Representation of General Invariants for Approximate Transformation Groups

Rafail K. Gazizov

Department of Computational and Applied Mathematics, University of the Witwatersrand, P.O. Wits 2050, Johannesburg, South Africa

Submitted by Robert L. Anderson

Received August 19, 1996

Invariants of approximate transformation groups are studied. It turns out that the infinitesimal criterion for them is similar to that of Lie's theory. Namely, the problem of invariants of approximate groups reduces to solving first-order partial differential equations with a small parameter. The problems of solvability, number of independent invariants, and a representation of general approximate invariants are discussed. © 1997 Academic Press

1. INTRODUCTION

Symmetry properties of differential equations with a small parameter are expediently investigated by using the concept of approximate transformation groups. Such groups were introduced and partly studied in [1] (see also [2] and references therein). In particular, a possibility of constructing approximate solutions of differential equations using their approximate symmetries was shown in [1]. In this way, a notion of invariants of approximate transformation groups was used.

In this paper, an infinitesimal criterion for invariants of approximate groups is studied. It is similar to that of Lie's theory. Namely, the problem of invariants of approximate groups is reduced to solving linear first-order partial differential equations (PDEs) with a small parameter. Therefore, solutions of such equations are considered in detail.

* On leave from Department of Mathematics, Ufa State Aviational Technical University, 12 K. Marx Str., Ufa, 450025, Russia. E-mail address: gazizov@math.ugatu.ac.ru.

0022-247X /97 $25.00
Copyright © 1997 by Academic Press
All rights of reproduction in any form reserved.
This paper is quite related to previous works on approximate transformation groups \([1, 2]\) and therefore the notation used there is utilized. Namely, hereafter, \(z = (z^1, \ldots, z^N) \in \mathbb{R}^N\) is the independent variable and \(\varepsilon\) is a small parameter. The equality

\[
F(z, \varepsilon) = o(\varepsilon^p),
\]

means that

\[
\lim_{\varepsilon \to 0} \frac{F(z, \varepsilon)}{\varepsilon^p} = 0,
\]
or, equivalently,

\[
F(z, \varepsilon) = \varepsilon^{p+1} \varphi(z, \varepsilon),
\]

where \(\varphi(z, \varepsilon)\) is an analytic function defined in a neighborhood of \(\varepsilon = 0\) and \(p\) is a positive integer.

If

\[
f(z, \varepsilon) - g(z, \varepsilon) = o(\varepsilon^p),
\]

we write

\[
f(z, \varepsilon) = g(z, \varepsilon) + o(\varepsilon^p),
\]
or, briefly

\[
f = g.
\]

The common rule on summation with respect to a repeated index is used. However, the \(\Sigma\) sign appears in the cases when summation limits are not evident.

For comparison, some results of the theory of invariants for exact transformation groups are presented (for details see, for example, \([3]\)).

Let \(G_i\) be a (local) Lie group of transformations

\[
z^{i'} = f_i(z, a), \quad i = 1, \ldots, N,
\]
in \(\mathbb{R}^N\) with a group parameter \(a = (a^1, \ldots, a^r) \in \mathbb{R}^r\) and let

\[
X_\alpha = \xi_\alpha(z) \frac{\partial}{\partial z^j}
\]
be base generators of the corresponding Lie algebra. A non-constant function \(I(z)\) is called an invariant of the group \(G_i\), if \(I(z') = I(z)\) for all transformations \(G_i\). A necessary and sufficient condition for \(I(z)\) to be an invariant of \(G_i\) is that

\[
X_\alpha I(z) = 0, \quad \alpha = 1, \ldots, r.
\] (1.1)
Equations (1.1) are linear first-order partial differential equations. A number of their solutions is defined by the generic rank $r_s(\xi)$ of the matrix $||\xi(z)||$. Namely, if $r_s < N$, then the group $G_s$ has $N - r_s$ functionally independent invariants $I^1(z), \ldots, I^{N - r_s}(z)$ and any other invariant can be represented in the form

$$I(z) = \varphi(I^1(z), \ldots, I^{N - r_s}(z))$$

for some function $\varphi$.

The main results of the paper are as follows.

In Section 2, the infinitesimal criterion for invariance of approximate functions under an approximate transformation group is proved (Theorem 1). It is shown that if

$$X_\alpha = \xi_\alpha(z, \varepsilon)\frac{\partial}{\partial z^\alpha}, \quad \alpha = 1, \ldots, r,$$

(1.2)

generate a basis of the corresponding approximate Lie algebra, then invariants $I(z, \varepsilon)$ of the approximate group can be found as solutions of the equations

$$X_\alpha I(z, \varepsilon) = 0, \quad \alpha = 1, \ldots, r.$$  

(1.3)

Equations (1.3) are linear first-order partial differential equations with a small parameter and we call them approximate equations (see Definition 3).

The other sections are devoted to investigation of such equations.

In Section 3, it is shown that a one-parameter group of transformations in $\mathbb{R}^N$ has exactly $N - 1$ independent invariants. A formula for the representation of the general invariant is proposed in Theorem 3.

In the case of multi-parameter approximate groups which are considered in Section 4, the solution of the system (1.3) of the approximate equations is equivalent to one of the systems $\Omega_0, \Omega_1, \ldots, \Omega_p$ of nonhomogeneous linear first-order partial differential equations independent of the small parameter (the systems (4.2)). However, unlike the theory of invariants of exact Lie transformation groups, the condition that operators (1.2) generate an approximate Lie algebra is not sufficient for completeness and compatibility of the systems $\Omega_0, \ldots, \Omega_p$. The problem of calculating invariants of multi-parameter groups is solved only after the reduction of the corresponding systems $\Omega_0, \Omega_1, \ldots, \Omega_p$ to solvable form (Proposition 2). Then, Proposition 3 yields forms of independent invariants, and a formula of representation of the general invariant of a multi-parameter approximate group follows from Theorem 4.

In Section 5, a sufficient condition of completeness of the systems $\Omega_0, \Omega_1, \ldots, \Omega_p$ is given.
2. INVARIANCE OF APPROXIMATE FUNCTIONS

Consider a smooth function $I(z, \varepsilon)$ of the form

$$I(z, \varepsilon) = I_0(z) + \varepsilon I_1(z) + \cdots + \varepsilon^p I_p(z) + o(\varepsilon^p).$$

**Definition 1.** A class of functions $J(z, \varepsilon)$ such that $J(z, \varepsilon) \approx I(z, \varepsilon)$ is called an approximate function $I(z, \varepsilon)$.

Let a family of approximate transformations $(T_a)$,

$$T_a: z^i \approx f^i(z, a, \varepsilon) = f^i_0(z, a) + \varepsilon f^i_1(z, a) + \cdots + \varepsilon^p f^i_p(z, a), \quad i = 1, \ldots, N, \quad (2.1)$$

in $\mathbb{R}^N$ form an $r$-parameter approximate transformation group $G$, with respect to a group parameter $a \in \mathbb{R}^N$, and let

$$X_a = \xi^i_a(z, \varepsilon) \frac{\partial}{\partial z^i}, \quad \alpha = 1, \ldots, r, \quad (2.2)$$

be a basis of the corresponding approximate Lie algebra.

**Definition 2.** An approximate function $I(z, \varepsilon)$ is called an invariant of the approximate group $G$ if and only if the approximate equations hold

$$X_a I(z, \varepsilon) = 0, \quad \alpha = 1, \ldots, r. \quad (2.4)$$

**Theorem 1.** An approximate function $I(z, \varepsilon)$ is an invariant of the approximate group $G$, with the base generators (2.2) if and only if the following approximate equations hold:

$$X_a I(z, \varepsilon) = 0, \quad \alpha = 1, \ldots, r. \quad (2.4)$$

**Proof.** The family $(T_a)$ of the approximate transformations forms a (local) Lie group [2]. Therefore for any element of this group, a one-parameter subgroup can be defined passing through this element (see, e.g., [3, 4]). Hence, an approximate function $I(z, \varepsilon)$ is an invariant of the $r$-parameter approximate group $G$, if and only if it is an invariant of any one-parameter subgroup $G_1$.

