J. Math. Anal. Appl. 390 (2012) 224–233

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On determining the domain of the adjoint operator

Michał Wojtylak [1]

Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

ARTICLE INFO

Article history:
Received 10 June 2011
Available online 20 January 2012
Submitted by J.A. Ball

Keywords:
Selfadjoint operator
Normal operator
H-selfadjoint operator
Commutator

ABSTRACT

A theorem that is of aid in computing the domain of the adjoint operator is provided. It may serve e.g. as a criterion for selfadjointness of a symmetric operator, for normality of a formally normal operator or for H-selfadjointness of an H-symmetric operator. Differential operators and operators given by an infinite matrix are considered as examples.

0. Introduction

In the literature there exist many criteria for selfadjointness of symmetric operators. As a root of the present research one should mention the paper by Driessler and Summers [8], which presents a criterion for selfadjointness connected with the notion of domination (relative boundedness) and the first commutator. Later on that result has been extended by Cichoń, Stochel and Szafraniec [5] and by the author of the present paper [25,26]. The aim of this note is to generalize this result in such way that it serves simultaneously as a criterion for normality of a formally normal operator as well as a criterion for selfadjointness of symmetric operator in a Krein space. Furthermore, an important issue will be illustrating this generalization with various examples.

Let us describe now the framework of the present research. Given a pair (A, A_0) of operators in a Hilbert space, with A closable and densely defined and $A_0 \subseteq A^*$, we want to provide a necessary condition for the equality $\bar{A}_0 = A^*$. This condition should not involve the operator A^* itself but the operators A and A_0 only. The main interest will lie in the following instances:

(a0) A is a symmetric operator, $A_0 = A$;
(a1) A is a formally normal operator, $A_0 = A^*|_{\mathcal{D}(A)}$;
(a2) A is a q-formally normal operator with $q \in (0, \infty)$, $A_0 = A^*|_{\mathcal{D}(A)}$;
(a3) $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and the graph norms of A and A^* are equivalent on $\mathcal{D}(A)$, $A_0 = A^*|_{\mathcal{D}(A)}$;
(a4) A is an H-symmetric operator, where $H \in \mathbf{B}(\mathcal{K})$ is selfadjoint and boundedly invertible, $A_0 = HAH^{-1}$.

Note that the equality $\bar{A}_0 = A^*$ means in the above cases that, respectively, $\bar{A}$ is selfadjoint, $\bar{A}$ is normal, $\bar{A}$ is q-normal, $\mathcal{D}(\bar{A}) = \mathcal{D}(A^*)$, and $\bar{A}$ is H-selfadjoint (see the Preliminaries for this and for definitions of the classes appearing above). After these explanations we can present the main result of the paper. The theorem is proved later on in a slightly stronger form as Theorem 3, cf. Remark 4. If S is an operator in a Hilbert space $\mathcal{K}$ then $\mathcal{D}(S)$ and $\mathcal{R}(S)$ denote, respectively, the domain and the range of S and WOT lim stands for the limit in the weak operator topology.

E-mail address: michal.wojtylak@gmail.com.

[1] The author gratefully acknowledges the assistance of the Polish Ministry of Higher Education and Science grant NN201 546438.

doi:10.1016/j.jmaa.2012.01.024

Theorem 1. *Let A be a closable, densely defined operator in a Hilbert space $\mathcal{K}$ and let $A_0 \subseteq A^*$. If there exists a sequence $(T_n)_{n=0}^{\infty} \subseteq \mathbf{B}(\mathcal{K})$ such that*

$$\operatorname{WOT}\lim_{n\to\infty} T_n = I_{\mathcal{K}},$$

$$\mathcal{R}(T_n) \subseteq \mathcal{D}(\bar{A}), \qquad \mathcal{R}\big(T_n^*\big) \subseteq \mathcal{D}(\bar{A}_0), \quad n \in \mathbb{N} \tag{1}$$

and

$$\sup_{n\in\mathbb{N}} \|\bar{A}T_n - T_n\bar{A}\| < +\infty, \tag{2}$$

then $\bar{A}_0 = A^*$.

In the classical literature like [2,13,21] one can find a technique of proving selfadjointness based on computing the relative bound. The method presented above is an alternative approach, based rather on the notions of commutativity and domination. An example of a first order symmetric differential operator from [26] shows the difference between those two approaches.

In the symmetric case (a0) the technique presented in the theorem above was already used in the literature in the context of differential operators on manifolds [4,9,10] and graphs [11]. In [6] one can find examples of applications of the domination techniques to symmetric integral operators. Therefore, in the present paper we do not focus our attention on the (a0) class, but show possible applications of the main result in the classes (a1)–(a4).

The content of the present paper is the following. Section 1 has a preliminary character, but already in the consecutive section we prove the main result of the paper. In Section 3 we consider the class (a1) of formally normal operators, extending the results from [26]. In Section 4 we will consider H-symmetric operators (class (a4)) given by infinite matrices.

In Sections 5 we make a link with a theory of commutative domination in the sense of [20,22,27]. Namely, the sequence $(T_n)_{n=0}^{\infty}$ in Theorem 1 above may be in many cases chosen as

$$T_n = n^m (S - \mathrm{i}\cdot n)^{-m}, \quad n \in \mathbb{N},$$

where S is a selfadjoint operator and $m \geqslant 1$, examples can be found in the already mentioned work [6]. However, this approach requires computing the commutator (2). In Theorem 11 we replace (2) by a condition involving the commutator $SA - AS$, the new assumption being stronger then (2) but nevertheless easier to calculate. Again, we formulate the result in the general setting of the pair (A_0, A), the case $SA - AS = 0$ is the announced link with commutative domination. In Section 6 we apply Theorem 11 to a first order differential operator A with nonconstant coefficients. A necessary conditions, expressed in terms of coefficients, for A being of class (a3) and for $\mathcal{D}(\bar{A}) = \mathcal{D}(A^*)$ are provided. As the reader noticed there are so far no applications of the main result to the class (a2).

