On topological entropy of transitive triangular maps

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Abstract

In the paper of Alsedà, Kolyada, Llibre and Snoha [L. Alsedà, S.F. Kolyada, J. Llibre, L’. Snoha, Entropy and periodic points for transitive maps, Trans. Amer. Math. Soc. 351 (1999) 1551–1573] there was—among others—proved that a nonminimal continuous transitive map \( f \) of a compact metric space \((X, \rho)\) can be extended to a triangular map \( F \) on \( X \times I \) (i.e., \( f \) is the base for \( F \)) in such a way that \( F \) is transitive and has the same entropy as \( f \). The presented paper shows that under certain conditions the extension of minimal maps is guaranteed, too: Let \((X, f)\) be a solenoidal dynamical system. Then there exist a transitive triangular map \( F \) such that \( h(F) = h(f) \).

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0. Introduction

Let \((X, \rho)\) be a compact metric space. Let \( f : X \to X \) be a continuous map. We say that \( f \) is (topologically) transitive if for any two nonempty open sets \( U \) and \( V \) in \( X \), there is a nonnegative integer \( k \) such that \( f^k(U) \cap V \neq \emptyset \). We say that a subset \( M \) of \( X \) is a minimal set for a map \( f \) if it is nonempty, closed and invariant and if no proper subset of \( M \) has the same properties. By \( f^n \)-minimal set we mean a minimal set for the map \( f^n \). It is easily seen that if \( X \) itself is minimal then \( f \) is transitive.

Now we recall the Bowen’s definition [3] of topological entropy. A subset \( E \) of \( X \) is called \((n, \epsilon)\)-separated if for every two different points \( x, y \in E \) there exists \( 0 \leq j < n \) with \( \rho(f^j(x), f^j(y)) > \epsilon \). A set \( E_1 \subset X \) \((n, \epsilon)\)-spans another set \( K \subset X \) provided that for each \( x \in K \) there is \( y \in E_1 \) for which \( \rho(f^j(x), f^j(y)) \leq \epsilon \) for all \( 0 \leq j < n \). For a compact set \( K \subset X \) let \( s_n(\epsilon, K) \) be the maximal possible cardinality of an \((n, \epsilon)\)-separated set \( E \) contained in \( K \) and let \( r_n(\epsilon, K) \) be the minimal possible cardinality of a set \( E_1 \) which \((n, \epsilon)\)-spans \( K \) (we will write \( s_n(\epsilon, K, f) \) and \( r_n(\epsilon, K, f) \) if we wish to stress the dependence on \( f \)). Further, let

\[
s(\epsilon, K, f) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K, f)
\]

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Then the entropy of \( f \) on the set \( K \) is defined by
\[
h_\rho(f, K) = \lim_{s \to 0} s(\varepsilon, K, F)
\]
and the topological entropy of \( f \) by
\[
h(f) = h_\rho(f, X).
\]
Denote by \( \mathbb{N} \) the set of positive integers. A point \( x \in X \) is said to be recurrent if for every neighborhood \( V \) of \( x \), there is a sequence of positive integers \( k_1 < k_2 < \cdots \) such that \( f^{k_i} \in V \), for each \( i \in \mathbb{N} \). The set of all recurrent points for \( f \) is denoted by \( \text{Rec}(f) \). The point \( x \in X \) is said to be regularly recurrent if for every neighborhood \( V \) of \( x \), there is an \( n \in \mathbb{N} \) such that, for every nonnegative integer \( k \), \( f^{kn}(x) \in V \). Note that a minimal set containing regularly recurrent point can have positive topological entropy.