Let us consider a one-parameter subgroup $G_1$ with the generator $X = \xi^i(z, \varepsilon)(\partial / \partial z^i)$. In this case, the necessary condition of the theorem is proved by the differentiation of (2.3) with respect to $a \in \mathbb{R}^1$ by means of the equality

$$\frac{dI(z', \varepsilon)}{da} = \frac{\partial I(z', \varepsilon)}{\partial z'^i} \frac{dz'^i}{da} \approx \xi^i(z', \varepsilon) \frac{\partial I(z', \varepsilon)}{\partial z'^i}, \quad (2.5)$$
where the last approximate equality is valid by virtue of Lie’s theorem for one-parameter approximate transformation groups [1, 2].

To prove sufficiency, we consider Eqs. (2.4) rewritten at the point 
\[ z' = f(z, a, \varepsilon) \]:
\[
\xi^i(z', \varepsilon) \frac{\partial I(z', \varepsilon)}{\partial z'^i} \approx 0.
\]

Then using (2.5), it follows that
\[
\frac{dI(z', \varepsilon)}{da} \approx 0. \quad (2.6)
\]

Hence, the function \( I(f(z, a, \varepsilon)) \), as a function of \( a \), satisfies the approximate differential equation (2.6) with the initial conditions
\[
I(z', \varepsilon)|_{a=0} = I(z, \varepsilon). \quad (2.7)
\]

The solution of the approximate Cauchy problem (2.6), (2.7) gives the approximate equation (2.3).

Since any generator \( X \) of the approximate Lie algebra can be represented in the form \( X = \lambda^a X_a \), where the \( X_a \) are base generators and \( \lambda^a = \text{const} \), we have (2.4). The theorem is proved.

**Remark 1.** Equations (2.4) are linear first-order partial differential equations with coefficients depending on a small parameter. Hence, the problem of constructing invariants of approximate transformation groups is reduced to solving such equations.

**Remark 2.** In (2.4), instead of the base operators, essential operators of the approximate Lie algebra can be considered. Recall that base operators of an approximate Lie algebra are obtained by the addition to essential operators some operators which are obtained by multiplying the essential operators by \( \varepsilon, \varepsilon^2, \ldots, \varepsilon^p \) and neglecting the terms of order \( o(\varepsilon^p) \) (see [2]).

3. **ONE-PARAMETER APPROXIMATE GROUPS:**

**INTEGRATION OF A FIRST-ORDER LINEAR PDE WITH A SMALL PARAMETER**

Let us consider a set of \( p + 1 \) smooth \( N \)-dimensional vector-functions
\[
\eta_0(z), \eta_1(z), \ldots, \eta_p(z)
\]
with coordinates
$$\eta_0^i(z), \eta_1^i(z), \ldots, \eta_p^i(z), \quad i = 1, \ldots, N,$$
and a class of equations
$$\eta^i(z, \varepsilon) \frac{\partial I_0}{\partial z^i} + \eta^2(z, \varepsilon) \frac{\partial I_1}{\partial z^2} + \cdots + \eta^N(z, \varepsilon) \frac{\partial I_N}{\partial z^N} = 0 \quad (3.1)$$
such that
$$\eta^i(z, \varepsilon) \approx \eta_0^i(z) + \varepsilon \eta_1^i(z) + \cdots + \varepsilon^p \eta_p^i(z). \quad (3.2)$$

**Definition 3.** The set of Eqs. (3.1) with coefficients $\eta^i(z, \varepsilon)$ satisfying the conditions (3.2) with fixed vector-functions $\eta_0(z), \ldots, \eta_p(z)$ is called an approximate (up to $o(\varepsilon^p)$) equation.

At first, we consider the case $\eta_0(z) = (\eta_0^1(z), \ldots, \eta_0^N(z)) \neq 0$. Then for some approximate equations of the form (3.1) with fixed coefficients $\eta^i(z, \varepsilon)$ satisfying the conditions (3.2), an approximate (up to $o(\varepsilon^p)$) solution $I(z, \varepsilon)$ can be found in the form
$$I(z, \varepsilon) = I_0(z) + \varepsilon I_1(z) + \cdots + \varepsilon^p I_p(z). \quad (3.3)$$
Substituting (3.3) and (3.2) into (3.1) and equating to zero the coefficients of powers of $\varepsilon$ on the left-hand side of the obtained equality yield the equations
$$\eta_0^i(z) \frac{\partial I_0}{\partial z^i} + \eta_0^2(z) \frac{\partial I_1}{\partial z^2} + \cdots + \eta_0^N(z) \frac{\partial I_N}{\partial z^N} = 0, \quad (3.4)$$
$$\eta_0^i(z) \frac{\partial I_1}{\partial z^i} + \cdots + \eta_0^N(z) \frac{\partial I_1}{\partial z^N} = - \eta_1^i(z) \frac{\partial I_0}{\partial z^i} - \cdots - \eta_1^N(z) \frac{\partial I_0}{\partial z^N}, \quad (3.5)$$
$$\quad \cdots \cdots \cdots$$
$$\eta_0^i(z) \frac{\partial I_p}{\partial z^i} + \cdots + \eta_0^N(z) \frac{\partial I_p}{\partial z^N} = - \sum_{k=1}^p \left( \eta_k^i(z) \frac{\partial I_{p-k}}{\partial z^i} + \cdots + \eta_k^N(z) \frac{\partial I_{p-k}}{\partial z^N} \right). \quad (3.6)$$
It is clear from this system that the functions $I_0(z), \ldots, I_p(z)$ are defined only by the vector-functions $\eta_0(z), \ldots, \eta_p(z)$ and they don't depend on a choice of the coefficients $\eta^i(z, \varepsilon)$ from (3.2). Therefore we use the following definition.
Definition 4. An approximate solution of the approximate differential equation \( (3.1), (3.2) \) is defined as a class of functions \( I(z, \varepsilon) \) such that

\[
I(z, \varepsilon) = I_0(z) + \varepsilon I_1(z) + \cdots + \varepsilon^p I_p(z),
\]

where the functions \( I_0(z), \ldots, I_p(z) \) are determined by the system (3.4)–(3.6).

Let us show that any approximate equation \( (3.1), (3.2) \) has \( N - 1 \) approximate solutions of the form (3.7) that are functionally independent when \( \varepsilon = 0 \). Indeed, the function \( I_0(z) \) is determined from Eq. (3.4) having \( N - 1 \) functionally independent solutions \( I_1^k(z), I_2^k(z), \ldots, I_{N-1}^k(z) \). For each of these functions \( I_0^k(z), k = 1, \ldots, N - 1 \), one can obtain a partial solution \( I_1^k(z) \) of the corresponding Eq. (3.5). Similarly, partial solutions of every equation of the system (3.4)–(3.6) are constructed. As a result, \( N - 1 \) approximate partial solutions

\[
I^k(z, \varepsilon) = I_0^k(z) + \varepsilon I_1^k(z) + \cdots + \varepsilon^p I_p^k(z), \quad k = 1, \ldots, N - 1
\]

(functionally independent when \( \varepsilon = 0 \)) of the approximate equation \( (3.1), (3.2) \) are obtained.

These solutions can be used for constructing the general approximate solution of the approximate equation \( (3.1), (3.2) \). The result is as follows.

Theorem 2. If the vector-function \( \eta_0(z) \neq 0 \) and \( I^1(z, \varepsilon), \ldots, I^{N-1}(z, \varepsilon) \) are functionally independent approximate solutions of the approximate equation \( (3.1), (3.2) \), then the general solution of this equation has the form

\[
I = \varphi_0(I^1, \ldots, I^{N-1}) + \varepsilon \varphi_1(I^1, \ldots, I^{N-1}) + \cdots + \varepsilon^p \varphi_p(I^1, \ldots, I^{N-1}),
\]

where \( \varphi_0, \varphi_1, \ldots, \varphi_p \) are arbitrary functions. That is, any approximate solution

\[
I(z, \varepsilon) = \psi_0(z) + \varepsilon \psi_1(z) + \cdots + \varepsilon^p \psi_p(z)
\]

of the approximate equation \( (3.1), (3.2) \) can be represented in the form (3.9).

Remark 3. In the formula (3.9), the arguments of the functions \( \varphi_i \), \( i = 0, \ldots, p \), are considered up to \( o(\varepsilon^{p-1}) \).

