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Chebyshev wavelets approach for nonlinear systems of Volterra integral equations

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ABSTRACT

In this paper, a new approach for solving nonlinear systems of Volterra integral equations has been proposed. The method is based on Chebyshev wavelets approximations. The method is described and after that the error is analyzed. At the end, some examples are presented to illustrate the ability and simplify of the method and the results reveal the effectiveness of the technique.

1. Introduction

Orthogonal functions and polynomials have been used by many authors for solving various functional equations. The main idea of using an orthogonal basis is that the problem under study reduces to a system of linear or nonlinear algebraic equations. This can be done by truncated series of orthogonal basis functions for the solution of problem and using the operational matrices. In this paper, Chebyshev wavelets basis, on the interval [0, 1], have been used. There are many applications of the Chebyshev wavelet method in the literature [1–3]. An extension of Chebyshev wavelets method for solving nonlinear systems of Volterra integral equations [4–6], is the novelty of this paper.

Mathematical modeling of many phenomena in different disciplines leads to a system of Volterra integral equations. So the solutions of these systems are of great interest for mathematicians and engineers. Systems of Volterra integral equations have been solved by some methods, the Adomian decomposition method [7,8], Homotopy perturbation method [9,10], Variational iteration method [11], Adomian–Pade technique [12], Runge–Kutta method [13], radial basis function networks [14] and block by block method [15]. The general form of these systems can be presented as follows

(a) The first kind

$$\sum_{j=1}^{m}\int_0^x k_{i,j}(x,t)G_{ij}(u_1(t),u_2(t),\ldots,u_n(t))dt = f_i(x), \quad 0\le x\le 1,\ i=1,2,\ldots,n,\ m=1,2,\ldots. \tag{1}$$

(b) The second kind

$$u_i(x) = f_i(x) + \sum_{j=1}^{m}\int_0^x k_{ij}(x,t)G_{ij}(u_1(t),u_2(t),\ldots,u_n(t))dt, \quad 0\le x\le 1,\ i=1,2,\ldots,n,\ m=1,2,\ldots. \tag{2}$$

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where $k_{ij}(x, t) \in L^2([0, 1] \times [0, 1])$ are the kernels, $f_i(x),\ i = 1, 2, \dots, n$, are known functions, G_{ij} are linear or non-linear vector functions of n unknown real functions $u_1(t), \dots, u_n(t)$.

2. Wavelets and Chebyshev wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet, [16–18]. When the dilation parameter, a and the translation parameter, b, vary continuously we have the following family of continuous wavelets as

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\, \psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R},\ a \neq 0. \tag{3}$$

If we take the dilation and translation parameters a^{-k}, and nba^{-k}, respectively where $a > 1$, $b > 0$, n, and k are positive integers, then we have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{\frac{k}{2}}\, \psi(a^k x - nb). \tag{4}$$

These functions are a wavelet basis for $L^2(\mathbb{R})$ and in special case $a = 2$, and $b = 1$, the functions $\psi_{k,n}(x)$ are an orthonormal basis.

Chebyshev wavelets $\psi_{n,m}(x) = \psi(k, n, m, x)$ have four arguments, $n = 1, 2, \dots, 2^{k-1}$, k is an arbitrary positive integer and m is the order of Chebyshev polynomials of the first kind. They are defined on the interval [0, 1], as follows:

$$\psi_{nm}(x) = \psi(k, n, m, x) = \begin{cases} 2^{\frac{k}{2}}\tilde{T}_m(2^k x - 2n + 1), & \dfrac{n-1}{2^{k-1}} \leq x < \dfrac{n}{2^{k-1}}, \\ 0, \quad \text{otherwise} \end{cases} \tag{5}$$

where

$$\tilde{T}_m(x) = \begin{cases} \dfrac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\dfrac{2}{\pi}}\, T_m(x), & m > 0. \end{cases} \tag{6}$$

and $m = 0, 1, \dots, M-1$ and $n = 1, 2, \dots, 2^{k-1}$. $T_m(x)$ are the famous Chebyshev polynomials of the first kind of degree m, which are orthogonal with respect to the weight function $W(x) = \frac{1}{\sqrt{1-x^2}}$, on the interval $[-1, 1]$, and satisfy the following recursive formula:

$$\begin{cases} T_0(x) = 1, \\ T_1(x) = x, \\ T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), \quad m = 1, 2, \dots. \end{cases} \tag{7}$$