Next, we recall the notion of distributional chaos which was introduced by Schweizer and Smítal in 1994 [7]. For any pair \((x, y)\) of points in \( X \) and any \( n \in \mathbb{N} \), define a distribution function \( \Phi_{xy}^{(n)} : \mathbb{R} \to [0, 1] \) by
\[
\Phi_{xy}^{(n)}(t) = \frac{1}{n} \#\{0 \leq i \leq n - 1; \rho(f^i(x), f^i(y)) < t\}.
\]
Obviously, \( \Phi_{xy}^{(n)}(t) \) is a nondecreasing function, \( \Phi_{xy}^{(n)}(t) = 0 \) for \( t \leq 0 \) and \( \Phi_{xy}^{(n)}(t) = 1 \) for \( t \) greater than the diameter of \( X \). Put
\[
\Phi_{xy}(t) = \lim_{n \to \infty} \Phi_{xy}^{(n)}(t), \quad \text{and} \quad \Phi_{xy}^*(t) = \limsup_{n \to \infty} \Phi_{xy}^{(n)}(t).
\]
The function \( \Phi_{xy} \) is called the lower distribution, and \( \Phi_{xy}^* \) the upper distribution of \( x \) and \( y \). Obviously, \( \Phi_{xy}(t) \leq \Phi_{xy}^*(t) \) for any real \( t \). If \( \Phi_{xy}(t) < \Phi_{xy}^*(t) \) for all \( t \) in an interval, we simply write \( \Phi_{xy} < \Phi_{xy}^* \) and we say that \( f \) is distributionally chaotic.

Let \( M_0 \supseteq M_1 \supseteq \cdots \) be minimal subsets of \( X \) for the maps \( f^{m_0}, f^{m_1}, \ldots \), respectively. Obviously \( m_{i+1} \) is a multiple of \( m_i \) for all \( i \). If \( m_i \to \infty \) then any invariant closed set \( S \subset X = \bigcap_{j \geq 0} \text{Orb}(M_j) \) is called a solenoidal set; if \( X \) is nowhere dense then we call \( X \) a solenoid.

Let \( X \) be a metric space. Let \( C \) denote the class of continuous maps \( f : X \to X \), and \( \Delta \) the class of triangular maps \( F : X \times I \to X \times I \), i.e., the continuous functions defined by
\[
F(x, y) = (f(x), g(x, y)) = (f(x), g_x(y)).
\]
The map \( f \in C \) is called the base for \( F \), and \( g_x : X \times I \to I \) is a family of continuous maps depending continuously on \( x \). Note that \( F \) transforms the layer \( I_x := \{x\} \times I \) into the layer \( I_{f(x)} \).

Triangular maps have much simpler dynamics than continuous maps of the square in general [6]. This is because the projection \( \pi_1 : (x, y) \mapsto x \) semiconjugates any \( F \in \Delta \) to its base \( f \), i.e., \( f \circ \pi_1 = \pi_1 \circ F \). Recall that, for example, the projection \( \pi_1 \) maps the class \( \text{Per}(F) \) of periodic points of \( F \) onto \( \text{Per}(f) \), and if \( M \) is a minimal set for \( F \in \Delta \), then \( \pi_1(M) \) is a minimal set for \( f \). But, on the other hand, a big difference between the dynamics of maps in \( C \) and in \( \Delta \) already appears in the simplest cases in which every periodic point of \( F \) is a fixed point and the base is linear, cf. [4,6]. Note also that if the layer maps \( g_x \) are monotone (i.e., their topological entropy is zero) then entropy of the triangular map \( F \) equals entropy of the base map \( f \).

The Main Theorem of this paper (Theorem 2.1) shows that a continuous transitive map \( f \) of an infinite minimal compact Hausdorff space \( X \) containing a regularly recurrent point (i.e., \( X \) is a solenoidal set—see Theorem 1.3) can be extended to a triangular map \( F \) on \( X \times I \) (i.e., \( f \) is the base for \( F \)) in such a way that \( F \) has the following two properties: (a) \( F \) is transitive; (b) \( F \) has the same topological entropy as \( f \). This result, among others, proves the validity of a theorem from [11] to some minimal spaces; there was proved that on a compact metric space \( (X, \rho) \) which is not minimal, a continuous transitive map \( f \) can be extended to a triangular map \( F \) in such a way that \( F \) remains transitive and its entropy does not increase (i.e., is the same as the entropy of \( f \)). The last result of the presented paper (Theorem 2.2) shows that \( F \) can be constructed in such a way that \( F \) is distributionally chaotic.
1. Solenoidal sets in compact Hausdorff spaces

In this section we introduce some results concerning solenoidal sets in compact Hausdorff spaces.

