Sample path large deviations for a class of random currents

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Abstract

We study long-time asymptotic behavior of the current-valued processes on compact Riemannian manifolds determined by the stochastic line integrals. Sample path large deviation estimates are proved, which induce the law of the iterated logarithm as a corollary. As their application, we give a probabilistic approach to the analysis on noncompact Abelian covering manifolds.

Keywords: Diffusion; Manifold; Large deviation; Random current; Stochastic line integral; The law of the iterated logarithm; Limit theorem; Abelian covering

1. Introduction

Asymptotic behaviors of the diffusion processes on Riemannian manifolds have been one of the central problems on intersection of the probability theory and the geometry. In this paper, along the course given by Ikeda (1985), Ikeda and Ochi (1986) and Ochi (1985), we give several limit theorems formulated as large deviation principles.

The essence of their approach is to regard the diffusion processes as semimartingales which take values in the space of 1-currents with the aid of the stochastic line integrals. When $M$ is compact, it is proved in Ochi (1985) that there is a continuous process $\{X_t\}_{t \geq 0}$ taking values in the topological dual of the space of the smooth 1-forms such that $\{X_t(\alpha)\}_{t \geq 0}$ realizes the stochastic line integral along $\alpha$ for every smooth 1-form $\alpha$ (see Proposition 2.1 below).
In this article, we study a large deviation principle for the current-valued process \( \{X_t\}_{t \geq 0} \) and its martingale part \( \{Y_t\}_{t \geq 0} \) in the framework above. For such current-valued processes, the law of large numbers and the central limit theorem have already been established in Ikeda (1985), Ikeda and Ochi (1986) and Ochi (1985) (see Theorems 2.2 and 2.4). The latter asserts that the family \( \{X^\lambda_{GA}\}_{GA \geq 0} \) or \( \{Y^\lambda_{GA}\}_{GA \geq 0} \) defined below converges in law as \( GA \to \infty \). On the basis of these limit theorems, we introduce our large deviation. Let us define another scaling \( g(GA) \) so that \( \lim_{GA \to \infty} g(GA) = \infty \) and \( \lim_{GA \to \infty} \sqrt{GA}/g(GA) = \infty \). Define \( \tilde{X}^\lambda \) and \( \tilde{Y}^\lambda \) by \( \tilde{X}^\lambda := g(GA)^{-1}X^\lambda \) and \( \tilde{Y}^\lambda := g(GA)^{-1}Y^\lambda \). Then, the law of \( \{\tilde{X}^\lambda_{GA}\}_{GA > 0} \) or \( \{\tilde{Y}^\lambda_{GA}\}_{GA > 0} \) satisfies the large deviation principle in the space of current-valued continuous functions on \([0, \infty)\). For instance, the large deviation for \( \{\tilde{Y}^\lambda_{GA}\}_{GA \geq 0} \) asserts that there is a rate function \( L \) which satisfies

\[
\limsup_{GA \to \infty} \frac{1}{g(GA)^2} \log \left( \sup_{x \in M} P_x [\tilde{Y}^\lambda_{GA} \in A] \right) \leq - \inf_{w \in A} L(w),
\]

\[
\liminf_{GA \to \infty} \frac{1}{g(GA)^2} \log \left( \inf_{x \in M} P_x [\tilde{Y}^\lambda_{GA} \in A] \right) \geq - \inf_{w \in A^0} L(w)
\]

for each Borel set \( A \) in the space of current-valued processes. This is our main theorem (Theorem 2.5). Moreover, we give an explicit representation of the rate function, which is very similar to that of the classical Schilder theorem for Brownian motion.

Our result is regarded as a generalization of Baldi’s (1991) work, which develops a large deviation estimate associated with the stochastic homogenization of periodic diffusions on \( \mathbb{R}^d \). As pointed out in Ochi (1985), the stochastic homogenization of periodic diffusions on \( \mathbb{R}^d \) follows from the central limit theorem of stochastic line integrals on the torus. Other problems connected with our framework is also arranged in Ikeda (1985), Ikeda and Ochi (1986) and Ochi (1985) and Manabe (1982). The latter one has dealt with asymptotics of the winding number of diffusion paths.

The organization of the paper is as follows. In Section 2 we shall introduce the precise formulation of our framework. We review the law of large numbers and the central limit theorem, and state our main result. The proof is given in Section 3, following the general theory of Baldi in large deviations. Section 4 is devoted to three topics related to our results. As the first one, we shall give a comparison with Manabe’s (1992) result which is the large deviation estimate in the same framework under the different scaling condition for \( g \). The second one is the Strassen law of the iterated logarithm, which is a standard application of sample path large deviation estimates. By using it, we give a result sharper than the law of large numbers. The third one is connected with the long-time asymptotics of the Brownian motion on noncompact Abelian covering manifolds.

2. Framework and main results

Let \( M \) be a \( d \)-dimensional, compact and connected Riemannian manifold. Take a differential operator \( A/2 + b \) on \( M \), where \( A \) is the Laplace–Beltrami operator and \( b \) a smooth vector field. The diffusion process on \( M \) associated with \( A/2 + b \) shall be denoted by \( \{z_t\}_{t \geq 0}, \{P_x\}_{x \in M} \). We denote the normalized invariant measure of \( \{z_t\} \)
by $m$. For simplicity, we abbreviate the initial point $x$ and denote $\mathbb{P} = \mathbb{P}_x$ when it is not so significant.

Let us define several function spaces which are the basis of our analysis. For 1-forms $\alpha$ and $\beta$ on $M$, $|\alpha|_p(x)$ (resp. $(\alpha, \beta)_p(x)$) means the cotangent norm of $\alpha(x)$ (resp. the inner product between $\alpha(x)$ and $\beta(x)$) on the cotangent space $T^*_xM$ at $x \in M$. Let $\mathcal{D}_{1,\infty}$ be the totality of smooth 1-forms on $M$ equipped with the Schwartz topology determined by the seminorms $\left\{ \|\cdot\|_p \right\}_{p \geq 0}$ given by

$$
\|x\|_p = \left\{ \int_M |(1 - A_1)^{p/2}x|^2 \, d\nu \right\}^{1/2},
$$

where $A_1$ is the Hodge–Kodaira Laplacian which acts on 1-forms, and $v$ the normalized Riemannian measure. This topology makes $\mathcal{D}_{1,\infty}$ a nuclear space (see Itô, 1984; Ochi, 1985; our seminorms are different from those in Ochi (1985). However, they induce the same topology). Let the Hilbert space $\mathcal{D}_{1,p}$ be the completion of $\mathcal{D}_{1,\infty}$ by $\|\cdot\|_p$. The space $\mathcal{D}_{1,-\infty}$ of 1-currents on $M$ is the dual space of $\mathcal{D}_{1,\infty}$ and $\mathcal{D}_{1,-p}$ the dual space of $\mathcal{D}_{1,p}$. We denote the operator norm on $\mathcal{D}_{1,-p}$ by $\|\cdot\|_{-p}$. Also we denote the dual pairing between $\mathcal{D}_{1,-p}$ and $\mathcal{D}_{1,p}$ by $\langle \cdot, \cdot \rangle_{-p}$. For each positive measure $\mu$ on $M$, let $L^2_\mathcal{I}(\mu)$ be the family of measurable 1-forms $\alpha$ with $\|\alpha\|^2_{L^2_\mathcal{I}(\mu)} < \infty$, where

$$
\|\alpha\|^2_{L^2_\mathcal{I}(\mu)} = (\alpha, \alpha)_{L^2_\mathcal{I}(\mu)} = \int_M (\alpha^1, \alpha^2) \, d\mu
$$

for measurable 1-forms $\alpha^1, \alpha^2$.

We denote the stochastic line integral of a smooth 1-form $\alpha$ along diffusion paths $\{z_s\}_{0 \leq s \leq t}$ by $X_t(\alpha)$ (see Ikeda and Manabe, 1979; Ikeda and Watanabe, 1989). Let us review some properties of $X_t(\alpha)$ which shall be used later. Firstly, if $\alpha = du$ for some $u \in C^\infty(M)$, then

$$
X_t(\alpha) = u(z_t) - u(z_0). \tag{2.1}
$$

Secondly, $X_t(\alpha)$ becomes a semimartingale and its decomposition is given by the following:

$$
X_t(\alpha) = Y_t(\alpha) + \int_0^t \langle b, \alpha \rangle - \frac{1}{2} \langle \delta \alpha \rangle(z_s) \, ds, \tag{2.2}
$$

where $\delta$ is the formal adjoint of the exterior derivative on $L^2(\nu)$ and $\langle \cdot, \cdot \rangle$ means the pointwise dual pairing between vector fields and 1-forms. In addition, the quadratic variation $\langle Y(\alpha) \rangle_t$ of the martingale $Y_t(\alpha)$ is given by

$$
\langle Y(\alpha) \rangle_t = \int_0^t |\alpha(z_s)|^2 \, ds. \tag{2.3}
$$

Let $\mathcal{C}_p = C([0, \infty) \rightarrow \mathcal{D}_{1,-p})$ be the Polish space of all $\mathcal{D}_{1,-p}$-valued continuous functions equipped with the compact uniform topology. Then we may regard $X = \{X_t(\alpha)\}_{t \geq 0, \alpha \in \mathcal{D}_{1,\infty}}$ and $Y = \{Y_t(\alpha)\}_{t \geq 0, \alpha \in \mathcal{D}_{1,\infty}}$ as $\mathcal{C}_p$-valued random variables for sufficiently large $p$ as follows.
Proposition 2.1 (Ikeda and Ochi, 1986; Ochi, 1985). (i) For \( p > d \), there exists a \( \mathcal{C}_p \)-valued random variable \( \hat{Y} = \{ \hat{Y}_t \}_{t \geq 0} \) such that for each \( t \geq 0 \)
\[
\hat{Y}_t(x) = Y_t(x), \quad x \in \mathcal{D}_{1,\infty} \text{ a.s.}
\]
(ii) For \( p > d + 1 \), there also exists a \( \mathcal{C}_p \)-valued random variable \( \hat{X} = \{ \hat{X}_t \}_{t \geq 0} \) such that for each \( t \geq 0 \)
\[
\hat{X}_t(x) = X_t(x), \quad x \in \mathcal{D}_{1,\infty} \text{ a.s.}
\]

For simplicity, we use the same symbols \( X \) and \( Y \) for these versions. Thus, we regard \( X \) and \( Y \) as \( \mathcal{C}_p \)-valued random variables in the sense above.

Under this formulation, some limit theorems have been achieved. For each \( \mathcal{G} \in \mathcal{D}_{1,p} \), \( \mathcal{G} \) is given as follows:
\[
e(\mathcal{G}) = \int_M (\langle b, \mathcal{G} \rangle - \frac{1}{2} \delta \mathcal{G}) \, dm.
\]

Note that the mapping \( e : \mathcal{G} \mapsto e(\mathcal{G}) \) belongs to \( \mathcal{D}_{1, -p} \) when \( p > d/2 + 1 \). Then the law of large numbers is given as follows.

Theorem 2.2 (Ikeda, 1985; Manabe, 1992). For sufficiently large \( p \), we have
\[
\lim_{\mathcal{G} \to \infty} \frac{1}{\mathcal{G}} X_{\mathcal{G}} = e, \quad \lim_{\mathcal{G} \to \infty} \frac{1}{\mathcal{G}} Y_{\mathcal{G}} = 0 \text{ a.s. in } \mathcal{D}_{1, -p}.
\]

Next, we state the central limit theorem. For the purpose, we give the definition of \( \mathcal{D}_{1, -\infty} \)-valued Wiener processes following Itô (1984).

