Abstract

In this note we prove that a Lyapunov stable map having the average-shadowing property from a compact metric space onto itself is topologically ergodic, but it is not topologically weakly mixing.

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1. Introduction

By a discrete dynamical system, we mean a pair \((X, f)\), where \(X\) is a metric space with metric \(d\) and \(f\) is a continuous map from \(X\) into itself.

It is known that a numerous class of real problems are modelled by a discrete dynamical system

\[ x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots \]

The basic goal of the theory of discrete dynamical systems is to understand the nature of all orbits \(x, f(x), f^2(x), \ldots, f^n(x)\) as \(n\) becomes large and, generally, this is an impossible task. In concrete situations, we are often unable to compute the initial condition \(x\) exactly. We just compute a value \(x_0\) close to \(x\). It may also be the case that we cannot compute \(f(x_0)\) exactly, but just a value \(x_1\) close to \(f(x_0)\). Then we compute a value \(x_2\) close to \(f(x_1)\) and so on. In this way, we obtain a sequence \(x_0, x_1, x_2, \ldots\) that can be thought of as the predicted behavior of the system \((X, f)\) at \(x\). It is natural to ask whether or not this predicted behavior is close to the actual behavior of the system. This leads to research on shadowing properties.

The pseudo-orbit tracing property is one of the most important notions in dynamical systems (see [2]), which is closely related to stability and chaos of systems, see, for instance, [5,7,8]. In [9], Yang discussed the relationship between the pseudo-orbit tracing property and topological ergodicity, and showed that a chain transitive system \((X, f)\) having the pseudo-orbit tracing property is topologically ergodic. In a recent work, Blank [1] introduced the notion of...
the average-shadowing property in studying chaotic dynamical systems, which is a good tool to characterize Anosov diffeomorphisms (see [4]). Now a natural question arises: which system having the average-shadowing property is topologically ergodic? In this note, we try to discuss this question and show that a Lyapunov stable system \((X, f)\) having the average-shadowing property is topologically ergodic. In addition, we also show that such a system is not topologically weakly mixing.

2. Some basic terminology

Let \((X, f)\) be a dynamical system with metric \(d\). If \(x \in X\) then the trajectory of \(x\) is the sequence \(O(x, f) = \{f^n(x)\}_0 \leq n < \infty\).

For \(\delta > 0\), a sequence \(\{x_i\}_{0 \leq i < \infty}\) of points in \(X\) is called a \(\delta\)-average-pseudo-orbit of \(f\) if there is a positive integer \(N = N(\delta) > 0\) such that for every integer \(n \geq N\) and every nonnegative integer \(k\),

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.
\]

A map \(f\) is said to have the average-shadowing property, if for any \(\varepsilon > 0\) there is a \(\delta > 0\) such that every \(\delta\)-average-pseudo-orbit \(\{x_i\}_{0 \leq i < \infty}\) is \(\varepsilon\)-shadowed in average by some point \(z \in X\), that is,

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \varepsilon.
\]

A point \(x \in X\) is said to be stable point of \(f\) if for any \(\varepsilon > 0\) there is a \(\delta > 0\) such that \(d(f^n(x), f^n(y)) < \varepsilon\) for every \(y \in X\) with \(d(x, y) < \delta\) and every positive integer \(n\). The \(f\) is called Lyapunov stable if every point of \(X\) is a stable point of \(f\). The \(f\) is called sensitive dependence on initial conditions if every point of \(X\) is not stable point of \(f\).

If \(U\) and \(V\) are two nonempty subsets of \(X\), then we let

\[
N(U,V) = \{n : f^n(U) \cap V \neq \emptyset, \ 0 \leq n < \infty\}.
\]

A map \(f\) is called topologically transitive if for any two nonempty open subsets \(U\) and \(V\) of \(X\), \(N(U,V) \neq \emptyset\).

The \(f\) is called topologically weakly mixing if \(f \times f\) is topologically transitive. The \(f\) is called topologically mixing if for any two nonempty open subsets \(U\) and \(V\) of \(X\) there is a positive integer \(N\) such that \(N(U,V) \supset \{N, N+1, \ldots\}\).

A map \(f\) is called topologically ergodic if for any two nonempty open subsets \(U\) and \(V\) of \(X\), \(N(U,V)\) has positive upper density, that is,

\[
\tilde{D}(N(U,V)) = \limsup_{n \to \infty} \frac{\text{Card}[N(U,V) \cap \{0, 1, \ldots, n-1\}]}{n} > 0,
\]

where \(\text{Card}(E)\) denotes the number of members in the finite set \(E\).

It is well known that

\[
\text{mixing} \quad \uparrow \quad \text{weakly mixing} \quad \downarrow \quad \text{ergodic} \quad \uparrow \quad \text{transitive}.
\]

3. Results and proofs

The main result of the paper is the following theorem.