Lemma 1.1. Let \( X \) be a minimal compact Hausdorff space with respect to some \( f \in C(X, X) \), and let \( X \) contain more than one point. Let \( \alpha \in X \) be a regularly recurrent point for \( f \). Then for some positive integer \( n \), \( \omega_{f^n}(\alpha) \) is a proper minimal subset of \( X \) for \( f^n \).

Proof. Let \( \alpha \in X \) be a regularly recurrent point for \( f \), let \( \beta = f(\alpha) \neq \alpha \). (The fact that \( \beta \neq \alpha \) follows immediately from the minimality of \( X \).) Take \( V_\alpha, V_\beta \) disjoint compact neighborhoods of \( \alpha \) and \( \beta \), respectively, and such that \( f(V_\alpha) \subset V_\beta \). Since \( \alpha \) is regularly recurrent, there exists a positive integer \( n \) such that \( f^{kn}(\alpha) \in V_\alpha \) for each positive integer \( k \).

So, \( M := \omega_{f^n}(\alpha) \) is a minimal set for \( f^n \) since \( \alpha \) is regularly recurrent for \( f^n \), and it is a proper subset of \( X \) since \( V_\alpha \cap V_\beta = \emptyset \). \( \square \)

In the proof of the next theorem we use an important result concerning maps on minimal sets:

Lemma 1.2. [2, Lemma 2.1] Let \( X \) be a compact Hausdorff space, let \( f \in C(X, X) \). Suppose that \( X \) is a minimal set for \( f \). Let \( n \) be a positive integer. Then for some positive integer \( t \leq n \) and some \( f^n \)-minimal set \( M \), we have each of the following:

1. \( X \) is the disjoint union of \( M, f(M), \ldots, f^{t-1}(M) \).
2. Each of the sets \( M, f(M), \ldots, f^{t-1}(M) \) is clopen.
3. The collection \( \{ M, f(M), \ldots, f^{t-1}(M) \} \) is the collection of all subsets of \( X \) which are \( f^n \)-minimal.

Also, for each \( x \in X \), the closure of the \( f^n \)-orbit of \( x \) is an \( f^n \)-minimal set.

Theorem 1.3. Let \( X \) be a minimal compact Hausdorff space with respect to some \( f \in C(X, X) \), let \( X \) contain more than one point, and let \( \alpha \in X \) be a regularly recurrent point for \( f \). Then \( X \) is a solenoidal set.

Proof. Let \( M_1 \subset X \) be a proper minimal set for \( f^n \), by Lemma 1.1. By Lemma 1.2 there is \( t_1 \leq n \) such that \( X \) is a disjoint union of \( M_1, f(M_1), \ldots, f^{t_1-1}(M_1) \).

Now, we can proceed by induction: Applying Lemma 1.1 to the minimal set \( M_1 \) and the map \( f^{t_1} \) we can find an increasing sequence \( \{t_k\}_{k=0}^\infty \) of positive integers and a decreasing sequence of compact sets \( \{ M_k \}_{k=0}^\infty \) such that \( t_0 = 1, M_0 = X, \) and \( M_k = \omega_{f^{t_k}}(\alpha) \) is an \( f^k \)-minimal set such that \( M_k, f^{t_k-1}(M_k), \ldots, (f^{t_k-1})^{t_k-2}(M_k) \) is a periodic decomposition of \( X \) into minimal sets with \( f^{t_k-1-t_k}(M_k) = M_k \). \( \square \)

2. A parametric class of triangular maps

Throughout this section we assume that \( X \) is an infinite compact Hausdorff space, and \( f \) is a minimal, and hence, transitive continuous map of \( X \), and that \( X \) contains a regularly recurrent point \( \alpha \). By Theorem 1.3 there is a sequence \( \{ p_i \} \) of positive integers such that \( X \) can be decomposed to \( p_1 \) clopen subsets on the first “level”, then to \( p_1 \cdot p_2 \) clopen subsets on the second one, etc.; let

\[
\underline{X} = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_i} \times \cdots = \prod_{i=1}^\infty Z_{p_i}.
\]

So, we may identify any element \( x \in X \) with the corresponding sequence \( x = x_1x_2 \ldots \in \underline{X}, x_i \in \{0, \ldots, p_i - 1\} \), for any \( i \in \mathbb{N} \).