The set of Chebyshev wavelets is an orthogonal set with respect to the weight function $W_n(x) = W(2^k x - 2n + 1)$. A function $f(x)$ defined on the interval [0, 1] may be presented as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x). \tag{8}$$

The series representation of $f(x)$ in (7) is called a wavelet series and the wavelet coefficients c_{nm} are given by $c_{nm} = (f(x), \psi_{nm}(x))_{W_n(x)}$. The convergence of the series (8), in $L^2[0, 1]$, means that

$$\lim_{s_1, s_2 \to \infty} \|f(x) - \sum_{n=1}^{s_1} \sum_{m=0}^{s_2} c_{nm}\psi_{nm}(x)\| = 0. \tag{9}$$

Therefore one can consider the following truncated series for series (8)

$$f(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = C^T \psi(x), \tag{10}$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C &= \left[c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}\right]^T \\ &= \left[c_1, c_2, \dots, c_M, c_{M+1}, \dots, c_{2^{k-1}M}\right]^T, \end{aligned} \tag{11}$$

and

$$\begin{aligned}\psi(x) &= \left[\psi_{10}(x), \psi_{11}(x), \ldots, \psi_{1,M-1}(x), \psi_{20}(x), \psi_{21}(x),\right.\\ &= \left.\ldots, \psi_{2,M-1}(x), \ldots, \psi_{2^{k-1}0}(x), \ldots, \psi_{2^{k-1},M-1}(x)\right]^T\\ &= \left[\psi_1(x), \psi_2(x), \ldots, \psi_M(x), \psi_{M+1}(x), \ldots, \psi_{2^{k-1}M}(x)\right]^T. \end{aligned} \tag{12}$$

The integration of the product of two Chebyshev wavelets vector functions with respect to the weight function $W_n(x)$, is derived as

$$\int_0^1 W_n(x)\psi(x)\psi^T(x)dx = I, \tag{13}$$

where I is an identity matrix.

A function $f(x, y)$ defined on $[0, 1] \times [0, 1]$ can be approximated as the following

$$f(x, y) \simeq \psi^T(x)K\psi(y). \tag{14}$$

Here the entries of matrix $K = [k_{ij}]_{2^{k-1}M\times 2^{k-1}M}$ will be obtain by

$$k_{i,j} = (\psi_i(x), (f(x, y), \psi_j(y))_{W_n(y)})_{W_n(x)}, \quad i, j = 1, 2, \ldots, 2^{k-1}M. \tag{15}$$

The integration of the vector $\psi(x)$, defined in (12), can be achieved as

$$\int_0^x \psi(t)dt = P\psi(x) \tag{16}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of integration [1,2]. This matrix is determined as follows.

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F\\ O & L & F & \ddots & \vdots\\ O & O & L & \ddots & F\\ \vdots & \ddots & \ddots & \ddots & F\\ O & \cdots & O & O & L \end{bmatrix}, \tag{17}$$

where L, F and O are $M \times M$ matrices given by

$$L = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \cdots & 0\\ -\frac{\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots & 0\\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & & & \vdots\\ \frac{\sqrt{2}}{2}(-1)^r\left(\frac{1}{r-2}-\frac{1}{r}\right) & \cdots & -\frac{1}{2(r-2)} & 0 & \frac{1}{2r} & \cdots & 0\\ \vdots & & \ddots & \ddots & \ddots & & \vdots\\ \frac{\sqrt{2}}{2}(-1)^M\left(\frac{1}{M-2}-\frac{1}{M}\right) & 0 & 0 & 0 & \cdots & -\frac{1}{2(M-2)} & 0 \end{bmatrix}, \tag{18}$$

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ -\frac{2\sqrt{2}}{3} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^r}{r}-\frac{1-(-1)^{r-2}}{r-2}\right) & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^M}{M}-\frac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0 \end{bmatrix}, \tag{19}$$

$$O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{20}$$

The property of the product of two Chebyshev wavelets vector functions will be as follows

$$\psi(x)\psi^T(x)Y \simeq \tilde{Y}\psi(x), \tag{21}$$

where Y is a given vector and $\tilde{Y}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix. This matrix is called the operational matrix of product.