**Theorem 3.1.** Let \(X\) be a compact metric space and \(f : X \to X\) be a Lyapunov stable map from \(X\) onto itself. If \(f\) has the average-shadowing property, then \(f\) is topologically ergodic.

**Proof.** Suppose that \(U\) and \(V\) are two nonempty open subsets of \(X\). We choose \(x \in U\), \(y \in V\) and \(\varepsilon > 0\) such that \(B(x, \varepsilon) \subset U\) and \(B(y, \varepsilon) \subset V\), where \(B(a, \varepsilon) = \{b \in X : d(a, b) < \varepsilon\}\).
Since $f$ is Lyapunov stable, every point $x \in X$ is a stable point of $f$, hence for any $\varepsilon > 0$ there is a $\delta_x > 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for every positive integer $n$ and every point $y \in B(x, \delta_x)$. By the compactness of $X$, there is a $\delta > 0$ such that for any $u, v \in X$, $d(u, v) < \delta$ implies $d(f^n(u), f^n(v)) < \varepsilon$ for every positive integer $n$. Since $f$ has the average-shadowing property, let $\delta_1 = \delta_1(\delta/2)$ be a positive number as in the definition of the average-shadowing property. Let $D = \text{diam}(X) = \sup\{d(x, y) : x, y \in X\}$ be the diameter of $X$. We choose a positive integer $N_0$ such that $3D/N_0 < \delta_1$.

Define a periodic sequence $\{w_i\}_{0 \leq i < \infty}$ such that

\[ w_i = x_{-N_0 + [i \mod 2N_0]} \quad \text{if} \quad [i \mod 2N_0] \in \{1, 2, \ldots, N_0\}, \]

\[ w_i = y_{-2N_0 + [i \mod 2N_0]} \quad \text{if} \quad [i \mod 2N_0] \in \{N_0 + 1, N_0 + 2, \ldots, 2N_0\}, \]

where $x_0 = x, y_0 = y$, and $x_{-i} \in f^{-1}(x_{-i+1}), y_{-i} \in f^{-1}(y_{-i+1})$, for $i = 1, 2, \ldots, N_0 - 1$. That is, the terms of the sequence from $i = 1$ to $2N_0$ are

\[ x_{-N_0 + 1}, x_{-N_0 + 2}, \ldots, x_{-1}, x_0, \]
\[ y_{-N_0 + 1}, y_{-N_0 + 2}, \ldots, y_{-1}, y_0. \]

It is easy to see that for $n \geq N_0$ and $0 \leq k < \infty$,

\[ \frac{1}{n} \sum_{i=0}^{n-1} d(f(w_{k+i}), w_{k+i+1}) < \frac{[n/N_0] \times 3D}{n} \leq \frac{3D}{N_0} < \delta_1. \]

Thus $\{w_i\}_{0 \leq i < \infty}$ is a periodic $\delta_1$-average-pseudo-orbit of $f$. Hence, it can be $\delta/2$-shadowed in average by some $w \in X$, that is,

\[ \lim_{n \to \infty} \sup \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(w), w_i) < \frac{\delta}{2}. \]

For $z \in \{x, y\}$, let

\[ J_z = \{i : w_i \in \{z_{-N_0 + 1}, z_{-N_0 + 2}, \ldots, z_{-1}, z\} \text{ and } d(f^i(w), w_i) < \delta\}. \]

We have the following claim:

**Claim.** $J_z$ has positive upper density, that is, $\tilde{D}(J_z) > 0$.

**Proof.** Without loss of generality, we assume $z = x$. Suppose on the contrary that $\tilde{D}(J_x) = 0$, then we have

\[ \lim_{n \to \infty} \frac{\text{Card}(J_x \cap [0, 1, \ldots, n-1])}{n} = 0. \]

Let

\[ J'_x = \{i : w_i \in \{x_{-N_0 + 1}, x_{-N_0 + 2}, \ldots, x_{-1}, x\} \text{ and } d(f^i(w), w_i) \geq \delta\}. \]

Then

\[ \lim_{n \to \infty} \frac{\text{Card}(J'_x \cap [0, 1, \ldots, n-1])}{n} = \frac{1}{2}. \]

Hence, for any $\rho \in (0, 1/2)$ there is a positive integer $N$ such that

\[ \frac{\text{Card}(J'_x \cap [0, 1, \ldots, n-1])}{n} > \frac{1}{2} - \rho. \]
for every $n \geq N$. Thus,
\[
\lim_{n \to \infty} \sup_{i=0}^{n-1} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(w), w_i) \geq \lim_{n \to \infty} \sup_{i \in J_n \cap \{0, 1, \ldots, n-1\}} \frac{1}{n} \sum_{i \in J_n \cap \{0, 1, \ldots, n-1\}} d(f^i(w), w_i) \\
\geq \delta \lim_{n \to \infty} \frac{\text{Card}(J_n \cap \{0, 1, \ldots, n-1\})}{n} \\
\geq \delta \left( \frac{1}{2} - \rho \right).
\]
Since $\rho$ is arbitrary, we have
\[
\lim_{n \to \infty} \sup_{i=0}^{n-1} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(w), w_i) \geq \frac{1}{2} \delta.
\]
This is a contradiction. Therefore $\tilde{D}(J_i) > 0$. The proof of claim is completed.

