Weak type estimates of square functions associated with quasiradial Bochner–Riesz means on certain Hardy spaces

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Abstract

Let $\varrho_d \in C^\infty (\mathbb{R}^n \setminus \{0\})$ be a non-radial homogeneous distance function of degree $d \in \mathbb{N}$ satisfying $\varrho_d(t\xi) = t^d \varrho_d(\xi)$. For $f \in \mathcal{S}(\mathbb{R}^n)$, we define square functions $\mathcal{G}_\delta f(x)$ associated with quasiradial Bochner–Riesz means $\mathfrak{R}_{\varrho_d,t} f$ of index $\delta$ by

$$
\mathcal{G}_\delta f(x) = \left( \int_0^\infty \left| \mathfrak{R}_{\varrho_d,t}^\delta f(x) - \mathfrak{R}_{\varrho_d,t} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}
$$

where $\mathfrak{R}_{\varrho_d,t}^\delta f(x) = \mathcal{F}^{-1}[(1 - \varrho_d / t^d)_{+}\hat{f}](x)$. If $\{\xi \in \mathbb{R}^n: \varrho_d(\xi) = 1\}$ is a smooth convex hypersurface of finite type, then we prove in an extremely easy way that $\mathcal{G}_\delta$ is well-defined on $H^p(\mathbb{R}^n)$ when $\delta = n(1/p - 1/2) - 1/2$ and $0 < p < 1$; moreover, it is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$. In addition, if $\varrho_d \in C^\infty (\mathbb{R}^n \setminus \{0\})$, then we also prove that $\mathcal{G}_\delta$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ when $\delta > n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

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1. Introduction

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space on $\mathbb{R}^n$. For $f \in \mathcal{S}(\mathbb{R}^n)$, we denote the Fourier transform of $f$ by

$$
\mathcal{F}[f](x) = \hat{f}(x) = \int_{\mathbb{R}^n} e^{-i(x,\xi)} f(\xi) \, d\xi.
$$

Then the inverse Fourier transform of $f$ is given by

$$
\mathcal{F}^{-1}[f](x) = \tilde{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x,\xi)} f(\xi) \, d\xi.
$$

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Let $M$ be a real-valued $n \times n$ matrix whose eigenvalues have positive real parts. Then we consider the dilation group $\{A_t\}_{t>0}$ in $\mathbb{R}^n$ generated by the infinitesimal generator $M$, where $A_t = \exp(M \log t)$ for $t > 0$. We introduce $A_t$-homogeneous distance functions $\varrho_d$ defined on $\mathbb{R}^n$ which is of degree $d \in \mathbb{N}$; that is, $\varrho_d : \mathbb{R}^n \to [0, \infty)$ is a continuous function satisfying $\varrho_d(A_t \xi) = t^d \varrho_d(\xi)$ for all $\xi \in \mathbb{R}^n$ and $t > 0$. If $A_t = tI$, then we call such $\varrho_d$ a homogeneous distance function of degree $d$. One can refer to [4] and [16] for its fundamental properties.

In the following we shall denote the unit sphere of $\varrho_d$ by $\Sigma_{\varrho_d} = \{ \xi \in \mathbb{R}^n : \varrho_d(\xi) = 1 \}$ and denote by $\mathbb{R}_d^n = \mathbb{R}^n \setminus \{0\}$.

We use the polar coordinates; given $x \in \mathbb{R}^n$, we write $x = r \theta$ where $r = |x|$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in S^{n-1}$. Given two quantities $A$ and $B$, we write $A \lesssim B$ or $B \gtrsim A$ if there is a constant $c$ (possibly depending on the dimension $n$ and the index $p$ to be given) such that $A \leq cB$. We also write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

In case that $\varrho$ is a non-radial homogeneous distance function of degree 1, Dappa and Trebels [4] proved that if $\varrho \in C^{n/2+1}(\mathbb{R}^n)$ is $A_1$-homogeneous distance function of degree 1, then $\varrho^\delta$ is bounded on $L^p(\mathbb{R}^n)$ for $p > 1$, and is of weak type $(1, 1)$. In case that $\varrho \in C^{\infty}(\mathbb{R}^n)$ is $A_{d}$-homogeneous distance function of degree 1 and the surface measure $d\sigma$ on $\Sigma_{\varrho}$ satisfies $d\sigma(\xi) = O(|\xi|^{-n})$ for $0 < n \leq (n-1)/2$, it was shown by Seeger [12] that if $\delta > n(1/2 - 1/p) - 1/2$, then $\varrho^\delta$ is bounded on $L^p(\mathbb{R}^n)$, which is a partial extension of Carbery’s result [2] in higher dimensions.

The purpose of this article is to obtain sharp weak type endpoint $(\delta(p) = n(1/p - 1/2) - 1/2)$ results of $\varrho_d^{\delta(p)}$ on $H^p(\mathbb{R}^n)$, $0 < p < 1$, under certain curvature condition on $\Sigma_{\varrho}$, where $\varrho_d^{\delta(p)}(\mathbb{R}^n)$ is a homogeneous distance function of degree $d \in \mathbb{N}$. That is to say, this result is to generalize that of [7] in $\mathbb{R}^n$ to non-radial cases (which is of finite type and convex) by using the arguments based on the result in [9]. Here $H^p(\mathbb{R}^n)$ denotes the standard real Hardy space as defined by E.M. Stein in [14].

In our first result we shall assume that $\varrho_d \in C^{\infty}(\mathbb{R}_d^n)$, $\varrho_d$ is a non-radial homogeneous distance function of degree $d \in \mathbb{N}$ and $\Sigma_{\varrho_d}$ is a smooth convex hypersurface of $\mathbb{R}^n$ which is of finite type, i.e. every tangent line makes finite order of contact with $\Sigma_{\varrho_d}$.

**Theorem 1.1.** Suppose that $\varrho_d \in C^{\infty}(\mathbb{R}_d^n)$ is a non-radial homogeneous distance function of degree $d \in \mathbb{N}$ and $\Sigma_{\varrho_d}$ is a smooth convex hypersurface of finite type. Then $\varrho_d^{\delta(p)}$ is well-defined on $H^p(\mathbb{R}^n)$ when $0 < p < 1$; moreover, $\varrho_d^{\delta(p)}$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^{p, \infty}(\mathbb{R}^n)$. That is, there is a constant $C = C(n, p, \Sigma_{\varrho_d}) > 0$ such that for any $f \in H^p(\mathbb{R}^n)$,

$$\left| \{ x \in \mathbb{R}^n : \varrho_d^{\delta(p)} f(x) > \lambda \} \right| \leq \frac{C}{\lambda^p} \| f \|_{H^p}, \quad \lambda > 0,$$

where $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}^n$. 

Theorem 1.2. The angular part $\Phi(\theta)$ This makes the problem more complicated and difficult than the radial case. But it turns out in Corollary 2.8 that the angular part $\Phi(\theta)$ of the decay of the kernel should be in $L^p(S^{n-1})$ for any $p$ with $0 < p < 1$. So it is possible to work it out in spite of badness of the kernel.

Remark. (i) As a matter of fact, we prove this result under more general surface condition than the finite type condition on $\Sigma_d$, which is to be called a spherically integrable condition of order $< 1$ in Section 2.

(ii) In case that $\rho_d$ is radial (in fact, $\rho_d(\xi) = \rho_d(1)|\xi|^d$ by the homogeneity condition), some properties of the Bessel functions could be used in order to estimate decay of the Bochner–Riesz kernel. However we cannot do that in case of non-radial $\rho_d$ and a convex hypersurface $\Sigma_{\rho_d}$ of finite type, which may cause some difficulty to obtain decay of the kernel. In fact, if $\xi(x)$ is the point of $\Sigma_{\rho_d}$ whose outer unit normal vector is in the direction $x$ and at which the Gaussian curvature vanishes, then the decay of the quasiradial Bochner–Riesz kernel is pretty bad in the direction $x$. This makes the problem more complicated and difficult than the radial case. But it turns out in Corollary 2.8 that the angular part $\Phi(\theta)$ of the decay of the kernel should be in $L^p(S^{n-1})$ for any $p$ with $0 < p < 1$. So it is possible to work it out in spite of badness of the kernel.

Our second result is to obtain that if $\delta > n(1/p - 1/2) - 1/2$ for $0 < p < 1$ then $\mathcal{G}_{\rho_d}$ admits $(H^p, L^p)$-estimate under no surface condition on $\Sigma_{\rho_d}$ where $\rho_d \in C^\infty(\mathbb{R}^n_0)$ is a non-radial homogeneous distance function of degree $d \in \mathbb{N}$.