3. Solution of systems of Volterra integral equations via Chebyshev wavelets method

Consider the systems of Volterra integral equations (1) and (2). Let's consider the following approximations for unknown functions $u_i, (x), \ i = 1, 2, \dots, n$.

$$u_i(x) \simeq C_i^T \psi(x), \quad i = 1, 2, \dots, n \tag{22}$$

where $C_i, \ i = 1, 2, \dots, n$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C_i &= \left[c^i_{10}, , c^i_{11}, \dots, c^i_{1M-1}, c^i_{20}, c^i_{21}, \dots, c^i_{2M-1}, \dots, c^i_{2^{k-1}0}, \dots, c^i_{2^{k-1}M-1}\right]^T \\ &= \left[c_{i,1}, c_{i,2}, \dots, c_{i,M}, c_{i,M+1}, \dots, c_{i,2^{k-1}M}\right]^T, \end{aligned} \tag{23}$$

and $\psi(x)$ is defined in (12). Also consider the following approximations

$$\begin{aligned} &f_i(x) \simeq F_i^T \psi(x), \qquad G_{i,j}(u_1(t), u_2(t), \dots, u_n(t)) \simeq Y_{ij}^T \psi(t), \\ &k_{ij}(x, t) \simeq \psi^T(x) K_{ij} \psi(t), \quad i = 1, 2, \dots, n, \ j = 1, 2, \dots, m. \end{aligned} \tag{24}$$

where K_{ij} are the $2^{k-1}M \times 2^{k-1}M$ matrices, F_i are the $2^{k-1}M \times 1$ matrices, and Y_{ij} are column vectors with the entries of the vectors C_i for $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Substitution of approximations (22) and (24) into the systems (1) and (2), will be resulted to:

$$\begin{aligned} F_i^T \psi(x) &= \sum_{j=1}^{m} \int_0^x \psi^T(x) K_{ij} \psi(t) \psi^T(t) Y_{ij}, dt, \\ &= \sum_{j=1}^{m} \psi^T(x) K_{ij} \left(\int_0^x \psi(t) \psi^T(t) Y_{ij}, dt \right), \\ &= \sum_{j=1}^{m} \psi^T(x) K_{ij} \tilde{Y}_{i,j} P \psi(x), \quad i = 1, 2, \dots, n, \ m = 1, 2, \dots \end{aligned} \tag{25}$$

and

$$\begin{aligned} C_i^T \psi(x) &= F_i^T \psi(x) + \sum_{j=1}^{m} \int_0^x \psi^T(x) K_{ij} \psi(t) \psi^T(t) Y_{ij} dt, \\ &= F_i^T \psi(x) + \sum_{j=1}^{m} \psi^T(x) K_{ij} \left(\int_0^x \psi(t) \psi^T(t) Y_{ij}, dt \right), \\ &= F_i^T \psi(x) + \sum_{j=1}^{m} \psi^T(x) K_{ij} \tilde{Y}_{i,j} P \psi(x), \quad i = 1, 2, \dots, n, \ m = 1, 2, \dots \end{aligned} \tag{26}$$

where $\tilde{Y}_{ij}$ are $2^{k-1}M \times 2^{k-1}M$ operational matrices for production and P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of integration [1–3].

According to the Galerkin method by multiplying $W_n(x)\psi^T(x)$, in both sides of the systems (25) and (26) and then applying $\int_0^1 (.)dx$, linear or non-linear systems in terms of the entries of $C_i, \ i = 1, 2, \dots, n$, will be obtained. The elements of vector functions $C_i, \ i = 1, 2, \dots, n$ can be computed by solving these systems.

