y = \{i \mid b_i \in Y\}$. Let I be a subset of $[\mathbf{k}]$ and $\bar{I} = [\mathbf{k}]\backslash I$. Define: $E_I = \{e_i \mid i \in I\}$, $A_I = \{a_i \mid i \in I\}$, $B_I = \{b_i \mid i \in I\}$. Let $I_1 \subseteq [\mathbf{k}]$ and $I_2 \subseteq [\mathbf{k}]$. Define: $I_1 - I_2 = I_1\backslash(I_1 \cap I_2)$, $I_1 \triangle I_2 = (I_1 - I_2) \cup (I_2 - I_1)$.

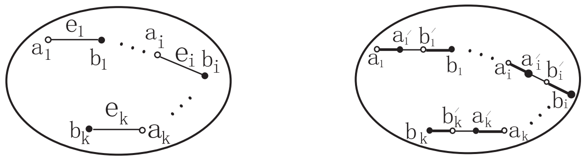
Theorem 3.2. *Suppose G is a plane weighted graph with even number of vertices and the weight of every edge e in G is denoted by ω_e. Let $e_1 = a_1b_1, e_2 = a_2b_2, \ldots, e_k = a_kb_k$ $(k \geqslant 2)$ be k independent edges in the boundary of a face f of G, and let vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order on f, and let $A = \{a_i \mid i = 1, 2, \ldots, k\}$, $B = \{b_i \mid i = 1, 2, \ldots, k\}$ and $E = \{e_i \mid i = 1, 2, \ldots, k\}$. Then, for any $j = 1, 2, \ldots, k$,*

$$\begin{aligned} &\sum_{\substack{W \subseteq B \\ |W| \text{ is even}}} \left(\prod_{e \in E_{I_W}} \omega_e \right) M(G - A_{I_W})M(G - E_{\overline{I_W}} - B_{I_W}) \\ &= \sum_{\substack{Y \subseteq B \\ |Y| \text{ is odd}}} \left(\prod_{e \in E_{\{j\} \triangle I_Y}} \omega_e \right) \left\{ M(G-E_{I_Y \cap \{j\}}-B_{\{j\}-I_Y}-A_{I_Y-\{j\}})M(G-E_{\overline{I_Y} \cap \overline{\{j\}}}-B_{\overline{\{j\}}-\overline{I_Y}}-A_{\overline{I_Y}-\overline{\{j\}}}) \right\}, \end{aligned} \tag{8}$$



(a) (b)

Fig. 3. (a) The weighted graph G in Lemma 3.1. (b) The weighted graph G' obtained from G in Lemma 3.1.



(a) (b)

Fig. 4. (a) The weighted graph G in the proof of Theorem 3.2. (b) The weighted graph G' obtained from G in the proof of Theorem 3.2.

where the first product is over all edges in E_{I_W}, the second product is over all edges in $E_{\{j\}\triangle I_Y}$, the first sum ranges over all even subsets of B, and the second sum ranges over all odd subsets of B.

Proof. Let G' be the graph obtained from G by deleting k edges $e_1, e_2, \ldots, e_k$ and adding $3k$ edges $a_ia_i', a_i'b_i', b_i'b_i$ with the weights $\sqrt{\omega_{e_i}}, 1, \sqrt{\omega_{e_i}}$ for $i = 1, 2, \ldots, k$, and leaving all other weights unchanged. Hence, the vertex set of G', denoted by $V(G')$, is $\{a_i', b_i' \mid 1 \leqslant i \leqslant k\} \cup V(G)$, and the edge set of G', denoted by $E(G')$, is $\{a_ia_i', a_i'b_i', b_i'b_i \mid i = 1, 2, \ldots, k\} \cup E(G) \backslash \{e_i \mid 1 \leqslant i \leqslant k\}$, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. For the sake of convenience, denote the edge $a_i'b_i'$ by $e_i' = a_i'b_i'$ for $i = 1, 2, \ldots, k$. Fig. 4(a) and (b) show this procedure.