Theorem 1.2. Suppose that $\rho_d \in C^\infty(\mathbb{R}^n_0)$ is a non-radial homogeneous distance function of degree $d \in \mathbb{N}$. If $\delta > \delta(p)$ for $0 < p < 1$, then $\mathcal{G}_{\rho_d}$ is a bounded operator from $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$; that is, there is a constant $C = C(n, p) > 0$ such that for any $f \in H^p(\mathbb{R}^n)$,

$$\|\mathcal{G}_{\rho_d} f \|_{L^p} \leq C \| f \|_{H^p},$$

provided that $\delta > n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

Remark. This problem is still left open on the critical index $\delta = n(1/p - 1/2) - 1/2$ and $0 < p < 1$.

The outline of the paper is as follows. In Section 2, we shall furnish preliminary estimates on smooth convex hypersurface $\Sigma_{\rho_d}$ of finite type and decay estimate for the quasiradial Bochner–Riesz type kernel $K_{\rho_d}^{(p)} = \mathcal{F}^{-1}[\rho_d(1 - \rho_d)^{\delta(p)}]$ whose proof is based on a certain decomposition of the multiplier $\rho_d(1 - \rho_d)^{\delta(p)}$ like that of the Bochner–Riesz multiplier $(1 - |\xi|^2)^{\delta(p)}$ used in Córdoba [3], and also give $(H^p, L^p, \infty)$-estimates of $\mathcal{G}_{\rho_d}$ for the case that $\rho_d \in C^\infty(\mathbb{R}^n_0)$ and $\delta = \delta(p)$, $0 < p < 1$. In Section 3, we shall obtain $(H^p, L^p)$-estimates of $\mathcal{G}_{\rho_d}$ for the case that $\rho_d \in C^\infty(\mathbb{R}^n_0)$ and $\delta > \delta(p)$, $0 < p < 1$.

2. $(H^p, L^p, \infty)$-estimate for the case that $\rho_d \in C^\infty(\mathbb{R}^n_0)$ and $\delta = \delta(p)$, $0 < p < 1$

In this section we shall focus on obtaining $(H^p, L^p, \infty)$-mapping properties of the square function $\mathcal{G}_{\rho_d}^{(p)}$ associated with the quasiradial Bochner–Riesz means $\mathcal{R}_{\rho_d}^{(p)}$ of index $\delta(p), 0 < p < 1$, under the condition that $\Sigma_{\rho_d}$ is a smooth convex hypersurface of finite type, where $\rho_d \in C^\infty(\mathbb{R}^n_0)$ is a non-radial homogeneous distance function of degree $d \in \mathbb{N}$.

Let $\Sigma$ be a smooth convex hypersurface of $\mathbb{R}^n$ and let $d\sigma$ be the induced surface area measure on $\Sigma$. Let $\mathcal{E}(\Sigma)$ be the set of points of $\Sigma$ at which the Gaussian curvature $\kappa$ vanishes, and let $\mathcal{N}(\Sigma) = \{n(\xi): \xi \in \mathcal{E}(\Sigma)\}$ where $n(\xi)$ denotes the outer unit normal to $\Sigma$ at $\xi \in \Sigma$. For $x \in \mathbb{R}^n$, denote by $d(x/|x|, \mathcal{N}(\Sigma))$ the geodesic distance on $S^{n-1}$ between $x/|x|$ and $\mathcal{N}(\Sigma)$, and by $\mathcal{B}(\xi(x), s)$ the spherical cap near $\xi(x) \in \Sigma$ cut off from $\Sigma$ by a plane parallel to $T_{\xi(x)}(\Sigma)$ (the affine tangent plane to $\Sigma$ at $\xi(x)$) at distance $s > 0$ from it; that is,

$$\mathcal{B}(\xi(x), s) = \{\xi \in \Sigma: \text{dist}(\xi, T_{\xi(x)}(\Sigma)) < s\},$$

where $\xi(x)$ is the point of $\Sigma$ whose outer unit normal vector is in the direction $x$ and $\text{dist}(\xi, T_{\xi(x)}(\Sigma))$ is the shortest distance between $\xi \in \Sigma$ and the tangent plane $T_{\xi(x)}(\Sigma)$. These spherical caps play an important role in furnishing the decay of the Fourier transform of the measure $d\sigma$. It is well known [10,14] that the function

$$\Phi(\theta) = \sup_{r > 0} \sigma(\mathcal{B}(\xi(r\theta), 1/r)) (1 + r)^{n-1} \quad (2.1)$$

is bounded on $S^{n-1}$ provided that $\Sigma$ has nonvanishing Gaussian curvature.
**Definition 2.1.** Let $Σ$ be a smooth convex hypersurface of $\mathbb{R}^n$. Then we say that $Σ$ satisfies a spherically integrable condition of order $< 1$ if $Φ ∈ L^p(S^{n-1})$ for any $p < 1$.

**Remark.** (i) B. Randol [10] proved that if $Σ$ is a real analytic convex hypersurface of $\mathbb{R}^n$ then $Φ ∈ L^p(S^{n-1})$ for some $p > 2$. Thus any real analytic convex hypersurface satisfies a spherically integrable condition of order $< 1$.

(ii) Let $Σ$ be a smooth convex hypersurface of finite type $k ≥ 2$ and suppose that $N(Σ)$ is a $m$-dimensional submanifold of $\mathbb{R}^n$ which is on $S^{n-1}$, where $m < [k(n-1)]/[2(k-1)]$. Then we see (refer to [8]) that $Σ$ satisfies a spherically integrable condition of order $< 1$. Moreover, it is not hard to see that $Σ$ satisfies a spherically integrable condition even for $m ≤ n-2$. We mention for reader that it can be shown by Lemma 2.8 [8] and the fact that $Σ$ is of finite type $P(k)$; i.e. there is some constant $C = C(Σ) > 0$ such that for any $θ ∈ S^{n-1}$,

$$Φ(θ) ≤ \frac{C}{d(θ, N(Σ))^{\frac{k-2}{m-1}(n-1)}}.$$  

Since $Σ$ is smooth and of finite type, it is absolutely impossible that $N(Σ)$ is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ which is on $S^{n-1}$.

(iii) More generally, it was shown by I. Svensson [18] that if $Σ$ is a smooth convex hypersurface of finite type $k ≥ 2$ then $Φ ∈ L^p(S^{n-1})$ for some $p > 2$.

Thus, by the above remark (iii), it is natural for us to obtain the following lemma.

**Lemma 2.2.** Any smooth convex hypersurface of finite type always satisfies a spherically integrable condition of order $< 1$.

Sharp decay estimates for the Fourier transform of surface measure on a smooth convex hypersurface $Σ$ of finite type $k ≥ 2$ has been obtained by Bruna, Nagel, and Wainger [1]; precisely speaking, $|\mathcal{F}[dσ](x)|$ is equivalent to $σ[\mathcal{B}(ξ(x), 1/|x|)]$. They define a family of anisotropic balls on $Σ$ by letting

$$\mathcal{B}(ξ_0, s) = \{ξ ∈ Σ: \text{dist}(ξ, Tξ_0(Σ)) < s\}$$

where $ξ_0 ∈ Σ$. We now recall some properties of the anisotropic balls $\mathcal{B}(ξ_0, s)$ associated with $Σ$. The proof of the doubling property in [1] makes it possible to obtain the following stronger estimate for the surface measure of these balls

$$σ[\mathcal{B}(ξ_0, γs)] ≤ \begin{cases} γ^{\frac{n+1}{n-1}} σ[\mathcal{B}(ξ_0, s)], & γ ≥ 1, \\ γ^{\frac{n+1}{n-1}} σ[\mathcal{B}(ξ_0, s)], & γ < 1. \end{cases} \quad (2.2)$$

It also follows from the triangle inequality and the doubling property [1] that there is a positive constant $C > 0$ independent of $s > 0$ such that

$$\frac{1}{C} σ[\mathcal{B}(ξ_0, s)] ≤ σ[\mathcal{B}(ξ, s)] ≤ C σ[\mathcal{B}(ξ_0, s)] \quad \text{for any } ξ ∈ \mathcal{B}(ξ_0, s). \quad (2.3)$$


**Lemma 2.3.** Let $0 < p < 1$. Suppose that $\{h_k\}$ is a sequence of measurable functions defined on $\mathbb{R}^n$ such that for all $k ∈ \mathbb{N}$,

$$\|h_k\|_{L^p,∞} ≤ 1.$$  

If $\{c_k\} ∈ ℓ^p$, then we have the following estimate

$$\left\| \sum_{k=1}^{∞} c_k h_k \right\|_{L^p,∞} ≤ \left(\frac{2 - p}{1 - p}\right)^{1/p} \|\{c_k\}\|_{ℓ^p}.$$
Lemma 2.4. (See [9].) Let $\Sigma$ be a smooth convex hypersurface of $\mathbb{R}^n$ which is of finite type $k \geq 2$. Then there is a constant $C = C(\Sigma) > 0$ such that for any $x \in B(0; s)$ and $y \in B(0; 2s)^\circ$, $0 < s \leq 1$,
\[ \xi(x - y) \in B(\xi(x), C/|x|) \]
where $\xi(x)$ is the point of $\Sigma$ whose outer unit normal is in the direction $x$.