4. Error analysis

Theorem 1. *Assume P be the number of vanishing moments for a wavelet $\psi_{nm}(x)$ and let $f(x) \in C^P[0, 1]$. Then the wavelet coefficient, c_{nm}, decays as follows*

$$|c_{nm}| \le C_P 2^{-n\left(P+\frac{1}{2}\right)} \underset{\xi\in[0,1]}{\mathrm{Max}} \left|f^{(p)}(\xi)\right|, \tag{27}$$

where C_P is an independent constant from n, m and $f(x)$.

The above theorem implies that wavelet coefficients are exponentially decayed with respect to P and by increasing P the decay increases.

Since the truncated Chebyshev wavelets series is approximate solution of a system, so one has an error function $error(f(x))$ for $f(x)$ as follows

$$error(f(x)) = \left|f(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right| \tag{28}$$

where setting $x = x_j$, $x_j \in [0, 1]$, the absolute error value of x_j can be obtained.

The error bound of the approximate solution by using Chebyshev wavelets series is given by the following theorem.

Theorem 2. *Suppose $f(x) \in C^P[0, 1]$ and $C^T\psi(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)$ is the approximate solution using Chebyshev wavelets method. Then the error bound would be obtained as follows*

$$\|error(f(x))\| \le \frac{1}{P!2^{P(k-1)}}\mathrm{Max}_{\xi\in[0,1]}|f^{(P)}(\xi)|. \tag{29}$$

Proof. Using the definition of norm in the inner product space, we have

$$\|error(f(x))\|^2 = \|f(x) - C^T\psi(x)\|^2 = \int_0^1 W(x)(f(x) - C^T\psi(x))^2 dx. \tag{30}$$

Because the interval $[0, 1]$ is divided into 2^{k-1} subintervals $I_n = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ that the function $f(x)$ is approximated on them by using Chebyshev wavelets method as a polynomial of the Pth degree at most with the least-square property, therefore would be as

$$\begin{aligned}\|error(f(x))\|^2 &= \int_0^1 W(x)(f(x) - C^T\psi(x))^2 dx = \sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)(f(x) - C^T\psi(x))^2 dx\\ &\le \sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)(f(x) - S_P(x))^2 dx,\end{aligned}$$

where $S_P(x)$ is any polynomial of degree P that interpolates $f(x)$ on I_n with the following error bound for interpolating

$$|f(x) - S_p(x)| \le \frac{1}{P!2^{P(k-1)}}\mathrm{Max}_{\xi_n\in I_n}|f^{(P)}(\xi_n)|. \tag{31}$$

Therefore, using (31) would be obtained

$$\begin{aligned}\|error(f(x))\|^2 &\le \sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\left(\frac{1}{P!2^{P(k-1)}}\mathrm{Max}_{\xi_n\in I_n}\left|f^{(P)}(\xi_n)\right|\right)^2 dx\\ &\le \sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} W_n(x)\left(\frac{1}{P!2^{P(k-1)}}\mathrm{Max}_{\xi\in[0,1]}\left|f^{(P)}(\xi)\right|\right)^2 dx\\ &= \int_0^1 \left(\frac{1}{P!2^{P(k-1)}}\mathrm{Max}_{\xi\in[0,1]}\left|f^{(P)}(\xi)\right|\right)^2 dx\\ &= \left\|\frac{1}{P!2^{P(k-1)}}\mathrm{Max}_{\xi\in[0,1]}\left|f^{(P)}(\xi)\right|\right\|^2. \quad\square\end{aligned}$$

5. Numerical examples

In this section, some examples of systems of Volterra integral equations are considered and will be solved by introduced method. These examples are solved for $k = 1$ and $M = 6$.