Proof. The function \( \psi_0(z) \) from (3.10) is a solution of the (exact) equation (3.4) (obtained from (3.1) when \( \varepsilon = 0 \)) and can be represented as a function of the functionally independent solutions \( I_0^1(z), \ldots, I_0^{N-1}(z) \) of
this equation in the form (see (3.8))

\[ \psi_0(z) = \varphi_0(I^1_0(z), \ldots, I^{N-1}_0(z)) \]

with an appropriately chosen function \( \varphi_0 \). Therefore the solution \( I(z, \varepsilon) \) can be rewritten as

\[
I(z, \varepsilon) = \varphi_0(I^1(z, \varepsilon), \ldots, I^{N-1}(z, \varepsilon)) + \varepsilon \hat{\psi}_1(z) + \cdots + \varepsilon^n \hat{\psi}_p(z)
\]

for some functions \( \hat{\psi}_1(z), \ldots, \hat{\psi}_p(z) \).

Here the function \( \psi_1(z) \) is also a solution of Eq. (3.4) and can be represented in the form

\[
\hat{\psi}_1(z) = \varphi_1(I^1_0(z), \ldots, I^{N-1}_0(z)).
\]

Therefore we have

\[
I(z, \varepsilon) = \varphi_0(I^1(z, \varepsilon), \ldots, I^{N-1}(z, \varepsilon)) + \varepsilon \hat{\psi}_1(z) + \cdots + \varepsilon^n \hat{\psi}_p(z)
\]

for some functions \( \hat{\psi}_1(z), \ldots, \hat{\psi}_p(z) \).

Repeating \( p \) times the above procedure, we complete the proof of Theorem 2.

**Example 1.** Consider the approximate (up to \( o(\varepsilon^2) \)) equation

\[
(1 + \varepsilon^2 x^2) \frac{\partial I}{\partial x} + \left( \varepsilon + \varepsilon^2 xy \right) \frac{\partial I}{\partial y} = 0.
\]

(3.11)

Its solution can be found in the form

\[
I(x, y, \varepsilon) = I_0(x, y) + \varepsilon I_1(x, y) + \varepsilon^2 I_2(x, y),
\]

where the functions \( I_0(x, y), I_1(x, y), I_2(x, y) \) are determined from the system

\[
\frac{\partial I_0}{\partial x} = 0, \\
\frac{\partial I_1}{\partial x} = -\frac{\partial I_0}{\partial y}, \\
\frac{\partial I_2}{\partial x} = -x^2 \frac{\partial I_0}{\partial x} - \frac{\partial I_1}{\partial y} - xy \frac{\partial I_0}{\partial y}.
\]
The first equation of this system has only one functionally independent solution: let
\[ I_1^0 = y. \]
Substituting this solution into the second equation, we obtain a partial solution
\[ I_1^s = -x. \]
As a partial solution of the third equation, one can choose
\[ I_2^s = -\frac{1}{2}x^2y. \]

Thus, Eq. (3.11) has an approximate solution
\[ I(x, y, \varepsilon) = y - \varepsilon x - \frac{\varepsilon^2}{2}x^2y \]
and according to Theorem 2, any approximate solution of Eq. (3.11) can be represented in the form
\[ I \approx \varphi_0(y - \varepsilon x - \frac{\varepsilon^2}{2}x^2y) + \varepsilon\varphi_1(y - \varepsilon x) + \varepsilon^2\varphi_2(y) \quad (3.12) \]
for appropriately chosen functions \( \varphi_0, \varphi_1, \) and \( \varphi_2. \)

Let us now consider the case, when in (3.2), \( \eta_0(z) = \eta_1(z) = \cdots = \eta_{p-1}(z) = 0, \) and \( \eta_p(z) \neq 0 \) for some \( q, \) \( 1 \leq q \leq p. \) Then after renumbering, the condition (3.2) can be written as
\[ \eta^*(z, \varepsilon) = \varepsilon^q(\eta_0(z) + \varepsilon\eta_1(z) + \cdots + \varepsilon^{p-q}\eta_{p-1-q}(z)) + o(\varepsilon^p), \quad (3.13) \]
where \( \eta_0(z) \neq 0. \) In this case, an approximate solution satisfying the approximate equation (3.1), (3.13) with precision \( o(\varepsilon^p) \) can be found in the form
\[ I(z, \varepsilon) = I_0(z) + \varepsilon I_1(z) + \cdots + \varepsilon^{p-q}I_{p-\theta}(z) + o(\varepsilon^{p-q}), \quad (3.14) \]
where the functions \( I_0(z), I_1(z), \ldots, I_{p-\theta}(z) \) are determined from the first \( p - q \) + 1 equations of the system (3.4)–(3.6). Thus, the following proposition is valid.

**Proposition 1.** The problem of the approximate integration of Eqs. (3.1), (3.13) with precision \( o(\varepsilon^p) \) is equivalent to that of Eq. (3.1) with precision \( o(\varepsilon^{p-q}) \) where the coefficients are obtained from (3.13) by dividing by \( \varepsilon^q. \)

In this case, Theorem 2 takes the form:

**Theorem 2’.** Let the functions \( \eta^*(z, \varepsilon) \) in (3.1) have the form (3.13), where \( \eta_0(z) \neq 0, \) and \( I_1^*(z, \varepsilon), \ldots, I_{N-1}^*(z, \varepsilon) \) are functionally independent approximate solutions of the form (3.14) of the approximate equation (3.1), (3.13). Then any approximate (up to \( o(\varepsilon^p) \) ) solution of this approximate equation can be represented in the form
\[ I = \varphi_0(I^1, \ldots, I^{N-1}) + \varepsilon\varphi_1(I^1, \ldots, I^{N-1}) + \cdots + \varepsilon^{p-q}\varphi_{p-q}(I^1, \ldots, I^{N-1}) + o(\varepsilon^{p-q}), \quad (3.15) \]
Remark 4. In formula (3.15), the arguments of the functions \( \varphi_i, i = 0, \ldots, p - q \), are considered up to \( o(\varepsilon^{p-q-i}) \) (cf. Remark 3 to Theorem 2).

Example 2. Consider the approximate (up to \( o(\varepsilon^2) \)) equation

\[
\frac{\partial I}{\partial x} + \varepsilon^2 \frac{\partial I}{\partial y} = 0
\]  

(3.16)

obtained from (3.9) by multiplying by \( \varepsilon \). According to Proposition 1 and Theorem 2, any approximate (up to \( o(\varepsilon^2) \)) solution of this equation can be represented in the form

\[
I = \varphi_0(y - \varepsilon x) + \varepsilon \varphi_1(y) + o(\varepsilon)
\]  

(3.17)

(cf. Example 1).

The following result for invariants of one-parameter approximate transformation groups is obtained from Theorems 2 and 2.’

Theorem 3. Any one-parameter approximate transformation group \( G_1 \) generated by the operator

\[
X = \xi^i(z, \varepsilon) \frac{\partial}{\partial z^i}
\]

with coordinates

\[
\xi^i(z, \varepsilon) \approx \varepsilon^q \left( \xi^j_0(z) + \varepsilon \xi^j_1(z) + \cdots + \varepsilon^{p-q} \xi^{j_p}_0(z) \right) + o(\varepsilon^p), \quad q = 0, \ldots, p,
\]

where the vector \( \xi^j_0(z) = (\xi^j_0(z), \ldots, \xi^j_0(z)) \neq 0 \) has exactly \( N - 1 \) invariants of the form

\[
I^k(z, \varepsilon) \approx I^k_0(z) + \varepsilon I^k_1(z) + \cdots + \varepsilon^{p-q} I^k_{p-q}(z),
\]

\[
I^k_0(z) \neq 0, k = 1, \ldots, N - 1.
\]

These invariants are functionally independent when \( \varepsilon = 0 \). Any invariant of \( G_1 \) can be represented in the form

\[
I(z, \varepsilon) = \varphi_0(I^1, \ldots, I^{N-1}) + \varepsilon \varphi_1(I^1, \ldots, I^{N-1}) + \cdots
\]

\[
+ \varepsilon^{p-q} \varphi_{p-q}(I^1, \ldots, I^{N-1}) + o(\varepsilon^{p-q}),
\]

where \( \varphi_0, \varphi_1, \ldots, \varphi_{p-q} \) are arbitrary functions.