Now, let $J_m(y) = \{i \in J_y : w_i = y-m\}$, for $0 \leq m \leq N_0 - 1$. Then, by claim, there is an integer $m_0$ with $0 \leq m_0 \leq N_0 - 1$ such that $\tilde{D}(J_{m_0}(y)) > 0$.

Choose $i_0 \geq N_0$ and $0 \leq k_0 \leq N_0 - 1$ such that $f^{i_0}(w) \in B(x-k_0, \delta)$. For any $j \in J_{m_0}(y)$ with $j \geq i_0 + k_0$, we have $f^j(w) \in B(y-m_0, \delta)$. Since $f$ is Lyapunov stable, we have
\[
f^{j_0+k_0} \in B(x, \varepsilon) \quad \text{and} \quad f^{j+k_0} \in B(y, \varepsilon).
\]
Let $n_j = (j + m_0) - (i_0 + k_0)$, then $f^{n_j}(B(x, \varepsilon)) \cap B(y, \varepsilon) \neq \emptyset$. So, $f^{n_j}(U) \cap V \neq \emptyset$. Hence,
\[
\tilde{D}(K(U, V)) \geq \tilde{D}(J_{m_0}(y)) > 0.
\]
This shows that $f$ is topologically ergodic.

The proof of Theorem 3.1 is completed.

The following theorem is due to Yang [9]. For completeness, we give a proof of this theorem.

**Theorem 3.2.** Let $X$ be a compact metric space containing at least two points and $f : X \to X$ be continuous map. If $f$ is topologically weakly mixing then $f$ is sensitive dependent on initial conditions.

**Proof.** Suppose on the contrary that $f$ is not sensitive dependent on initial conditions. There is at least one stable point $z$ of $f$.

Suppose that $a$ and $b$ are two distinct points of $X$. We choose two open subsets $U$ and $V$ of $X$ such that $a \in U \subset \overline{U}$, $b \in V \subset \overline{V}$ and $U \cap V = \emptyset$. Let $d = d(U, \overline{V})$. Clearly, $d > 0$. For any $\varepsilon \in (0, d)$, there is $\delta > 0$ such that $d(z, z') < \delta$ then we have $d(f^n(z), f^n(z')) < \varepsilon/2$ for all positive integer $n$.

Since $f$ is topologically weakly mixing, there is a positive integer $N$ such that
\[
(f \times f)^N(B(z, \delta) \times B(z, \delta)) \cap (U \times V) \neq \emptyset.
\]
It follows that $f^N(B(z, \delta)) \cap U \neq \emptyset$ and $f^N(B(z, \delta)) \cap V \neq \emptyset$. Hence, there are $x \in B(z, \delta)$ such that $f^N(x) \in U$ and $y \in B(z, \delta)$ such that $f^N(y) \in V$. Since $z$ is stable point for $f$, we have
\[
d(f^N(x), f^N(z)) < \frac{\varepsilon}{2} \quad \text{and} \quad d(f^N(y), f^N(z)) < \frac{\varepsilon}{2}.
\]
Thus,
\[
d(f^N(x), f^N(y)) \leq d(f^N(x), f^N(z)) + d(f^N(z), f^N(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Note that $f^N(x) \in U$ and $f^N(y) \in V$, we have $d(U, V) < \varepsilon$. On the other hand, $d(U, V) \geq d(U, V) = d > \varepsilon$. This is a contradiction. Therefore, $f$ is sensitive dependent on initial conditions.

The proof of Theorem 3.2 is completed.
**Remark 3.3.** Lardjane [3] studied the relationship between topological mixing and sensitive dependence on initial conditions and proved that if \( f \) is topologically mixing then it is sensitive dependent on initial conditions. This conclusion of Lardjane is a corollary of Theorem 3.2, because topological mixing implies topologically weak mixing.

**Theorem 3.4.** Let \( X \) be a compact metric space and \( f : X \to X \) be a Lyapunov stable map from \( X \) onto itself having the average-shadowing property, then \( f \) is topologically ergodic, but \( f \) is not topologically weakly mixing. Further, \( f \) is not topologically mixing.

**Proof.** It is obtained directly from Theorems 3.1 and 3.2. □

4. Conclusion

We have known from Xiong [6] that topological transitivity is strictly weaker than topological ergodicity. In this paper we show that topological ergodicity is strictly weaker than topological mixing. Moreover, we also show that topological ergodicity is different to topologically weak mixing, although both properties lie between topological transitivity and topological mixing.

**References**