Table 1
Numerical results of Example 1.

x	$u(exact)$	$u(CWM)$	$error(u(x))$	$v(exact)$	$v(CWM)$	$error(v(x))$
0	0	−0.0000260703 5437	0.00002607035 437	0	0.0002222967327	0.0002222967327
0.2	0.04	0.03998703593	0.00001296407437	0.2	0.1999646030	0.0000353966836
0.4	0.16	0.1599826154	0.00001738456777	0.4	0.4000415581	0.0000415581197
0.6	0.36	0.3600478974	0.00004789734703	0.6	0.5999773361	0.0000226638703
0.8	0.64	0.6399664612	0.00003353875937	0.8	0.7999977346	0.0000022653373
1	1	1.000185856	0.0001858557196	1	0.999922349	0.0000776850473

Example 1. Consider the following nonlinear system of Volterra integral equations of the first kind with the exact solutions $u(x) = x^2$ and $v(x) = x$ [7,14].

$$\begin{cases} \int_0^x (1 - x^2 + t^2)(u(t) + v^3(t))dt = -\frac{1}{12}x^6 - \frac{2}{15}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3, \\ \int_0^x (5 + x - t)(u^3(t) - v(t))dt = \frac{1}{56}x^8 + \frac{5}{7}x^7 - \frac{1}{6}x^3 - \frac{5}{2}x^2, \quad 0 \le x \le 1. \end{cases} \tag{32}$$

Let's

$$\begin{aligned} &u(x) \simeq C_1^T \psi(x), \qquad v(x) \simeq C_2^T \psi(x), \\ &u^3(x) \simeq Y_1^T \psi(x), \qquad v^3(x) \simeq Y_2^T \psi(x), \\ &-\frac{1}{12}x^6 - \frac{2}{15}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 \simeq F_1^T \psi(x), \\ &\frac{1}{56}x^8 + \frac{5}{7}x^7 - \frac{1}{6}x^3 - \frac{5}{2}x^2 \simeq F_2^T \psi(x), \\ &(1 - x^2 + t^2) \simeq \psi^T(x) K_1 \psi(t), \\ &(5 + x - t) \simeq \psi^T(x) K_2 \psi(t). \end{aligned}$$

Substitution into the system (29), leads to the following system

$$\begin{cases} F_1^T \psi(x) = \psi^T(x) K_1 \int_0^x \psi(t)\psi^T(t)(C_1 + Y_2)dt = \psi^T(x) K_1 \tilde{Y}_1 P \psi(x), \\ F_2^T \psi(x) = \psi^T(x) K_2 \int_0^x \psi(t)\psi^T(t)(Y_1 - C_2)dt = \psi^T(x) K_2 \tilde{Y}_2 P \psi(x). \end{cases} \tag{33}$$

Multiply $W_n(x)\psi^T(x)$, on both sides of the system (33), apply $\int_0^1 (.)dx$, and then solve the system. The elements of vector functions C_1 and C_2 can be obtained as follows

$$\begin{aligned} C_1 &= [0.4700121513, 0.4431403268, 0.1108030029, 0.00002342689770, \\ &\quad 0.00003248339560, 0.00004361621557]^T, \\ C_2 &= [0.6266750242, 0.4430699839, 0.00002475338584, -0.00003950954832, \\ &\quad 0.00002662956111, -0.00004993764185]^T. \end{aligned}$$

Therefore, the following solutions will result.

$$\begin{aligned} u(x) \simeq C_1^T \psi(x) &= 0.02519840204x^5 - 0.05830434600x^4 + 0.04658408778x^3 + 0.9851318713x^2 \\ &\quad + 0.001601910954x - 0.00002607035437 \\ v(x) \simeq C_2^T \psi(x) &= -0.02885048049x^5 + 0.07597237616x^4 - 0.07222939205x^3 + 0.02971053030x^2 \\ &\quad + 0.9950969843x + 0.0002222967327. \end{aligned}$$

Table 1 shows some values of the solutions and absolute errors at some x's and plots of the exact and approximate solutions are shown in Fig. 1. Comparison between the obtained solutions by the Adomian decomposition method in [7] and the results of this paper show that the absolute error of the Chebyshev wavelets method is less than the absolute error of the Adomian decomposition method.