Obviously, by the definition of G', G' is a plane weighted graph with even number of vertices. Furthermore, vertices $a_1', b_1', a_2', b_2', \ldots, a_k', b_k'$ appear in a cyclic order on a face of G'. Let $A' = \{a_i' \mid 1 \leqslant i \leqslant k\}$ and $B' = \{b_i' \mid 1 \leqslant i \leqslant k\}$. By Theorem 2.2, we have

$$\sum_{\substack{W' \subseteq B' \\ |W'| \text{ is even}}} M(G' - W')M(G' - A' - \overline{W'})$$

$$= \sum_{\substack{Y' \subseteq B' \\ |Y'| \text{ is odd}}} M(G' - a_j' - Y')M(G' - A' \backslash \{a_j'\} - \overline{Y'}) \tag{9}$$

for any $j = 1, 2 \ldots, k$, where the first sum ranges over all even subsets W' of B' and the second sum is over all odd subsets Y' of B', and $\overline{Y'} = B' \backslash Y'$, $\overline{W'} = B' \backslash W'$.

Let Y' be an odd subset of B'. By our notation defined above, $I_{Y'} = \{i \mid b_i' \in Y'\}$. Let $Y = \{b_i \mid i \in I_{Y'}\}$. Hence $I_Y = I_{Y'}$. Note that

$$M(G' - a_j' - Y') = M(G' - \{a_i', b_i' \mid i \in I_Y \cap \{j\}\} - \{a_i' \mid i \in \{j\} - I_Y\} - \{b_i' \mid i \in I_Y - \{j\}\}).$$

By Lemma 3.1, after removing the forced edges we have

$$M(G' - a_j' - Y') = \left(\prod_{e \in E_{(\{j\} - I_Y) \cup (I_Y - \{j\})}} \sqrt{\omega_e}\right) M(G - E_{I_Y \cap \{j\}} - B_{\{j\} - I_Y} - A_{I_Y - \{j\}})$$

$$= \left(\prod_{e \in E_{\{j\} \triangle I_Y}} \sqrt{\omega_e}\right) M(G - E_{I_Y \cap \{j\}} - B_{\{j\} - I_Y} - A_{I_Y - \{j\}}). \tag{10}$$

Similarly, we have

$$M(G' - A'\backslash\{a'_j\} - \overline{Y'}) = M(G' - \{a'_i, b'_i \mid i \in \overline{\{j\}} \cap \overline{I_Y}\} - \{a'_i \mid i \in \overline{\{j\}} - \overline{I_Y}\} - \{b'_i \mid i \in \overline{I_Y} - \overline{\{j\}}\})$$
$$= \left(\prod_{e \in E_{\overline{\{j\}} \triangle \overline{I_Y}}} \sqrt{\omega_e}\right) M(G - E_{\overline{I_Y} \cap \overline{\{j\}}} - B_{\overline{\{j\}} - \overline{I_Y}} - A_{\overline{I_Y} - \overline{\{j\}}}). \tag{11}$$

It is not difficult to prove the following two claims:

Claim 1.

$$\{j\} \triangle I_Y = \overline{\{j\}} \triangle \overline{I_Y}.$$

Claim 2. *The mapping $\phi : \{b'_i \mid i \in I_{Y'}\} \longmapsto \{a_i \mid i \in I_{Y'}\}$ is a bijection between the set of the odd subsets of B' and the set of the odd subsets of A.*

By Claims 1–2 and (10)–(11), the following claim is obvious:

Claim 3.

$$\sum_{\substack{Y' \subseteq B' \\ |Y'| \text{ is odd}}} M(G' - a'_j - Y') M(G' - A'\backslash\{a'_j\} - \overline{Y'})$$
$$= \sum_{\substack{Y \subseteq B \\ |Y| \text{ is odd}}} \left(\prod_{e \in E_{\{j\} \triangle I_Y}} \omega_e\right) \left\{ M(G - E_{I_Y \cap \{j\}} - B_{\{j\} - I_Y} - A_{I_Y - \{j\}}) M(G - E_{\overline{I_Y} \cap \overline{\{j\}}}) - B_{\overline{\{j\}} - \overline{I_Y}} - A_{\overline{I_Y} - \overline{\{j\}}}) \right\}.$$

Let W' be an even subset of B'. By our notation defined above, $I_{W'} = \{i \mid b'_i \in W'\}$. Let $W = \{b_i \mid b'_i \in W'\}$, $I_W = I_{W'}$. As in the proof of Claim 3 we can prove the following claim:

Claim 4.

$$\sum_{\substack{W' \subseteq B' \\ |W| \text{ is even}}} M(G' - W') M(G' - A' - \overline{W'}) = \sum_{\substack{W \subseteq B \\ |W| \text{ is even}}} \left(\prod_{e \in E_{I_W}} \omega_e\right) M(G - A_{I_W}) M(G - E_{\overline{I_W}} - B_{I_W})$$

The theorem is immediate from Claims 3–4 and (9). □

If we set $k = 2$, it is not difficult to see that the following corollary holds.