Let us consider the metric space \( (\underline{X}, \rho) \), where \( \rho(\underline{a}, \underline{b}) = \max\{1/i; a_i \neq b_i\} \) for any distinct \( \underline{a} = a_1a_2 \ldots, \underline{b} = b_1b_2 \ldots \) in \( \underline{X} \), and \( \rho(\underline{a}, \underline{b}) = 0 \) otherwise. Let \( \lambda = \{ p_1, p_2, p_3, \ldots \} \) be a sequence of integers greater than \( 1 \) which characterizes the solenoidal set \( X \) generated by a regularly recurrent point \( \alpha \), cf. Theorem 1.3.
Assume that the function \( f : X \to X \) acts on \( X \) as an adding machine with the base \( \lambda \), i.e., for \( a \in X \), \( f(a) \equiv b = a + 1000 \ldots \) where the adding is in a changing base given by the sequence \( \lambda \) from the left to right, i.e., \( b_1 = a_1 + 1 \) if \( a_1 + 1 < p_1 \), otherwise \( b_1 = 0 \) and we carry 1 to the next position. The terms \( b_2, b_3, \ldots \) are successively determined in the same fashion—we only have to change the base \( p_i \) in each step (e.g., for \( \lambda = \{2, 3, 5, 2, 6, \ldots \} \) we have \( f(11304 \ldots) = 02304 \ldots, f(12400 \ldots) = 00010 \ldots \)).

It is easy to see that \( \omega_f(x) = X \) for any \( x \in X \).

Now, we can start with the definition of a class of triangular maps which will be needed in the proof of our main theorem. This construction was inspired by a similar and simpler one from [5].

Denote by \( T \) the class of maps \( F : X \times I \to X \times I \), where \( X \) is a solenoidal set and \( F(x, y) = (f(x), g(x, y)) \), where \( f : X \to X \) is the adding machine with the base \( \lambda \), \( g(x, \cdot) : X \times I \to I \) is continuous and nondecreasing for any \( x \in X \) and the family \( g(x, \cdot) \) depends continuously on \( x \) with respect to the uniform metric. Thus \( F \) is continuous on \( X \times I \).

Note that each map \( F \in T \) (and obviously also its monotonic extension \( \tilde{F} \in \Delta \)) has topological entropy \( h(F) = h(f) \). Indeed, we have (see [6])

\[
\sup \{ h(F, I_x); \ x \in X \} + h(f) \geq h(F),
\]

where \( h(F, I_x) \) denotes the topological entropy of the map \( F : X \times I \to X \times I \) with respect to the compact subset \( I_x \), i.e., the entropy \( h(F, I_x) \) is computed only for trajectories starting from \( I_x \). But since \( F^i \) is monotonic on \( I_x \) for any \( i \), we have clearly \( h(F, I_x) = 0 \). Thus, \( h(F) = h(f) \).

Now, we introduce a special subclass \( T_1 \) of \( T \). Let \( \{k_i\}_{i=1}^{\infty} \) be an increasing sequence of positive integers with \( k_i - i \to \infty \). For any \( x = x_1 x_2 \ldots x_i \ldots \) in \( X \), the digits \( x_{k_1}, x_{k_2}, \ldots \) are called control digits of \( x \). Let \( \{\varphi_i\}_{i=1}^{\infty} \) be a sequence of mappings from \( I \) into \( I \) of the form

\[
\varphi_i(t) = t^{k_i}, \quad \text{with } s_i > 0, \quad \lim_{i \to \infty} s_i = 1.
\]

We define a function \( F : X \times I \to X \times I \) as follows:

If the first zero control digit of \( x \) is \( x_{k_a} \),

\[
F(x, y) = (f(x), \varphi_{k_a}(y));
\]

otherwise \( F(x, y) = (f(x), y) \). The condition \( \lim_{i \to \infty} s_i = 1 \) assures the continuity of \( F \). Moreover, it is easy to recognize that \( F \) is a homeomorphism of \( X \times I \) into itself.