Lemma 2.5. Let $\Sigma$ be a smooth convex hypersurface of $\mathbb{R}^n$ which is of finite type $k \geq 2$. Then there is a constant $C = C(\Sigma) > 0$ such that for any $x$, $y \in \mathbb{R}^n$ with $|x| > 2|y| > 0$,
\[ \Phi \left( \frac{x - y}{|x - y|} \right) \leq C \Phi \left( \frac{x}{|x|} \right) \]
where $\Phi$ is the function defined as in (2.1).

Proof. It easily follows from (2.3), the definition of $\Phi$, and Lemma 2.4 that for any $y \in B(0; s)$ and $x \in B(0; 2s)^\circ$, $0 < s \leq 1$,
\[
\Phi \left( \frac{x - y}{|x - y|} \right) \leq \sup_{r > 0} \left[ \mathcal{B}(\xi(x - y), 1/r) \right] (1 + r)^{\frac{d - 1}{2}} \\
\leq \sup_{r > 0} \left[ \mathcal{B}(\xi(x), 1/r) \right] (1 + r)^{\frac{n - 1}{2}} = \Phi \left( \frac{x}{|x|} \right).
\]
Thus this implies that for any $x$, $y \in \mathbb{R}^n$ with $|x| > 2|y| > 0$,
\[
\Phi \left( \frac{x - y}{|x - y|} \right) = \Phi \left( \frac{x/|y| - y/|y|}{|x/|y| - y/|y||} \right) \leq \Phi \left( \frac{x/|y|}{|x/|y||} \right) = \Phi \left( \frac{x}{|x|} \right). \quad \Box
\]

We shall employ a decomposition of the multiplier $\varrho_d(1 - \varrho_d)^{\delta(p)}$ like that of the Bochner–Riesz multiplier $(1 - |\xi|^2)^{\delta(p)}$ used in Córdoba [3], where $\varrho_d \in C^\infty(\mathbb{R}^n_0)$ is a non-radial homogeneous distance function of degree $d \in \mathbb{N}$. Let $\varphi \in C_c^\infty(1/2, 2)$ be a function satisfying $\sum_{k \in \mathbb{Z}} \varphi(2^k t) = 1$ for $t > 0$. For $k \in \mathbb{N}$, let
\[
\Phi_k^{\delta(p)}(\xi) = \varphi(2^{k+1}(1 - \varrho_d)\varrho_d(1 - \varrho_d)^{\delta(p)}
\]
and $\Phi_0^{\delta(p)}(\xi) = \varrho_d(1 - \varrho_d)^{\delta(p)} - \sum_{k \in \mathbb{N}} \Phi_k^{\delta(p)}$. Then we note $\sum_{k \in \mathbb{N}} \Phi_k^{\delta(p)} = \varphi \varrho_d(1 - \varrho_d)^{\delta(p)}$ a.e., where $\varphi \in C_c^\infty(\mathbb{R}^n)$ is a function supported in the closed annulus
\[
\{ \xi \in \mathbb{R}^n : 1/2 < \varrho_d(\xi) < 2 \}
\]
such that
\[
\varphi(\xi) = \sum_{k \in \mathbb{N}} \Phi_k^{\delta(p)}(1 - \varrho_d(\xi)) \quad (2.4)
\]
on the open annulus $\{ \xi \in \mathbb{R}^n : 1/2 < \varrho_d(\xi) < 1 \}$. We now introduce a partition of unity $\mathcal{E}_\ell$, $\ell = 1, 2, \ldots, L_0$, on the unit sphere $\Sigma_{\varrho_d}$, which we extend to $\mathbb{R}^n$ by way of $\Pi_\ell(t \xi) = \mathcal{E}_\ell(\xi)$, $t > 0$, $\xi \in \Sigma_{\varrho_d}$, and which satisfies the following properties; by compactness of $\Sigma_{\varrho_d}$, there are a sufficiently large finite number of points $\zeta_1, \zeta_2, \ldots, \zeta_{L_0} \in \Sigma_{\varrho_d}$ such that for $\ell = 1, 2, \ldots, L_0$,

(a) $\sum_{\ell=1}^{L_0} \Pi_\ell(\zeta) = 1$ for all $\zeta \in \Sigma_{\varrho_d}$,
(b) $\mathcal{E}_\ell(\zeta) = 1$ for all $\zeta \in \Sigma_{\varrho_d} \cap B(\zeta; 2^{-M_0/2})$,
(c) $\mathcal{E}_\ell$ is supported in $\Sigma_{\varrho_d} \cap B(\zeta; 2^{1-M_0/2})$,
(d) $|\mathcal{D}_\alpha \Pi_\ell(\zeta)| \lesssim 2^{k_0|M_0/2}$ for any multi-index $\alpha$, if $1/2 < \varrho_d(\zeta) < 2$,
(e) $L_0 \lesssim 2^{(n-1)M_0/2}$ for some sufficiently large fixed $M_0$ (to be chosen later),
where $B(\zeta_0; s)$ denotes the ball in $\mathbb{R}^n$ with center $\zeta_0 \in \Sigma_{\varrho_d}$ and radius $s > 0$. For each $\ell = 1, 2, \ldots, L_0$, let $K_{\varrho_d \ell}^{\delta(p)} = F^{-1}[\varphi \Pi_{\ell} \varrho_d(1 - \varrho_d + \delta(p)_{\ell})]$ and $K_{0}^{\delta(p)} = F^{-1}[\Phi_0^{\delta(p)}].$

Next we invoke a simple observation to obtain decay estimate for kernels $K_{\varrho_d \ell}^{\delta(p)}$, $K_{0}^{\delta(p)}$ corresponding to the decomposition of the Bochner–Riesz type multiplier defined above. Without loss of generality, we can assume that $\varrho_d \in C^\infty(\mathbb{R}^n)$ because we can replace $\varrho_d$ by $\varrho_d^{N_1}$ for sufficiently large $N_1 > 0$ by a subordination argument in [3]. Then we easily see that the kernel $K_{0}^{\delta(p)}$ has a nice decay, and so its corresponding square function admits $(H^p, L^{p, \infty})$-estimate for the critical index $\delta(p) = n(1/p - 1/2) - 1/2, 0 < p < 1$. Thus we concentrate upon obtaining the decay estimate for the kernel $K_{\varrho_d \ell}^{\delta(p)}$; in particular, in the first half of this section we shall show that the following summation of $N^{th}$ derivatives

$$\sum_{|\alpha|=N} \frac{1}{\alpha!} D^{\alpha} K_{\varrho_d \ell}^{\delta(p)}(x)$$

of the kernel $K_{\varrho_d \ell}^{\delta(p)}$ has the same decay as that of the kernel $K_{\varrho_d \ell}^{\delta(p)}$ with the constant not depending upon $N \in \mathbb{N}$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ is a multi-index, $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$, and

$$D^{\alpha} K_{\varrho_d \ell}^{\delta(p)}(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} K_{\varrho_d \ell}^{\delta(p)}(x).$$

We observe that for any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$,

$$D^{\alpha} K_{\varrho_d \ell}^{\delta(p)}(x) = F^{-1}\left[i^{(|\alpha|)} \xi^\alpha \varphi(\xi) \Pi_{\ell}(\xi) \varrho_d(\xi) (1 - \varrho_d(\xi))_{\ell}^{\delta(p)}(\xi)\right](x),$$

(2.5)

where we denote by $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$.

Now we introduce polar coordinates with respect to a homogeneous distance function $\varrho_d \in C^\infty(\mathbb{R}^n_0)$ as follows; the diffeomorphism

$$\mathbb{R}_+ \times \Sigma_{\varrho} \rightarrow \mathbb{R}^n_0, (\varrho, \xi) \mapsto \varrho \xi = \xi, \quad \xi \in \Sigma_{\varrho_d}$$

defines polar coordinates with respect to $\varrho$ by way of

$$d\xi = \varrho^{n-1}(\xi, n(\xi))d\varrho d\sigma(\xi),$$

where $d\sigma(\xi)$ denotes the surface area measure on $\Sigma_{\varrho_d}$ and $n(\xi)$ is the outer unit normal vector to $\Sigma_{\varrho_d}$ at $\xi \in \Sigma_{\varrho_d}$. Now fix $\xi_0 \in \Sigma_{\varrho_d}$. Then the unit sphere $\Sigma_{\varrho_d}$ can be parametrized near $\xi_0 \in \Sigma_{\varrho_d}$ by a map

$$w \mapsto P(w), \quad w \in \mathbb{R}^{n-1}, \quad |w| < 1,$$

such that $P(0) = \xi_0$. Then there is a neighborhood $U_0$ of $\xi_0$ with compact closure and a neighborhood $V_0$ of the origin in $\mathbb{R}^{n-1}$ so that the map

$$Q : (1/2, 3/2) \times V_0 \rightarrow U_0, \quad (\varrho, w) \mapsto Q(\varrho, w) = \varrho P(w)$$