Remark 5. The last formula can be rewritten in the equivalent form

\[
I(z, \varepsilon) = \varphi(I^1, I^2, \ldots, I^{N-1}, \varepsilon),
\]

where \( \varphi \) is an arbitrary function of \( n \) arguments.
Example 3. Consider the approximate (up to $o(\varepsilon^2)$) transformation group on the plane $(x, y)$,

\[ x' = x + a + \varepsilon^2 \left( x^2 a + xa^2 + \frac{a^3}{3} \right), \quad y' = y + \varepsilon a + \varepsilon^2 \left( xy + \frac{1}{2} ya^2 \right) \]

with the generator

\[ X = (1 + \varepsilon^2 x^2) \frac{\partial}{\partial x} + (\varepsilon + \varepsilon^2 xy) \frac{\partial}{\partial y}. \tag{3.18} \]

The invariants of this group are determined by the equation

\[ (1 + \varepsilon^2 x^2) \frac{\partial I}{\partial x} + (\varepsilon + \varepsilon^2 xy) \frac{\partial I}{\partial y} = o(\varepsilon^2) \]

and, according to Example 1, the general invariant is

\[ I = \varphi_0 \left( y - \varepsilon x - \frac{\varepsilon^2}{2} x^2 y \right) + \varepsilon \varphi_1 (y - \varepsilon x) + \varepsilon^2 \varphi_2 (y), \]

where $\varphi_0$, $\varphi_1$, and $\varphi_2$ are arbitrary functions.

Any invariant of the exact transformation group generated by the operator (3.18) has the form

\[ I(x, y) = \varphi \left( \frac{y - \varepsilon x}{\sqrt{1 + \varepsilon^2 x^2}} \right), \]

where $\varphi$ is an arbitrary function and $\varepsilon$ is a constant. The Taylor expansion (in $\varepsilon$) of the argument gives

\[ \frac{y - \varepsilon x}{\sqrt{1 + \varepsilon^2 x^2}} = y - \varepsilon x - \frac{\varepsilon^2}{2} x^2 y + o(\varepsilon^2). \]

4. Multi-parameter approximate groups:
Integration of a system of linear first-order PDEs with a small parameter

Consider a system of $r$ approximate linear homogeneous first-order partial differential equations

\[ X_{a_0^*} I = (X_{a_0^*}; 0 + \varepsilon X_{a_0^*} 1 + \cdots + \varepsilon^p X_{a_0^*} p) I = 0, \]

\[ X_{a_1^*} I = (\varepsilon X_{a_1^*}; 0 + \cdots + \varepsilon^n X_{a_1^*} p-1) I = 0, \]
4.1

\[ X_{\alpha_r}I = \left( \varepsilon^k X_{\alpha_r,0} + \cdots + \varepsilon^p X_{\alpha_r,p-k} \right) I = 0, \]

including \( r \) equations of \( k \)th-order in \( \varepsilon \) \((k = 0, \ldots, p, \alpha_k = 1, \ldots, r, \) and \( r_0 + r_1 + \cdots + r_p = r \). Here the operators \( X_{\alpha_r,j} \) have the form

\[ X_{\alpha_r,j} = \xi_{\alpha_r,j}^i(z) \frac{\partial}{\partial z^i}. \]

One may seek approximate solutions of system (4.1) in the form

\[ I(z, \varepsilon) = I_0(z) + \varepsilon I_1(z) + \cdots + \varepsilon^p I_p(z). \]

After substitution of this equality in the system, the following sequence of systems \( \Omega_0, \Omega_1, \ldots, \Omega_p \) for \( I_0, I_1, \ldots, I_p \) is obtained:

system \( \Omega_0 \),

\[ X_{\alpha_0,0}I_0 = 0, \]

\[ X_{\alpha_1,0}I_0 = 0, \]

\[ \cdots \]

\[ X_{\alpha_r,0}I_0 = 0; \]

system \( \Omega_1 \),

\[ X_{\alpha_0,0}I_1 + X_{\alpha_0,1}I_0 = 0, \]

\[ X_{\alpha_1,0}I_1 + X_{\alpha_1,1}I_0 = 0, \]

\[ \cdots \]

\[ X_{\alpha_p,0}I_1 + X_{\alpha_p,1}I_0 = 0; \]

system \( \Omega_2 \),

\[ X_{\alpha_0,0}I_2 + X_{\alpha_0,1}I_1 + X_{\alpha_0,2}I_0 = 0, \]

\[ X_{\alpha_1,0}I_2 + X_{\alpha_1,1}I_1 + X_{\alpha_1,2}I_0 = 0, \]

\[ \cdots \]

\[ X_{\alpha_p,0}I_2 + X_{\alpha_p,1}I_1 + X_{\alpha_p,2}I_0 = 0; \]

system \( \Omega_p \),

\[ X_{\alpha_0,0}I_p + X_{\alpha_0,1}I_{p-1} + \cdots + X_{\alpha_0,p}I_0 = 0. \]
We will consider the system $V$ of homogeneous equations as one for determining the function $I_q(z)$, and each of the systems $V_{q_0}, q = 1, \ldots, p$ of nonhomogeneous equations as one for determining the function $I_q(z)$ when the functions $I_0(z), \ldots, I_{q-1}(z)$ are known and satisfy the systems $\Omega_0, \ldots, \Omega_{q-1}$. For investigation of such systems, the methods of solving systems of linear first-order partial differential equations (see, e.g., [5]) are used. Accordingly, as the first step, conditions of compatibility and completeness have to be checked.

For investigating the condition of completeness of a system $\Omega_q, q = 0, \ldots, p$, we have to calculate the Jacobi bracket for all pairs of equations from $\Omega_q$. As has been shown in the Appendix, thereafter we obtain a linear first-order partial differential equation for $I_0(z), \ldots, I_m(z), m \le q$. If this equation cannot be represented as a linear function of the equations of the systems $\Omega_0, \ldots, \Omega_m$, we add this equation to the system $\Omega_m$, $m \le q$ and repeat the procedure. Otherwise, the system is complete (on the solutions of the systems $\Omega_0, \ldots, \Omega_{q-1}$).

For investigating the compatibility of a system $\Omega_q, q = 1, \ldots, p$, we will consider it as a system of first-order algebraic equations with respect to $\partial I_q/\partial z^1, \ldots, \partial I_q/\partial z^N$. Then there are two possibilities.

1. The system $\Omega_q$ is compatible on the solutions of the systems $\Omega_0, \ldots, \Omega_{q-1}$.

2. A new additional equation for $I_0(z), \ldots, I_m(z), m < q$, is obtained. In this case, we add this equation to the system $\Omega_m$ and repeat the described procedure.

Since any one of the systems $\Omega_0, \ldots, \Omega_p$ contains not more than $N$ independent equations, after a finite number of steps, we obtain the following result.

**Proposition 2.** The sequence (4.2) of the systems $\Omega_0, \Omega_1, \ldots, \Omega_p$ can be reduced to new systems $\Omega_0, \Omega_1, \ldots, \Omega_p$ having the following properties: every system $\Omega_q, 0 \le q \le p$, is complete and compatible on the solutions of the systems $\Omega_0, \ldots, \Omega_{q-1}$.

Let the system $\Omega_0$ of linear homogeneous equations contain $r_0^s$ independent equations. Then this system has $s_0 = N - r_0^s$ independent solutions

$I_{1_0}^1(z), \ldots, I_{s_0}^1(z)$.

For each function $I_k^1(z), k = 1, \ldots, s_0$, we can construct a partial solution $I_k^1(z)$ of the system $\Omega_1$.

Let the system $\Omega_1^s$ of homogeneous equations corresponding to $\Omega_1$ contain $s_1^s (\le r_0^s)$ independent equations. (It is obvious that all equations
of the system \( \Omega_1^0 \) are included in the system \( \Omega_0 \). Therefore the system \( \Omega_1^0 \) has \( s_1 = N - r_1^2 \) independent solutions and we can choose these solutions as

\[
I_0^1(z), \ldots, I_0^{s_1}(z), I_0^{s_1+1}(z), \ldots, I_0^2(z).
\]

Consider now the system \( \Omega_2 \). For each pair of the functions \((I_0(z), I_1(z))\) of the form

\[
(I_0^k(z), I_1^k(z)), \quad k = 1, \ldots, s_2; \quad (0, I_1^l(z)), \quad l = s_0 + 1, \ldots, s_1
\]

we can construct a corresponding partial solution

\[
I_0^k(z), \quad i = 1, \ldots, s_1
\]

of the system \( \Omega_1 \). As partial solutions of the corresponding system \( \Omega_2^0 \) of homogeneous equations we can choose

\[
I_0^1(z), \ldots, I_0^{s_1}(z), I_0^{s_1+1}(z), \ldots, I_0^{s_2}(z), I_0^{s_2+1}(z), \ldots, I_0^{s_1}(z),
\]

where \( s_2 = N - r_2^2, r_2^2 \) is a number of independent equations of \( \Omega_2^0 \).