Definition 2.3. A continuous \( \mathcal{D}_{1, -\infty} \)-valued stochastic process \( \{ W_t \}_{t \geq 0} \) with stationary independent increments and \( W_0 = 0 \) is called a \( \mathcal{D}_{1, -\infty} \)-valued Wiener process. It is characterized by its mean functional \( \zeta \) and covariance functional \( \sigma \) given by
\[
\langle \zeta, \mathcal{G} \rangle_{\mathcal{D}_{1,\infty}} = \mathbb{E}[W_1(\mathcal{G})],
\]
\[
\sigma(x^1, x^2) = \mathbb{E}[(W_1(x^1) - \langle \zeta, x^1 \rangle_{\mathcal{D}_{1,\infty}})(W_1(x^2) - \langle \zeta, x^2 \rangle_{\mathcal{D}_{1,\infty}})].
\]

We consider the differential equation
\[
(\frac{1}{2} A + b)u(x) = (\langle b, x \rangle - \frac{1}{2} \delta x)(x) - e(x), \quad x \in M
\]
for each \( x \in \mathcal{D}_{1,p} \) with \( p > d \). We denote by \( u_\lambda \) the unique solution of (2.4) up to an additive constant (see Hörmander, 1963, for example) and set \( Q_\lambda = du_\lambda \). Then \( Q : \mathcal{D}_{1, p} \to \mathcal{D}_{1, p} \) becomes a continuous linear projection. In addition, the drift \( b \) is the gradient of a function, ranges of \( Q \) and \( 1 - Q \) are orthogonal in \( L^2(m) \) each other. Let \( Q^* \) be the adjoint projection on \( \mathcal{D}_{1, -p} \). We use the same symbol \( Q^* \) for the naturally extended projection on \( \mathcal{C}_p \).

For \( \lambda > 0 \), let us define scaled processes \( X^\lambda \) and \( Y^\lambda \) in the following:
\[
X^\lambda_t(x) = \frac{1}{\sqrt{\lambda}} (X_{\lambda t}(x) - \lambda e(x)), \quad Y^\lambda_t(x) = \frac{1}{\sqrt{\lambda}} Y_{\lambda t}(x)
\]
for \( t \in [0, \infty) \) and \( x \in \mathcal{D}_{1,\infty} \).
Then the central limit theorem is known in the following sense.

**Theorem 2.4** (Ikeda, 1985; Ikeda and Ochi, 1986). (i) Pick \( p > d \). If \( \lambda \) tends to \( \infty \), the probability law of \( Y^\lambda \) on \( \mathcal{C}_p \) converges weakly to that of the \( \mathcal{D}_{1,-\infty} \)-valued Wiener process \( W^1 \) with the mean functional \( \zeta^1 = 0 \) and the covariance functional

\[
\sigma^1(x^1, x^2) = (x^1, x^2)_{L^2_1(m)}.
\]

(ii) Pick \( p > d + 1 \). If \( \lambda \) tends to \( \infty \), the law of \( X^\lambda \) converges weakly to that of the \( \mathcal{D}_{1,-\infty} \)-valued Wiener process \( W^2 \) with the mean functional \( \zeta^2 = 0 \) and the covariance functional

\[
\sigma^2(x^1, x^2) = ((1 - Q)x^1, (1 - Q)x^2)_{L^2_1(m)}.
\]

Note that we can take \( W^1 \) and \( W^2 \) as elements of \( \mathcal{C}_p \) for each corresponding \( p \), respectively, in a similar way as Proposition 2.1.

Let us begin to describe our main theorem. First we define the rate functions. Recall that \( w \in C([0, \infty) \rightarrow L^2_1(m)) \) is absolutely continuous if and only if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for each partition \( 0 \leq a_1 < b_1 \leq \cdots \leq a_n < b_n \) with \( \sum_{i=1}^n (b_i - a_i) < \delta \), \( \sum_{i=1}^n \|w_{b_i} - w_{a_i}\|_{L^2_1(m)} < \varepsilon \) holds. In this case, the Radon–Nikodym theorem implies that there is a Bochner-integrable function \( \dot{w} \) which takes values in \( L^2_1(m) \) such that

\[
w_t - w_s = \int_s^t \dot{w} u \, du
\]

holds. Note that the Radon–Nikodym theorem for vector valued measures is valid in this case since \( L^2_1(m) \) is a Hilbert space (see Diestel and Uhl, 1977). Let \( \mathcal{H} \) be the space of all absolutely continuous functions \( w \) in \( C([0, \infty) \rightarrow L^2_1(m)) \) with \( w_0 = 0 \). Now we define the rate functions in the following:

\[
L(w) = \begin{cases}
\frac{1}{2} \int_0^\infty \|\dot{w}_t\|_{L^2_1(m)}^2 \, dt & \text{if } w \in \mathcal{H}, \\
\infty & \text{otherwise},
\end{cases}
\]

(2.5)

\[
\hat{L}(w) = \inf_{(1 - Q^*)\eta = w} L(\eta).
\]

(2.6)

Take \( g = g(\lambda) \) such that \( g(\lambda) \rightarrow \infty \) and \( \sqrt{\lambda}/g(\lambda) \rightarrow \infty \) holds as \( \lambda \rightarrow \infty \). By using this new scaling \( g \) we define

\[
\hat{X}^\lambda_i = \frac{1}{g(\lambda)} X_i^\lambda, \quad \hat{Y}^\lambda_i = \frac{1}{g(\lambda)} Y_i^\lambda.
\]

Now we are ready to state our main theorem.
Theorem 2.5. (i) Pick $p > d$. For any Borel sets $A \subset \mathcal{C}_p$,

\[
\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{P}_x[\tilde{Y}^\lambda \in A] \right) \leq - \inf_{w \in A} L(w),
\]

\[
\liminf_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{P}_x[\tilde{Y}^\lambda \in A] \right) \geq - \inf_{w \in \hat{A}^o} L(w),
\]

where $\hat{A}$ (resp. $A^o$) is the closure of $A$ (resp. the interior of $A$) in $\mathcal{C}_p$.

(ii) Pick $p > d + 1$. For any Borel sets $A \subset \mathcal{C}_p$,

\[
\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{P}_x[\tilde{X}^\lambda \in A] \right) \leq - \inf_{w \in A} \hat{L}(w),
\]

\[
\liminf_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{P}_x[\tilde{X}^\lambda \in A] \right) \geq - \inf_{w \in \hat{A}^o} \hat{L}(w).
\]

Remark 2.6. We would like to take $p$ as small as possible since the estimate becomes more precise as $p$ becomes smaller. Indeed, if we take $p' < p$, the $\mathcal{C}_{p'}$-topology is stronger than the $\mathcal{C}_p$-topology and the rate functions take finite values only on $C([0, \infty) \to L^2_2(m))$. Note that the lower bound of $p$ in Theorem 2.5 essentially comes from the assumption of Proposition 2.1.

On the other hand, as a direct consequence of Theorem 2.5, we obtain the large deviation estimates subordinate to the compact uniform topology of $C([0, \infty) \to \mathcal{D}_{1, -\infty})$ since the topology of $C([0, \infty) \to \mathcal{D}_{1, -\infty})$ is weaker than that of $\mathcal{C}_p$.

Remark 2.7. In almost the same way as Theorem 2.5, we can prove the large deviation principle for the current-valued Wiener processes $\{g(\lambda)^{-1}W^1\}_{\lambda > 0}$ and $\{g(\lambda)^{-1}W^2\}_{\lambda > 0}$, where $W^1$ and $W^2$ appear in Theorem 2.4. Then their rate functions coincide with $L$ or $\hat{L}$, respectively.

It is intuitively obvious that the coincidence occurs when the decay parameter $g$ of $\tilde{Y}^\lambda$ or $\tilde{X}^\lambda$ increases much slower than $\lambda$, which causes the weak convergence. However, if we tried to prove this fact as a consequence of Theorem 2.4, we needed to investigate the precise rate of the weak convergence. Thus Theorem 2.5 asserts that when $\sqrt{\lambda}/g(\lambda) \to \infty$ the growth of $g$ is sufficiently slow to cause the coincidence of rate functions. In fact, under the assumption of $\sqrt{\lambda} = g(\lambda)$, $\{\tilde{Y}^\lambda\}_{\lambda > 0}$ also satisfies large deviation. But the rate function is different from $L$ (see Kuwada).

3. Proof of Theorem 2.5

For the proof, we shall use the following theorem due to Baldi (1988) and Dembo and Zeitouni (1998), which is a generalization of Gärtner’s (1977) result to infinite-dimensional state spaces. Since our aim is the uniform large deviation, we need a slight
extension of the original one. It is given as follows:

**Theorem 3.1** (Baldi, 1988; Dembo and Zeitouni, 1998). Let \( \{\mu_{\lambda, \tau}\}_{\lambda > 0, \tau \in \Gamma} \) be a family of 2-parameter probability measures on a topological vector space \( \mathcal{V} \). We denote by \( \mathcal{V}' \) the dual topological vector space of \( \mathcal{V} \). Take an increasing function \( g: \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \lim_{\lambda \to \infty} g(\lambda) = +\infty \) and assume the following properties:

(i) There exists
\[
A(\beta) := \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in \mathcal{V}} \int_{\mathcal{V}'} \exp(g(\lambda)^2 \langle \beta, x \rangle_{\mathcal{V}'} \, d\mu_{\lambda, \tau}(x)) \right)
\]
for each \( \beta \in \mathcal{V}' \) and it is finite in some neighborhood of 0. Furthermore,
\[
A(\beta) = \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in \mathcal{V}} \int_{\mathcal{V}'} \exp(g(\lambda)^2 \langle \beta, x \rangle_{\mathcal{V}'} \, d\mu_{\lambda, \tau}(x)) \right)
\]
also holds.

(ii) \( \{\mu_{\lambda, \tau}\}_{\lambda > 0, \tau \in \Gamma} \) is exponentially tight uniformly in \( \tau \in \Gamma \). In other words, for each large \( R > 0 \), we can take a compact set \( \mathcal{K} \subset \mathcal{V} \), which is independent of \( \lambda \) and \( \tau \), such that
\[
\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{\tau \in \Gamma} \mu_{\lambda, \tau}(\mathcal{K}^c) \right) \leq - R.
\]

(iii) Let \( I \) be the Legendre transform of \( A \), given by
\[
I(x) = \sup_{\beta \in \mathcal{V}'} (\langle \beta, x \rangle_{\mathcal{V}'} - A(\beta)).
\]
The set of points where \( I \) is strictly convex is denoted by \( \mathcal{F} \). That is, \( x \in \mathcal{F} \) if and only if there exists \( x = \mathcal{F}(x) \in \mathcal{V}' \) so that
\[
I(y) > I(x) + \langle \mathcal{F}, y - x \rangle_{\mathcal{V}'}
\]
holds for all \( y \neq x \).

Then \( \inf_{x \in \mathcal{G} \cap \mathcal{F}} I(x) = \inf_{x \in \mathcal{G}} I(x) \) holds for any open sets \( \mathcal{G} \).

Then for every Borel set \( A \subset \mathcal{V} \),
\[
\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{\tau \in \Gamma} \mu_{\lambda, \tau}(A) \right) \leq - \inf_{x \in A} I(x),
\]
\[
\liminf_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{\tau \in \Gamma} \mu_{\lambda, \tau}(A) \right) \geq - \inf_{x \in A^c} I(x).
\]

Theorem 3.1 is proved along the same line as the original one. Hence we omit the proof. Applying Theorem 3.1 to the case that \( \mathcal{V} = \mathcal{M} \), \( \mu_{\lambda, \tau} \) is the law of \( \tilde{Y}^\lambda \) (resp. \( \tilde{X}^\lambda \)) under \( \mathbb{P}_x \) (where \( \tau = x \)) and \( \mathcal{V}' = \mathcal{C}_p \), we shall obtain a large deviation estimate.