Example 2. Consider the following nonlinear system of Volterra integral equations of the second kind

$$\begin{cases} u(x) = \sin x - x + \int_0^x \left(u^2(t) + v^2(t)\right) dt, \\ v(x) = \cos x - \frac{1}{2}\sin^2 x + \int_0^x u(t)v(t)dt, \quad 0 \le x \le 1. \end{cases} \tag{34}$$

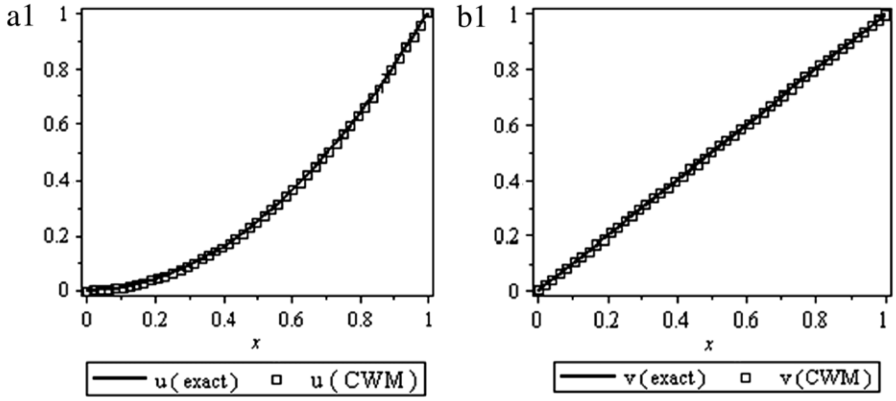

Fig. 1. (a1) and (b1) comparison of the exact and approximate solutions of Example 1.

Table 2
Numerical results of Example 2.

x	$u(exact)$	$u(CWM)$	$error(u(x))$	$v(exact)$	$v(CWM)$	$error(v(x))$
0	0	−0.000001403 703684	0.000001403703 684	1	1.000000319	0.000000319
0.2	0.1986693308	0.1986696393	0.0000003085	0.9800665778	0.9800669460	0.0000003682
0.4	0.3894183423	0.3894165787	0.00000017636	0.9210609940	0.9210604974	0.0000004966
0.6	0.5646424734	0.5646433964	0.000000923	0.8253356149	0.8253349025	0.0000007124
0.8	0.7173560909	0.7173557641	0.0000003268	0.6967067093	0.6967067839	0.0000000746
1	0.84144709848	0.8414719842	0.0000009994	0.5403023059	0.5403020838	0.0000002221

With the exact solutions $u(x) = \sin x$ and $v(x) = \cos x$ [10,15].

The vectors C_1 and C_2 are computed by solving the system of nonlinear equations for six unknowns, via the Maple package, as follows

$$\begin{aligned} C_1 &= [0.5638986783, 0.3768420409, -0.02600615260, -0.003987867721, \\ &\quad 0.0001365184253, 0.00001401577453]^T \\ C_2 &= [1.032209955, -0.2058701368, -0.04760382879, 0.002178573519, \\ &\quad 0.0002499015756, -0.6913472440 \times 10^{-5}]^T. \end{aligned}$$

Therefore, we have the following approximate solutions

$$\begin{aligned} u(x) &= 0.008097427401x^5 - 0.0005258577420x^4 - 0.1657167707x^3 - 0.0004456850651x^2 \\ &\quad + 1.000064274x - 0.1403703684 \times 10^{-5} \\ v(x) &= -0.003994087527x^5 + 0.04607914088x^4 - 0.002260687350x^3 - 0.4994799014x^2 \\ &\quad - 0.00004269977464 + 1.000000319. \end{aligned}$$

Table 2 shows some values of the solutions and absolute errors at some x's and plots of the exact and approximate solutions are shown in Fig. 2.