Corollary 3.3. *Let G be a plane weighted graph with even number of vertices. Let $e_1 = a_1b_1$ and $e_2 = a_2b_2$ be two independent edges on the boundary of a face f of G and a_1, b_1, a_2, b_2 appear in a cyclic order on a face of G. Then*

$$M(G)M(G - e_1 - e_2) + \omega_{e_1}\omega_{e_2} M(G - a_1 - a_2) M(G - b_1 - b_2)$$
$$= M(G - e_1) M(G - e_2) + \omega_{e_1}\omega_{e_2} M(G - a_1 - b_2) M(G - a_2 - b_1),$$

where ω_e denotes the weight of edge e.

Corollary 3.4. *Let $G = (U, V)$ be a plane weighted bipartite graph, in which $|U| = |V| = n$ and the weight of every edge e in G is denoted by ω_e. Let $e_1 = a_1b_1, e_2 = a_2b_2, \ldots, e_k = a_kb_k$ $(2 \leqslant k \leqslant n)$ be k independent edges in the boundary of a face f of G and let vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order on f. If $A = \{a_i \mid 1 \leqslant i \leqslant k\} \subseteq U$ and $B = \{b_i \mid 1 \leqslant i \leqslant k\} \subseteq V$, then for any $j = 1, 2, \ldots, k$*

$$M(G)M(G - e_1 - e_2 - \cdots - e_k)$$
$$= M(G - e_j) M(G - \overline{\{e_j\}}) + \sum_{\substack{1 \leqslant i \leqslant k \\ i \neq j}} \omega_{e_i}\omega_{e_j} M(G - a_i - b_j) M(G - a_j - b_i - E_{\overline{\{i,j\}}}), \tag{12}$$

where $\overline{\{e_j\}} = \{e_1, e_2, \ldots, e_k\}\backslash\{e_j\}$ and $E_{\overline{\{i,j\}}} = \{e_t \mid t \in [\mathbf{k}]\backslash\{i, j\}\}$.

Proof. Note that if W is a nonempty even subset of B or Y is an odd subset of A such that $|Y|\geqslant 3$ then $M(G-A_{I_W})=0$ and $M(G-E_{I_Y\cap\{j\}}-B_{\{j\}-I_Y}-A_{I_Y-\{j\}})=0$ in (8) in Theorem 3.2 (since G is a bipartite graph, and $A\subseteq U$, $B\subseteq V$). Hence the corollary is immediate from Theorem 3.2. □

One direct corollary of Corollaries 3.4 is the following result:

Corollary 3.5. *Let $G=(U,V)$ be a plane weighted bipartite graph in which $|U|=|V|$. Let $e_1=a_1b_1$ and $e_2=a_2b_2$ be two independent edges on the boundary of a face f of G and a_1,b_1,a_2,b_2 appear in a cyclic order on a face of G.*
(1) *If $\{a_1,a_2\}\subseteq U$ and $\{b_1,b_2\}\subseteq V$, then*

$$M(G)M(G-e_1-e_2)=M(G-e_1)M(G-e_2)+\omega_{e_1}\omega_{e_2}M(G-a_1-b_2)M(G-a_2-b_1).$$

(2) *If $a_1\in U$ and $a_2\in V$ or $a_1\in V$ and $a_2\in U$, then*

$$M(G)M(G-e_1-e_2)=M(G-e_1)M(G-e_2)-\omega_{e_1}\omega_{e_2}M(G-a_1-a_2)M(G-b_1-b_2),$$

where ω_e denotes the weight of edge e.