Now let us start the proof of Theorem 2.5. First we deal with the assertion about \( \tilde{Y}^\lambda \). For later use, we provide a lemma about the index of regularity order.
Lemma 3.2. Let $p > d$. Then we can take $p_0$ and $q$ with $0 < q < p_0 < p$ which satisfy the following:

(i) $Y$ takes its values in $C_{p_0}$,
(ii) there exists $C > 0$ such that $\sup_{x \in M} |x|(x) \leq C \|x\|_q$ for all $x \in D_{1,\infty}$,
(iii) the canonical embedding $D_{1,-p_0} \to D_{1,-p}$ is compact,
(iv) there is an orthonormal basis $\{x_n\}_{n=1}^\infty$ of $D_{1,p_0}$ so that $\sum_{n=1}^\infty n^r \|x_n\|_q^2 < \infty$ holds for some $\gamma > 0$.

Proof. By Proposition 2.1, (i) follows for $p_0 > d$. The Sobolev lemma on compact manifolds implies that if $q = d$ then (ii) holds. From the compactness of the power of the resolvent operator $(1 - A_1)^{p_0 - p}$, (iii) holds for any $p > p_0$.

As for (iv), let $\lambda_n$ be the $n$th eigenvalue of $-A_1$ and $\omega_n \in D_{1,\infty}$ an eigenform corresponding to $\lambda_n$ which makes a complete orthonormal system of $L^2_1(v)$. Then for $x \in D_{1,\infty}$ and $r > 0$,

$$\|x\|_r = \left\{ \sum_{n=1}^\infty (1 + \lambda_n)^r (x, \omega_n)_L^2 \right\}^{1/2}.$$

Thus $(1 + \lambda_n)^{-p_0} \omega_n$ forms a complete orthonormal system of $D_{1,p_0}$. The Weyl asymptotic formula (see Gaffney, 1958) implies that $\lambda_n^{d/2}/n$ is bounded as $n \to \infty$. Thus, if we take $p_0 - q > d/2$, which is consistent with the condition for (i)–(iii), then (iv) holds.

The first treatment is the exponential tightness (Proposition 3.4). The following lemma is almost the same as Lemma 2.1 in Manabe (1992).

Lemma 3.3. There are a constant $C_1 > 0$ and $\kappa > 0$ such that

$$\sup_{x \in M} \mathbb{P}_x \left[ \sup_{0 \leq s \leq t} \|\tilde{\gamma}_s^x\|_{-p_0} \geq \rho \right] \leq \kappa \exp \left( -\frac{g(\lambda)^2 \rho^2}{C_1 t} \right), \quad t > 0.$$

Proof. For each $x \in D_{1,\infty}$, the martingale representation theorem implies $Y_t(x) = B_{(Y_t(x))}$, where $B$ is a Brownian motion on $\mathbb{R}$. Thus, by using (2.3) and (ii) of Lemma 3.2, we have for any $\rho > 0$,

$$\mathbb{P} \left[ \sup_{0 \leq s \leq t} \|\tilde{\gamma}_s^x(x)\| \geq \rho \right] = \mathbb{P} \left[ \sup_{0 \leq s \leq t} |B_{(\tilde{\gamma}_s^x(x))}| \geq \rho \right] \leq \mathbb{P} \left[ \sup_{0 \leq s \leq \epsilon C_2^2 \rho^2 / g(\lambda)^2} |B_s| \geq \rho \right] \leq 2 \exp \left( -\frac{g(\lambda)^2 \rho^2}{2C_1 t} \right).$$

The last inequality follows from the exponential inequality of the Brownian motion. Take $N := \sum_{n=1}^\infty n^r \|x_n\|_q^2 < \infty$ and $b_n := n^r \|x_n\|_q^2 / N$, where $\gamma$, $q$ and $\{x_n\}_{n \in \mathbb{N}}$ are as
in (iv) of Lemma 3.2. Letting \( r = g(\lambda)^2 \rho^2 / 2C^2 Nt \), we have

\[
\mathbb{P}_x \left[ \sup_{0 \leq s \leq t} \| \tilde{Y}_s \| - p_0 > \rho \right] = \mathbb{P}_x \left[ \sup_{0 \leq s \leq t} \left( \sum_{n=1}^{\infty} \tilde{Y}_s(x_n)^2 \right) > \rho^2 \right] \\
\leq \sum_{n=1}^{\infty} \mathbb{P}_x \left[ \sup_{0 \leq s \leq t} \tilde{Y}_s(x_n)^2 > \rho^2 b_n \right] \\
\leq 2 \sum_{n=1}^{\infty} \exp \left( \frac{g(\lambda)^2 \rho^2 b_n}{2C^2 \|x_n\|^2 q t} \right) \\
= 2 \sum_{n=1}^{\infty} \exp \left( -r \|x_n\|^2 q t \right) \\
\leq 2e^{-r} + 2 \int_1^{\infty} e^{-rs} ds \\
= 2e^{-r} \left( 1 + \frac{1}{\gamma r} \int_0^{\infty} \left( \frac{s}{r} + 1 \right)^{(1-\gamma)/\gamma} e^{-s} ds \right).
\]

Since the right-hand side of the last equality is independent of the choice of \( x \), we obtain the desired result for large \( r \). As for small \( r \), since the left-hand side of the stated inequality is less than 1, we can take \( \kappa \) large enough to obtain desired estimate. \( \square \)

By using this lemma, we shall prove the exponential tightness.

**Proposition 3.4.** For each \( R > 0 \), there exists a compact set \( \mathcal{K} \subset \mathcal{C}_p \) such that

\[
\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in \mathcal{M}} \mathbb{P}_x \left[ \tilde{Y}_{s} \in \mathcal{K} \right] \right) \leq -R.
\]

**Proof.** Given \( N \in \mathbb{N} \) and a sequence \( \{c_k\}_{k \in \mathbb{N}} \) decreasing to 0, we set

\[
A_k := \left\{ w \in \mathcal{C}_{p_0} : \sup_{|t-s| \leq c_k, 0 \leq t, s \leq k} \| w_t - w_s \| - p_0 \leq \frac{1}{k}, \ w_0 = 0 \right\}, \quad k \in \mathbb{N}
\]

and \( A = \bigcap_{k=1}^{\infty} A_k \). For each \( T > 0 \), let \( \nu_T : \mathcal{C}_{p_0} \to \mathcal{C}_{p_0}^T := C([0, T] \to \mathcal{D}_{1, -p_0}) \) be the canonical restriction. Then \( \nu_T(A) \) is uniformly bounded, equicontinuous family of functions in \( \mathcal{C}_{p_0}^T \). Thus if we take arbitrary \( \varepsilon > 0 \), then there exists \( \delta > 0 \) with \( \delta^{-1} T \in \mathbb{N} \) such that if \( |t-s| < \delta \) then \( \| w_t - w_s \| - p_0 < \varepsilon \) for all \( w \in \nu_T(A) \). Let \( h := \sup \{ \| w_s \| - p_0 : w \in A, s \in [0, T] \} \) and \( B_h := \{ \xi \in \mathcal{D}_{1, -p_0} : \| \xi \| - p_0 \leq h \} \). Then, by (iii) of Lemma 3.2, there exists a finite set \( \{ \xi_1, \ldots, \xi_n \} \subset B_h \) such that for each \( \xi \in B_h \) there is \( j \in \{1, \ldots, n\} \) which satisfies \( \| \xi_j - \xi \| - p < \varepsilon \). Then we can easily show that there is
a constant $C_2$ such that a family
\[
\bigcup_{i \in \{1, \ldots, n\}} \bigg\{ w_i = \begin{cases} 
1 - \frac{\tau}{\delta} & \text{if } t = k\delta + \tau, \tau \in [0, \delta), \\
0 & \text{if } k = 0, \ldots, \delta^{-1}T - 1
\end{cases}
\bigg\}
\]
forms $C_2\varepsilon$-net of $\nu_T(A)$ in $\mathcal{C}_p^T$.

Consequently $\nu_T(A)$ is precompact in $\mathcal{C}_p^T$ and therefore so is $A$ in $\mathcal{C}_p$ by diagonal method. Thus it suffices to show
\[
\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{P}_x [\tilde{Y}^\lambda \in A^c] \right) \leq -R
\]
for each $R > 0$ by taking $N$ and $c_k$ as it satisfies.

Let $E_{t,k} := \{ w \in \mathcal{C}_p^T; \sup_{t \leq s \leq t + c_k} \| w_s - w_t \|_{-p_0} > 1/(3k) \}$. Then we have $A_k^c \subset \bigcup_{\ell \in Z, 0 \leq \ell < \kappa^{-1} - k} E_{\ell,c_k,k}$. Thus the Markov property and Lemma 3.3 imply
\[
\sup_{x \in M} \mathbb{P}_x [\tilde{Y}^\lambda \in E_{\ell,c_k,k}] = \sup_{x \in M} \mathbb{P}_x [\mathbb{P}_{\pi_x} [\tilde{Y}^\lambda \in E_{0,k}]] \leq \kappa \exp \left( -\frac{g(\lambda)^2}{9C_1k^2c_k} \right).
\]
Therefore we have
\[
\sup_{x \in M} \mathbb{P}_x [\tilde{Y}^\lambda \in A_k^c] \leq \frac{k\kappa}{c_k} \exp \left( -\frac{g(\lambda)^2}{9C_1k^2c_k} \right).
\]
Now we take $c_k = k^{-3}$ and fix $N$ sufficiently large such that for all $k \geq N$
\[
\exp \left( -\frac{g(\lambda)^2 k}{18C_1} \right) \leq \frac{1}{k^2} \exp \left( -\frac{g(\lambda)^2 k}{18C_1} \right) \leq \frac{1}{k^2} \exp(-Rg(\lambda)^2)
\]
for sufficiently large $\lambda$. Consequently,
\[
\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{P}_x [\tilde{Y}^\lambda \in A^c] \right) \leq \limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sum_{\lambda = N}^{\infty} \sup_{x \in M} \mathbb{P}_x [\tilde{Y}^\lambda \in A_k^c] \right) \leq -R. \quad \square
\]

Let us define a functional $H : \mathcal{C}_p^T \to \mathbb{R}$ as follows:
\[
H(\mu) := \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log (\mathbb{E}_\mu[\exp(g(\lambda)^2 \langle \mu, \tilde{Y}^\lambda \rangle_{\mathcal{C}_p^T})]),
\]
where $\langle \cdot, \cdot \rangle_{\mathcal{C}_p^T}$ means the dual pairing between $\mathcal{C}_p^T$ and $\mathcal{C}_p$. Next we calculate $H(\mu)$. First, we treat the case $\mu = \pi \delta_t$, or $\mu$ is a vector-valued Dirac measure for some $\pi \in \mathcal{D}_{1,p}$ and $t \in [0, 1]$. That is, $\langle \mu, w \rangle_{\mathcal{C}_p^T} = \langle w_t, \pi \rangle_{\mathcal{C}_p}$ holds for any $w \in \mathcal{C}_p$. The case $t = 0$ is trivial, we assume $t > 0$.