1. Preliminaries

Through the whole paper $(\mathcal{K}, \langle\cdot,-\rangle)$ stands for a Hilbert space. The sum and the product of unbounded operators is understood in a standard way, see e.g. [7]. We put

$$\operatorname{ad}(S,T) := ST - TS.$$

We say that an operator S in $\mathcal{K}$ is *bounded* if $\|Sf\| \leqslant c\|f\|$ for all $f \in \mathcal{D}(S)$ and some $c \geqslant 0$. We write $\mathbf{B}(\mathcal{K})$ for the space of all bounded operators with domain equal $\mathcal{K}$, stressing the fact that *not* every bounded operator is in $\mathbf{B}(\mathcal{K})$.

Let A be a closable, densely defined operator. We say that A is *symmetric* if $A \subseteq A^*$, *selfadjoint* if $A = A^*$.

Let $q \in (0, +\infty)$, we say that A is *q-formally normal* if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*f\| = \sqrt{q}\|Af\|$ for $f \in \mathcal{D}(A)$. We say that A is *q-normal* if A is q-formally normal and $\mathcal{D}(A) = \mathcal{D}(A^*)$. We refer the reader to [18,19] for a treatment on q-normals and related classes of operators. Note that (a2) together with $\bar{A}_0 = A^*$ gives q-normality of $\bar{A}$. Indeed, since the graph norms of $A_0 = A^*|_{\mathcal{D}(A)}$ and A are equivalent on $\mathcal{D}(A)$, we get $\mathcal{D}(\bar{A}) = \mathcal{D}(A^*)$, i.e. $\bar{A}$ is q-normal. We call A *formally normal* (*normal*) if it is 1-formally normal (1-normal, respectively).

Let $H \in \mathbf{B}(\mathcal{K})$ be selfadjoint and boundedly invertible. We say that A is *H-symmetric* if $A \subseteq H^{-1}A^*H$, *H-selfadjoint* if $A = H^{-1}A^*H$. If we introduce an indefinite inner product on $\mathcal{K}$ by $[f,g] = \langle Hf, g\rangle$, $f, g \in \mathcal{K}$, then $(\mathcal{K}, [\cdot,-])$ is a Krein space, see [1,3]. Defining A^+ as the adjoint of A with respect to $[\cdot,-]$ we easily see that $A^+ = H^{-1}A^*H$. Hence, Theorem 1 can suite as a criterion for selfadjointness of a closed symmetric operator in a Krein space, cf. [26]. Nevertheless, we will not use neither the indefinite inner product nor the operator A^+ in the present paper.

We also say that A is *essentially selfadjoint* (respectively, essentially q-normal, essentially H-selfadjoint) if $\bar{A}$ is selfadjoint (respectively, q-normal, H-selfadjoint).

The following facts will be frequently used later on. If S and T are densely defined operators in $\mathcal{K}$ and ST is densely defined then $(ST)^* \supseteq T^*S^*$. If additionally $S \in \mathbf{B}(\mathcal{K})$ then

$$(ST)^* = T^*S^*. \tag{3}$$

We say that an operator A *dominates an operator* B *on a linear space* $\mathcal{E}$ if $\mathcal{E} \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ and there exists $c > 0$ such that

$$\|Bf\| \leqslant c\big(\|Af\| + \|f\|\big), \quad f \in \mathcal{E}.$$

If $\mathcal{E} = \mathcal{D}(B)$ then we say that A *dominates* B. Note that if both operators are closed, then A dominates B if and only if $\mathcal{D}(B) \subseteq \mathcal{D}(A)$, by the closed graph theorem.

2. Approximate units for an unbounded operator. Main result

Let A be closable and densely defined, let $A_0 \subseteq A^*$ and let $T \in \mathbf{B}(\mathcal{K})$. Consider the following conditions:

(f1) the commutator $\operatorname{ad}(T, \bar{A})$ is densely defined and bounded in $\mathcal{K}$;
(f2) $T^*\mathcal{D}(A^*) \subseteq \mathcal{D}(\bar{A}_0)$.

If a sequence $(T_n)_{n=0}^{\infty} \subseteq \mathbf{B}(\mathcal{K})$ tends in the weak operator topology to $I_{\mathcal{K}}$ and is such that each of the operators T_n ($n \in \mathbb{N}$) satisfies (f1), (f2) we will call it an (f)-approximate unit for the pair (A, A_0). This notion has some connections with quasicentral approximate units and the unbounded derivation, see [23] and the papers quoted therein.

Proposition 2. *Let A be closable and densely defined, let $A_0 \subseteq A^*$ and let $(T_n)_{n=0}^{\infty} \subseteq \mathbf{B}(\mathcal{K})$ be an* (f)*-approximate unit for (A, A_0). Then the following conditions are equivalent*:

$$\sup_{n\in\mathbb{N}} \big\|\operatorname{ad}(T_n, \bar{A})\big\| < +\infty; \tag{4}$$

$$\sup_{n\in\mathbb{N}} \big\|\overline{\operatorname{ad}\big(T_n^*, A^*\big)}\big\| < +\infty; \tag{5}$$

$$\operatorname{WOT}\lim_{n\to\infty} \overline{\operatorname{ad}(T_n, \bar{A})} = 0; \tag{6}$$

$$\operatorname{WOT}\lim_{n\to\infty} \overline{\operatorname{ad}\big(T_n^*, A^*\big)} = 0. \tag{7}$$

Proof. Fix $n \in \mathbb{N}$. The operator $\operatorname{ad}(A^*, T_n^*)$ is densely defined by (f2) and is contained in $\operatorname{ad}(T_n, \bar{A})^*$. By (f1) the operator $\operatorname{ad}(T_n, \bar{A})^*$ belongs to $\mathbf{B}(\mathcal{K})$. Hence,

$$\overline{\operatorname{ad}\big(A^*, T_n^*\big)} = \operatorname{ad}(T_n, \bar{A})^*.$$

This shows the equivalences (4) $\Leftrightarrow$ (5) and (6) $\Leftrightarrow$ (7).