Continuing similar constructions for the systems \( \Omega_3, \ldots, \Omega_p \), we obtain the following.

**Proposition 3.** Let a number of independent equations of the system \( \Omega_k^0 \) be equal to \( r_k^2 \) \((k = 0, \ldots, p)\), and \( s_k = N - r_k^2 \). Then the system (4.1) has \( s_p \) independent approximate solutions of the form

\[
I_i^k(z, \varepsilon) = I_0^k(z) + \varepsilon I_1^k(z) + \varepsilon^2 I_2^k(z) + \cdots + \varepsilon^p I_p^k(z), \quad i_0 = 1, \ldots, s_0,
\]

\[
I_i^k(z, \varepsilon) = \varepsilon I_0^k(z) + \varepsilon^2 I_1^k(z) + \cdots + \varepsilon^p I_p^k(z), \quad i_1 = s_0 + 1, \ldots, s_1,
\]

\[
I_i^k(z, \varepsilon) = \varepsilon^2 I_0^k(z) + \cdots + \varepsilon^p I_p^k(z), \quad i_2 = s_1 + 1, \ldots, s_2,
\]

\[
\vdots
\]

\[
I_i^k(z, \varepsilon) = \varepsilon^p I_p^k(z), \quad i_p = s_{p-1} + 1, \ldots, s_p,
\]

where the functions \( I_0^k(z), k = 1, \ldots, s_p \) are functionally independent.
Let us introduce the following notation:

\[ J^{i_0}(z, \varepsilon) = I^{i_0}_0(z) + \varepsilon I^{i_0}_1(z) + \varepsilon^2 I^{i_0}_2(z) + \cdots \]

\[ + \varepsilon^p I^{i_0}_p(z), \quad i_0 = 1, \ldots, s_0, \]

\[ J^{i_1}(z, \varepsilon) = I^{i_1}_0(z) + \varepsilon I^{i_1}_1(z) + \cdots + \varepsilon^{p-1} I^{i_1}_{p-1}(z), \quad i_1 = s_0 + 1, \ldots, s_1, \]

\[ J^{i_2}(z, \varepsilon) = I^{i_2}_0(z) + \cdots + \varepsilon^{p-2} I^{i_2}_{p-2}(z), \quad i_2 = s_1 + 1, \ldots, s_2, \]

\[ \ldots \]

\[ J^{i_p}(z, \varepsilon) \approx I^{i_p}_0(z), \quad i_p = s_{p-1} + 1, \ldots, s_p. \]

**Theorem 4.** Any approximate solution of the system (4.1) can be represented in the form

\[ I(z, \varepsilon) \approx \varphi_0(J^1, \ldots, J^{i_0}) + \varepsilon \varphi_1(J^1, \ldots, J^{i_1}) \]

\[ + \varepsilon^2 \varphi_2(J^1, \ldots, J^{i_2}) + \cdots + \varepsilon^p \varphi_p(J^1, \ldots, J^{i_p}), \quad (4.3) \]

for some functions \( \varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_p. \)

**Proof.** Let a solution of the system (4.1) have the form

\[ I(z, \varepsilon) \approx \psi_0(z) + \varepsilon \psi_1(z) + \varepsilon^2 \psi_2(z) + \cdots + \varepsilon^p \psi_p(z). \]

Then the functions \( \psi_0(z), \psi_1(z), \psi_2(z), \ldots, \psi_p(z) \) satisfy the systems \( \Omega_0, \Omega_1, \ldots, \Omega_p \), and hence do the corresponding complete and compatible systems \( \Omega_0^*, \Omega_1^*, \ldots, \Omega_p^* \).

According to the theory of systems of linear homogeneous first-order PDEs [5], the function \( \psi_0(z) \) satisfying the system \( \Omega_0^* \) can be represented as a function of the functionally independent solutions \( I^{i_0}_0(z), \ldots, I^{i_0}_p(z) \) of \( \Omega_0^* \) in the form

\[ \psi_0(z) = \varphi_0(I^{i_0}_0(z), \ldots, I^{i_0}_p(z)) \]

with appropriately chosen function \( \varphi_0. \) Therefore the solution \( I(z, \varepsilon) \) can be rewritten as

\[ I(z, \varepsilon) \approx \varphi_0(I^1(z, \varepsilon), \ldots, I^{i_0}(z, \varepsilon)) + \varepsilon \tilde{\psi}_1(z) + \cdots + \varepsilon^p \tilde{\psi}_p(z) \]

for some functions \( \tilde{\psi}_1(z), \ldots, \tilde{\psi}_p(z). \)
It is obvious that the function \( \tilde{\psi}_3(z) \) satisfies the system \( \Pi_1^{0} \) (i.e., the system of the homogeneous equations corresponding to \( \Pi_1 \)) and therefore can be represented in the form

\[
\tilde{\psi}_3(z) = \varphi_3(I_0^0(z), \ldots, I_0^n(z)).
\]

Since the function

\[
\varepsilon \varphi_3(J^1, \ldots, J^{i_1})
\]

is a solution of the system (4.1), the function \( I(z, \varepsilon) \) can be rewritten in the form

\[
I(z, \varepsilon) \approx \varphi_0(J^1(z, \varepsilon), \ldots, J^{i_2}(z, \varepsilon)) + \varepsilon \varphi_3(J^1(z, \varepsilon), \ldots, J^{i_1}(z, \varepsilon))
\]

\[
+ \varepsilon^2 \varphi_2(z) + \cdots + \varepsilon^{i_2} \varphi_p(z)
\]

for some functions \( \varphi_2(z), \ldots, \varphi_p(z) \).

Repeating the above procedure we complete the proof of the theorem.

Remark 6. Consider the following two systems of approximate (up to \( o(\varepsilon^{p-1}) \)) equations obtained from (4.1), namely, the system

\[
X_{n_0} I = \left( X_{n_0, 0} + \varepsilon X_{n_0, 1} + \cdots + \varepsilon^{p-1} X_{n_0, p-1} \right) I = o(\varepsilon^{p-1}),
\]

\[
X_{n_1} I = \left( \varepsilon X_{n_1, 0} + \cdots + \varepsilon^{p-1} X_{n_1, p-2} \right) I = o(\varepsilon^{p-1}),
\]

\[
\cdots
\]

\[
X_{n_{p-1}} I = \varepsilon^{p-1} X_{n_{p-1}, 0} I = o(\varepsilon^{p-1}),
\]

obtained from (4.1) by neglecting the terms of order \( o(\varepsilon^p) \), and the system

\[
X_{n_0} I = \left( X_{n_0, 0} + \varepsilon X_{n_0, 1} + \cdots + \varepsilon^{p-1} X_{n_0, p-1} \right) I = o(\varepsilon^{p-1}),
\]

\[
X_{n_1} I = \left( X_{n_1, 0} + \varepsilon X_{n_1, 1} + \cdots + \varepsilon^{p-1} X_{n_1, p-1} \right) I = o(\varepsilon^{p-1}),
\]

\[
\cdots
\]

\[
X_{n_p} I = \varepsilon^{p-1} X_{n_p, 0} I = o(\varepsilon^{p-1})
\]

obtained by dividing the equations of non-zero order (i.e., the equations \( X_{n_0}, \ldots, X_{n_p} \)) by \( \varepsilon \) and by neglecting in \( X_{n_0} \) the terms of order \( o(\varepsilon^p) \).

The solution (4.3) yields the following solutions to these systems: a solution

\[
I(z, \varepsilon) = \varphi_0(J^1, \ldots, J^{i_2}) + \varepsilon \varphi_3(J^1, \ldots, J^{i_1})
\]

\[
+ \cdots + \varepsilon^{i_1} \varphi_{i_2}(J^1, \ldots, J^{i_2}) + o(\varepsilon^{p-1})
\]

(4.6)
of the system (4.4), and a solution
\[
I(z, \varepsilon) = \varphi_0(J^1, \ldots, J^n) + \varepsilon \varphi_1(J^1, \ldots, J^n) + \cdots \\
+ \varepsilon^{p-1} \varphi_{p-1}(J^1, \ldots, J^{n-1}) + o(\varepsilon^{p-1}) \tag{4.7}
\]
of the system (4.5).

In the general case, the solutions (4.6) and (4.7) do not provide the general solutions of the corresponding systems. However if, in particular, the sequence (4.2) of the systems \(\Omega_0, \ldots, \Omega_p\) corresponding to (4.1) are complete and compatible, then the formulae (4.6), (4.7) yield the general solutions.