Let $\mathcal{L}_t = \int_0^t \pi \delta_s \, ds$ be the mean occupation time of $\pi_s$ and $\mathcal{Q}_t^\lambda = \mathcal{Q}_t^\lambda$. We can regard $\mathcal{L}_t$ and $\mathcal{Q}_t^\lambda$ as probability measures on $M$. Then, by (2.3), $\langle \tilde{Y}_t(x) \rangle_t = \int_M |x|^2 \, d\mathcal{Q}_t^\lambda$ holds. In order to estimate the asymptotic behavior of the quadratic variation, we shall use the upper estimate of the following Donsker–Varadhan large deviation principle.
Lemma 3.5 (Donsker and Varadhan, 1975). For any Borel sets $A$ in the space of probability measures on $M$,

$$\limsup_{\lambda \to \infty} \frac{1}{A} \log \left( \sup_{x \in M} P_x[\mathcal{L}_t^\lambda \in A] \right) \leq - \inf I(v).$$

Note that the rate function $I$ attains its minimum or zero only at $m$.

By using this lemma, we shall prove

Proposition 3.6. If $\mu = z\delta_z$ for some $z \in \mathcal{D}_{1,p}$ and $t > 0$, then we have

$$H(\mu) = \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{E}_x[\exp(g(\lambda)^2\langle \mu, \tilde{Y}^\lambda_t \rangle_{\varrho_p})] \right)$$

$$= \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{E}_x[\exp(g(\lambda)^2\langle \mu, \tilde{Y}^\lambda_t \rangle_{\varrho_p})] \right) = \frac{1}{2} \| z \|_{L^2(\mu)}^2 t.$$

Proof. Take $\varepsilon > 0$. We set $A_\varepsilon := \{|\langle Y^\lambda(x) \rangle_t - \| z \|_{L^2(\mu)}^2 | < \varepsilon \}$ and divide the expectation into two parts, the main term and the remainder term.

As a first step, we estimate the remainder term. By using the Schwarz inequality, we have

$$\mathbb{E}[\exp(g(\lambda)^2\langle \mu, \tilde{Y}^\lambda_t \rangle_{\varrho_p}); A_\varepsilon^c] = \mathbb{E}[\exp(g(\lambda)^2Y^\lambda_t(x)); A_\varepsilon^c]$$

$$\leq \mathbb{E}[\exp(2g(\lambda)^2Y^\lambda_t(x))]^{1/2} P[A_\varepsilon^c]^{1/2}$$

$$\leq \mathbb{E}[\exp(2g(\lambda)^2Y^\lambda_t(x) - 2g(\lambda)^2\langle Y^\lambda(x) \rangle_t)]^{1/2}$$

$$\times \exp(g(\lambda)^2tC^2\| z \|_{\varrho_p}^2 P[A_\varepsilon^c]^{1/2}$$

$$= \exp(g(\lambda)^2tC^2\| z \|_{\varrho_p}^2 P[A_\varepsilon^c]^{1/2}.$$

Lemma 3.5 implies that there exists $c > 0$ such that $\log(\sup_{x \in M} P_x[A_\varepsilon^c]) \leq -c\lambda$ for sufficiently large $\lambda$. Since $\sqrt{\lambda}/g(\lambda) \to \infty$ holds as $\lambda \to \infty$ by assumption, we have

$$\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{E}_x[\exp(g(\lambda)^2\langle \mu, \tilde{Y}^\lambda_t \rangle_{\varrho_p}); A_\varepsilon^c] \right) = -\infty. \quad (3.1)$$

Next let us turn to the estimate of the main term. Apparently, we have

$$\liminf_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{E}_x[\exp(g(\lambda)^2\langle \mu, \tilde{Y}^\lambda_t \rangle_{\varrho_p}); A_\varepsilon] \right)$$

$$\geq \liminf_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{E}_x \left[ \exp \left( g(\lambda)^2Y^\lambda_t(x) - \frac{g(\lambda)^2}{2}\langle Y^\lambda(x) \rangle_t \right); A_\varepsilon \right] \right)$$

$$+ \frac{1}{2} \| z \|_{L^2(\mu)}^2 t - \frac{\varepsilon}{2} \quad (3.2)$$

and

$$\limsup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{E}_x[\exp(g(\lambda)^2\langle \mu, \tilde{Y}^\lambda_t \rangle_{\varrho_p}); A_\varepsilon] \right) \leq \frac{1}{2} \| z \|_{L^2(\mu)}^2 t + \frac{\varepsilon}{2}. \quad (3.3)$$
Now the following lemma, which follows from easy calculation, gives the final touch of this estimate.

**Lemma 3.7.** Suppose that \( \{a(\lambda)\}_{\lambda > 0} \subset \mathbb{R}_+ \) and \( \{b(\lambda)\}_{\lambda > 0} \subset \mathbb{R} \) satisfy the following properties:

(i) There exist \( \eta > 0 \) and \( a \in \mathbb{R} \) such that \( |g(\lambda)^{-2} \log a(\lambda) - a| < \eta \) holds for sufficiently large \( \lambda \),

(ii) \( \lim_{\lambda \to \infty} g(\lambda)^{-2} \log |b(\lambda)| = -\infty \).

Then \( g(\lambda)^{-2} \log (a(\lambda) + b(\lambda)) \in (a - 2\eta, a + 2\eta) \) for sufficiently large \( \lambda \).

Indeed, by applying Lemma 3.7 for \( a(\lambda) \equiv 1 \) and

\[
b(\lambda) = -\sup_{x \in M} \mathbb{E}_x \left[ \exp \left( g(\lambda)Y_1^t(x) - \frac{g(\lambda)^2}{2} \langle Y_1^t(x) \rangle_t \right) ; A_e^c \right],
\]

we conclude that

\[
\lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{E}_x \left[ \exp \left( g(\lambda)Y_1^t(x) - \frac{g(\lambda)^2}{2} \langle Y_1^t(x) \rangle_t \right) ; A_e \right] \right) = 0
\]

since we can prove \( \lim_{\lambda \to \infty} g(\lambda)^{-2} \log |b(\lambda)| = -\infty \) in the similar way as (3.1). Thus the right-hand side in (3.2) is equal to \( (\|z\|_{L^2(m)}^2 t - \varepsilon)/2 \). Finally, we use Lemma 3.7 again with

\[
a(\lambda) = \inf_{x \in M} \mathbb{E}_x [\exp(g(\lambda)^2 \langle \mu, \tilde{Y}_1^t \rangle_q) ; A_e],
\]

\[
b(\lambda) = \inf_{x \in M} \mathbb{E}_x [\exp(g(\lambda)^2 \langle \mu, \tilde{Y}_1^t \rangle_q) ; A_e^c]
\]

and with

\[
a(\lambda) = \sup_{x \in M} \mathbb{E}_x [\exp(g(\lambda)^2 \langle \mu, \tilde{Y}_1^t \rangle_q) ; A_e],
\]

\[
b(\lambda) = \sup_{x \in M} \mathbb{E}_x [\exp(g(\lambda)^2 \langle \mu, \tilde{Y}_1^t \rangle_q) ; A_e^c].
\]

Note that \( a(\lambda) \) and \( b(\lambda) \) satisfy the assumption of Lemma 3.7 in each case by (3.1)–(3.3). Then we conclude

\[
\frac{1}{2} \|z\|_{L^2(m)}^2 t - 2\varepsilon \leq \liminf_{\lambda \to \infty} -\frac{1}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{E}_x [\exp(g(\lambda)^2 \langle \mu, \tilde{Y}_1^t \rangle_q)] \right)
\]

\[
\leq \limsup_{\lambda \to \infty} -\frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{E}_x [\exp(g(\lambda)^2 \langle \mu, \tilde{Y}_1^t \rangle_q)] \right)
\]

\[
\leq \frac{1}{2} \|z\|_{L^2(m)}^2 t + 2\varepsilon. \quad \square
\]

By using the Markov property, we obtain a similar estimate when \( \mu \) is written by finite sum of vector-valued Dirac measures.
Corollary 3.8. If \( \mu = \sum_{k=1}^{n} \varepsilon_k \delta_{t_k} \) for \( \varepsilon_k \in \mathcal{D}_{1,p} \), \( k = 1, \ldots, n \) and \( 0 \leq t_1 \leq \cdots \leq t_n \), then we have

\[
H(\mu) = \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in \mathcal{M}} \mathbb{E}_x[\exp(g(\lambda)^2 \langle \mu, \tilde{Y}^{\lambda} \rangle_{\mathcal{F}_p})] \right)
= \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in \mathcal{M}} \mathbb{E}_x[\exp(g(\lambda)^2 \langle \mu, \tilde{Y}^{\lambda} \rangle_{\mathcal{F}_p})] \right) = \frac{1}{2} \int_{0}^{\infty} ||\mu_s||_{L_1^{2}(m)}^2 \, ds
\]

where \( \mu_s = \sum_{t_k > s} \varepsilon_k \).

Pick \( \mu \in \mathcal{C}^T_{p} \). We show \( \mu \) to be \( \mathcal{D}_{1,p} \)-valued measure on \([0, \infty)\). Recall that \( \mathcal{C}^T_{p} = C([0, T] \to \mathcal{D}_{1,-p}) \) and \( v_T \) is the canonical restriction mapping from \( \mathcal{C}_{p} \) to \( \mathcal{C}^T_{p} \). Since \( \mathcal{C}_p \) is the projective limit of \( \mathcal{C}^T_{p} \) as \( T \to \infty \), we can regard \( \mu \) as an element of \( \mathcal{C}^T_{p} \) for some \( T \). That is, there exists \( \bar{\mu} \in \mathcal{C}^T_{p} \) for some \( T \) such that the equality \( \langle \mu, w \rangle_{\mathcal{C}_p} = \langle \bar{\mu}, v_T(w) \rangle_{\mathcal{C}^T_{p}} \) holds. Hence we can identify \( \mu \) with \( \bar{\mu} \). Then \( \mu \) is characterized as \( \mathcal{D}_{1,p} \)-valued measure whose support is contained in \([0, T] \). Indeed, since the bilinear mapping \( C([0, T] \to \mathbb{R}) \times \mathcal{D}_{1,-p} \ni (\phi, \beta) \mapsto \phi \beta \in \mathcal{C}_p \) is continuous, \( \mu \) also belongs to continuous bilinear functionals on \( C([0, T] \to \mathbb{R}) \times \mathcal{D}_{1,-p} \) or continuous linear operators from \( C([0, T] \to \mathbb{R}) \) to \( \mathcal{D}_{1,p} \). Therefore \( \mu \) is identified with \( \mathcal{D}_{1,p} \)-valued measure by Theorem 2 in VI 7.2 of Dunford and Schwartz (1958).

Let us define \( \mathcal{C}^{T}_{h} \) by

\[
\mathcal{C}^{T}_{h} := \left\{ \mu = \sum_{k=1}^{n} \varepsilon_k \delta_{t_k} : \text{for some } n \in \mathbb{N}, \{\varepsilon_k\}_{k=1}^{n} \subset \mathcal{D}_{1,p}, \{t_k\}_{k=1}^{n} \subset [0, \infty) \right\}.
\]

The following lemma is a slight modification of that in the basic measure theory on \([0, \infty)\):

Lemma 3.9. For each \( \mu \in \mathcal{C}^{T}_{p} \), there exists a sequence \( \{\mu^n\}_{n \in \mathbb{N}} \subset \mathcal{C}^{T}_{h} \) such that \( \mu^n \) converges to \( \mu_s \) in \( \mathcal{D}_{1,p} \) as \( n \to \infty \) uniformly in \( s \in [0, \infty) \). In particular,

\[
\lim_{n \to \infty} \sup_{x \in \mathcal{M}, s \in [0, T]} ||\mu^n_s - \mu_s|| = 0 \text{ for all } T > 0.
\]

The last assertion of Lemma 3.9 is a consequence of (ii) of Lemma 3.2.