Suppose now that (4) is satisfied. The weak convergence of $(T_n)_{n=0}^{\infty}$ to identity implies that for $f \in \mathcal{D}(\operatorname{ad}(T_n, \bar{A}))$, $g \in \mathcal{D}(A^*)$ one has

$$\big\langle \operatorname{ad}(T_n, \bar{A}) f, g\big\rangle = \big\langle \bar{A} f, T_n^* g\big\rangle - \big\langle T_n f, A^* g\big\rangle \xrightarrow{n\to\infty} \langle \bar{A} f, g\rangle - \big\langle f, A^* g\big\rangle = 0.$$

Since $\mathcal{D}(\operatorname{ad}(T_n, \bar{A}))$ and $\mathcal{D}(A^*)$ are dense in $\mathcal{K}$ we have (6) by a standard triangle inequality argument.

The implication (6) $\Rightarrow$ (4) holds, since every sequence convergent in the weak operator topology is bounded in the norm, by the uniform boundedness principle. $\square$

After these preparations we can easily derive the main result of the paper.

Theorem 3. *Let A be closable and densely defined, let $A_0 \subseteq A^*$ and let $(T_n)_{n=0}^{\infty} \subseteq \mathbf{B}(\mathcal{K})$ be an* (f)*-approximate unit for (A, A_0). If*

$$\sup_{n\in\mathbb{N}} \big\|\operatorname{ad}(T_n, \bar{A})\big\| < +\infty,$$

then $\bar{A}_0 = A^*$.

Proof. Fix an arbitrary $f \in \mathcal{D}(A^*)$ and consider the sequence $f_n := T_n^* f$ ($n \in \mathbb{N}$), which is contained in $\mathcal{D}(\bar{A}_0)$ by (f2). Observe that

$$\langle f_n, g\rangle \to \langle f, g\rangle \quad (n \to \infty), \quad g \in \mathcal{K}, \tag{8}$$

since T_n^* tends to $I_{\mathcal{K}}$ in the weak operator topology. Furthermore, note that

$$\big\langle A^* f_n, g\big\rangle \to \big\langle A^* f, g\big\rangle \quad (n \to \infty), \quad g \in \mathcal{K}. \tag{9}$$

Indeed, since f belongs to $\mathcal{D}(A^*)$, which is contained in $\mathcal{D}(\operatorname{ad}(A^*, T_n^*))$ by (f2), we have

$$\big\langle A^* T_n^* f, g\big\rangle = \big\langle T_n^* A^* f, g\big\rangle + \big\langle \operatorname{ad}\big(A^*, T_n^*\big) f, g\big\rangle, \quad g \in \mathcal{K}.$$

The first summand tends with $n \to \infty$ to $\langle A^* f, g\rangle$ by the convergence of T_n^*, the second summand goes to zero by Proposition 2. Hence, (9) is shown.

Consider now the graph norm $\|\cdot\|_{A^*}$ on $\mathcal{D}(A^*)$, which makes $\mathcal{D}(A^*)$ a Banach space. Formulas (8) and (9) and the fact that $f \in \mathcal{D}(A^*)$ was taken arbitrary imply that $\mathcal{D}(\bar{A}_0)$ is weakly dense in $(\mathcal{D}(A^*), \|\cdot\|_{A^*})$. Since $\mathcal{D}(\bar{A}_0)$ is a linear space, it is dense in $\mathcal{D}(A^*)$ in the $\|\cdot\|_{A^*}$-topology as well. But $\mathcal{D}(\bar{A}_0)$ is closed in $\|\cdot\|_{A^*}$-topology as $A_0 \subseteq A^*$. Hence, $\bar{A}_0 = A^*$. □

Remark 4. Observe that the following condition

(e) $\mathcal{R}(T) \subseteq \mathcal{D}(\bar{A})$, $\mathcal{R}(T^*) \subseteq \mathcal{D}(\bar{A}_0)$

implies (f1), (f2). Indeed, if (e) holds then $\mathcal{D}(\bar{A}) \subseteq \mathcal{D}(\operatorname{ad}(T, \bar{A}))$. Furthermore, $\bar{A}T \in \mathbf{B}(\mathcal{K})$, by the closed graph theorem. Since $\mathcal{R}(T^*) \subseteq \mathcal{D}(\bar{A}_0) \subseteq \mathcal{D}(A^*)$, we have $A^*T^* \in \mathbf{B}(\mathcal{K})$, again by the closeness of the graph. By $(T\bar{A})^* = A^*T^*$, the operator $T\bar{A}$ is bounded. Hence (f1) is showed, (f2) is obvious. Therefore Theorem 1 is proved as well.

Remark 5. It was shown in [26] that in the (a4) case conditions

(d1) the operators TA and AT are bounded and the domain of the commutator $\mathcal{D}(\operatorname{ad}(T, A))$ is dense in $\mathcal{K}$,
(d2) the operator A_0T^* is densely defined

(presented here in an equivalent form) imply (e), see Proposition 2 and the consecutive remarks. Hence, (d1), (d2) together with (a4) imply (f1), (f2). Therefore, Theorem 3 of [26] can be seen as a special case of Theorem 3 above.

3. Some normal operators

In this section we will concentrate on the (a1) class. We begin with a proposition that unifies Theorem 6 of [26] and Proposition 1 of [17]. If E is the spectral measure of a normal operator N and $\mathbb{D}$ is the closed unit disc then we set

$$\mathcal{B}(N) := \bigcup_{n\in\mathbb{N}} \mathcal{R}\big(E(n\mathbb{D})\big).$$

Proposition 6. *Let $\mathcal{K}$ be a Hilbert space, and let A be a formally normal operator in $\mathcal{K}$. If there exists a normal operator N in $\mathcal{K}$ such that $\mathcal{B}(N) \subseteq \mathcal{D}(A)$ and the spectral measure E of N satisfies the condition*

$$\sup_{n\in\mathbb{N}} \big\|\operatorname{ad}\big(A, E(n\mathbb{D})\big)\big\| < +\infty,$$

then A is essentially normal.