**Theorem 2.1** (Main Theorem). Let \( (X, f) \) be a solenoidal dynamical system. Then there exist a transitive triangular map \( F \) with \( h(F) = h(f) \).

**Proof.** We show that \( F \in T_1 \). Since the functions \( \varphi_i \) commute, the value \( F^n(Q, y_0) = : (f^n(Q), y_m) \) depends only on the number of times any function \( \varphi_i \) is applied. Given an \( r \in \mathbb{N} \), take \( n \) so that \( k_n \leq r < k_{n+1} \), and denote by \( \pi(r) = p_1 \cdot p_2 \cdot \ldots \cdot p_r \). Then the points \( f^i(Q), 0 \leq i < \pi(r) \) are represented by all the \( \pi(r) \) sequences

\[
a_1 \ldots a_0 0 \ldots, \quad a_i \in \{0, \ldots, p_i - 1\}, \quad 1 \leq i \leq r
\]

which have the \((n + 1)\)th control digit equal zero and so the only functions that may enter in the expression of \( y_i, 1 \leq i \leq \pi(r) \), are \( \varphi_1, \ldots, \varphi_{n+1} \). The number of times the function \( \varphi_i, 1 \leq i \leq n + 1 \), enters the expression of \( y_{\pi(r)} \) equals the number \( N(r, i) \) of sequences \( a_1 \ldots a_0 0 \ldots \) having \( a_{k_i} = 0 \) and \( a_{k_x} \neq 0 \) for all \( 1 \leq s < i \). It is easy to see that \( N(r, i) = \pi(r) \cdot v(n, i) \) where

\[
v(n, i) = \frac{p_{k_1} - 1}{p_{k_1}} \cdot \frac{p_{k_2} - 1}{p_{k_2}} \cdot \ldots \cdot \frac{p_{k_i - 1} - 1}{p_{k_i - 1}} \cdot \frac{1}{p_{k_i}} \quad \text{if } 1 \leq i \leq n,
\]

and

\[
v(n, n + 1) = \frac{p_{k_1} - 1}{p_{k_1}} \cdot \frac{p_{k_2} - 1}{p_{k_2}} \cdot \ldots \cdot \frac{p_{k_n} - 1}{p_{k_n}}.
\]

So we have

\[
y_{\pi(k_n)} = \varphi_1^{\pi(k_n)v(n, 1)} \circ \varphi_2^{\pi(k_n)v(n, 2)} \circ \ldots \circ \varphi_{n+1}^{\pi(k_n)v(n, n+1)}(y_0).
\]
Since
\[ f^{\pi(k_n)}(0) = 0 \ldots 0 100 \ldots \]
for the next \( \pi(k_n) \) iterations we use exactly the same functions as starting from 0. We may proceed in this way until the \( k_{n+1} \)th digit stays zero. This means that we can repeat this process \( \pi(k_{n+1} - 1)/\pi(k_n) \) times. Thus, for any \( m \) with \( 0 \leq m \leq \pi(k_{n+1} - 1)/\pi(k_n) \),
\[ y_{m\pi(k_n)} = \varphi_1^{m\pi(k_n)v(n,1)} \circ \varphi_2^{m\pi(k_n)v(n,2)} \circ \cdots \circ \varphi_{m\pi(k_n)v(n,n+1)}(y_0). \]

In order to construct the function \( F \) we start by imposing on the sequence \( \{s_i\} \) the additional condition
\[ s_{2i-1}s_{2i-1}^{-1} = 1, \quad \text{for } i \geq 1, \]
or equivalently,
\[ \varphi_{2i-1}^{s_{2i-1}} \circ \varphi_{2i}^{-s_{2i-1}} = \text{Id}, \quad \text{for } i \geq 1. \]