Now we complete the calculation of \( H(\mu) \).

Proposition 3.10.

\[
H(\mu) = \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \sup_{x \in \mathcal{M}} \mathbb{E}_x[\exp(g(\lambda)^2 \langle \mu, \tilde{Y}^{\lambda} \rangle_{\mathcal{F}_p})] \right)
= \lim_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in \mathcal{M}} \mathbb{E}_x[\exp(g(\lambda)^2 \langle \mu, \tilde{Y}^{\lambda} \rangle_{\mathcal{F}_p})] \right) = \frac{1}{2} \int_{0}^{\infty} ||\mu_s||_{L_1^{2}(m)}^2 \, ds
\]

for all \( \mu \in \mathcal{C}^{T}_{p} \), where \( \mu_s = \mu((s, \infty)) \in \mathcal{D}_{1,p} \).
Proof. First we remark about some upper estimate. That is,
\[
\frac{1}{g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{E}_x \left[ \exp \left\{ g(\lambda)^2 \left\langle \mu, \tilde{Y}^\lambda \right\rangle_x \right\} \right] \right) \leq \tilde{H}(\mu),
\]
where
\[
\tilde{H}(\mu) = \frac{1}{2} \int_0^\infty \sup_{x \in M} |\mu_s(x)|^2 \, ds.
\]
In order to obtain (3.4), take the orthonormal basis \( \{ z_n \}_{n \in \mathbb{N}} \) in \( \mathcal{D}_{1,\infty} \) given by (iv) of Lemma 3.2 and its dual basis \( \{ \beta_n \}_{n \in \mathbb{N}} \) in \( \mathcal{D}_{1,-p_0} \). Then the integration-by-parts formula for semimartingales implies
\[
\langle \mu, Y \rangle_{\mathcal{C}_p} = \sum_{n=1}^\infty \int_0^\infty Y_s(z_n) \, d\mu^\beta_n(s)
\]
where \( \mu^\beta \) is an \( \mathbb{R} \)-valued measure determined by \( \mu^\beta(A) = \langle \mu(A), \beta \rangle_{\mathcal{P}} \) for any Borel sets \( A \) and \( \mu^\beta_s = \mu^\beta((s, \infty)) \). By using it,
\[
\mathbb{E}[\exp \{ g(\lambda)^2 \left\langle \mu, \tilde{Y}^\lambda \right\rangle_x \}] = \mathbb{E} \left[ \exp \left\{ g(\lambda) \sum_{n=1}^\infty \int_0^\infty \mu^\beta_n(s) \, dY_s^\lambda(z_n) \right\} \right]
\]
\[
= \mathbb{E} \left[ \exp \left\{ g(\lambda)^2 \sum_{n,k=1}^{\infty} \int_0^\infty \mu^\beta_n(s) \, d\langle Y^\lambda(z_n), Y^\lambda(z_k) \rangle_s \right\} \right]
\]
\[
\times \mathbb{E} \left[ \exp \left\{ g(\lambda) \sum_{n=1}^\infty \int_0^\infty \mu^\beta_n(s) \, dY_s^\lambda(z_n) \right\} \right]
\]
\[
- \frac{g(\lambda)^2}{2} \sum_{n,k=1}^{\infty} \int_0^\infty \mu^\beta_n(s) \, d\langle Y^\lambda(z_n), Y^\lambda(z_k) \rangle_s \right\} \right].
\]
Note that the infinite series \( \sum_{n=1}^\infty \mu^\beta_n z_n \) converges to \( \mu_s \) uniformly on \( M \) and in \( \mathcal{D}_{1,p_0} \) for each fixed \( s \). Indeed, by virtue of (ii) and (iv) of Lemma 3.2, we have
\[
\lim_{k \to \infty} \sup_{N \gg k} \sum_{n=k}^N |\mu^\beta_n| \sup_{x \in M} |z_n(x)| \leq C \lim_{k \to \infty} \sup_{N \gg k} \sum_{n=k}^N |\mu^\beta_n| \|z_n\|_{q}
\]
\[
\leq \lim_{k \to \infty} \left\{ \sum_{n=k}^\infty |\mu^\beta_n|^2 \right\}^{1/2} \left\{ \sum_{n=k}^\infty \|z_n\|^2_{q} \right\}^{1/2} = 0.
\]
Hence we have
\[
\sum_{n,k=1}^{\infty} \int_0^\infty \mu_s^{b_n} \mu_s^{b_k} \, \langle Y^i(z_n), Y^j(z_k) \rangle_s \, ds = \int_0^\infty \sum_{n,k=1}^{\infty} \mu_s^{b_n} \mu_s^{b_k} (z_n, z_k) (z_{ij}) \, ds
\]
\[
= \int_0^\infty |\mu_s(z_{ij})|^2 \, ds \leq \int_0^\infty \sup_{x \in M} |\mu_s(x)|^2 \, ds.
\]
Thus we conclude
\[
\mathbb{E}[\exp\{g(\lambda)^2 \langle \mu, \tilde{Y}^i \rangle_{\mathcal{C}_p}\}] \leq \exp\left\{ \frac{g(\lambda)^2}{2} \int_0^\infty \sup_{x \in M} |\mu_s(x)|^2 \, ds \right\} \leq \exp\{g(\lambda)^2 \tilde{H}(\mu)\}.
\]
Take $\hat{\mu} \in \mathcal{G}'$. For $a > 1$ and $b > 1$ with $a^{-1} + b^{-1} = 1$, the Hölder inequality and (3.4) imply
\[
\frac{1}{a g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{E}_x[\exp\{g(\lambda)^2 \langle \mu, \tilde{Y}^i \rangle_{\mathcal{C}_p}\}] \right) \leq \frac{1}{a g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{E}_x[\exp\{ag(\lambda) \langle \mu - \hat{\mu}, Y^i \rangle_{\mathcal{C}_p}\}] \right) + \frac{1}{b g(\lambda)^2} \log \left( \sup_{x \in M} \mathbb{E}_x[\exp\{g(\lambda) \langle \hat{\mu}, Y^i \rangle_{\mathcal{C}_p}\}] \right)
\]
\[
\leq a \tilde{H}(\mu - \hat{\mu}) + b \frac{1}{(b g(\lambda))^2} \log \left( \sup_{x \in M} \mathbb{E}_x[\exp\{bg(\lambda) \langle \hat{\mu}, Y^i \rangle_{\mathcal{C}_p}\}] \right).
\]
Applying Corollary 3.8 by using $bg$ instead of $g$, we have
\[
\lim_{\lambda \to \infty} \frac{1}{(b g(\lambda))^2} \log \left( \sup_{x \in M} \mathbb{E}_x[\exp\{bg(\lambda) \langle \hat{\mu}, Y^i \rangle_{\mathcal{C}_p}\}] \right) = H(\hat{\mu}).
\]
Note that
\[
\sup_{s \in [0, \infty)} \|\mu_s\|_{L^2(m)} \leq C \sup_{s \in [0, \infty)} \|\mu_s\|_p < \infty
\]
holds and the support of $\mu_s$ is compact. Hence, approximating $\mu$ by $\hat{\mu}$, we conclude that Lemma 3.9 implies that $\tilde{H}(\mu - \hat{\mu})$ tends to 0 and $H(\hat{\mu})$ tends to $H(\mu)$. Thus we obtain the upper bound by taking $b \downarrow 1$.

As to lower bound, the following estimate
\[
\frac{b^2}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{E}_x \left[ \exp \left\{ \frac{g(\lambda)}{b} \langle \hat{\mu}, Y^i \rangle_{\mathcal{C}_p} \right\} \right] \right) \leq a \tilde{H}(\hat{\mu} - \mu) + b \frac{1}{g(\lambda)^2} \log \left( \inf_{x \in M} \mathbb{E}_x[\exp\{g(\lambda) \langle \mu, Y^i \rangle_{\mathcal{C}_p}\}] \right)
\]
given by the Hölder inequality implies the conclusion in a similar way. \qed
Now the analysis of $H(\mu)$ has finished and we need the calculation of the Legendre transform of $H$ to complete the proof of (i) of Theorem 2.5. Recall that $L$ is given by (2.5).

**Proposition 3.11.** Define $L_0(w) := \sup_{\mu \in \mathcal{C}_p} (\langle \mu, w \rangle_{\mathcal{Q}_p} - H(\mu))$. Then $L_0 = L$ holds.

**Proof.** First, we treat the case that there exists a point $t \in [0, \infty)$ so that $w_t \not\in L^2_t(m)$. Then we can choose elements $\{z_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}_{1,\infty}$ so that $\|z_n\|_{L^2_t(m)} = 1$ and $\langle w_t, z_n \rangle_p > n$.

Then for $\mu^n = z_n \delta_t$ we have $\langle \mu^n, w \rangle_{\mathcal{Q}_p} - H(\mu)) > n - t/2$. Thus $L_0(w) = \infty$ holds and we may assume that $w_t \in L^2_t(m)$ for all $t \in [0, \infty)$ for the rest of the proof.

If $w$ is not absolutely continuous with respect to $\| \cdot \|_{L^2_t(m)}$, then $L_0(w) = \infty$ holds. Indeed, we may assume that there exists $\varepsilon > 0$ so that for any $\rho > 0$ we can take a partition $0 = a_1 < b_1 < \cdots < a_n < b_n$ with $\sum_{i=1}^n (b_i - a_i) \leq \rho$ and $\sum_{i=1}^n \| w_{b_i} - w_{a_i} \|^2_{L^2_t(m)} \geq \varepsilon$. Also we take $\{\theta_i\}_{i=1}^n \subset \mathcal{D}_{1,\infty}$ so that $\| \theta_i \|^2_{L^2_t(m)} = 1$ for $i = 1, \ldots, n$ and

$$\| w_{b_i} - w_{a_i} - w_{b_i} - w_{a_i} \|^2_{L^2_t(m)} \leq \frac{\varepsilon}{2n}.$$ 

Let us define $\mu \in \mathcal{C}_p$ as follows:

$$\mu = \sum_{i=1}^n \frac{\theta_i}{\sqrt{\rho}} (\delta_{b_i} - \delta_{a_i}).$$

Then we have

$$\int_0^\infty \| \mu_s \|^2_{L^2_t(m)} \, ds = \frac{1}{\rho} \sum_{i=1}^n (b_i - a_i) \leq 1,$$

$$\langle \mu, w \rangle_{\mathcal{Q}_p} = \frac{1}{\sqrt{\rho}} \sum_{i=1}^n (w_{b_i} - w_{a_i}, \theta_i)_{L^2_t(m)} \geq \frac{\varepsilon}{2\sqrt{\rho}}.$$ 

Since we can take $\rho$ arbitrary small for fixed $\varepsilon > 0$, we conclude that $L_0(w) = \infty$. Thus we may assume that $w$ is absolutely continuous with respect to $\| \cdot \|_{L^2_t(m)}$. We denote the Radon–Nikodym density of $w$ by $\dot{w}$.