Proof. We set $T_n := E(n\mathbb{D})$ $(n \in \mathbb{N})$ and apply Theorem 1. □

Next let us provide an analogue of Theorem 7 of [26], see also there for references to works on selfadjoint Dirac operators. Take the Hilbert space $\mathcal{K} := (L^2(\mathbb{R}^m))^k$, where $k, m \in \mathbb{N}$ and let $\mathcal{C}_0^\infty(\mathbb{R}^m)$ denote the complex space of infinitely differentiable functions on $\mathbb{R}^m$ with compact supports. Consider the differential operator A in $\mathcal{H}$ given by

$$Au := \mathrm{i}^{-1} \sum_{l=1}^m \alpha_l \frac{\partial u}{\partial x_l} + Qu, \quad u \in \mathcal{D}(A) = \big(\mathcal{C}_0^\infty\big(\mathbb{R}^m\big)\big)^k,$$

where $\alpha_1, \dots, \alpha_m$ are complex $k \times k$ matrices and $Q : \mathbb{R}^m \to \mathbb{C}^{k\times k}$ is a locally integrable matrix-valued function. Note that $(\mathcal{C}_0^\infty(\mathbb{R}^m))^k \subseteq \mathcal{D}(A^*)$ and

$$A^*u = \mathrm{i}^{-1} \sum_{l=1}^m \alpha_l^* \frac{\partial u}{\partial x_l} + Q^*u, \quad u \in \big(\mathcal{C}_0^\infty\big(\mathbb{R}^m\big)\big)^k.$$

A direct calculation shows that the following conditions

$$\begin{aligned} &\alpha_l^*\alpha_r = \alpha_l\alpha_r^*, && \text{for } r, l = 1, \dots, m; \\ &Q(x)Q^*(x) = Q^*(x)Q(x) && \text{for a.e. } x \in \mathbb{R}^m; \\ &\alpha_l^*Q(x) = \alpha_l Q^*(x), && \text{for a.e. } x \in \mathbb{R}^m,\ l = 1, \dots, m \end{aligned} \tag{10}$$

imply $\|Au\| = \|A^*u\|$ for $u \in (\mathcal{C}_0^\infty(\mathbb{R}^m))^k$, i.e. formal normality of A. We say that Q *satisfies the local Hölder condition* if for every $n \in \mathbb{N}$ there exists a $b_n \in (0,1]$ such that

$$\sup_{|x|,|y| \leqslant n,\ x \neq y} \frac{|Q(x) - Q(y)|}{|x-y|^{b_n}} < \infty,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$ and $\mathbb{R}^k$.

Proposition 7. *Assume that conditions* (10) *hold and that the function Q satisfies the local Hölder condition. Then A is essentially normal in $\mathcal{K}$.*

Sketch of the proof. We apply Theorem 1 to the (a1) instance. The construction of the sequence T_n follows exactly the same lines as in the proof of Theorem 7 of [26]. □

4. Infinite *H*-selfadjoint matrices

In [6] Cichoń, Stochel and Szafraniec investigated symmetric integral and matrix operators. The main tools were the domination techniques from their previous paper [5] based on the computation of the first and second commutator. The discussion on applicability of these criteria in the Jacobi matrix case can be found in [5], in the present work we will show how the first commutator reasonings can be applied to H-symmetric operators, restricting to the matrix operators on $\ell^2 = \ell^2(\mathbb{N})$ ($\mathbb{N} = \{1,2,\ldots\}$). By ℓ^2_0 we denote the space of all complex sequences with finite number of nonzero entries. Given a matrix $[a_{k,l}]_{k,l\in\mathbb{N}}$, we define the matrix operator $\tilde{A}$ by

$$\mathcal{D}(\tilde{A}) = \left\{ \{\xi_k\}_{k\in\mathbb{N}} \in \ell^2 \colon \sum_{k\in\mathbb{N}} \left| \sum_{l\in\mathbb{N}} |a_{k,l}\xi_l| \right|^2 < +\infty \right\},$$

$$\tilde{A}\{\xi_k\}_{k\in\mathbb{N}} = \left\{ \sum_{l\in\mathbb{N}} a_{k,l}\xi_l \right\}_{k\in\mathbb{N}}.$$

Let us suppose that the matrices $[h_{k,l}]_{k,l\in\mathbb{N}}$ and $[g_{k,l}]_{k,l\in\mathbb{N}}$ have the following properties:

(h1) $[h_{k,l}]_{k,l\in\mathbb{N}}$ and $[g_{k,l}]_{k,l\in\mathbb{N}}$ are Hermitian-symmetric matrices;
(h2) $[g_{k,l}]_{k,l\in\mathbb{N}}$ is a band matrix, i.e. there exists a $p \in \mathbb{N}$ such that $g_{k,l} = 0$ for $|k-l| > p$;
(h3) $s_g := \sup_{k,l\in\mathbb{N}} |g_{k,l}| < +\infty$ and $[h_{k,l}]_{k,l\in\mathbb{N}}$ defines a bounded operator;
(h4) $\sum_{j\in\mathbb{N}} h_{k,j} g_{j,l} = \sum_{j\in\mathbb{N}} g_{k,j} h_{j,l} = \delta_{k,l}$ for $k,l \in \mathbb{N}$.

Then $[h_{k,l}]_{k,l\in\mathbb{N}}$ and $[g_{k,l}]_{k,l\in\mathbb{N}}$ define bonded, selfadjoint operators on ℓ^2, which will be called G and H, respectively; obviously $G = H^{-1}$. An example of such a matrix $[h_{k,l}]_{k,l\in\mathbb{N}}$, additionally equal to $[g_{k,l}]_{k,l\in\mathbb{N}}$, is a block-diagonal matrix with each block on the diagonal being of the anti-diagonal form

$$\pm \begin{pmatrix} 0 & \ldots & 1 \\ \vdots & & \vdots \\ 1 & \ldots & 0 \end{pmatrix},$$

and with the size of all blocks being bounded from above. The proposition below is an H-symmetric version of Theorem 13 of [6].