(2.6)

is a diffeomorphism with $Q(1, 0) = \xi_0$. The Jacobian of $Q$ is given by

$$J(\varrho, w) = \varrho^{n-1}|P(w), n(P(w))|R(w),$$

where $R(w)$ is positive and

$$[R(w)]^2 = \det\left(\begin{bmatrix} \frac{d\varrho}{dw} \\ \frac{dP}{dw} \end{bmatrix}\right).$$

\textbf{Lemma 2.6.} Suppose that $\varrho_d \in C^\infty(\mathbb{R}^n_0)$ is a non-radial homogeneous distance function of degree $d \in \mathbb{N}$. Let $\xi_0, U_0, \varrho$ be as above. If $\delta(p) = n(1/p - 1/2) - 1/2$ for $0 < p < 1$, then for each $M \in \mathbb{N}$, there are $\eta = \eta(M) > 0$, $\mu = \mu(M) > 0$, a sufficiently large $M_0 > 0$ in (a)–(e), $\varepsilon_0 > 0$, a neighborhood $U_1$ of $\xi_0$ with $\text{supp}(\varphi \Pi_{\ell}) \subset \overline{U_1} \subset U_0$, and a neighborhood $V_1$ of the origin in $\mathbb{R}^{n-1}$ so that (2.6) holds and such that for any $\alpha \in (\mathbb{N} \cup \{0\})^n$,

$$|D^{\alpha} K_{\varrho_d \ell}^{\delta(p)}(x)| \leq \eta \mu^{\alpha}(1 + |x|)^{-M} \left|\left|\frac{x}{|x|}, n(\xi_0)\right|\right| \leq 1 - \varepsilon_0$$

(2.7)
and
\[
\mathcal{D}_\alpha^\delta K_{\partial \ell}^\delta (x) = \sum_{j=0}^{M-1} \mathcal{h}_j (x) + O\left(|x|^{-M}\right) \quad \text{if} \quad \left|\left< x, \nu (\xi, \ell) \right>\right| \geq 1 - \epsilon_0, \tag{2.8}
\]
where
\[
\mathcal{h}_j (x) = |x|^{-\delta (p) - 1 - j} \int_{\Sigma_{\delta d}} e^{i|x|^{\langle 1/|\eta|, \nu \rangle}} \mathcal{t}_j (\xi, x/|x|) \, d\sigma (\xi)
\]
and \(\mathcal{t}_j \in C_0^\infty (\mathcal{P}(\partial_0) \times S^{n-1})\) for \(j = 0, 1, 2, \ldots, M - 1\). In particular,
\[
\mathcal{t}_0 (\xi, x/|x|) = \Gamma (\delta (p) + 1) e^{-i\pi (\delta (p) + 1)/2} \kappa^\alpha \Pi \xi (\xi, n (\xi)) \left[\left< x, \xi, \ell \right>\right]^{-\delta (p) - 1}.
\]

**Proof.** Applying generalized polar coordinates that we introduced above, we have that
\[
\mathcal{D}_\alpha^\delta K_{\partial \ell}^\delta (x) = \int\int e^{i|x|^{\langle 1/|\eta|, \nu \rangle}} (1 - \epsilon)^{\delta (p) + 1} \kappa^\alpha \phi (\xi) \Pi \xi (\xi, n (\xi)) d\epsilon d\sigma (\xi)
\]
\[
= \int\int e^{i|x|^{\langle 1/|\eta|, \nu \rangle}} w^\alpha \Pi \xi (\xi, n (\xi)) \Pi \xi (\xi, n (\xi)) d\epsilon d\sigma (\xi)
\]
\[
\leq \mathcal{H}_\xi^\delta (x, \epsilon)^{n + |\alpha|} (1 - \epsilon)^{\delta (p)} \, d\epsilon.
\]

We note that if \(|\langle \theta, n (\xi) \rangle| < 1\), then we have that
\[
\nabla_w \theta (\xi, \mathcal{P}(w)) \bigg|_{w=0} \neq 0.
\]
Combining this with the homogeneity condition on the distance function \(\rho_d\) and choosing a sufficiently large \(M_0 > 0\) in (a)–(e), we may select \(\epsilon_0 > 0\), a neighborhood \(\mathcal{U}_1\) of \(\xi\) with \(\text{supp} (\phi \Pi \xi) \subset \mathcal{U}_1 \subset \mathcal{U}_0\), and a neighborhood \(\mathcal{V}_1\) of the origin in \(\mathbb{R}^{n-1}\) so that (2.6) holds, and such that for all \((w, \rho) \in \mathcal{V}_1 \times [1/2, 1],
\]
\[
\nabla_w \theta (\xi, \mathcal{P}(w)) \bigg|_{w=0} \neq 0.
\]
and
\[
c_1 \leq \left|\frac{\partial}{\partial \rho} \theta (\xi, \mathcal{P}(w))\right| \leq c_2 \quad \text{if} \quad \left|\langle \theta, n (\xi) \rangle\right| \geq 1 - \epsilon_0 \quad \tag{2.10}
\]
for some \(c_0 > 0, c_1 > 0,\) and \(c_2 > 0\). We choose some \(\epsilon \in S^{n-2}\) so that for all \((w, \rho) \in \mathcal{V}_1 \times [1/2, 1],
\]
\[
\left|\nabla_w \theta (\xi, \mathcal{P}(w))\right| \geq \frac{1}{2} \left|\nabla_w \theta (\xi, \mathcal{P}(w))\right| \geq \frac{1}{2} c_0 \quad \text{if} \quad \left|\langle \theta, n (\xi) \rangle\right| \leq 1 - \epsilon_0. \tag{2.11}
\]
If \(|\langle \theta, n (\xi) \rangle| \leq 1 - \epsilon_0\), we apply the integration of \(\mathcal{H}_\xi^\delta (x, \epsilon)\) by parts with respect to \(w\)-variable \(N\)-times to obtain that
\[
\mathcal{H}_\xi^\delta (x, \epsilon) = \int e^{i|x|^{\langle 1/|\eta|, \nu \rangle}} (\mathcal{D}_\epsilon^\nu \Pi \xi (\xi, n (\xi)) \Pi \xi (\xi, n (\xi)) d\epsilon d\sigma (\xi)
\]
\[
\leq \mathcal{H}_\xi^\delta (x, \epsilon)^{n + |\alpha|} (1 - \epsilon)^{\delta (p)} \, d\epsilon. \tag{2.12}
\]
where \(\mathcal{D}_\epsilon^\nu\) denotes the transpose of the differential operator
\[
\mathcal{D}_\epsilon^\nu g = \mathcal{D}_\epsilon^\nu [\mathcal{D}_\epsilon^\nu (\mathcal{P}(w)) \mathcal{D}_\epsilon^\nu (\mathcal{P}(w))] \mathcal{D}_\epsilon^\nu g; \quad \text{i.e.} \quad \mathcal{D}_\epsilon^\nu g = -\mathcal{D}_\epsilon^\nu \left(\frac{g}{\mathcal{D}_\epsilon^\nu (\mathcal{P}(w))}\right).
\]
Thus, it follows from (2.9), (2.11), and (2.12) that for each \(M \in \mathbb{N}\), there are constants \(\eta = \eta (M), \mu = \mu (M) > 0\) such that for any \(\alpha \in (\mathbb{N} \cup \{0\})^n,
\]
\[
|\mathcal{D}_\alpha^\delta K_{\partial \ell}^\delta (x)| \leq \eta \mu |\alpha| \left(1 + |x|\right)^{-M}. \tag{2.13}
\]
If \(|\langle \theta, n (\xi) \rangle| \geq 1 - \epsilon_0\), then by (2.10), the above (2.8) follows from the asymptotic result (see [5]) of (2.9) with respect to \(\rho\)-variable. Therefore we complete the proof. □
As we observed in Lemma 2.6, the main contribution to the decay of the kernel $\mathcal{D}_{\delta(p)}^{n}K_{\delta(p)}(x)$ comes from points near the horizontal part of $\sum_{0}d$, i.e. from points $x \in \mathbb{R}^n$ whose unit vector $x/|x|$ is almost parallel to $n(\zeta)$.