Take an orthonormal basis $\{\eta_n\}_{n \in \mathbb{N}}$ of $L^2_t(m)$ which consists of elements in $\mathcal{D}_{1,\infty}$. Pick $N > 0$ and take $\mu \in \mathcal{C}_p^{t,N} := \{v \in \mathcal{C}_p, v^{\eta_n} = 0 \text{ for all } n > N\}$. Then we have

$$L_0(w) \geq \sup_{\mu \in \mathcal{C}_p^{t,N}} (\langle \mu, w \rangle_{\mathcal{Q}_p} - H(\mu)).$$

Note that the calculation of the right-hand side of the last inequality has been reduced to the classical finite-dimensional case. If $w_0 \neq 0$, there exists $n_0 \in \mathbb{N}$ which satisfies $\langle w_0, \eta_{n_0} \rangle_p \neq 0$. Accordingly, by taking $N > n_0$, $L_0(w) = \infty$ follows. In the case of $w_0 = 0$, we have

$$\sup_{\mu \in \mathcal{C}_p^{t,N}} (\langle \mu, w \rangle_{\mathcal{Q}_p} - H(\mu)) = \frac{1}{2} \sum_{n=1}^N \int_0^\infty |\langle \dot{w}_s, \eta_n \rangle_p|^2 \, ds,$$

where the right-hand side may diverge. Thus letting $N \to \infty$ we obtain

$$L_0(w) \geq \frac{1}{2} \int_0^\infty \sum_{n=1}^\infty |\langle \dot{w}_s, \eta_n \rangle_p|^2 \, ds = \frac{1}{2} \int_0^\infty \| \dot{w}_s \|^2_{L^2_t(m)} \, ds. \quad (3.5)$$
Thus $L_0(w) = \infty$ holds if the most right-hand side of (3.5) diverges. Finally, we consider the case $\int_0^\infty \|\tilde{w}_s\|^2_{L^2_1(m)} \, ds < \infty$. For $\mu \in \mathcal{C}_p'$ whose support is contained in $[0, T]$, the integration-by-parts formula implies

$$\langle \mu, w \rangle_{\mathcal{C}_p} - H(\mu) = \sum_{n=1}^{\infty} \left( \frac{1}{2} \int_0^T |\langle \tilde{w}_s, \eta_n \rangle_p|^2 \, ds - \frac{1}{2} \int_0^T |\langle \hat{w}_s, \eta_n \rangle_p - \mu_n^0|^2 \, ds \right)$$

$$\leq \frac{1}{2} \int_0^T \sum_{n=1}^{\infty} |\langle \tilde{w}_s, \eta_n \rangle_p|^2 \, ds.$$

Hence, combining with (3.5), we obtain the desired result. \( \square \)

Now let us define $\mathcal{H}_1$ the subset of $\mathcal{H}$ such that $w$ is in $\mathcal{H}_1$ if and only if $L(w) < \infty$, the support of $w$ is compact, $\hat{w}$ is absolutely continuous with respect to $\| \cdot \|_{L^2_1(m)}$, and the Radon–Nikodym derivative $B[\hat{w}]$ of $\hat{w}$ takes values in $\mathcal{D}_1, p$ for a.e. $s$. Then $L$ is strictly convex at $w$ for all $w \in \mathcal{H}_1$. Indeed, for any $w' \in \mathcal{C}_p$ with $L(w') < \infty$ and $w \in \mathcal{H}_1$, by taking $\mu \in \mathcal{C}_p'$ which is determined by $\mu_x = \tilde{w}_s$, we obtain

$$L(w') - L(w) + \langle \mu, w' - w \rangle_{\mathcal{C}_p} = \frac{1}{2} \int_0^\infty \|\tilde{w}'_s - \tilde{w}_s\|^2_{L^2_1(m)} \, ds \geq 0$$

by easy calculation. Obviously, the equality holds if and only if $w' = w$.

Completing the proof of (i) of Theorem 2.5, we show that $\inf_{x \in G \cap \mathcal{H}_1} L(x)$ is equal to $\inf_{x \in G} L(x)$ for all open set $G \subset \mathcal{C}_p$. Then it is sufficient to prove that for all $w \in \mathcal{C}_p$ there exists a sequence $\{w^n\}_{n \in \mathbb{N}} \subset \mathcal{H}_1$ such that $w^n$ tends to $w$ in $\mathcal{C}_p$ and $L(w^n)$ tends to $L(w)$ as $n$ goes to $\infty$. We prepare the following lemma for the proof.

**Lemma 3.12.** Let $S$ be a separable Hilbert space and $\tilde{S}$ the dense subspace of $S$. We define $\mathcal{I}$ the subset of $L^2([0, \infty) \rightarrow S)$ such that $w \in \mathcal{I}$ if and only if $w_0 = 0$, $w$ is absolutely continuous with respect to $\| \cdot \|_S$, the support of $w$ is compact, and $w_s \in \tilde{S}$ for almost every $s$. Then $\mathcal{I}$ is dense in $L^2([0, \infty) \rightarrow S)$.

**Proof.** We prove the orthogonal complement $\mathcal{I}^\perp$ of $\mathcal{I}$ is equal to $\{0\}$. Pick $w \in \mathcal{I}^\perp$. For $0 < a < b < \infty$ and $\alpha \in \tilde{S}$, we take $\tilde{\eta}^n$ such that

$$\tilde{\eta}^n_t = \begin{cases} \alpha, & t \in [a - \frac{1}{n}, a), \\ 0, & t \in [a, a - \frac{1}{n}) \cup [a, b) \cup [b + \frac{1}{n}, \infty), \\ -\alpha, & t \in [b, b + \frac{1}{n}). \end{cases}$$

Then $\eta^n \in \mathcal{I}$ is defined by $\eta^n_t = \int_0^t \tilde{\eta}^n_s \, ds$. Thus we have $\int_0^\infty (w_s, \eta^n_s)_S \, ds = 0$ and taking $n$ to $\infty$, we obtain $\int_a^b (w_s, \alpha)_S \, ds = 0$. Since $a < b$ is arbitrary, $(w_s, \alpha)_S = 0$ for almost every $s$. Since $S$ is separable, we can take a countable dense subset $\{\alpha_n\}_{n \in \mathbb{N}} \subset \tilde{S}$ and we conclude that $(w_s, \alpha_n)_S = 0$ holds for all $n$ for almost every $s$. Hence $w = 0$. \( \square \)
For \( w \in \mathcal{G}_p \) with \( L(w) < \infty \), Lemma 3.12 implies that we can take absolutely continuous functions \( \hat{w}_n \in L^2([0, \infty) \to L^1_1(m)) \) with compact supports which take values in \( \mathcal{Q}_{1, p} \). We define \( w^n \in \mathcal{H}_1 \) by \( w^n = \int_0^1 \hat{w}_n \, ds \). Then we have
\[
\|w^n - w_t\|_{-p} \leq C \|w^n - w_t\|_{L^1_1(m)} \leq C(2L(w^n - w_t))^{1/2}.
\]
By definition, \( \lim_{n \to \infty} L(w^n - w) = \lim_{n \to \infty} L(w^n) - L(w) = 0 \) holds and hence \( w^n \) converges to \( w \) in \( \mathcal{G}_p \). Thus we complete the proof of (i) of Theorem 2.5.

Next we show the second claim of Theorem 2.5. Indeed, (2.4) implies
\[
\langle (1 - Q)X \rangle_{\infty} \overset{\text{L}}{\longrightarrow} (1 - Q)^* X = \langle (1 - Q)^* Y \rangle_{\infty}.
\]
Thus we obtain
\[
\hat{X}_t^i(\alpha) = \hat{Y}_t^i((1 - Q)\alpha) + \frac{1}{g(\hat{\lambda})\sqrt{\hat{\lambda}}} (u_\alpha(z^i_{\infty}) - u_\alpha(z^i_0))
\]
by using (2.1) and (2.2). The contraction principle (see Dembo and Zeitouni, 1998, for example) shows that \( \{(1 - Q)^* \hat{Y}_t^i\}_{t > 0} \) satisfies the large deviation principle with the rate function \( \hat{L} \). Note that \( \sup_{x \in M} |u_\alpha(x)| \leq C \|x\| \) follows from the Sobolev lemma and the hypoellipticity of \( \Lambda/2 + b \) (see Hörmander, 1963). Hence \( \{(1 - Q)^* \hat{X}_t^i\}_{t > 0} \) also satisfies the large deviation principle with the rate function \( \hat{L} \) since \( \{(1 - Q^*) \hat{Y}_t^i\}_{t > 0} \) and \( \{(1 - Q^*) \hat{X}_t^i\}_{t > 0} \) are exponentially equivalent (Dembo and Zeitouni, 1998, Theorem 4.2.13).

**Remark 3.13.** As we remarked before, ranges of \( Q \) and \( (1 - Q) \) are orthogonal in \( L^2_1(m) \) each other when \( b \) is the gradient of a function. In the case, the explicit form of \( \hat{L} \) is given as follows:
\[
\hat{L}(w) = \begin{cases} \hat{L}(w) & \text{if } w \in \mathcal{H} \cap \text{Range}(1 - Q^*), \\ \infty & \text{otherwise}. \end{cases}
\]
Indeed, by taking completion in \( L^2_1(m) \),
\[
L^2_1(m) = \text{Range}(1 - Q)^{L^2_1(m)} \oplus \text{Range}(Q)^{L^2_1(m)}
\]
follows. Then, since \( \text{Range}(1 - Q^*) \) (resp. \( \text{Range}(Q^*) \)) annihilates on \( \text{Range}(Q) \) (resp. \( \text{Range}(1 - Q) \)), we have
\[
\text{Range}(1 - Q^*) \cap L^2_1(m) \subset \text{Range}(1 - Q)^{L^2_1(m)},
\]
\[
\text{Range}(Q^*) \cap L^2_1(m) \subset \text{Range}(Q)^{L^2_1(m)}.
\]
For \( w \in \text{Range}(1 - Q^*) \cap L^2_1(m) \) and \( \eta \in L^2_1(m) \) with \( (1 - Q^*) \eta = w \), we have
\[
\int_M |\eta|^2 \, dm = \int_M |w + Q^* \eta|^2 \, dm = \int_M |w|^2 \, dm + \int_M |Q^* \eta|^2 \, dm.
\]
Hence we obtain
\[
\inf_{(1 - Q^*) \eta = w} \int_M |\eta|^2 \, dm = \int_M |w|^2 \, dm
\]
and therefore (3.6) holds.
4. Related topics

4.1. Comparison with previous results

We introduce another large deviation estimate given by Manabe (1992) and compare with Theorem 2.5.