Proposition 8. *Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers and $m \geqslant 0$ be an integer such that the matrices*

$$\left[\frac{|a_{k,l+q}|}{1+|c_l|^m} \right]_{k,l\in\mathbb{N}}, \quad q \in \{-p,\ldots,p\} \tag{11}$$

(*with $a_{k,r} := 0$ for $r \leqslant 0$*) *and*

$$\left[|a_{k,l}| \frac{|c_k - c_l|}{1+|c_k|+|c_l|} \right]_{k,l\in\mathbb{N}} \tag{12}$$

define bounded operators on $\mathcal{K}$. If

$$\sum_{q=-p}^{p} a_{k,l+q} g_{l+q,l} = \sum_{q=-p}^{p} g_{k,k+q} \bar{a}_{l,k+q}, \quad k,l \in \mathbb{N}, \tag{13}$$

then the operator $A = \tilde{A}|_{\ell^2_0(\mathbb{N})}$ is essentially H-selfadjoint and $\tilde{A} = \bar{A}$.

Note that (11) implies that $\ell_0^2 \subseteq \mathcal{D}(A)$ and condition (13) obviously means that A is H-symmetric. For the proof of the proposition we will need the following lemma, also to be used in the next section. As usually, $\rho(S)$ stands for the resolvent set of S.

Lemma 9. *Let A be a closable, densely defined operator, let S be a closed densely defined operator and let $z \in \rho(S)$. If*

$$\mathcal{D}(S^m) \subseteq \mathcal{D}(\bar{A}), \qquad \mathcal{D}(S^{*m}) \subseteq \mathcal{D}(A^*) \quad \text{for some } m \in \mathbb{N},$$

then $\operatorname{ad}((S-z)^{-1}, \bar{A})$ is a closable densely defined operator. If it is additionally bounded then

$$\left\|\operatorname{ad}\left((S-z)^{-m}, \bar{A}\right)\right\| \leqslant m\left\|(S-z)^{-1}\right\|^{m-1}\left\|\operatorname{ad}\left((S-z)^{-1}, \bar{A}\right)\right\|. \tag{14}$$

Proof. First note that since $z \in \rho(S)$, one has $\bar{z} \in \rho(S^*)$. Hence, the operators S^m and S^{*m} are closed and densely defined with nonempty resolvent sets [7, Thm. VII.9.7]. Since $(S-z)^{-1}\mathcal{D}(S^m) = \mathcal{D}(S^{m+1}) \subseteq \mathcal{D}(\bar{A})$, the commutator $\operatorname{ad}((S-z)^{-1}, \bar{A})$ is densely defined. Furthermore, note that

$$\operatorname{ad}\left((S-z)^{-1}, \bar{A}\right)^* \supseteq \operatorname{ad}\left(A^*, \left(S^* - \bar{z}\right)^{-1}\right).$$

The domain of the operator on the right-hand side contains $\mathcal{D}(S^{*m})$, which is dense in $\mathcal{K}$. By von Neumann's theorem $\operatorname{ad}((S-z)^{-1}, \bar{A})$ is closable. Suppose now that it is also bounded. Since $\mathcal{D}(S^m)$ is dense in $\mathcal{K}$, the formula ([5, Prop. 2(i)])

$$\operatorname{ad}\left((S-z)^{-m}, \bar{A}\right)f = \sum_{j=0}^{m-1}(S-z)^{-j}\operatorname{ad}\left((S-z)^{-1}, \bar{A}\right)(S-z)^{-m+1+j}f, \quad f \in \mathcal{D}\left(S^m\right)$$

gives the desired estimate. $\square$

Proof of Proposition 8. First note that by (11) we have $\ell_0^2 \subseteq \mathcal{D}(A)$. Now define the selfadjoint operator S by the diagonal matrix $[\delta_{k,l}c_l]_{k,l\in\mathbb{N}}$. By (11) with $q = 1$ we obtain that $A(S^m - z)^{-1}$ is a bounded operator, hence $\bar{A}(S^m - z)^{-1} \in \mathbf{B}(\ell^2)$ and consequently

$$\mathcal{D}\left(S^m\right) \subseteq \mathcal{D}(\bar{A}). \tag{15}$$

Let now $\{\xi_l\}_{l\in\mathbb{N}} \in \ell_0^2$, then $G(S^m - \bar{z})^{-1}\{\xi_l\}_{l\in\mathbb{N}} \in \ell_0^2 = \mathcal{D}(A)$ and

$$\begin{aligned} AG\left(S^m - \bar{z}\right)^{-1}\{\xi_l\}_{l\in\mathbb{N}} &= AG\left\{\frac{\xi_l}{c_l^m - \bar{z}}\right\}_{l\in\mathbb{N}} \\ &= \left\{\sum_{k\in\mathbb{N}} a_{r,k} \sum_{|k-l|\leqslant p} g_{k,l}\frac{\xi_l}{c_l^m - \bar{z}}\right\}_{r\in\mathbb{N}} = \sum_{q=-p}^{p}\left\{\sum_{k\in\mathbb{N}} a_{r,l+q}\, g_{l+q,l}\frac{\xi_l}{c_l^m - \bar{z}}\right\}_{r\in\mathbb{N}}. \end{aligned}$$

It follows now easily from (11) and the assumption (h3) that the operator $C := AG(S^m - \bar{z})^{-1}$ is bounded on ℓ_0^2. Since $C^* \supseteq (S^{*m} - \bar{z})^{-1}GA^*$, C is closable. Hence, $\bar{A}G(S^m - \bar{z})^{-1} = \bar{C} \in \mathbf{B}(\ell^2)$ and consequently

$$\mathcal{D}\left(S^m\right) \subseteq \mathcal{D}(\bar{A}_0), \tag{16}$$

where $A_0 = HAG$, according to (a4). This together with (15) implies that assumption (1) is satisfied with

$$T_n = n^m(S - n\mathrm{i})^{-m}, \quad n \in \mathbb{N}.$$

Obviously, T_n tends with $n \to \infty$ to I_{ℓ^2} in the strong operator topology. To apply Theorem 1 one needs to show that $(T_n)_{n\in\mathbb{N}}$ and A satisfy (2). Observe that for $\xi = \{\xi_k\}_{k\in\mathbb{N}} \in \ell_0^2$ one has

$$\operatorname{ad}\left((S - n\mathrm{i})^{-1}, \bar{A}\right)\xi = \left\{\sum_{l\in\mathbb{N}} \frac{a_{kl}\xi_l(c_k - c_l)}{(n\mathrm{i} - c_k)(n\mathrm{i} - c_l)}\right\}_{k\in\mathbb{N}}.$$