Note that in (8) we have powers of \( s_{2i-1} \) and \( s_{2i} \) (with exponents \( p_{2i} \) and \( p_{2i-1} - 1 \), respectively) while in (9) iterates of functions \( \varphi_{2i-1} \) and \( \varphi_{2i} \), with the same exponents. By (3)–(7) this implies
\[ \varphi_{2i-1}^{m\pi(r)v(l,2i-1)} \circ \varphi_{2i}^{m\pi(r)v(l,2i)} = \text{Id}, \quad \text{if } r \geq k_{2i}, \ l \geq 2i, \ i \geq 1. \]

Consequently, by (7),
\[ y_{m\pi(k_{2n})} = \varphi_{2n+1}^{m\pi(k_{2n})v(2n,2n+1)}(y_0), \quad \text{i.e.,} \]
\[ y_{m\pi(k_{2n})} = y_0^{(s_{2n+1})m\pi(k_{2n})v(2n,2n+1)}, \quad 0 \leq m \leq \frac{\pi(k_{2n+1} - 1)}{\pi(k_{2n})}. \]

We want to show that it is possible to choose the sequence of parameters \( \{s_n\} \) satisfying (8) such that
\[ \omega_F((0, y_0)) \supseteq I_0, \quad \text{for any } y_0 \in (0, 1). \]

Since the \( \omega \)-limit sets are strongly \( F \)-invariant, and \( F \) maps any fibre \( I_x \) onto the fibre \( I_{f(x)} \) (13) implies \( \omega_F((0, y_0)) = X \times I \). Since \( \lim_{n \to \infty} f^{\pi(k_n)}(0) = 0 \) to prove (13) it is enough to assure that the values given by (12), with \( m = 1 \), are dense in \( I \) and this is equivalent to requiring that
\[ \pi(k_{2n})v(2n, 2n + 1) \log(s_{2n+1}) \quad \text{is dense in } (-\infty, \infty). \]

Thus, we are looking for a sequence \( \{s_i\}_{i=1}^\infty \) satisfying (8), (14), and the last condition from (1), i.e.,
\[ \lim_{i \to \infty} s_i = 1. \]

To satisfy (14) we define the sequence \( \{s_{2n+1}\}_{n=1}^\infty \) by
\[ \log(s_{2n+1}) = \frac{\sigma_n}{\pi(k_{2n})v(2n, 2n + 1)} \]
where \( \{\sigma_n\}_{n=1}^\infty \) is a sequence dense in \( (-\infty, \infty) \) satisfying
\[ \left| \frac{\sigma_n}{\pi(k_{2n})v(2n, 2n + 1)} \right| < \frac{1}{n}. \]

Note that we can always meet this condition provided the sequence \( \{k_i\}_{i=1}^\infty \) increases sufficiently rapidly. Thus, there is a sequence \( \{\eta_n\}_{n=1}^\infty \) such that (17) is satisfied provided
\[ k_{2n} \geq \eta_n, \quad n = 1, 2, \ldots. \]

Now if the sequence \( \{s_{2n+1}\}_{n=1}^\infty \) satisfies (16) and (17), we have \( \lim_{n \to \infty} s_{2n+1} = 1 \), and the sequence \( \{s_{2n}\}_{n=1}^\infty \) which is uniquely determined by \( \{s_{2n+1}\}_{n=1}^\infty \) via (8), also converges to 1 and consequently, (15) is satisfied. So (i) is proved provided the sequence \( \{k_{2n}\} \) satisfies condition (18). \( \Box \)
Theorem 2.2. Let \((X, f)\) be the solenoidal dynamical system, let \(F\) be the map from the Main Theorem. Then for any \(u \in \{0\} \times (0, 1)\) and \(v = (0, 0)\) or \(v = (0, 1)\),
\[
\Phi^*_{uv}(t) = 1, \quad \Phi_{uv}(t) = 0, \quad t \in (0, 1);
\]
hence \(F\) is distributionally chaotic.