If $|\langle \theta, n(\zeta) \rangle| \geq 1 - \varepsilon_0$, then by (2.8) we can deduce the following estimate; for any multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$,

$$
\mathcal{D}_{\delta(p)}^{n}K_{\delta(p)}(x) \sim |x|^{-\delta(p)-1} \int_{\Sigma_{\delta(p)}} e^{i|x||\tilde{n}_{\zeta}/|n_{x}|} t_{0}(\zeta, x/|x|) d\sigma(\zeta),
$$

where

$$
t_{0}(\zeta, x/|x|) = \Gamma(\delta(p) + 1) e^{-i \pi (\delta(p)+1)/2} \zeta^{\alpha} \Pi(\zeta)[n(\zeta)\left[|x/|x|, \zeta\right]]^{-\delta(p)-1}.
$$

Without loss of generality, we can assume that $r_{\delta} \equiv \sup \{|\zeta| : \zeta \in \Sigma_{\delta(p)}| \leq 1$; for, by the change of variable, we may choose a sufficiently large $N_1 > 0$ so that $r_{\delta} \leq 1$ and

$$
K_{\delta(p)}(x) = N_1^{-d} \mathcal{F}^{-1}\left[\mathcal{F}(\zeta) (1 - \mathcal{F}(\zeta))^{\delta(p)} \right](N_1 x),
$$

where $K_{\delta(p)}(x) = \mathcal{F}^{-1}[\lambda_{d}(\zeta) (1 - \lambda_{d}(\zeta))^{\delta(p)}](x)$ and $\lambda_{d}(\zeta) = \lambda_{d}(N_1 \zeta)$. Since $|\zeta^\alpha| \leq 1$ by $r_{\delta} \leq 1$, it easily follows from (2.1), the fact that $|\mathcal{F}[d\sigma](x)| \sim \sigma(\mathcal{B}(\xi(x), 1/|x|))$ mentioned above, and (2.14) that for any multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$,

$$
|\mathcal{D}_{\delta(p)}^{n}K_{\delta(p)}(x)| \leq \frac{C}{(1 + |x|)^{n/p}} \Phi \left(\frac{x}{|x|}\right).
$$

Hence, by (2.15) and the multinomial theorem, we obtain that for any $N \in \mathbb{N}$,

$$
\sum_{|\alpha| = N} \frac{1}{\alpha!} \left|\mathcal{D}_{\delta(p)}^{n}K_{\delta(p)}(x)\right| \leq \sum_{|\alpha| = N} \frac{1}{\alpha!} \frac{1}{(1 + |x|)^{n/p}} \Phi \left(\frac{x}{|x|}\right)
$$

$$
= \frac{n^N}{N!} \frac{1}{(1 + |x|)^{n/p}} \Phi \left(\frac{x}{|x|}\right)
$$

$$
\leq \frac{1}{(1 + |x|)^{n/p}} \Phi \left(\frac{x}{|x|}\right).
$$

If $|\langle \theta, n(\zeta) \rangle| \leq 1 - \varepsilon_0$, then it easily follows from (2.7) and the multinomial theorem that for any $N > d\mu([n/p + 1])$,

$$
\sum_{|\alpha| = N} \frac{1}{\alpha!} \left|\mathcal{D}_{\delta(p)}^{n}K_{\delta(p)}(x)\right| \leq \sum_{|\alpha| = N} \frac{1}{\alpha!} \frac{\eta([n/p + 1]) \mu([n/p + 1])]^{|\alpha|}}{(1 + |x|)^{n/p}}
$$

$$
= \frac{n^N \mu([n/p + 1])]^N}{N!} \eta([n/p + 1])
$$

$$
\leq \frac{\eta([n/p + 1])}{(1 + |x|)^{n/p}}.
$$

Therefore we can easily obtain the following corollaries.

**Corollary 2.7.** Suppose that $\varrho_{d} \in C^{\infty}(\mathbb{R}^n)$ is a non-radial homogeneous distance function of degree $d \in \mathbb{N}$. Let $\zeta_{\ell}$, $\zeta_0$, and $\epsilon_0$ be as above. If $\delta(p) = n(1/p - 1/2) - 1/2$ for $0 < p < 1$, then there are constants $\mu_0 = \mu([n/p + 1]) > 0$ and $\eta_0 = \eta([n/p + 1]) > 0$ given in (2.13), a sufficiently large $M_0 > 0$ in (a)–(e), $\varepsilon_0 > 0$, a neighborhood $U_{1}$ of $\zeta_{\ell}$ with $\text{supp}(\varphi \Pi_{\ell}) \subset U_{1} \subset U_{0}$, and a neighborhood $V_{1}$ of the origin in $\mathbb{R}^{n-1}$ so that (2.6) holds and such that for any $N > d\mu_0$,

$$
\sum_{|\alpha| = N} \frac{1}{\alpha!} \left|\mathcal{D}_{\delta(p)}^{n}K_{\delta(p)}(x)\right| \leq \frac{\eta_0}{(1 + |x|)^{n/p}} \Phi \left(\frac{x}{|x|}, n(\zeta_{\ell})\right) \leq 1 - \varepsilon_0
$$

and

$$
\sum_{|\alpha| = N} \frac{1}{\alpha!} \left|\mathcal{D}_{\delta(p)}^{n}K_{\delta(p)}(x)\right| \leq \frac{1}{(1 + |x|)^{n/p}} \Phi \left(\frac{x}{|x|}, n(\zeta_{\ell})\right) \geq 1 - \varepsilon_0,
$$

where $\alpha \in (\mathbb{N} \cup \{0\})^n$ is a multi-index.
Corollary 2.8. Suppose that \( \varrho_d \in C^\infty(\mathbb{R}_+^d) \) is a non-radial homogeneous distance function of degree \( d \in \mathbb{N} \). If we set 
\[
K_{\varrho_d}^{\delta(p)}(x) = \mathcal{F}^{-1}[\varrho_d(1 - \varrho_d^\delta)\hat{1}](x) \quad \text{for} \quad 0 < p < 1,
\]
then we have the following uniform estimate; there is a constant \( C = C(n, p) > 0 \) such that for any \( N \in \mathbb{N} \),
\[
\left| K_{\varrho_d}^{\delta(p)}(x) \right| + \sum_{\left| \alpha \right| = N} \frac{1}{\alpha!} \left| \mathcal{T}^\alpha K_{\varrho_d}^{\delta(p)}(x) \right| \leq \frac{C}{(1 + |x|)^{n/p}} \Phi \left( \frac{x}{|x|} \right),
\]
where \( \alpha \in (\mathbb{N} \cup \{0\})^n \) is a multi-index. Here the constant \( C > 0 \) is independent of \( N \in \mathbb{N} \).

We now introduce the real Hardy space \( H^p(\mathbb{R}^n) \) defined in terms of atomic decompositions along the pattern of Stein [14]. Let \( 0 < p < 1 \). For \( \mu \geq n(1/p - 1) \), a function \( \alpha \in L^\infty(\mathbb{R}^n) \) is called a \((p, \mu)\)-atom centered at \( x_0 \in \mathbb{R}^n \) if it satisfies

(i) there is a ball \( B(x_0; s) \) with \( \text{supp}(\alpha) \subset B(x_0; s) \),
(ii) \( \| \alpha \|_{L^\infty} \leq |B(x_0; s)|^{-1/p} \), and
(iii) \( \int_{\mathbb{R}^n} \alpha(x) x^\alpha \, dx = 0 \) for \( |\alpha| \leq \mu \),

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) is an \( n \)-tuple of nonnegative integers and \( |\alpha| = \sum_{i=1}^n \alpha_i \leq \mu \). If \( f = \sum_{k=1}^\infty c_k \alpha_k \) where the \( \alpha_k \)'s are \((p, \mu)\)-atoms and \( \{c_k\} \in l^p \), then \( f \in H^p(\mathbb{R}^n) \) and \( \|f\|_{H^p} \lesssim \sum_k |c_k|^p \) and the converse inequality also holds (see [14]).