By the method similar to that in the proof of Theorem 3.2, we can prove the following result:

Theorem 3.6. *Let G be a plane weighted graph with even number of vertices. Let a_1 and b_1 be two vertices of G and $e=a_2b_2$ an edge of G. If the four vertices a_1,b_1,a_2,b_2 appear in a cyclic order on a face of G, then*

$$\begin{aligned}&M(G)M(G-a_1-b_1-e)\\&\quad=M(G-a_1-b_1)M(G-e)+\omega_eM(G-a_1-a_2)M(G-b_1-b_2)-\omega_eM(G-a_1-b_2)M(G-a_2-b_1).\end{aligned}$$

A direct corollary of Theorem 3.6 is the following result:

Corollary 3.7. *Let $G=(U,V)$ be a plane weighted bipartite graph in which $|U|=|V|$. Let a_1 and b_1 be two vertices of G with different colors and $e=a_2b_2$ an edge of G. If a_1,b_1,a_2,b_2 appear in a cyclic order of a face of G, then*
(i) *if $\{a_1,b_2\}\subseteq U$ and $\{a_2,b_1\}\subseteq V$ (or $\{a_1,b_2\}\subseteq V$ and $\{a_2,b_1\}\subseteq U$) then*

$$M(G)M(G-a_1-b_1-e)=M(G-a_1-b_1)M(G-e)+\omega_eM(G-a_1-a_2)M(G-b_1-b_2);$$

(ii) *if $\{a_1,a_2\}\subseteq U$ and $\{b_1,b_2\}\subseteq V$ or $\{a_1,a_2\}\subseteq V$ and $\{b_1,b_2\}\subseteq U$ then*

$$M(G)M(G-a_1-b_1-e)=M(G-a_1-b_1)M(G-e)-\omega_eM(G-a_2-b_1)M(G-a_1-b_2);$$

where ω_e is the weight of edge $e=a_2b_2$.

Remark 4. Let $G=(U,V)$ be a plane weighted graph with even number of vertices. Let a_i and b_i for $i=1,2,\ldots,s$ be $2s$ vertices of G, and let $e_i=a_{s+i}b_{s+i}$ for $i=1,2,\ldots,t$ be t edges of G ($6\leqslant s+t\leqslant n$). If vertices $a_1,b_1,a_2,b_2,\ldots,a_{s+t},b_{s+t}$ appear in the boundary of a face f of G (which may appear in different order of f), we can consider the problems similar to Theorem 3.6.

4. Weighted Aztec diamonds

In this section, we use Corollary 3.5 to give a new proof of one identity concerning perfect matchings of the weighted Aztec diamond in Yan and Zhang [16], which implies a formula on the sum of weights of perfect matchings of the weighted Aztec diamond in [4,14].

The Aztec diamond of order n, denoted AD_n, is defined to be the graph whose vertices are the white squares of a $(2n+1)\times(2n+1)$ chessboard with black corners, and whose edges connect precisely those pairs of white squares that are diagonally adjacent (Fig. 5(a) illustrates AD_4). In [6], four proofs are presented that $M(AD_n)=2^{n(n+1)/2}$.

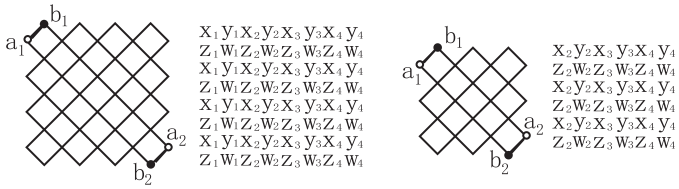


Fig. 5. (a) The weighted Aztec diamond $(AD_4; 1 \leqslant i \leqslant 4)$. (b) The weighted Aztec diamond $(AD_3; 2 \leqslant i \leqslant 4)$.

Ciucu [4] showed that $M(AD_n) = 2^n M(AD_{n-1})$, which clearly implies the previous formula (since $M(AD_1) = 2$). By two different methods, Kuo [12] and Yan and Zhang [16] proved that

$$M(AD_n) = \frac{2M(AD_{n-1})^2}{M(AD_{n-2})} \tag{13}$$

which, in turn, implies that $M(AD_n) = 2^{n(n+1)/2}$. Recently, Eu and Fu [7] and Brualdi and Kirkland [1] gave independently a new method to prove this formula.