Since (cf. [6, p. 769])

$$\frac{n}{|n\mathrm{i} - c_k||n\mathrm{i} - c_l|} \leqslant \frac{\sqrt{3}}{1 + |c_k| + |c_l|}, \quad n, k, l \in \mathbb{N}$$

we conclude that

$$n\left\|\operatorname{ad}\left((S - n\mathrm{i})^{-1}, A\right)\xi\right\|^2 \leqslant 3\|K\|^2\|\xi\|^2, \quad \xi \in \ell_0^2,\ n \in \mathbb{N},$$

where K is the bounded operator given by (12). Thanks to (15) and (16) we can apply Lemma 9 and obtain that the commutator $\operatorname{ad}((S - n\mathrm{i})^{-1}, \bar{A})$ is closable. Hence, it is bounded and

$$n\left\|\operatorname{ad}\left((S - n\,\mathrm{i})^{-1}, \bar{A}\right)\right\| \leqslant \sqrt{3}\|K\|, \quad n \in \mathbb{N}.$$

By the second part of Lemma 9

$$\sup_{n\in\mathbb{N}} n^m \left\| \operatorname{ad}\left((S-n\mathrm{i})^{-m}, \bar{A}\right)\right\| < +\infty,$$

which is the desired inequality (2). Applying Theorem 1 we get $\bar{A}_0 = A^*$, i.e. the operator A is essentially H-selfadjoint. Since $\tilde{A}$ is H-symmetric and contains $\bar{A}$, one has $\tilde{A} = \bar{A}$. □

Proposition 10. *Suppose that we are given real numbers $d \geqslant 0$, $s \geqslant 0$, $\alpha > 2$. If (13) is satisfied and*

$$|a_{k,l}| \leqslant \begin{cases} d(1+k+l)/(|k-l|^{\alpha}), & k \neq l, \\ d(k+1)^s, & k = l, \end{cases} \quad k,l \geqslant 0 \tag{17}$$

the operator $A = \tilde{A}|_{\ell^2_0(\mathbb{N})}$ is essentially H-selfadjoint and $\tilde{A} = \bar{A}$.

This proposition has again its symmetric origin in [6], namely of Proposition 14. Note that besides the assumption of H-symmetry in (13) the matrices $[g_{kl}]_{kl\in\mathbb{N}}$ and $[g_{kl}]_{kl\in\mathbb{N}}$ are not involved in the assumptions.

Proof. We need to show that (cf. [6])

$$\sum_{k,l\in\mathbb{N}} \frac{|a_{k,l+q}|^2}{(1+|c_l|^m)^2} < +\infty, \quad q = -p, \dots, p, \tag{18}$$

with $c_l = l$, which will guarantee boundedness of all operators in (11). It was shown in [6] that

$$\sum_{k\in\mathbb{N}} |a_{k,l}|^2 \leqslant \mathcal{O}\left(l^2 + l^{2s}\right).$$

Hence,

$$\sum_{k\in\mathbb{N}} |a_{k,l+q}|^2 \leqslant \mathcal{O}\left((l+q)^2 + (l+q)^{2s}\right) = \mathcal{O}\left(l^2 + l^{2s}\right), \quad q = -p, \dots, p$$

and (18) holds with $m > s+3/2$. Boundedness of the operator in (12) follows the same lines as in the proof of Proposition 14 of [6]. □

5. Towards commutative domination

In this section we will show a relation between the results on commutative [22,24,27] and noncommutative domination [5,6,25,26]. One should mention here the work by Nelson [16], which deals with the symmetric case and analytic vectors. Nevertheless, the aim of the present paper is to consider classes different then symmetric operators using simple graph arguments only. We say that $\mathcal{E} \subseteq \mathcal{D}(S)$ is a *core for S* if the graph of S is contained in the closure of the graph $S|_{\mathcal{E}}$. The symbol $\mathcal{D}^{\infty}(A)$ stands for $\bigcap_{n=0}^{\infty} \mathcal{D}(A^n)$.

Theorem 11. *Let A be a closable, densely defined operator, let $A_0 \subseteq A^*$ and let S be a closed densely defined operator such that there exists a sequence $(z_n)_{n=0}^{\infty} \subseteq \rho(S)$ satisfying*

$$\operatorname*{WOT\,lim}_{n\to\infty} z_n (S - z_n)^{-1} = I_{\mathcal{K}}. \tag{19}$$

Assume that

(i) $\mathcal{D}^{\infty}(S) \subseteq \mathcal{D}(\bar{A})$, $\mathcal{D}^{\infty}(S^*) \subseteq \mathcal{D}(\bar{A}_0)$,
(ii) $\operatorname{ad}(A_0, S^*)$ *is densely defined,*
(iii) *there exists a linear subspace $\mathcal{D} \subseteq \mathcal{D}(\operatorname{ad}(S, \bar{A}))$, which is a core for S and S dominates* $\operatorname{ad}(S, \bar{A})$ *on $\mathcal{D}$,*

then $\bar{A}_0 = A^*$. If, additionally, the resolvent set of A is nonempty and

(i′) $\mathcal{D}(S) \subseteq \mathcal{D}(\bar{A})$,
(iii′) $\operatorname{ad}(S, \bar{A}) f = 0$ *for $f \in \mathcal{D}$,*

then the resolvents of A and S commute.

The problem of existence of a sequence $(z_n)_{n=0}^{\infty}$ satisfying (19) was discussed in [26] in detail. In case S is (similar to) a selfadjoint operators in Hilbert spaces such a sequence exists. Note that precise knowledge of the sequence is not necessary to apply the theorem.