Proof. In this proof we use the notation from the proof of the Main Theorem. In order to assure (2) we take a sequence \(\{\sigma_n\}\), dense in \((-\infty, \infty)\) and such that
\[
\sigma_n < 0 \quad \text{for } n \text{ odd, and } \sigma_n > 0 \quad \text{for } n \text{ even}
\]
and we show how to define recursively the sequence \(\{k_n\}\) in order to meet condition (18).

We start the recursive process by taking \(k_1\) arbitrarily, \(s_1 = s_2 = 1\) and \(k_2 > k_1\) satisfying (18). Assume now we have constructed \(k_i\), for \(i \leq 2n\), so that (18) is satisfied.

If \(n\) is even take \(\rho_n \in (0, 1/2n)\) such that
\[
y_j < \frac{1}{2n} \quad \text{if } y_0 \leq \rho_n \text{ and } 0 < j < \pi(k_{2n}).
\]
This is possible since only a finite number of continuous functions \(\varphi_i\) enter in the expression of \(y_j\) and for them both points 0 and 1 are fixed. Since \(n\) is even, \(\sigma_n > 0\), by (19). Hence, by (16), \(s_{2n+1}^{\pi(k_{2n})} < 1\) and consequently, there is a positive integer \(b(n)\) such that
\[
\varphi_{2n+1}^{b(n)\pi(k_{2n})}(y_0) = y_{0+1}^{s_{2n+1}^{\pi(k_{2n})}} < \rho_n, \quad \text{for } y_0 = 1 - (1/2n).
\]
Similarly, if \(n\) is odd we take \(\rho_n \in (0, 1/2n)\) such that
\[
y_j > 1 - \frac{1}{2n} \quad \text{if } y_0 \geq 1 - \rho_n \text{ and } 0 \leq j < \pi(k_{2n}).
\]
Since \(\sigma_n < 0\) there is a positive integer \(b(n)\) such that
\[
\varphi_{2n+1}^{b(n)\pi(k_{2n})}(y_0) > 1 - \rho_n, \quad \text{for } y_0 = 1/2n.
\]
Now, we choose \(k_{2n+1}\) such that
\[
\frac{b(n)\pi(k_{2n})}{\pi(k_{2n+1} - 1)} < \frac{1}{2n}
\]
and \(k_{2n+2} > k_{2n+1}\) satisfying (18).

Now we show that the function \(F\) constructed in this way satisfies (2). Take \(y_0 \in (0, 1)\) and \(n_0\) such that \(y_0 \in [1/2n_0, 1 - (1/2n_0)]\). Let \(n\) be an even integer greater than \(n_0\). For every \(r\), \(b(n)\pi(k_{2n}) \leq r \leq \pi(k_{2n+1} - 1)\), we can write \(r = m\pi(k_{2n}) + j\) with \(b(n) \leq m \leq \pi(k_{2n+1} - 1)/\pi(k_{2n})\) and \(0 \leq j < \pi(k_{2n})\). So, by (12), (20) and (21)
\[
y_m\pi(k_{2n}) \leq y_{b(n)\pi(k_{2n})} < \rho_n \quad \text{and} \quad y_j < \frac{1}{2n}.
\]
Thus
\[
\# \begin{Bmatrix}
0 \leq i < \pi(k_{2n+1} - 1) \text{ and } y_i < \frac{1}{2n}
\end{Bmatrix} \geq \pi(k_{2n+1} - 1) - b(n)\pi(k_{2n}),
\]
and so, by (24)
\[
\Phi_{uv}^{\pi(k_{2n+1})} \left( \frac{1}{2n} \right) \geq 1 - \frac{b(n)\pi(k_{2n})}{\pi(k_{2n+1} - 1)} > 1 - \frac{1}{2n}.
\]
Hence we conclude that \(\Phi_{uv}^*(t) = 1\) for \(t \in (0, 1)\). Similarly, if we take \(n\) odd, we get
\[
\Phi_{uv}^{\pi(k_{2n+1})} \left( 1 - \frac{1}{2n} \right) \leq \frac{1}{2n},
\]
and so \(\Phi_{uv}(t) = 0\) for \(t \in (0, 1)\). \(\Box\)
References