Proof of Theorem 1.1. Fix \( 0 < p < 1 \). Let \( \alpha \) be a \((p, n(1/p - 1))\)-atom supported in the ball \( B(x_0; s) \) with center \( x_0 \in \mathbb{R}^n \) and radius \( s > 0 \). Then we see that
\[
\mathcal{G}_{\varrho_d}^{\delta(p)} \alpha(x) = \left( \int_0^\infty |k_{\varrho_d}^{\delta(p), t} \ast \alpha(x)|^2 \frac{dt}{t} \right)^{1/2},
\]
where \( k_{\varrho_d}^{\delta(p)}(x) = t^n K_{\varrho_d}^{\delta(p)}(tx) \) and \( K_{\varrho_d}^{\delta(p)}(x) = \mathcal{F}^{-1}[\varrho_d(1 - \varrho_d^\delta)\hat{1}](x) \). Then it follows from changing the order of integration and the Plancherel’s theorem that
\[
\left\| \mathcal{G}_{\varrho_d}^{\delta(p)} \alpha \right\|_{L^2}^2 = \int_{\mathbb{R}^n} \left| \hat{\alpha}(\xi) \right|^2 \int_0^\infty \left( \frac{\varrho_d(\xi)}{t^d} \right)^2 \left( 1 - \frac{\varrho_d(\xi)}{t^d} \right)^{2\delta(p)} \frac{dt}{t} \, d\xi
\]
\[
= \left( \frac{1}{d} \int_0^1 t(1 - t)^{2\delta(p)} \, dt \right) \left\| \hat{\alpha} \right\|_{L^2}^2 \lesssim \| \alpha \|_{L^2}^2, \quad \delta(p) > -1/2.
\]
By Plancherel’s theorem, (2.21), and Hölder’s inequality with \( p/2 + 1/q = 1 \), we have that
\[
\int_{B(x_0; 2s)} \left| \mathcal{G}_{\varrho_d}^{\delta(p)} \alpha(x) \right|^p \, dx \lesssim \left\| \mathcal{G}_{\varrho_d}^{\delta(p)} \alpha \right\|_{L^2}^p \left| B(x_0; 2s) \right|^{1/q} \lesssim C.
\]
Thus by Chebyshev’s inequality we have that for all \( \lambda > 0 \),
\[
\left| \left\{ x \in B(x_0; 2s) : \left| \mathcal{G}_{\varrho_d}^{\delta(p)} \alpha(x) \right| > \lambda/2 \right\} \right| \lesssim \lambda^{-p}.
\]
Thus it suffices to show that
\[
\left| \left\{ x \in B(x_0; 2s) : \left| \mathcal{G}_{\varrho_d}^{\delta(p)} \alpha(x) \right| > \lambda/2 \right\} \right| \lesssim \lambda^{-p}, \quad \lambda > 0.
\]
Since \( k_{\varrho_d}^{\delta(p)} \ast \alpha \) is translation invariant, we may assume that \( x_0 = 0 \). If we set \( b(x) = s^{n/p} \alpha(tx) \), then \( b \) is clearly a \((p, n(1/p - 1))\)-atom that is supported in the unit ball \( B(0; 1) \). We observe that
provided that
\[
K^{\delta(p)}_{e_{d,t}} \ast a(x) = s^{-n/p} \frac{1}{K^{\delta(p)}_{e_{d,s}} \ast b(x/s)},
\]
\[
\mathcal{S}^{\delta(p)}_{e_{d}} a(x) = s^{-n/p} \left( \int_{0}^{\infty} \left| K^{\delta(p)}_{e_{d,t}} \ast b(x/s) \right|^2 dt \right)^{1/2}.
\] (2.22)

If \( x \in B(0; 2) \), then it follows from Lemma 2.5, Corollary 2.8, and Minkowski’s integral inequality that
\[
\left( \int_{0}^{\infty} \left| K^{\delta(p)}_{e_{d,t}} \ast b(x/s) \right|^2 dt \right)^{1/2} \lesssim \int_{B(0; 1)} \Phi \left( \frac{x-y}{|x-y|} \right) \left( \int_{1}^{\infty} t^{2n} \frac{dt}{(1+t|x|)^{2n/p}} \right)^{1/2} dy
\]
\[
\lesssim \Phi \left( \frac{x}{|x|} \right) \left( \int_{1+|x|}^{\infty} t^{2n-1-\frac{2n}{p}} dt \right)^{1/2}
\]
\[
\lesssim \frac{1}{(1+|x|)^{n/p}} \Phi \left( \frac{x}{|x|} \right).
\] (2.23)

Thus by (2.22) and (2.23) we have that
\[
s^{-n/p} \left( \int_{0}^{\infty} \left| K^{\delta(p)}_{e_{d,t}} \ast b(x/s) \right|^2 dt \right)^{1/2} \lesssim \frac{1}{(s+|x|)^{n/p}} \Phi \left( \frac{x}{|x|} \right).
\] (2.24)

provided that \( x \in B(0; 2s) \). Let \( N_0 \) be an integer satisfying \( N_0 < n/(1/p - 1) \leq N_0 + 1 \), i.e. \( n/(n + N_0 + 1) \leq p < n/(n + N_0) \). If \( x \in B(0; 2) \), let \( Q_{x,x}(y) \) be the \( N_0 \)th order Taylor polynomial of the function \( y \to K^{\delta(p)}_{e_{d}} (t(x-y)) \) expanded near the origin. If \( x \in B(0; 2) \), then it follows from the moment condition on the atom \( b \), Taylor’s theorem, Minkowski’s integral inequality, Lemma 2.5, and Corollary 2.8 that
\[
\left( \int_{0}^{\infty} \left| K^{\delta(p)}_{e_{d,t}} \ast b(x/s) \right|^2 dt \right)^{1/2} \lesssim \left( \int_{0}^{\infty} \left| \frac{1}{B(0; 1)} \int_{|y|=N_0+1} \frac{1}{\alpha!} |D^{\alpha} K^{\delta(p)}_{e_{d}} (t(x-y))| dy \right|^2 t^{2n+2(N_0+1)-1} dt \right)^{1/2}
\]
\[
\lesssim \left( \int_{0}^{\infty} \left( \int_{B(0; 1)} \frac{1}{(1+t|x-y|)^{n/p}} \Phi \left( \frac{x-y}{|x-y|} \right) dy \right)^2 t^{2n+2(N_0+1)-1} dt \right)^{1/2}
\]
\[
\lesssim \left( \int_{0}^{\infty} \left( \int_{B(0; 1)} \frac{1}{(1+t|x-y|)^{n/p}} \Phi \left( \frac{x-y}{|x-y|} \right) dy \right)^2 t^{2n+2(N_0+1)-1} dt \right)^{1/2} d\tau
\]
\[
\lesssim \int_{0}^{\infty} \Phi \left( \frac{x-y}{|x-y|} \right) \left( \int_{0}^{\infty} \frac{t^{2n+2(N_0+1)-1}}{(1+t|x-y|)^{2n/p}} dt \right)^{1/2} dy d\tau
\]
\[
\lesssim \Phi \left( \frac{x}{|x|} \right) \left( \int_{0}^{\infty} \frac{t^{2n+2(N_0+1)-1}}{(1+t|x|)^{2n/p}} dt \right)^{1/2}
\]
because \( n + (N_0 + 1) - n/p \geq 0 \). Combining this with (2.22), we have that

\[
S^{-n/p} \left( \int_0^1 |K^{(p)}_{\delta d} \ast b(x/s)|^2 \frac{dt}{t} \right)^{1/2} \lesssim \frac{1}{(s + |x|)^{n/p}} \Phi \left( \frac{x}{|x|} \right),
\]

(2.25)

provided that \( x \in B(0; 2s) \). Thus by (2.22), (2.24), and (2.25) we conclude that

\[
\Phi^{(p)}(r) \leq \frac{1}{(s + |x|)^{n/p}} \Phi \left( \frac{x}{|x|} \right),
\]

whenever \( x \in B(0; 2s) \). Hence we have the following estimate

\[
\int_{\{x \in B(0; 2s) : \Phi^{(p)}(r) > \lambda \}} dx \lesssim \int \left( \int_{2s} r^{n-1} dr \right) d\theta \lesssim \lambda^{-p}
\]

because \( \Omega \in L^p(S^{n-1}) \) for any \( p < 1 \) by Lemma 2.2. Therefore by Lemma 2.3 we complete the proof.

3. \((H^p, L^p)\)-estimate for the case that \( \varrho_d \in C^\infty(\mathbb{R}_0^n) \) and \( \delta > \delta(p) \), \( 0 < p < 1 \)

We shall adopt another decomposition of the Bochner–Riesz type multiplier \( \varrho_d(1 - \varrho_d)_{+}^\delta \) as in Córdoba [3] where \( \varrho_d \in C^\infty(\mathbb{R}_0^n) \) is a non-radial homogeneous distance function of degree \( d \in \mathbb{N} \). Let a function \( \phi \in C^\infty_c(1/2, 2) \) satisfy

\[
\sum_{k \in \mathbb{Z}} \phi(2^k t) = 1 \quad \text{for all } t > 0.
\]

For \( k \in \mathbb{N} \), let \( \Phi_k^\delta = \phi(2^{k+1}(1 - \varrho_d))\varrho_d(1 - \varrho_d)_+^\delta \) and \( \Phi_0^\delta = \varrho_d(1 - \varrho_d)_+^\delta - \sum_{k \in \mathbb{N}} \Phi_k^\delta \). For each \( k \in \mathbb{Z} \), we now introduce a partition of unity \( \Sigma_{\varrho_d} \), \( \ell = 1, 2, \ldots, N_k \), on the unit sphere \( \Sigma_{\varrho_d} \) which we extend to \( \mathbb{R}^n \) by way of \( \Pi_{\varrho_d}(A; \xi) = \Sigma_{\varrho_d}(\xi), t > 0, \xi \in \Sigma_{\varrho_d}, \delta \) and which satisfies the following properties; there are a finite number of points \( \varsigma_{k1}, \varsigma_{k2}, \ldots, \varsigma_{kN_k} \in \Sigma_{\varrho_d} \) such that for \( \ell = 1, 2, \ldots, N_k \),