Example 3. Consider

$$\begin{cases} \displaystyle\int_0^x \left((5+x-t)u(t) + \left(\frac{x^2}{2}+t\right)v(t)w(t)\right)dt = \frac{1}{48}x^6 + \frac{19}{270}x^5 + \frac{19}{72}x^4 + \frac{7}{6}x^3 + x^2 + 5x, \\ \displaystyle\int_0^x \left((\frac{x^2}{2}+t)u(t) + (3+x-t)v(t) + \frac{1}{4}(x^2-t^2)w(t)\right)dt = \frac{1}{24}x^5 + \frac{35}{288}x^4 + \frac{17}{18}x^3 + \frac{5}{4}x^2 + \frac{9}{2}x, \\ \displaystyle\int_0^x \left(tu(t)v(t) - xtv^2(t) - 5w(t)\right)dt = -\frac{1}{54}x^7 + \frac{1}{72}x^6 - \frac{1}{4}x^5 + \frac{17}{96}x^4 - \frac{9}{8}x^3 - \frac{1}{2}x^2 - \frac{10}{3}x, \quad 0 \le x \le 1. \end{cases} \tag{35}$$

With the exact solution $u(x) = \frac{1}{4}x^2 + 1$, $v(x) = \frac{1}{3}x^2 + \frac{3}{2}$, and $w(x) = \frac{1}{2}x + \frac{2}{3}$, [9].

By applying the Chebyshev wavelets method and solving the resulted nonlinear system, the following results would be achieved.

$$\begin{aligned} C_1 &= [1.370909608, 0.1107744910, 0.02796071583, -0.3904883363 \times 10^{-5}, -0.3849125038 \times 10^{-5}, \\ &\quad -0.2901199063 \times 10^{-5}]^T, \end{aligned}$$

Table 3
Numerical results of Example 3.

x	$u(exact)$	$u(CWM)$	$error(u(x))$	$v(exact)$	$v(CWM)$	$error(v(x))$	$w(exact)$	$w(CWM)$	$error(w(x))$
0	1	1.000001158	0.000001158	1.5	1.499999358	0.000000642	0.6666666667	0.6666569385	0.0000097282
0.2	1.01	1.01000096	0.00000095997788	1.513333333	1.513332873	0.000000460547730	0.7666666667	0.766658447	0.000000821952071
0.4	1.04	1.039999975	0.00000002495013	1.553333333	1.553333413	0.00000007928579	0.8666666667	0.8666636607	0.0000030059964
0.6	1.09	1.089997698	0.00000230162855	1.62	1.620001073	0.00000107265340	0.9666666667	0.966673803	0.000007136243
0.8	1.16	1.160004458	0.00000445835934	1.713333333	1.713330858	0.00000247532844	1.066666667	1.066658534	0.000008132458
1	1.25	1.249977054	0.00002294593658	1.833333333	1.83346595	0.00001326126583	1.166666667	1.166710616	0.000043949641

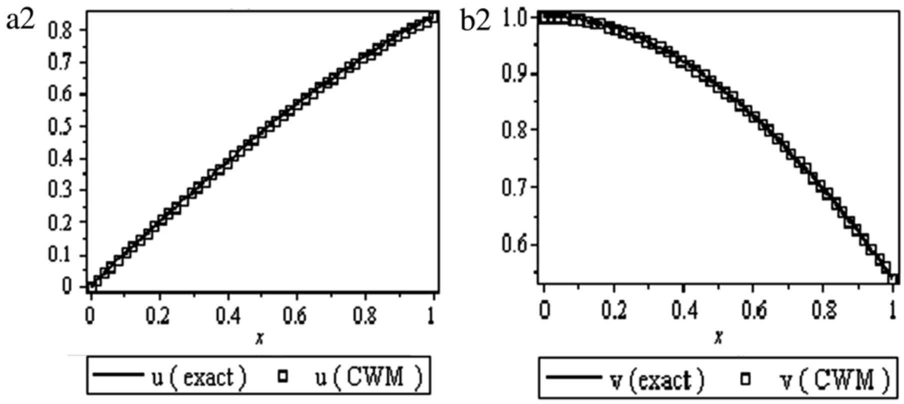

Fig. 2. (a2) and (b2) comparison of the exact and approximate solutions of Example 2.