Proof. By assumption (ii) and von Neumann's theorem we get $\operatorname{ad}(S, A)$ closable. Standard domination technique (see e.g. Lemma 1 of [26]) gives

$$\mathcal{D}(S) = \mathcal{D}(\overline{S|_{\mathcal{D}}}) \subseteq \mathcal{D}\big(\overline{\operatorname{ad}(S, \bar{A})}\big).$$

Hence, S dominates $\operatorname{ad}(S, \bar{A})$, i.e. for some $c \geqslant 0$ we have

$$\big\|\overline{\operatorname{ad}(S, \bar{A})} f\big\| \leqslant c\big(\|f\| + \|Sf\|\big), \quad f \in \mathcal{D}(S). \tag{20}$$

We apply (20) to $f := (S - z_n)^{-1} g \in \mathcal{D}(S)$ with arbitrary $n \in \mathbb{N}$ and $g \in \mathcal{K}$, getting

$$\begin{aligned}\big\|\overline{\operatorname{ad}(S, \bar{A})}(S - z_n)^{-1} g\big\| &\leqslant c\big(\big\|(S - z_n)^{-1} g\big\| + \big\|S(S - z_n)^{-1} g\big\|\big)\\ &\leqslant c\big(\big\|(S - z_n)^{-1}\big\| + \big\|z_n(S - z_n)^{-1} + I\big\|\big)\|g\|.\end{aligned} \tag{21}$$

It is now apparent that there exists a constant $d \geqslant 0$, such that

$$\big\|\overline{\operatorname{ad}(S, \bar{A})}(S - z_n)^{-1}\big\| \leqslant d, \quad n \in \mathbb{N}. \tag{22}$$

Fix $z \in \rho(S)$, then

$$\begin{aligned}\operatorname{ad}\big((S - z)^{-1}, \bar{A}\big) &\supseteq (S - z)^{-1} \bar{A}(S - z)(S - z)^{-1} - (S - z)^{-1}(S - z)\bar{A}(S - z)^{-1}\\ &= (S - z)^{-1} \operatorname{ad}\big(\bar{A}, (S - z)\big)(S - z)^{-1} = (S - z)^{-1} \operatorname{ad}(\bar{A}, S)(S - z)^{-1} =: C.\end{aligned}$$

By (21) the operator C is bounded, furthermore, it is also densely defined. Indeed, the linear space $\mathcal{F} = (S - z)\mathcal{D}$ is contained in $\mathcal{D}(C)$ because $(S - z)^{-1}\mathcal{F} = \mathcal{D} \subseteq \mathcal{D}(\operatorname{ad}(\bar{A}, S))$ and $\mathcal{F}$ is dense in $\mathcal{K}$ because $z \in \rho(S)$ and $\mathcal{D}$ is a core for S.

By Proposition 8.1 of [25] there exists $m \in \mathbb{N}$ such that $\mathcal{D}(S^m) \subseteq \mathcal{D}(\bar{A})$ and $\mathcal{D}(S^{*m}) \subseteq \mathcal{D}(\bar{A}_0)$. By Lemma 9 the commutator $\operatorname{ad}((S - z)^{-1}, \bar{A})$ is closable. Since it contains the densely defined and bounded operator C, its closure belongs to $\mathbf{B}(\mathcal{K})$. By (22) we have

$$|z_n| \big\|\operatorname{ad}\big((S - z_n)^{-1}, \bar{A}\big)\big\| \leqslant |z_n| \big\|(S - z_n)^{-1}\big\| \big\|\operatorname{ad}(\bar{A}, S)(S - z_n)^{-1}\big\| \leqslant td, \tag{23}$$

with $t = \sup_{n \in \mathbb{N}} \|z_n(S - z_n)^{-1}\|$, which is finite because of (19). By the second part of Lemma 9 we have

$$\sup_{n \in \mathbb{N}} |z_n|^m \big\|\operatorname{ad}\big((S - z_n)^{-m}, \bar{A}\big)\big\| < \infty. \tag{24}$$

By Theorem 1 applied to $T_n = z_n^m (S - z_n)^{-m}$ we get $\bar{A}_0 = A^*$.

To prove the second statement of the theorem fix $z \in \rho(S)$ and $w \in \rho(A)$. One can easily check, that (iii′) implies that $Cf = 0$ for $f \in \mathcal{D}(C)$, consequently $\overline{\operatorname{ad}((S - z)^{-1}, A)} = 0$. Observe that

$$(A - w)^{-1} \operatorname{ad}\big((S - z)^{-1}, A\big)(A - w)^{-1} = \operatorname{ad}\big((A - w)^{-1}, (S - z)^{-1}\big),$$

where both operators are in $\mathbf{B}(\mathcal{K})$ by (i′). In consequence both of them are zero. □

6. Differential operators

As an application of Theorem 11 consider the differential operator

$$Au := \mathrm{i}^{-1} \sum_{l=1}^{m} Q_l \frac{\partial u}{\partial x_l}, \quad u \in \mathcal{D}(A) = \big(\mathcal{C}_0^{\infty}\big(\mathbb{R}^m\big)\big)^k,$$

in the Hilbert space $\mathcal{K} = (L^2(\mathbb{R}^m))^k$ ($k, m \in \mathbb{N}$). We assume that $Q_1, \ldots, Q_m : \mathbb{R}^m \to \mathbb{C}^{k \times k}$ are $\mathcal{C}^2$-functions. First let us also note, that if P_1, P_2 are complex polynomials of m variables then the operator $P_1(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m})$ dominates $P_2(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m})$ on $(\mathcal{C}_0^{\infty}(\mathbb{R}^m))^k$ if and only if for some $c > 0$

$$\big|P_2(\zeta)\big| \leqslant c\big(1 + \big|P_1(\zeta)\big|\big), \quad \zeta \in \mathbb{R}^m. \tag{25}$$

Indeed, the case $k = 1$ is well known (see e.g. [14]) and the multidimensional case is a simple consequence of the one-dimensional one. For other types of domination inequalities for differential operators we refer the reader to [12,15] and the papers quoted therein. Let us introduce the following notation:

$$\begin{aligned}Q(x) &= \big(Q_r^*(x) Q_l(x)\big)_{r,l=1}^{m} \in \mathbb{C}^{mk \times mk}, \quad x \in \mathbb{R}^m,\\ Q^{(*)}(x) &= \big(Q_r(x) Q_l^*(x)\big)_{r,l=1}^{m} \in \mathbb{C}^{mk \times mk}, \quad x \in \mathbb{R}^m.\end{aligned}$$