**Example 4.** Consider the following system of linear approximate (up to \(o(\varepsilon^2)\)) first-order PDE:
\[
\frac{\partial I}{\partial z^1} + \varepsilon \frac{\partial I}{\partial z^2} + \varepsilon^2 \frac{\partial I}{\partial z^3} = 0, \\
\varepsilon z^4 \frac{\partial I}{\partial z^1} - \varepsilon^2 z^3 \frac{\partial I}{\partial z^4} = 0. \tag{4.8}
\]
We seek the solutions of the form
\[
I(z, \varepsilon) = I_0(z) + \varepsilon I_1(z) + \varepsilon^2 I_2(z).
\]
The systems \(\Omega_0, \Omega_1, \text{ and } \Omega_2\) have the form:

**system \(\Omega_0,\)**
\[
\frac{\partial I_0}{\partial z^1} = 0, \\
z^4 \frac{\partial I_0}{\partial z^4} = 0;
\]

**system \(\Omega_1,\)**
\[
\frac{\partial I_1}{\partial z^1} + \frac{\partial I_0}{\partial z^2} = 0, \\
z^4 \frac{\partial I_1}{\partial z^1} - z^1 \frac{\partial I_0}{\partial z^4} = 0;
\]

**system \(\Omega_2,\)**
\[
\frac{\partial I_2}{\partial z^1} + \frac{\partial I_1}{\partial z^2} + \frac{\partial I_0}{\partial z^3} = 0.
\]
To check the completeness of the system $\Omega_1$, we calculate the Jacobi bracket for the equations of the system $\Omega_1$:

\[
\left( \frac{\partial I_1}{\partial z^1} + \frac{\partial I_0}{\partial z^2}, z^4 \frac{\partial I_1}{\partial z^1} - z^1 \frac{\partial I_0}{\partial z^4} \right)
\]

\[
= \frac{\partial}{\partial z^1} \left( z^4 \frac{\partial I_1}{\partial z^1} - z^1 \frac{\partial I_0}{\partial z^4} \right) - z^4 \frac{\partial}{\partial z^1} \left( \frac{\partial I_1}{\partial z^1} + \frac{\partial I_0}{\partial z^2} \right) = - \frac{\partial I_0}{\partial z^4}.
\]

Hence, the system $\Omega_1$ is complete, if $I_0(z)$ satisfies the additional equation

\[
\frac{\partial I_0}{\partial z^4} = 0.
\]

Add this equation to the system $\Omega_0$.

From the compatibility of the system $\Omega_1$, the additional equation for the function $I_0(z)$ is obtained:

\[
z^4 \frac{\partial I_0}{\partial z^2} + z^1 \frac{\partial I_0}{\partial z^4} = 0.
\]

As a result, we have the following systems:

system $\Omega_0$,

\[
\frac{\partial I_0}{\partial z^1} = 0, \quad \frac{\partial I_0}{\partial z^2} = 0, \quad \frac{\partial I_0}{\partial z^4} = 0;
\]

system $\Omega_1$,

\[
\frac{\partial I_1}{\partial z^1} = 0;
\]

system $\Omega_2$,

\[
\frac{\partial I_2}{\partial z^1} + \frac{\partial I_1}{\partial z^2} + \frac{\partial I_0}{\partial z^3} = 0.
\]

These systems yield the following three particular solutions of (4.8):

$I^1(z, \varepsilon) = z^3 - \varepsilon^2 z^1$, \quad $I^2(z, \varepsilon) = \varepsilon z^2 - \varepsilon^2 z^1$, \quad $I^3(z, \varepsilon) = \varepsilon z^4$.

The general solution of the system (4.8) is

$I(z, \varepsilon) \approx \varphi_0(z^3 - \varepsilon^2 z^1) + \varepsilon \varphi_1(z^3, z^2 - \varepsilon z^1, z^4) + \varepsilon^2 \varphi_2(z^3, z^2, z^4)$. 

The described procedure can be used for calculating invariants of multi-parameter approximate groups. But, unlike the theory of invariants of exact Lie transformation groups, the condition that operators $X_1, \ldots, X_n$ generate an approximate Lie algebra is not sufficient for completeness and compatibility of the systems $\Omega_0, \ldots, \Omega_p$. The following two examples illustrate this statement.

**Example 5.** Consider approximate (up to $o(\epsilon)$) operators

$$X_1 = \epsilon \frac{\partial}{\partial z_1}, \quad X_2 = \epsilon \left( \frac{\partial}{\partial z_2} + z^2 \frac{\partial}{\partial z_3} \right)$$

which generate the approximate (up to $o(\epsilon)$) Lie algebra $[X_1, X_2] = o(\epsilon)$, i.e., here $[X_1, X_2] = 0$. The corresponding system $\Omega_0$ has the form

$$\frac{\partial I_0}{\partial z_1} = 0,$$

$$\frac{\partial I_0}{\partial z_2} + z^2 \frac{\partial I_0}{\partial z_3} = 0$$

and is not complete. Indeed, forming the Poisson bracket of the left-hand sides, we obtain the equation

$$z^2 \frac{\partial I_0}{\partial z_3} = 0,$$

which is independent of (4.9).

**Example 6.** Let us consider the operators

$$X_1 = z^2 \frac{\partial}{\partial z_1} + z^2 \frac{\partial}{\partial z_3},$$

$$X_2 = (z^2 \ln z^1 + \epsilon z^2) \frac{\partial}{\partial z_1} + (z^2 \ln z^1 + \epsilon z^2) \frac{\partial}{\partial z_3}.$$

These operators satisfy the condition

$$[X_1, X_2] = X_1$$
and consequently they generate an approximate (up to $o(\varepsilon)$) Lie algebra. The corresponding systems $\Omega_0, \Omega_1$ have the form

system $\Omega_0$,

$$z^2 \frac{\partial I_0}{\partial z^1} + z^2 \frac{\partial I_0}{\partial z^2} = 0,$$

$$z^2 \ln z^2 \frac{\partial I_0}{\partial z^1} + z^2 \ln z^2 \frac{\partial I_0}{\partial z^2} = 0;$$

system $\Omega_1$,

$$z^2 \frac{\partial I_1}{\partial z^1} + z^2 \frac{\partial I_1}{\partial z^2} = 0,$$

$$z^2 \ln z^2 \frac{\partial I_1}{\partial z^1} + z^2 \ln z^2 \frac{\partial I_1}{\partial z^2} = -z^2 \frac{\partial I_0}{\partial z^1} - z^2 \frac{\partial I_0}{\partial z^2}.$$

The systems $\Omega_0$ and $\Omega_1$ are complete (the system $\Omega_1$ is complete by virtue of the system $\Omega_0$), but the condition of compatibility of the system $\Omega_1$ yields the additional equation for $I_0(z)$, i.e.,

$$\frac{\partial I_0}{\partial z^1} + \frac{\partial I_0}{\partial z^2} = 0.$$

Thus, the problem of determining a number of independent invariants of an approximate transformation group is solved only after investigation of completeness and compatibility of the corresponding systems $\Omega_0, \Omega_1, \ldots, \Omega_p$. In Section 5, we present a sufficient condition of completeness for the systems $\Omega_0, \Omega_1, \ldots, \Omega_p$. Here the simplest sufficient condition of compatibility of such systems is given.

**Proposition 4.** Let the systems $\Omega_0, \Omega_1, \ldots, \Omega_p$ of the sequence (4.2) be complete, and all equations of the system $\Omega_0$ be independent. Then the systems $\Omega_1, \ldots, \Omega_p$ are compatible.