**Theorem 4.1 (Manabe, 1992).** Assume $b = 0$. Then for any Borel sets $A \subset \mathcal{D}_{1, -p}$, we have

$$
\limsup_{t \to \infty} \frac{1}{t} \log \left( \sup_{x \in M} \mathbb{P}_x \left[ \frac{1}{t} X_t \in A \right] \right) \leq - \inf_{\zeta \in A} \tilde{L}(\zeta),
$$

$$
\liminf_{t \to \infty} \frac{1}{t} \log \left( \inf_{x \in M} \mathbb{P}_x \left[ \frac{1}{t} X_t \in A \right] \right) \geq - \inf_{\zeta \in A^\circ} \tilde{L}(\tilde{\zeta}).
$$

Here the rate function $\tilde{L}$ is given by the following:

$$
\tilde{L}(\zeta) = \sup_{\|\zeta\|_p = 1} \inf_{f \in \mathcal{A}} \frac{1}{2} \left( \frac{\langle \zeta, \alpha \rangle_p^2}{\sigma^2(\alpha, f)} + \int_M |d f|^2 dv \right),
$$

where we set

$$
\mathcal{A} = \left\{ f \in C^2(M); \quad f > 0, \quad \int_M f^2 dv = 1 \right\},
$$

$$
\sigma^2(\alpha, f) = \inf_{\psi \in C^2(M)} \int_M |\alpha + d\psi|^2 f^2 dv.
$$

Theorem 2.5 implies that the law of $g(\lambda)^{-1} \lambda^{-1/2} X_\lambda$ on $\mathcal{D}_{1, -p}$ under $\lambda \to \infty$ also satisfies the large deviation principle if $\sqrt{\lambda}/g(\lambda) \to \infty$ holds. By (3.6), it is governed by the rate function $L'$ which takes finite value $L'(\tilde{\zeta}) = \|\tilde{\zeta}\|_{L^2(v)}^2 / 2$ only when $\tilde{\zeta} \in \text{Range}(1 - Q^*) \cap L^2_1(v)$. Theorem 4.1 treats the case $g(\lambda) = \sqrt{\lambda}$, and it is an interesting question whether these rate functions $L'$ and $\tilde{L}$ coincide or not. We can easily show that $\tilde{L}(\tilde{\zeta}) = L'(\tilde{\zeta}) = \infty$ if $\tilde{\zeta} \notin \text{Range}(1 - Q^*)$. Indeed, there is $\alpha \in \text{Range}(Q)$ with $\langle \tilde{\zeta}, \alpha \rangle_p > 0$ in this case. Since $\sigma(\alpha, f) = 0$ for all $f \in \mathcal{A}$, $\tilde{L}(\tilde{\zeta}) = \infty$ holds. Note that, for any $\alpha \in \mathcal{D}_{1, \infty}$, $Q\alpha$ coincides with the exact component of the Hodge–Kodaira orthogonal decomposition (see Warner, 1983). Another relation the author knows is the domination $\tilde{L}(\tilde{\zeta}) \leq L'(\tilde{\zeta})$. Actually, pick $\tilde{\zeta} \in L^2_1(v)$ with $Q^*\tilde{\zeta} = 0$. If we take $f \equiv 1$, then

$$
\frac{1}{2} \left( \frac{\langle \tilde{\zeta}, \alpha \rangle_p^2}{\sigma^2(\alpha, f)} + \int_M |d f|^2 dv \right) = \frac{\langle \tilde{\zeta}, \alpha \rangle_p^2}{2\|\alpha\|_{L^2_1(v)}^2} = \frac{\langle \tilde{\zeta}, (1 - Q)\alpha \rangle_{L^2_1(v)}^2}{2\|\alpha\|_{L^2_1(v)}^2} \leq \frac{1}{2} \|\zeta\|_{L^2_1(v)}^2.
$$
4.2. The law of the iterated logarithm

Recall that our result is considered to be a generalization of Baldi (1991). As in Baldi (1991), we can prove the law of the iterated logarithm in our framework as an application of the sample path large deviation estimate.

**Theorem 4.2.** Let \( g(\lambda) = \sqrt{\log \log \lambda} \).

(i) For \( p > d \), the family \( \{\tilde{Y}_t^{\lambda}\}_{\lambda > 0} \) is almost surely relatively compact in \( \mathcal{C}_p \) and the limit set

\[
\left\{ w \in \mathcal{C}_p; \text{there exists } \{\lambda_n\}_{n \in \mathbb{N}} \text{ with } \lambda_n \to \infty \text{ such that } \lim_{n \to \infty} \tilde{Y}_t^{\lambda_n} = w \right\}
\]

almost surely coincides with \( \mathcal{K} \) given by \( \mathcal{K} := \{w \in \mathcal{C}_p; L(w) \leq 1\} \).

(ii) For \( p > d + 1 \), the family \( \{\tilde{X}_t^{\lambda}\}_{\lambda > 0} \) is almost surely relatively compact in \( \mathcal{C}_p \) and the limit set

\[
\left\{ w \in \mathcal{C}_p; \text{there exists } \{\lambda_n\}_{n \in \mathbb{N}} \text{ with } \lambda_n \to \infty \text{ such that } \lim_{n \to \infty} \tilde{X}_t^{\lambda_n} = w \right\}
\]

almost surely coincides with \( \hat{\mathcal{K}} \) given by \( \hat{\mathcal{K}} := \{w \in \mathcal{C}_p; \hat{L}(w) \leq 1\} \).

We shall give only the outline of the proof of the first assertion here. The second one is similarly proved. Taking care for the fact that \( L \) is good rate function, we can prove the following proposition in the same way as in Baldi (1991).

**Proposition 4.3.** For every \( \varepsilon > 0 \) and \( T > 0 \), there exists a positive real number \( \lambda_0 \) almost surely such that for any \( \lambda > \lambda_0 \), we have

\[
\inf_{w \in \mathcal{K}} \sup_{0 \leq t \leq T} \| \tilde{Y}_t^{\lambda} - w \|_{-p} \leq \varepsilon.
\]

Thus it is proved that the limit set of \( \tilde{Y}_t^{\lambda} \) is contained in \( \mathcal{K} \). It is enough to prove the following proposition for the inverse inclusion.

**Proposition 4.4.** Take an arbitrary \( w \in \mathcal{K} \) with \( L(w) < 1 \). Then for any \( \varepsilon > 0 \), there exists \( c > 1 \) such that

\[
\mathbb{P} \left[ \limsup_{n \to \infty} \left\{ \sup_{0 \leq t \leq T} \| \tilde{Y}_t^{c^n} - w \|_{-p} \leq \varepsilon \right\} \right] = 1.
\]

For the proof, we use the uniformity of the large deviation and the fact that \( Y \) is an additive functional of \( \{z_t\}_{t \geq 0} \) instead of the Markov property used in Baldi (1991).

**Proof of Proposition 4.4.** By a version of the Borel–Cantelli lemma, it suffices to show that

\[
\sum_{n=1}^{\infty} \mathbb{P} \left[ \sup_{0 \leq t \leq T} \| \tilde{Y}_t^{c^n} - w \|_{-p} \leq \varepsilon \right] = \mathcal{F}_{c^{n-1}T} = \infty,
\]

where \( \mathcal{F}_t = \sigma \{z_s; 0 \leq s \leq t\} \).
Now we have
\[
\left\{ \sup_{0 \leq t \leq T} \| \tilde{Y}_t^{c^n} - w_t \|_p \leq \varepsilon \right\} = \left\{ \sup_{0 \leq t \leq c^{-1}T} \| \tilde{Y}_t^{c^n} - w_t \|_p \leq \varepsilon \right\} \cap \left\{ \sup_{c^{-1}T \leq t \leq T} \| \tilde{Y}_t^{c^n} - w_t \|_p \leq \varepsilon \right\}
\]
\[
\sup_{0 \leq t \leq c^{-1}T} \| \tilde{Y}_t^{c^n} - w_t \|_p \leq \frac{\varepsilon}{2}
\]
\[
\sup_{c^{-1}T \leq t \leq T} \left\{ \| (\tilde{Y}_t^{c^n} - \tilde{Y}_{c^{-1}T}^{c^n}) - (w_t - w_{c^{-1}T}) \|_p \leq \frac{\varepsilon}{2} \right\}
\]
\[
= A_{1}^{(n)} \cap A_{2}^{(n)}.
\]
Then the Markov property of \( z_t \) implies
\[
P \left[ \sup_{0 \leq t \leq T} \| \tilde{Y}_t^{c^n} - w_t \|_p \leq \frac{\varepsilon}{2}, \tilde{Y}_{c^{-1}T}^{c^n} \in E_c \right] \geq A_{1}^{(n)} P_{z_{n-1}T} [ \tilde{Y}_{c^n}^{c^n} \in E_c ],
\]
where
\[
E_c = \left\{ \eta \in \mathcal{G}_p; \sup_{0 \leq t \leq (1-1^{-1})T} \| \eta_t - (w_{t+c^{-1}T} - w_{c^{-1}T}) \|_p \leq \frac{\varepsilon}{2} \right\}.
\]
Then there is a constant \( C_3 \) so that
\[
\| w_t \|_p^2 \leq C_3 \| w_t \|_{L^2_t(m)}^2 \leq C_3 \left\{ \int_0^t \| w'_s \|^2_{L^2_t(m)} ds \right\}^2 \leq 2C_3 tL(w) \leq 2C_3 t,
\]
and Proposition 4.3 implies that for sufficiently large \( n \) we have
\[
\sup_{0 \leq t \leq c^{-1}T} \| \tilde{Y}_t^{c^n} \|_p \leq \frac{\sqrt{e^n-1}g(c^n-1)}{\sqrt{e^n}g(c^n)} \sup_{0 \leq t \leq T} \| \tilde{Y}_t^{c^n-1} \|_p
\]
\[
\leq \frac{1}{\sqrt{c}} \left( \sup_{0 \leq t \leq T} \| \varphi_t \|_p + 1 \right) \leq \frac{1}{\sqrt{c}} (2C_3 T + 1).
\]
Thus if we fix \( c > 0 \) large enough, the event \( A_{1}^{(n)} \) occurs for all sufficiently large \( n \).

On the other hand, there is \( \psi \in E_c \) so that \( L(\psi) < 1 \). Indeed, we can take \( \psi \) as follows:
\[
\psi_t = \begin{cases} w_{t+c^{-1}T} - w_{c^{-1}T}, & t \in [0, (1-c^{-1})T), \\ w_T - w_{c^{-1}T}, & t \in [(1-c^{-1})T, \infty). \end{cases}
\]
Hence Theorem 2.5 implies that there exists \( 0 < \beta < 1 \) so that
\[
P_{z_{n-1}T} [ E_c ] \geq \exp \{ -\beta g(c^n)^2 \} \quad (4.1)
\]
holds for sufficiently large $n$. Since the right-hand side of (4.1) is not summable, the conclusion follows. □

As an application, Theorem 4.2 refines Theorem 2.2 as follows.

**Theorem 4.5.** Let us define $\tilde{X}_t \in \mathcal{E}_p$ by $\tilde{X}_t = X_{\lambda t} - \lambda t e$. Then
\[
\lim_{t \to \infty} \frac{1}{h(\lambda)} \tilde{X}_t = 0 \quad \text{in } \mathcal{E}_p
\]
holds $\mathbb{P}_x$-almost surely for all $x \in M$ if $\lim_{\lambda \to \infty} h(\lambda) \sqrt{\lambda \log \log \lambda} = \infty$ holds. In particular,
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \tilde{X}_t = 0 \quad \text{in } \mathcal{E}_p
\]
holds $\mathbb{P}_x$-almost surely for all $x \in M$ if and only if $r > 1/2$.

### 4.3. Long-time asymptotics of the Brownian motion on Abelian covering manifolds

Let $N$ be a noncompact Riemannian covering manifold of $M$ with its covering transformation group $\Gamma$ being Abelian. In this subsection, we remark that our theorems are related to the long-time asymptotics of the Brownian motion on $N$.