Stanley weighted the Aztec diamond of order n as follows. Weight every 4-cycle in the ith column by assigning the variables x_i, y_i, w_i and z_i to its four edges, starting with the northwestern edge and going clockwise. We denote this weighted Aztec diamond of order n by $(AD_n; 1 \leqslant i \leqslant n)$. The case $n = 4$, i.e. $(AD_4, 1 \leqslant i \leqslant 4)$, is illustrated in Fig. 5(a), and the array on the right indicates the weight pattern on edges. We can also weight every 4-cycle of AD_n in the ith column by assigning the variables x_{i+1}, y_{i+1}, w_{i+1} and z_{i+1} to its four edges, starting with northwestern edge and going clockwise. Denote this weight Aztec diamond of order n by $(AD_n; 2 \leqslant i \leqslant n+1)$. The case $n = 3$, i.e. $(AD_3, 2 \leqslant i \leqslant 4)$, is illustrated in Fig. 5(b), and the array on the right indicates the weight pattern on the edges).

Based on the method on the graphical vertex-condensation Yan and Zhang [16] proved that

$$\begin{aligned} &M(AD_n; 1 \leqslant i \leqslant n) M(AD_{n-2}; 2 \leqslant i \leqslant n-1) \\ &\quad = (x_1 w_n + y_n z_1) M(AD_{n-1}; 1 \leqslant i \leqslant n-1) M(AD_{n-1}; 2 \leqslant i \leqslant n), \end{aligned} \tag{14}$$

which implies the following theorem by induction on n, which was previously proved by Stanley [14] and Ciucu [4].

Theorem 4.1 (*Stanley [14] and Ciucu [4]*). *The sum of weights of perfect matchings of the weighted Aztec diamond $(AD_n; 1 \leqslant i \leqslant n)$ of order n*

$$M(AD_n; 1 \leqslant i \leqslant n) = \prod_{1 \leqslant i \leqslant j \leqslant n} (x_i w_j + z_i y_j).$$

Now we use Corollary 3.5 to give a new proof of (14) as follows.

Let $G = (AD_n; 1 \leqslant i \leqslant n)$. For the sake of convenience, we rotate clockwise AD_n by 45° so that their edges are horizontal and vertical. Let a_1 and b_1 be the two vertices which are the left and right vertices of the horizontal edge in the northern corner, and let a_2 and b_2 be the two vertices which are the right and left vertices of the horizontal edge in the southern corner, respectively. The cases $n = 3$ and 4 rotated by 45° are illustrated in Fig. 5(b) and (a), respectively. Obviously, two edges $e_1 = a_1 b_1$ and $e_2 = a_2 b_2$ appear the boundary of the unbounded face of G. Particularly, a_1 and a_2 share one color, and b_1 and b_2 have another color. Then, by Corollary 3.5, we have

$$M(G)M(G-e_1-e_2) = M(G-e_1)M(G-e_2) + \omega_{e_1}\omega_{e_2} M(G-a_1-b_2)M(G-a_2-b_1). \tag{15}$$

Note that, after the removing the forced edges, we have

$$M(G-e_1-e_2) = (y_n z_1)^{n-1}(y_1 y_2 \ldots y_n)(z_1 z_2 \ldots z_n) M(AD_{n-2}; 2 \leqslant i \leqslant n-1), \tag{16}$$

$$M(G-e_1) = z_1^n (y_1 y_2 \ldots y_n) M(AD_{n-1}; 2 \leqslant i \leqslant n), \tag{17}$$

$$M(G-e_2) = y_n^n (z_1 z_2 \ldots z_n) M(AD_{n-1}; 1 \leqslant i \leqslant n-1), \tag{18}$$

$$M(G - a_1 - b_2) = y_n^{n-1}(y_1 y_2 \dots y_n) M(AD_{n-1}; 2 \leqslant i \leqslant n), \tag{19}$$

$$M(G - a_2 - b_1) = z_1^{n-1}(z_1 z_2 \dots z_n) M(AD_{n-1}; 1 \leqslant i \leqslant n-1). \tag{20}$$

Note that $\omega_{e_1} = x_1$ and $\omega_{e_1} = w_n$. Hence (14) is immediate from (15)–(20).

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