**5. SUFFICIENT CONDITION OF COMPLETENESS OF THE SYSTEMS $\Omega_0, \ldots, \Omega_p$**

Consider the set of $r$ approximate operators

$$X_{a_0} = X_{a_0,0} + \varepsilon X_{a_0,1} + \cdots + \varepsilon^p X_{a_0,p},$$

$$X_{a_1} = \varepsilon X_{a_1,0} + \cdots + \varepsilon^p X_{a_1,p-1},$$

$$X_{a_p} = \varepsilon^p X_{a_p,0},$$

(5.1)
of the system (4.1). Here \( \alpha_k = 1, \ldots, r_k \) \( (k = 0, \ldots, p) \), \( r_0 + \cdots + r_p = r \),
\( X_{\alpha_k j} = \delta^i_{\alpha_k j}(z)(\partial/\partial z^i) \). Using these operators, for every \( q = 0, \ldots, p \) we construct the operators

\[
Y^q_{a_k} = X_{a_k 0} + \varepsilon X_{a_k 1} + \cdots + \varepsilon^q X_{a_k q},
\]
\[
Y^q_{a_1} = X_{a_1 0} + \varepsilon X_{a_1 1} + \cdots + \varepsilon^q X_{a_1 q},
\]
\[
\vdots
\]
\[
Y^q_{a_{p-q}} = X_{a_{p-q} 0} + \varepsilon X_{a_{p-q} 1} + \cdots + \varepsilon^q X_{a_{p-q} q},
\]
\[
Y^q_{a_{p-q+1}} = \varepsilon X_{a_{p-q+1} 0} + \varepsilon^2 X_{a_{p-q+1} 1} + \cdots + \varepsilon^q X_{a_{p-q+1} q-1},
\]
\[
\vdots
\]
\[
Y^q_{a_{0-1}} = \varepsilon^{q-1} X_{a_{0-1} 0} + \varepsilon^q X_{a_{0-1} 1},
\]
\[
Y^q_{0} = \varepsilon^q X_{0 0}
\]

which are exact operators for \( q = 0 \) and approximate operators for any \( q = 1, \ldots, p \).

**Theorem 5.** If for any \( q = 0, \ldots, p \) the commutators of the operators \( Y^q_{a_k}, \ldots, Y^q_{a_{p-q}} \) can be represented (with an error \( o(\varepsilon^q) \)) as linear functions of the operators \( Y^q_{a_k}, \ldots, Y^q_{a_{p-q}} \), then each of the system \( \Omega_i \) of the sequence (4.2) is complete (on the solutions of the systems \( \Omega_0, \Omega_1, \ldots, \Omega_{t-1}, t = 0, \ldots, p \)).

**Proof.** For \( q = 0 \), the operators (5.2) are reduced to the exact operators, i.e.,

\[
Y^0_{a_k} = X_{a_k 0}, \quad k = 0, \ldots, p,
\]

and the condition of the theorem is rewritten in the form\(^1\)

\[
\left[ X_{a_k 0} X_{\beta_l 0} \right] = \varphi^{\gamma_m}_{a_k \beta_l}(z) X_{\gamma_m 0}, \quad k, l, m = 0, \ldots, p,
\]

with some functions \( \varphi^{\gamma_m}_{a_k \beta_l}(z) \). This condition is equivalent to that of completeness of the system \( \Omega_0 \).

\(^1\) According to the convention on repeated indices, \( \varphi^{\gamma_m}_{a_k \beta_l}(z) X_{\gamma_m 0} = \sum_{m=0}^{p} \sum_{\gamma_m=1}^{\gamma_m} \varphi^{\gamma_m}_{a_k \beta_l}(z) X_{\gamma_m 0} \).
Consider the case \( q = 1 \). Then

\[
Y_{a_0}^1 = X_{a_2,0} + \varepsilon X_{a_2,1}, \\
Y_{a_1}^1 = X_{a_1,0} + \varepsilon X_{a_1,1}, \\
\vdots \\
Y_{a_{p-1}}^1 = X_{a_{p-1},0} + \varepsilon X_{a_{p-1},1}, \\
Y_{a_p}^1 = \varepsilon X_{a_p,0}.
\]

(5.3)

In this case, the condition of the theorem has the form

\[
\left[ Y_{a_i}^1, Y_{b_i}^1 \right] = \varphi_{a_i, \beta_i}^p(z) Y_{a_i}^1 + \varphi_{a_i, \beta_i}^\gamma(z) Y_{b_i}^1 + o(\varepsilon), \quad k, l, m = 0, \ldots, p - 1,
\]

or, taking account of (5.3),

\[
\left[ X_{a_i,0} + \varepsilon X_{a_i,1}, X_{b_i,0} + \varepsilon X_{b_i,1} \right] = \varphi_{a_i, \beta_i}^p(z) (X_{a_i,0} + \varepsilon X_{a_i,1}) + \varphi_{a_i, \beta_i}^\gamma(z) \varepsilon X_{b_i,0} + o(\varepsilon).
\]

Equating the coefficients of \( \varepsilon^0 \) and \( \varepsilon^1 \) on the left- and right-hand sides, respectively, yields the equalities

\[
\left[ X_{a_i,0}, X_{b_i,0} \right] = \varphi_{a_i, \beta_i}^\gamma(z) X_{a_i,0}, \\
\left[ X_{a_i,0}, X_{b_i,1} \right] + \left[ X_{a_i,1}, X_{b_i,0} \right] = \varphi_{a_i, \beta_i}^\gamma(z) X_{a_i,1} + \varphi_{a_i, \beta_i}^\gamma(z) X_{b_i,0}.
\]

(5.4)

The Jacobi bracket for the equations of the system \( \Omega_1 \) has the form (see the Appendix, formula (A.2))

\[
\{ X_{a_2,0}I_1 + X_{a_2,1}I_0, X_{b_i,0}I_1 + X_{b_i,1}I_0 \}
\]

\[
= \left[ X_{a_2,0}, X_{b_i,0} I_1 + \left( [X_{a_2,0}, X_{b_i,1}] + [X_{a_2,1}, X_{b_i,0}] \right) I_0. \right.
\]

Substituting here the conditions (5.4), we get

\[
\left[ X_{a_2,0}I_1 + X_{a_2,1}I_0, X_{b_i,0}I_1 + X_{b_i,1}I_0 \right]
\]

\[
= \varphi_{a_2, \beta_i}^\gamma(z) (X_{a_2,0}I_1 + X_{a_2,1}I_0) + \varphi_{a_2, \beta_i}^\gamma(z) X_{b_i,0}I_0,
\]

\( k, l, m = 0, \ldots, p - 1, \)
i.e., the linear function of the equations of the systems \( \Omega_0 \) and \( \Omega_1 \). Hence, the system \( \Omega_1 \) is complete on the solutions of the system \( \Omega_0 \).

Let us consider the case of arbitrary \( q (\leq p) \). Then the condition of the theorem has the form

\[
[Y^q_{\alpha}, Y^q_{\beta}] = \sum_{m=0}^{p} \sum_{k,l=0}^{p-q} \varphi^m_{\alpha_k, \beta_l}(z) Y^q_{\gamma_m}, \quad k, l = 0, \ldots, p - q.
\]

By virtue of (5.2), the left-hand side of this equality has the form

\[
\begin{aligned}
&[X_{\alpha, 0} + \varepsilon X_{\alpha, 1} + \cdots + \varepsilon^q X_{\alpha, q}, X_{\beta, 0} + \varepsilon X_{\beta, 1} + \cdots + \varepsilon^q X_{\beta, q}] \\
&= \sum_{i=0}^{q} \varepsilon^i \sum_{j=0}^{q} \left[ X_{\alpha, j}, X_{\beta, i-j} \right],
\end{aligned}
\]

and the right-hand side can be rewritten as

\[
\begin{aligned}
&\sum_{m=0}^{p-q} \varphi^m_{\gamma, \beta}(z) \left( X_{\gamma, 0} + \varepsilon X_{\gamma, 1} + \cdots + \varepsilon^q X_{\gamma, q} \right) \\
&+ \varphi^m_{\gamma, \beta+1}(z) \left( \varepsilon^2 X_{\gamma+1, 1} + \cdots + \varepsilon^q X_{\gamma+q, 1} \right) \\
&+ \cdots + \varphi^m_{\gamma, \beta}(z) \varepsilon^q X_{\gamma, 0}.
\end{aligned}
\]

Equating the coefficients of powers of \( \varepsilon \) in these expressions, we obtain the equations

\[
\begin{aligned}
\varepsilon^0: \left[ X_{\alpha, 0}, X_{\beta, 0} \right] &= \sum_{m=0}^{p-q} \varphi^m_{\gamma, \beta}(z) X_{\gamma, 0}^q; \\
\varepsilon^1: \left[ X_{\alpha, 0}, X_{\beta, 1} \right] &= \sum_{m=0}^{p-q} \varphi^m_{\gamma, \beta}(z) X_{\gamma, 1}^q + \varphi^{m-1}_{\gamma, \beta+1}(z) X_{\gamma+1, 1}^q; \\
\varepsilon^i: \sum_{j=0}^{i} \left[ X_{\alpha, j}, X_{\beta, i-j} \right] &= \sum_{m=0}^{p-q} \varphi^m_{\gamma, \beta}(z) X_{\gamma, i}^q + \sum_{j=1}^{i} \varphi^{m-1}_{\gamma, \beta+1}(z) X_{\gamma+1, i-j}^q,
\end{aligned}
\]

where \( i \leq q \).
For equations of the system $\Omega_q$, the Jacobi bracket has the form (see the Appendix)

$$\left\{ \sum_{i=0}^{q} X_{\alpha_k, i} I_{q-i} + \sum_{j=0}^{q} X_{\beta_k, j} I_{q-j} \right\} = \sum_{i=0}^{q} \sum_{j=0}^{i} \left[ X_{\alpha_k, i} X_{\beta_k, i-j} \right] I_{q-i}.$$ 