For the purpose, we give some preparations following Kotani and Sunada (2000). Let $\pi$ be the canonical projection from $N$ to $M$. There is a surjection $\rho$ from $\pi_1(M)$, the fundamental group of $M$, to $\Gamma \cong \pi_1(M)/\pi_4(\pi_1(N))$. Since $\Gamma$ is Abelian, it induces the surjective mapping $\bar{\rho}$ from the first homology group $H_1(M; \mathbb{Z}) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)]$ to $\Gamma$. Here $[\pi_1(M), \pi_1(M)]$ is the commutator subgroup of $\pi_1(M)$. Moreover we obtain the extended map $\bar{\rho} : H_1(M; \mathbb{R}) \to \Gamma \otimes \mathbb{R}$ by taking tensor product with $\mathbb{R}$. For simplicity, we denote the extension $\tilde{\rho} : H_1(M; \mathbb{R}) \to \Gamma \otimes \mathbb{R}$ by the same symbol $\tilde{\rho}$. Then we obtain the adjoint injective mapping $\tilde{\rho}^\ast : H^1(\Gamma, \mathbb{R}) \to H^1(M; \mathbb{R})$. Note that the first cohomology group $H^1(M; \mathbb{R})$ is identified with the totality of harmonic 1-forms $H_1(M)$ (see Warner, 1983, for example). Since $H_1(M)$ is regarded as a subspace of $\mathcal{D}_{1,p}$, we can pull-back the norm on $\text{Hom}(\Gamma, \mathbb{R})$. Note that the induced topology on $\text{Hom}(\Gamma, \mathbb{R})$ coincides with what comes from $L^2_v(v)$ since $\|x\|_\rho = \|x\|_{L^2(v)}$ holds for each $x \in H_1(M)$. Then it decides the dual norm on $\Gamma \otimes \mathbb{R}$ that makes $\Gamma \otimes \mathbb{R}$ a normed space. For fixed $x_0 \in N$, we define the map $\varphi : N \to \Gamma \otimes \mathbb{R}$ by
\[
r_{\otimes \mathbb{R}}(\varphi(x), x)_{\text{Hom}(\Gamma, \mathbb{R})} = \int_{c} \pi^\ast(\tilde{\rho}^\ast(x))
\]
for each $x \in \text{Hom}(\Gamma, \mathbb{R})$. Here $\pi^\ast(\tilde{\rho}^\ast(x))$ is the pull-back of $\tilde{\rho}^\ast(x)$ by $\pi$ and $c$ is a piecewise smooth path from $x_0$ to $x$. Note that the line integral in (4.2) is independent of the choice of $c$.

According to Kotani and Sunada (2000), through the spectral-geometric approach, we know the following precise long time asymptotics of the heat kernel $p(t, x, y)$ associated with $\Delta/2$ on $N$:
\[
\lim_{t \to \infty} \{(2\pi t)^{d/2} p(t, x, y) - C(N) \exp\{-d_{\Gamma}(x, y)^2/(2t)\}\} = 0
\]
uniformly in $x$ and $y$. Here $r = \dim \Gamma$ and $C(N)$ is a constant determined explicitly in terms of $\Gamma$ and the Riemannian metric. $d_{\Gamma}$ is determined by $d_{\Gamma}(x, y) := |\varphi(x) - \varphi(y)|_{\Gamma \otimes \mathbb{R}}$. Roughly speaking, this asymptotic behavior indicates us that the heat kernel $p(t, x, y)$ approaches to the pull-back of that on $\Gamma \otimes \mathbb{R}$ by $\varphi$. Thus we guess that there is a connection between the long-time asymptotics of the heat kernel and the asymptotics of $\varphi(B_t)$. We would like to give a probabilistic approach to this problem and our main theorem gives some information about the asymptotics of $\varphi(B_t)$.

To verify it, we prove that $\Gamma \otimes \mathbb{R} \langle \varphi(B_t), z \rangle_{\text{Hom}(\Gamma, \mathbb{R})}$ coincides with the stochastic line integral of $z$ along $\pi(B_t)$ under $\tilde{\mathbb{P}}_{x_0}$. Note that $\pi(B_t)$ is a Brownian motion on $\Gamma$ starting at $\pi(x_0)$ under $\tilde{\mathbb{P}}_{x_0}$.

Fix $t > 0$. The approximation theorem of stochastic line integrals (Ikeda and Manabe, 1979, Theorem 6.1) guarantees the existence of the sequence of random paths $\{c^{(\ell)}\}_{\ell \in \mathbb{N}}$ which are all piecewise geodesics and $c^{(\ell)} = \pi(B_t)$ so that

$$
\lim_{\ell \to \infty} \mathbb{E}_{x_0}\left[ \sup_{0 \leq s \leq t} \left| \int_{c^{(\ell)}[0, s]} \tilde{\varphi}(s) - \int_{\pi(B)[0, s]} \tilde{\varphi}(s) \right|^2 \right] = 0,
$$

where $\int_{\pi(B)[0, t]} \tilde{\varphi}(s)$ is the stochastic line integral of $\tilde{\varphi}(s) \in \mathcal{H}_1(\Gamma) \otimes \mathbb{R}$ along $\pi(B)$.

Now we have

$$
\Gamma \otimes \mathbb{R} \langle \varphi(B_t), z \rangle_{\text{Hom}(\Gamma, \mathbb{R})} = \int_{c^{(\ell)}[0, t]} z
$$

for all $\ell$ by virtue of the independence of the choice of the path $c$ in (4.2). Thus we obtain

$$
\Gamma \otimes \mathbb{R} \langle \varphi(B_t), z \rangle_{\text{Hom}(\Gamma, \mathbb{R})} = \int_{\pi(B)[0, t]} z \quad t \geq 0, \; \text{a.s.}
$$

(4.3)

Let us define $\tilde{\varphi} : \mathcal{D}_{1, -p} \to \Gamma \otimes \mathbb{R}$ by

$$
\Gamma \otimes \mathbb{R} \langle \tilde{\varphi}(w), z \rangle_{\text{Hom}(\Gamma, \mathbb{R})} = \langle w, \tilde{\varphi}(z) \rangle_p.
$$

Then, for the current-valued process $X_t$ associated with $\pi(B_t)$, $\varphi(B_t) = \tilde{\varphi}(X_t)$ holds almost surely since the left-hand side of (4.3) is clearly continuous with respect to $z$. Moreover $\tilde{\varphi}(X_t) = \tilde{\varphi}(Y_t)$ holds by (2.2), the explicit form of the bounded variation part of $X_t$.

Note that $\tilde{\varphi}$ is continuous since so is $\tilde{\varphi}(z)$ as a map from Hom($\Gamma$, $\mathbb{R}$) to $\mathcal{D}_{1, p}$.

Accordingly we can apply our main theorem with the aid of the contraction principle and obtain the following estimate for $\varphi(B_t)$.

**Corollary 4.6.** We set $V_t^{\lambda} := g(\lambda)^{-1} \lambda^{-1/2} \varphi(B_{\lambda t})$. Assume $\sqrt{\lambda}/g(\lambda) \to \infty$ as $\lambda \to \infty$. Then we have

$$
\lim \sup_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \mathbb{P}_{x_0}[V_t^{\lambda} \in A] \leq - \inf_{w \in A} L_1(w),
$$

$$
\lim \inf_{\lambda \to \infty} \frac{1}{g(\lambda)^2} \log \mathbb{P}_{x_0}[V_t^{\lambda} \in A] \geq - \inf_{w \in d^*} L_1(w)
$$
for any Borel set $A \subset C([0, \infty) \to \Gamma \otimes \mathbb{R})$. Here

$$L_1(w) = \begin{cases} \frac{1}{2} \int_0^\infty |\dot{w}_s|_{F \otimes \mathbb{R}}^2 \, ds & \text{if } w_0 = 0 \text{ and } w \text{ is absolutely continuous,} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. All we need to prove is $\inf_{\tilde{\rho}(\eta) = w} \bar{L}(\eta) = L_1(w)$. It follows in the similar way as Remark 3.13. Indeed, let $P$ be the orthonormal projection on $L_1^2(v)$ to the range of $\tilde{\rho}$. Then for $w \in \Gamma \otimes \mathbb{R}$ and $\eta \in L_1^2(v)$ with $\tilde{\rho}(\eta) = w$, we have

$$\|\eta\|^2_{L_1^2(v)} = \|P\eta\|^2_{L_1^2(v)} + \|(1 - P)\eta\|^2_{L_1^2(v)}$$

and

$$\|P\eta\|^2_{L_1^2(v)} = \sup_{\|\eta\|_{L_1^2(v)} = 1} \eta, \eta \rangle_{L_1^2(v)} = \sup_{\|\eta\|_{\text{Hom}(\Gamma, \mathbb{R})} = 1} |\langle \tilde{\rho} \tilde{\rho} (\tilde{v}), \gamma \rangle_{\text{Hom}(\Gamma, \mathbb{R})}|.$$

Next, we will construct a element $\eta_0 \in \mathcal{D}_{1, -p} \cap L_1^2(v)$ with $(1 - P)\eta_0 = 0$ and $\tilde{\rho}(\eta_0) = w$. Let $\tilde{w} \in \text{Hom}(\Gamma, \mathbb{R})$ be a element corresponding to $w \in \Gamma \otimes \mathbb{R}$. We define $\tilde{\rho}(\eta_0)$ as follows:

$$\langle \eta_0, \tilde{w} \rangle = \int_M (\tilde{\rho} (\tilde{w}), \tilde{z} \rangle \, dv$$

for $z \in \mathcal{D}_{1, -p}$. Then $(1 - P)\eta_0 = 0$ clearly holds. In addition,

$$\tilde{G} \otimes \mathbb{R} \langle \tilde{\rho} (\eta_0), \gamma \rangle_{\text{Hom}(\Gamma, \mathbb{R})} = \int_M (\tilde{\rho} (\tilde{w}), \tilde{\rho} (\gamma)) \, dv = \langle \tilde{w}, \gamma \rangle_{\text{Hom}(\Gamma, \mathbb{R})}$$

Thus we obtain $\inf_{\tilde{\rho}(\eta) = w} \|\eta\|^2_{L_1^2(v)} = |w|_{L_1^2(v)}^2$. □

In the same manner as Section 4.2, we can prove the following law of the iterated logarithms as a corollary of Corollary 4.6.

Corollary 4.7. Let $g(\lambda) = \sqrt{\log \log \lambda}$. Then the limit set of $\{V^x_\lambda \}_{\lambda > 0}$ coincides with the compact set $\mathcal{K}_1$ given by

$$\mathcal{K}_1 := \{ w \in C([0, \infty) \to \Gamma \otimes \mathbb{R}) ; L_1(w) \leq 1 \}.$$

Note that there is a following relation between $d_{\Gamma}(\cdot, \cdot)$ and the Riemannian distance $\text{dist}(\cdot, \cdot)$ (see Kotani and Sunada, 2000): there are positive constants $c_1, c_2$ and $c_3$ such that for all $x, y \in N$

$$c_1 d_{\Gamma}(x, y) \leq \text{dist}(x, y) \leq c_2 d_{\Gamma}(x, y) + c_3. \quad (4.4)$$

Now Corollary 4.7 gives us some information about the divergence order of the Brownian motion on $N$. 
Corollary 4.8. There is a nonrandom constant $c$ with $c_1 < c < c_2$ so that
\[
\limsup_{t \to \infty} \frac{\text{dist}(B_t, B_0)}{\sqrt{2t \log \log t}} = c \] 
\[\mathbb{P}_{x_0}\text{-almost surely.}\]

Proof. Note that the invariant $\sigma$-field of the Brownian motion on $N$ is trivial. It is a consequence of Kaimanovich (1986). This fact ensures the existence of the constant $c$. By virtue of (4.4), it suffices to show
\[
\limsup_{t \to \infty} \frac{d_t(B_t, B_0)}{\sqrt{2t \log \log t}} = \limsup_{t \to \infty} \frac{\varphi(B_t)}{\sqrt{2t \log \log t}} = 1.
\]
But Corollary 4.6 asserts that
\[
\limsup_{t \to \infty} \frac{\varphi(B_t)}{\sqrt{2t \log \log t}} = \frac{1}{\sqrt{2}} \sup_{w \in \mathcal{F}} |w|_{\mathcal{P}_{\mathbb{R}}} \leq \sup_{w \in \mathcal{F}} \sqrt{L(w)} = 1.
\]
The inequality above comes from the Schwarz inequality and we can easily show that the equality actually holds. \qed

Remark 4.9. Equality (4.3) was also pointed out by Manabe (1982). He used this fact to investigate the asymptotics of the homological behavior of the diffusion processes on compact Riemannian manifolds.

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