(i) \( \sum_{\ell=1}^{N_k} \Pi_{\varrho_d}(\varsigma) = 1 \) for all \( \varsigma \in \Sigma_{\varrho_d} \),

(ii) \( \Sigma_{\varrho_d}(\varsigma) = 1 \) for all \( \varsigma \in \Sigma_{\varrho_d} \cap B(\varsigma_{k\ell}; 2^{-k/2}) \),

(iii) \( \Sigma_{\varrho_d} \) is supported in \( \Sigma_{\varrho_d} \cap B(\varsigma_{k\ell}; c_3 2^{-k/2}) \),

(iv) \( |D^\alpha \Pi_{\varrho_d}(\varsigma)| \leq c_2 2^{\alpha|k/2|} \) for any multi-index \( \alpha \), if \( 1/2 \leq \varrho_d(\varsigma) \leq 2 \),

(v) \( N_k \leq c_3 2^{(n-1)/2} \) for fixed \( k \),

where \( B(\varsigma; s) \) denotes the ball in \( \mathbb{R}^n \) with center \( \varsigma \in \Sigma_{\varrho_d} \) and radius \( s > 0 \) and the positive constants \( c_1, c_2, c_3 \) do not depend upon \( k \). For each \( k \in \mathbb{Z} \), let \( \mathcal{H}_{\varrho_d, k\ell}^\delta = \mathcal{F}^{-1} \Phi_k^\delta \Pi_{\varrho_d} \) and \( \mathcal{H}_0 = \mathcal{F}^{-1} \Phi_0^\delta \).

Next we invoke a simple observation as in Section 2 to obtain decay estimate for kernels \( \mathcal{H}_{\varrho_d, k\ell} \), \( \mathcal{H}_0 \) corresponding to the decomposition of the Bochner–Riesz multiplier defined above. Without loss of generality, we can assume that \( \varrho_d \in C^\infty(\mathbb{R}^n) \) because we can replace \( \varrho_d \) by \( \varrho_d^N \) for sufficiently large \( N > 0 \) by a subordination argument in [4]. Then we easily see that the kernel \( \mathcal{H}_0 \) has a nice decay, and so its corresponding square function admits \((H^p, L^p)\)-estimate for the index \( \delta > n(1/p - 1/2) - 1/2 \) and \( 0 < p < 1 \). Thus we concentrate upon obtaining the decay estimate for the kernels \( \mathcal{H}_{\varrho_d, k\ell}^\delta \).
Lemma 3.1. Suppose that \( \varrho_d \in C^\infty(\mathbb{R}^d_0) \) is a non-radial homogeneous distance function of degree \( d \in \mathbb{N} \). For fixed \( k \in \mathbb{N} \) and for \( \ell = 1, 2, \ldots, N_k \), let \( T_{\xi_k}(\Sigma_{\varrho_d}) \) be the tangent space of \( \Sigma_{\varrho_d} \) at \( \xi_k \in \Sigma_{\varrho_d} \), \( \{\varrho_d^{j_n} \} \) be an orthonormal basis of \( T_{\xi_k}(\Sigma_{\varrho_d}) \), and \( \varrho_d^{j_n} \) be the outer unit normal vector to \( \Sigma_{\varrho_d} \) at \( \xi_k \in \Sigma_{\varrho_d} \). Then we have the following estimate

\[
|H_{\varrho_d, k\ell}(x)| \leq \frac{C_N 2^{-k(\delta + 1 + (n-1)/2)}}{(1 + 2^{-k}|x, \varrho_d^0|)|n|^{-1} (1 + 2^{-k/2}|x, \varrho_d^0|)|N|^{-1}}
\]

for any \( N \in \mathbb{N} \).

**Proof.** We need the following simple observation: let \( \varrho_d \in C^\infty(\mathbb{R}^d_0) \) and \( F \in C^N(\mathbb{R}^+) \). For \( e \in S^{n-1} \), let \( D_e f \) be the directional derivative \( \langle e, \nabla f \rangle \). Then by simple calculation one can have the following formula

\[
D_e^N (F \circ \varrho_d) = \sum_{v=1}^{N} \sum_{\beta \in \gamma^N} c_{N, \beta} D_e^\beta \varrho_d
\]

where \( \gamma^N = \{\beta; \sum_{m=1}^{v} \beta_m = N, \text{ at least } v - N \} \) of the numbers \( \beta_m \) are equal to 1, \( \beta = (\beta_1, \ldots, \beta_v) \) is a multi-index, and \( c_{N, \beta} \)'s are some constants. For \( k \in \mathbb{N} \), let \( F_k(t) = \phi(2^{k+1}(1-t))-(1-t)\delta \). Then it follows from simple computation that

\[
F_k^{(v)}(t) = (-1)^v \sum_{i=0}^{v} C(v, i) C(\delta, v-i) 2^i (k+1) \phi(i) (2^{k+1}(1-t))(1-t)^{\delta-v+i}
\]

where \( C(v, i) = \frac{v(v-1)(v-2)\cdots(v-i+1)}{i!} \) for positive integers \( v, i \), and \( C(v, 0) = 1 \). If we set \( G_k(t) = \phi(2^{k+1}(1-t))t(1-t)^\delta \) for \( k \in \mathbb{N} \), then we have that

\[
G_k^{(v)}(t) = vF_k^{(v-1)}(t) + t F_k^{(v)}(t).
\]

For fixed \( k, \ell \), by (3.1), (3.2), and (3.3), we have the estimate

\[
\left\| D_{\xi_k}^N \left[ \Phi_k^\delta \Lambda_{k\ell} \right] \right\|_{L^1} \leq c 2^{-k(\frac{\delta+1}{2})} 2^{-k\delta} 2^{kN}
\]

for any \( N \in \mathbb{N} \). Since we have the better estimate \( |D_{\xi_k}^N \varrho_d| \leq c 2^{-k/2} \) on the support of \( F[H_{\varrho_d, k\ell}^\delta] \) for fixed \( j, k, \ell \), it follows from (3.1) and Taylor’s theorem that

\[
\left\| D_{\xi_k}^N \left[ \Phi_k^\delta \Lambda_{k\ell} \right] \right\|_{L^1} \leq c 2^{-k(\frac{\delta+1}{2})} 2^{-k\delta} 2^{kN/2}
\]

for any \( N \in \mathbb{N} \). Using the integration by parts, it follows from (3.4) and (3.5) that

\[
|H_{\varrho_d, k\ell}(x)| \leq \frac{C_N 2^{-k(\delta + 1 + (n-1)/2)}}{(1 + 2^{-k}|x, \varrho_d^0|)|n|^{-1} (1 + 2^{-k/2}|x, \varrho_d^0|)|N|^{-1}}
\]

for any \( N \in \mathbb{N} \). □

**Lemma 3.2.** Suppose that \( \varrho_d \in C^\infty(\mathbb{R}^d_0) \) is a non-radial homogeneous distance function of degree \( d \in \mathbb{N} \). If \( \delta > n(1/p - 1/2) - 1/2 \) for \( 0 < p < 1 \), let a positive number \( p' < p \) be chosen so that \( \delta = n(1/p' - 1/2) - 1/2 \). For fixed \( k \in \mathbb{N} \) and for \( \ell = 1, 2, \ldots, N_k \), let \( T_{\xi_k}(\Sigma_{\varrho_d}) \) be the tangent space of \( \Sigma_{\varrho_d} \) at \( \xi_k \in \Sigma_{\varrho_d} \), \( \{\varrho_d^{j_n} \} \) be an orthonormal basis of \( T_{\xi_k}(\Sigma_{\varrho_d}) \), and \( \varrho_d^{j_n} \) be the outer unit normal vector to \( \Sigma_{\varrho_d} \) at \( \xi_k \in \Sigma_{\varrho_d} \). Then we have the following estimate

\[
|H_{\varrho_d, k\ell}(x)| + |\nabla H_{\varrho_d, k\ell}(x)| \leq \frac{C_p 2^{-k(\frac{\delta+1}{2p})}}{\prod_{j=0}^{n-1} (1 + |x, \varrho_d^{j_n}|)^{1/p'}} \leq C_p 2^{-k(\frac{\delta+1}{2p'})} Q_{k\ell}(x).
\]
Proof. This can easily be obtained by choosing \( \delta = n(1/p' - 1/2) - 1/2 \) and \( N = 1/p' \) in Lemma 2.1. We also observe that \( \nabla t H_{\delta, k} = \varphi \ast H_{\delta, k} \) for some \( \varphi \in \mathcal{S}(\mathbb{R}^n) \). ∎

For \( f \in \mathcal{G}(\mathbb{R}^n) \), \( \delta \in \mathbb{R} \), \( k \in \mathbb{N} \), and \( \ell = 1, 2, \ldots, N_k \), let

\[
\mathcal{G}_{\delta, k} f(x) = \left( \int_0^\infty \left| \mathcal{H}_{\delta, k} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}
\]

where \( \mathcal{H}_{\delta, k} f(x) = t^n \mathcal{H}_{\delta, k}^\delta(t x) \). We now need an elementary inequality to obtain the decay estimate of square functions \( \mathcal{G}_{\delta, k} \) corresponding to such kernels \( \mathcal{H}_{\delta, k}^\delta \) which act on certain atoms. We shall now state it without proof.