$$C_2 = [2.036637098, 0.1477068045, 0.03692842611, 0.2315201546 \times 10^{-5}, 0.2139368296 \times 10^{-5}, 0.1528559747 \times 10^{-5}]^T,$$

$$C_3 = [1.148875490, 0.2215642788, 0.5944442817 \times 10^{-5}, 0.7599016308 \times 10^{-5}, 0.6251581812 \times 10^{-5}, 0.8638907550 \times 10^{-5}]^T.$$

Therefore, we have the following approximate solutions

$$u(x) \simeq C_1^T \psi(x) = -0.001676110122x^5 + 0.003634336424x^4 - 0.002695611180x^3 + 0.2507910494x^2 - 0.00007776845858x + 1.000001158,$$

$$v(x) \simeq C_2^T \psi(x) = 0.0008830950267x^5 - 0.001898743184x^4 + 0.001397379211x^3 + 0.3329250622x^2 + 0.00004044331183x + 1.499999358,$$

$$w(x) \simeq C_3^T \psi(x) = 0.004990957212x^5 - 0.01157446124x^4 + 0.009386241600x^3 - 0.003128439331x^2 + 0.5003793796x + 0.6666569385.$$

Some values of exact, approximate solutions and absolute errors are presented in Table 3 and the plots of exact and approximate solutions are shown in Fig. 3.

6. Conclusion

The aim of this paper is to develop Chebyshev wavelets method for obtaining the solutions of nonlinear systems of Volterra integral equations. Illustrative examples are included to demonstrate that the method is a very effective and useful technique for finding approximate solutions of these systems. In [9,10], examples (2) and (3) were solved by the Homotopy perturbations method and comparison between the obtained absolute error values in [9,10] and this paper shows that the absolute error values of the Chebyshev wavelets method are less than the absolute error values of the Homotopy perturbations method. Research for finding more applications of this method and other orthogonal basis functions is one of the goals of our research group. Here, the computations associated with these examples are performed by the package Maple 13.

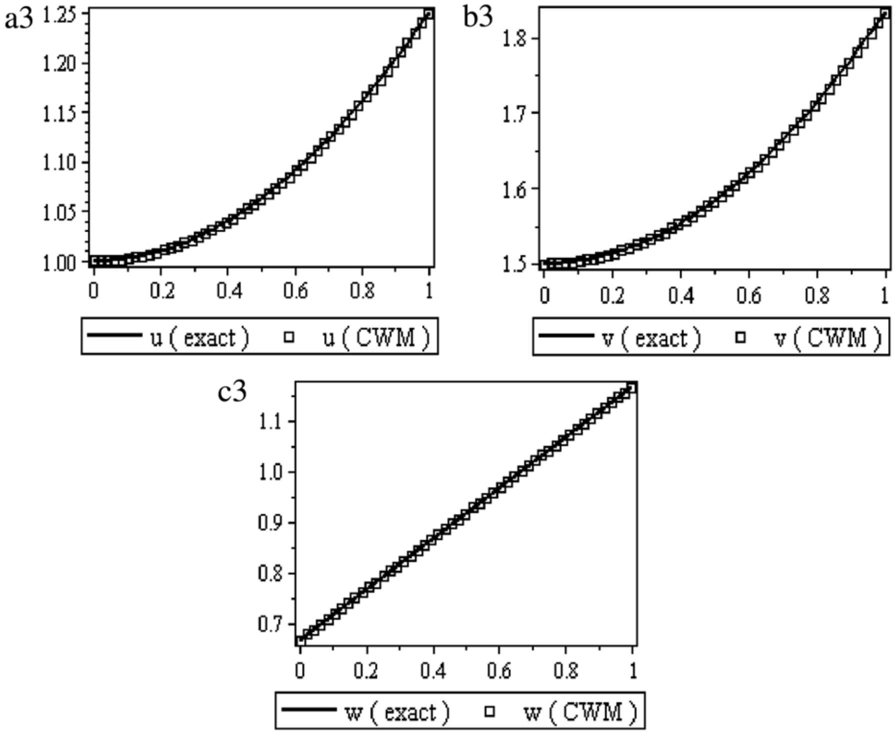

Fig. 3. (a3), (b3) and (c3) comparison of the exact and approximate solutions of Example 3.

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