Proposition 12. *If*

$$\frac{\partial Q_j}{\partial x_i}, \frac{\partial^2 Q_j}{\partial x_i x_h}, \in L^\infty\big(\mathbb{R}^m\big), \quad h, i, j = 1, \dots, m \tag{26}$$

and for some $c_1 > 0$ one has

$$c_1^{-1} Q(x) \leqslant Q^{(*)}(x) \leqslant c_1 Q(x), \quad x \in \mathbb{R}^m \tag{27}$$

and for some $c_2 > 0$

$$Q(x) \leqslant c_2 I_{\mathbb{C}^{mk\times mk}}, \quad x \in \mathbb{R}^m, \tag{28}$$

then $\mathcal{D}(\bar{A}) = \mathcal{D}(A^*)$.

Proof. First we will show that the graph norms of A and A^* are equivalent on $(\mathcal{C}_0^\infty(\mathbb{R}^m))^k$. Denoting by $\langle\cdot,-\rangle$ the standard inner product in $\mathbb{C}^k$ and $\mathbb{C}^{mk}$ and setting

$$\partial u := \left(\frac{\partial u_1}{\partial x_1}, \dots, \frac{\partial u_k}{\partial x_1}, \dots, \frac{\partial u_1}{\partial x_m}, \dots, \frac{\partial u_k}{\partial x_m}\right)$$

one has

$$\begin{aligned}\|Au\|^2 &= \int_{\mathbb{R}^m} \sum_{l,r=1}^m \left\langle Q_r^*(x) Q_l(x) \frac{\partial u}{\partial x_l}(x), \frac{\partial u}{\partial x_r}(x)\right\rangle dx \\ &= \int_{\mathbb{R}^m} \big\langle Q(x)\partial u(x), \partial u(x)\big\rangle dx.\end{aligned} \tag{29}$$

Furthermore, note that

$$A^* u = \mathrm{i}^{-1} \sum_{l=1}^m \frac{\partial}{\partial x_l} Q_l^* u = \mathrm{i}^{-1} \sum_{l=1}^m \frac{\partial Q_l^*}{\partial x_l} u + \mathrm{i}^{-1} \sum_{l=1}^m Q_l^* \frac{\partial u}{\partial x_l} u.$$

By (26) the first summand on the right-hand side is a bounded operator of u. Thus the graph norms of A^* and $B = \sum_{l=1}^m Q_l^* \frac{\partial u}{\partial x_l}$ are equivalent on $(\mathcal{C}_0(\mathbb{R}^m))^k$. Furthermore, for $u \in (\mathcal{C}_0(\mathbb{R}^m))^k$ one has

$$\begin{aligned}\|Bu\|^2 &= \int_{\mathbb{R}^m} \sum_{l,r=1}^m \left\langle Q_r(x) Q_l^*(x) \frac{\partial u}{\partial x_l}(x), \frac{\partial u}{\partial x_r}(x)\right\rangle dx \\ &= \int_{\mathbb{R}^m} \big\langle Q^{(*)}(x)\partial u(x), \partial u(x)\big\rangle dx,\end{aligned}$$

which, together with (27) and (29) implies the equivalence of graph norms of B and A, and hence A^* and A, on $(\mathcal{C}_0(\mathbb{R}^m))^k$.

Consider the essentially selfadjoint, nonnegative operator

$$Su = -\frac{\partial^2 u}{\partial x_1^2} - \dots - \frac{\partial^2 u}{\partial x_m^2}, \quad u \in \big(\mathcal{C}_0^\infty\big(\mathbb{R}^m\big)\big)^k = \mathcal{D}(S).$$

Note that by (28) and (29) one has

$$\|Au\|^2 \leqslant c_2 \int_{\mathbb{R}^m} \big\langle \partial u(x), \partial u(x)\big\rangle dx = \langle Su, u\rangle = \big\|S^{1/2}u\big\|^2.$$

Therefore $S^{1/2}$, and in consequence S, dominates A on $(\mathcal{C}_0^\infty(\mathbb{R}^m))^k$. Furthermore,

$$\begin{aligned}\mathrm{i}\cdot\mathrm{ad}(S, A)u &= -\sum_{l,r=1}^m \frac{\partial^2}{\partial x_l^2}\left(Q_r \frac{\partial u}{\partial x_r}\right) + \sum_{l,r=1}^m Q_l \frac{\partial}{\partial x_l}\left(\frac{\partial^2 u}{\partial x_r^2}\right) \\ &= -\sum_{l,r=1}^m \frac{\partial Q_r}{\partial x_l} \frac{\partial^2 u}{\partial x_l x_r} - \sum_{l,r=1}^m \frac{\partial^2 Q_r}{\partial x_l^2} \frac{\partial u}{\partial x_r}.\end{aligned}$$

Application of (25) to the right-hand side of the inequality

$$\left\|\mathrm{ad}(S,A)u\right\| \leqslant \sum_{l,r=1}^{m}\left\|\frac{\partial Q_r}{\partial x_l}\right\|_{L^\infty(\mathbb{R}^m)}\left\|\frac{\partial^2 u}{\partial x_l x_r}\right\| + \sum_{l,r=1}^{m}\left\|\frac{\partial^2 Q_r}{\partial x_l^2}\right\|_{L^\infty(\mathbb{R}^m)}\left\|\frac{\partial u}{\partial x_r}\right\|,$$

shows that S dominates $\mathrm{ad}(S,A)$ on $(\mathcal{C}_0^\infty(\mathbb{R}^m))^k$. Hence, by Theorem 11 we get $\mathcal{D}(\bar{A}) = \mathcal{D}(A^*)$. □

Acknowledgments

The author is indebted to Doctor Dariusz Cichoń, Professor Jan Stochel and Professor Franciszek Hugon Szafraniec for an inspiration and valuable discussions.

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