Substitution of the expressions (5.5) in the right-hand side of this equality yields

$$\sum_{i=0}^{q} \left( \sum_{m=0}^{p-q} \varphi_{\gamma_k, m}^{\epsilon_k, \beta}(z) X_{\gamma_m, i} + \sum_{j=1}^{i} \varphi_{\gamma_k, q-i}^{\epsilon_k, \beta}(z) X_{\gamma_{p-q-i}, i-j} \right) I_{q-i}$$

or, after interchanging the order of summation

$$\sum_{m=0}^{p-q} \varphi_{\gamma_k, m}^{\epsilon_k, \beta}(z) \sum_{i=0}^{q} X_{\gamma_m, i} I_{q-i} + \sum_{j=1}^{i} \varphi_{\gamma_k, q-i}^{\epsilon_k, \beta}(z) \sum_{i=j}^{q} X_{\gamma_{p-q-i}, i-j} I_{q-i}.$$ 

This sum can be rewritten in the form

$$\sum_{m=0}^{p-q} \varphi_{\gamma_k, m}^{\epsilon_k, \beta}(z) \left( \sum_{i=0}^{q} X_{\gamma_m, i} I_{q-i} \right) + \sum_{n=0}^{q-1} \varphi_{\gamma_k, n}^{\epsilon_k, \beta}(z) \left( \sum_{j=0}^{n} X_{\gamma_{p-q-j}, n-j} I_{n-j} \right),$$

where the expressions in the first brackets are equations of the system $\Omega_q'$, and in the second ones of the systems $\Omega_q, \ldots, \Omega_{q-1}$ (see (A.1)). Hence, the system $\Omega_q$ is complete on the solutions of the systems $\Omega_q, \ldots, \Omega_{q-1}$. The theorem is proved.

**Remark 7.** If the operators $X_{\alpha_0}, X_{\alpha_1}, \ldots, X_{\alpha_p}$ are obtained as solution of some approximate determining equations, then they satisfy the condition of Theorem 5. Moreover, in this case the operators $Y_{\alpha_0}, Y_{\alpha_1}, \ldots, Y_{\alpha_p}$ form an exact Lie algebra for $q = 0$, and an approximate (up to $o(\varepsilon^q)$) Lie algebra for any $q = 1, \ldots, p$.

**APPENDIX: JACOBI BRACKETS FOR EQUATIONS OF THE SYSTEM $\Omega_q$**

Consider the sequence (4.2) of the systems $\Omega_0, \Omega_1, \ldots, \Omega_p$. Each system $\Omega_q, q = 0, \ldots, p$, contains $r_0 + \cdots + r_{p-q}$ equations of the form

$$\sum_{i=0}^{q} X_{\alpha_k, i} I_{q-i} = 0, \quad k = 0, \ldots, p - q, \quad \alpha_k = 1, \ldots, r_k,$$

(A.1)
for determining $I_q(z)$ provided that the functions $I_{q-1}(z), \ldots, I_0(z)$ are known and are defined from the systems $\Omega_{q-1}, \ldots, \Omega_0$. We will now show that the Jacobi bracket for any pair of equations from $\Omega_q$ gives the expression containing only the first derivatives of the functions $I_q(z), \ldots, I_0(z)$.

Consider equations

\[
\sum_{i=0}^q X_{\alpha_i,j} I_{q-i} = X_{\alpha_0,j} I_q + X_{\alpha_1,j} I_{q-1} + \cdots + X_{\alpha_q,j} I_0 = 0
\]

and

\[
\sum_{j=0}^q X_{\beta_j,j} I_{q-j} = X_{\beta_0,j} I_q + X_{\beta_1,j} I_{q-1} + \cdots + X_{\beta_q,j} I_0 = 0
\]

from the system $\Omega_q$ ($k, l = 0, \ldots, p - q$). The Jacobi bracket $\{\cdot, \cdot\}$ for these equations has the form

\[
\left\{ \sum_{i=0}^q X_{\alpha_i,j} I_{q-i}, \sum_{j=0}^q X_{\beta_j,j} I_{q-j} \right\} = X_{\alpha_0,0} \left( \sum_{j=0}^q X_{\beta_j,j} I_{q-j} \right) - X_{\beta_0,0} \left( \sum_{i=0}^q X_{\alpha_i,j} I_{q-i} \right) = [X_{\alpha_0,0}, X_{\beta_0,0}] + X_{\alpha_0,0} \left( \sum_{j=1}^q X_{\beta_j,j} I_{q-j} \right) - X_{\beta_0,0} \left( \sum_{i=1}^q X_{\alpha_i,j} I_{q-i} \right).
\]

Here

\[
[X_{\alpha_0,0}, X_{\beta_0,0}] = X_{\alpha_0,0} X_{\beta_0,0} - X_{\beta_0,0} X_{\alpha_0,0}
\]

is the commutator of the operators $X_{\alpha_0,0}, X_{\beta_0,0}$ and consequently a differential operator of the first order. Other expressions include differential operators of the second order. Let us write them in the form

\[
\sum_{i=1}^q \left( [X_{\alpha_0,0}, X_{\beta_i,j}] + X_{\beta_i,j} X_{\alpha_0,0} - [X_{\beta_0,0}, X_{\alpha_i,j}] - X_{\alpha_i,j} X_{\beta_0,0} \right) I_{q-i}.
\]

By substitution of the equations of the systems $\Omega_{q-1}, \ldots, \Omega_0$, this expression is transformed into

\[
\sum_{i=1}^q \left( [X_{\alpha_0,0}, X_{\beta_i,j}] + [X_{\alpha_i,j}, X_{\beta_0,0}] \right) I_{q-i}
\]
Interchanging the order of summation in the last expression, we obtain the expression
\[
\sum_{i=1}^{q} \left( \left[ X_{\alpha_{i},0}, X_{\beta_{i},i} \right] + \left[ X_{\alpha_{i},i}, X_{\beta_{i},0} \right] \right) I_{q-i}
\]
containing only first derivatives.

After simple transformations, the following representation of the Jacobi bracket for equations of the system \( \Omega_{q} \) is obtained:
\[
\left\{ \sum_{i=0}^{q} X_{\alpha_{i},i} I_{q-i}, \sum_{j=0}^{q} X_{\beta_{j},j} I_{q-j} \right\}
\]

\[
= \sum_{i=0}^{q} \sum_{j=0}^{i} \left[ X_{\alpha_{i},j} X_{\beta_{i},i-j} \right] I_{q-i}
\]

\[
\left[ X_{\alpha_{i},0} X_{\beta_{i},0} \right] I_{q} + \left( \left[ X_{\alpha_{i},0} X_{\beta_{i},1} \right] + \left[ X_{\alpha_{i},1} X_{\beta_{i},0} \right] \right) I_{q-1}
\]

\[
+ \left( \left[ X_{\alpha_{i},0} X_{\beta_{i},2} \right] + \left[ X_{\alpha_{i},2} X_{\beta_{i},1} \right] + \left[ X_{\alpha_{i},2} X_{\beta_{i},0} \right] \right) I_{q-2} + \cdots
\]

\[
+ \left( \left[ X_{\alpha_{i},0} X_{\beta_{i},q} \right] + \cdots + \left[ X_{\alpha_{i},q} X_{\beta_{i},0} \right] \right) I_{0}.
\]  

(A.2)

ACKNOWLEDGMENTS

I thank the Department of Computational and Applied Mathematics, University of the Witwatersrand, where this work was completed, for their hospitality and support. The work was also supported by an FRD research grant for 1996. I am grateful to Professors V. A. Baikov, N. H. Ibragimov, and F. M. Mahomed for useful discussions.

REFERENCES

5. N. M. Günter, “Integration of First-Order Partial Differential Equations,” ONTI, GTTI, Leningrad, Moscow, 1934. [In Russian]