Lemma 3.3. For any \( a \geq 1 \) and \( b \geq 0 \), we have that \( \frac{1}{a^p} \leq \left( \frac{1 + b}{a+b} \right)^{1/p} \) for \( p > 0 \).

Lemma 3.4. Suppose that \( \varphi_d \in C^\infty(\mathbb{R}^n) \) is a non-radial homogeneous distance function of degree \( d \in \mathbb{N} \). If \( \delta > n(1/p - 1/2) - 1/2 \) for \( 0 < p < 1 \), let a positive number \( p' < p \) be chosen so that \( \delta = n(1/p' - 1/2) - 1/2 \). Suppose that \( a \) is a \((p, n(1/p' - 1))\)-atom on \( \mathbb{R}^n \) which is supported in the ball \( B(x_0; s) \) with center \( x_0 \in \mathbb{R}^n \) and radius \( s > 0 \). Then there is a constant \( C = C(n, p) > 0 \) such that

(a) \( \mathcal{G}_{\delta, k} a(x) \leq C s^{-n/p} 2^{-k(n-1)/p'} Q_k \delta(\frac{|x-y|}{s}) \) for any \( x \in B(x_0; 2s) \),

(b) \( \| (\mathcal{G}_{\delta, k} a) \chi_{B(x_0; 2s)} \|_{L^p} \leq C 2^{-k(n-1)/p'} \),

where \( Q_k(\delta) \) is the function given in Lemma 3.2.

Proof. (a) We first assume that \( a \) is a \((p, n(1/p' - 1))\)-atom which is supported in the unit ball \( B(0; 1) \) centered at the origin. If \( x \in B(0; 2s) \), then it easily follows from Lemmas 3.2, 3.3, and Minkowski’s integral inequality that

\[
\left( \int_{B(0;1)} \left| \mathcal{H}_{\delta, k}^\delta(t x - y) \right|^2 \frac{dt}{t} \right)^{1/2} \leq 2^{-k(n-1)/p'} \int_{B(0;1)} \sum_{j=0}^{n-1} \frac{(1 + |\langle y, e_{k \ell}^j \rangle|)^{1/p'}}{(1 + |x - y, e_{k \ell}^j|)} dy
\]

\[
\leq 2^{-k(n-1)/p'} Q_k(\delta)(x).
\]

because \( n(1 - 1/p') < 0 \) and \( |\langle x, e_{k \ell}^j \rangle| \leq |\langle x - y, e_{k \ell}^j \rangle| + |\langle y, e_{k \ell}^j \rangle| \). Let \( N_1 \in \mathbb{N} \) be an integer satisfying that \( N_1 < n(1/p' - 1) \leq N_1 + 1 \), i.e. \( n/(n + N_1 + 1) \leq p' < n/(n + N_1) \). If \( x \in B(0; 2s) \), let \( \mathcal{Q}_{t, x}(y) \) be the \( N_1 \)th order Taylor polynomial of the function \( y \rightarrow \mathcal{H}_{\delta, k}^\delta(t (x - y)) \) expanded near the origin. If \( x \in B(0; 2s) \), then it follows from the moment condition on the atom \( a \), Taylor’s theorem, Minkowski’s integral inequality, Lemmas 3.2, and 3.3 that
Then it easily follows from the change of variable and (3.10) that whenever $n + (N_1 + 1) - n/p' \geq 0$ and $\|\{x, e^j_{k\ell}\} \leq |\{x - \tau y, e^j_{k\ell}\}| + \tau |\{y, e^j_{k\ell}\}|$ for $0 < \tau < 1$. Combining this with (3.7), we have that
\[
\mathcal{G}^\delta_{\tilde{R}^\ell} a(x) \lesssim 2^{-k(D^*_{\tilde{R}^\ell})} \mathcal{Q}_{k\ell}(x), \quad \text{(3.8)}
\]
whenever $x \in \mathcal{B}(0; 2)^c$.

Finally, let $a$ be a $(p, n(1/p' - 1))$-atom supported in the ball $B(x_0; s)$. If we set $b(x) = s^{n/p} a(s(x - x_0))$, then $b$ is clearly a $(p, n(1/p' - 1))$-atom supported in the unit ball $B(0; 1)$. We also observe that
\[
\mathcal{H}^\delta_{\tilde{R}^\ell} \ast a(x) = s^{-n/p} \mathcal{H}^\delta_{\tilde{R}^\ell} \ast b((x - x_0)/s),
\]
\[
\mathcal{G}^\delta_{\tilde{R}^\ell} a(x) = s^{-n/p} \left( \int_0^1 |\mathcal{H}^\delta \ast b((x - x_0)/s)|^2 \frac{dt}{t} \right)^{1/2}.
\quad \text{(3.9)}
\]

Thus by (3.8) and (3.9) we can complete the part (a).

(b) We observe that there is a constant $C = C(n, p) > 0$ such that for any $x_0 \in \mathbb{R}^n$ and for any $k \in \mathbb{N}$, $\ell = 1, 2, \ldots, N_k$,
\[
\|\mathcal{Q}_{k\ell}(\cdot - x_0)/s\|_{L^p} \leq Cs^{n/p}. \quad \text{(3.10)}
\]

Then it easily follows from the change of variable and (3.10) that
\[
\left\| \mathcal{G}^\delta_{\tilde{R}^\ell} a(x) \mathcal{X}_{B(x_0; 2s)} \right\|_{L^p} \leq Cs^{-n/p} \frac{1}{p} \left\|\mathcal{Q}_{k\ell}(\cdot - x_0)/s\right\|_{L^p} \leq C 2^{-k(D^*_{\tilde{R}^\ell})}. \quad \square
\]

**Proof of Theorem 1.2.** First of all, we prove that if $\delta > n(1/p - 1/2) - 1/2$ for $0 < p < 1$ then $\mathcal{G}^\delta_{\tilde{R}^\ell} a \in L^p(\mathbb{R}^n)$ for any $(p, n(1/p' - 1))$-atom on $\mathbb{R}^n$ where $p' < p$ is a positive number satisfying $\delta = n(1/p' - 1/2) - 1/2$, and moreover there is a constant $C > 0$ independent of such atoms such that $\|\mathcal{G}^\delta_{\tilde{R}^\ell} a\|_{L^p} \leq C$. Let $a$ be a $(p, n(1/p' - 1))$-atom supported in the ball $B(x_0; s)$ with center $x_0 \in \mathbb{R}^n$ and radius $s > 0$. Then it follows from Plancherel’s theorem, (2.21), and Hölder’s inequality with $p/2 + 1/q = 1$ that
\[
\int_{B(x_0; 2s)} \left|\mathcal{G}^\delta_{\tilde{R}^\ell} a(x)\right|^p dx \lesssim \|\mathcal{G}^\delta_{\tilde{R}^\ell} a\|_{L^2}^p \|B(x_0; 2s)\|_1^{1/q} \leq C. \quad \text{(3.11)}
\]

Since $0 < p < 1$, it easily follows from (3.11), (v) of p. 276, and (b) of Lemma 3.4 that
\[
\|\mathcal{G}^\delta_{\tilde{R}^\ell} a\|_{L^p}^p = \left\| \left(\mathcal{G}^\delta_{\tilde{R}^\ell} a\right) \mathcal{X}_{B(x_0; 2s)} \right\|_{L^p}^p + \left\| \left(\mathcal{G}^\delta_{\tilde{R}^\ell} a\right) \mathcal{X}_{B(x_0; 2s)^c} \right\|_{L^p}^p
\lesssim 2^{n/q} + \sum_{k=1}^{\infty} \sum_{\ell=1}^{N_k} \left\| \left(\mathcal{G}^\delta_{\tilde{R}^\ell} a\right) \mathcal{X}_{B(x_0; 2s)^c} \right\|_{L^p}^p
\lesssim 2^{n/q} + \sum_{k=1}^{\infty} 2^{-k(D^*_{\tilde{R}^\ell})} \leq C. \quad \text{(3.12)}
\]
Finally, if \( f = \sum_{j=1}^{\infty} c_j a_j \) where the \( a_j \)'s are \((p, n(1/p' - 1))\)-atoms and \( \{c_j\} \in \ell^p \), then by (3.12) we have the estimate
\[
\left\| \mathcal{G}_{\delta}^{\varrho} f \right\|_{L^p} \leq \sum_{j} |c_j|^p \left\| \mathcal{G}_{\delta}^{\varrho} a_j \right\|_{L^p} \lesssim \sum_{j} |c_j|^p.
\]
Hence this completes the proof. \( \square \)

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References