Direct and converse results for multivariate generalized Bernstein polynomials

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Abstract

In this paper, the following generalization of multivariate Bernstein polynomials has been studied:

\[ B_n(f; x_1, x_2, \ldots, x_m) = \sum_{k_1, k_2, \ldots, k_m = 0}^{n} f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_m}{b_n} \right) \prod_{j=1}^{m} \left( \binom{n}{k_j} x_j^{k_j} (1-x_j)^{n-k_j} \right), \]

where \((b_n)\) is a sequence of positive numbers such that \(\lim_{n \to \infty} n/b_n = 1\) and direct and inverse theorems for this polynomials have been presented.

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1. Introduction

Direct and inverse theorems for Bernstein type linear positive operators have been investigated by many authors [1–13]. Direct theorems concern the rate of converges of these operators. Inverse theorems concern the task of determining the class of functions. These problems have been solved by several authors by means of different techniques of the main tools being the classical modulus of continuity. Some of them are those written in [1,2,4,13].

Some inverse theorems have been proved in [2] for \(B_n f\) denoting the, Bernstein polynomials on the two-dimensional simplex \(S = \{(x, y) : x + y \leq 1, x \geq 0, y \geq 0\}\).

The aim of this paper is to prove direct and inverse theorems and to give some examples of numerical solutions for the multivariate generalized Bernstein polynomials in (1).

Let \(D = [0, 1] \times [0, 1] \times \cdots \times [0, 1] = [0, 1]^m, m \in \mathbb{N}\) and \(C_I\) is a real-valued continuous functions space on \(D\). \(C_I(D)\) is a linear normed space with the uniform norm

\[ \| f \|_{C_I(D)} = \max_{(x_1, x_2, \ldots, x_m) \in D} | f(x_1, x_2, \ldots, x_m) |. \]

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Definition 1. Let \( f \in C_{IR}^m(D) \) and \( \delta \) is a positive number. Full continuity modulus of function \( f \) is

\[
\omega(f; \delta) = \max_{(x_1, x_2, \ldots, x_m), (y_1, y_2, \ldots, y_m) \in D} \sqrt{\sum_{j=1}^{m} (x_j - y_j)^2} \leq \delta
\]

and its partial continuity moduli with respect to \( x_i \) is

\[
\omega^{(i)}(f; \delta) = \max_{0 \leq x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m \leq 1} \max_{|x_j - y_j| \leq \delta} \left| f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m) \right|, \quad i = 1, 2, \ldots, m.
\]

The best known of two of features of continuity modulus are as such below

\[
\lim_{\delta \to 0} \omega(f; \delta) = 0
\]

and for any \( \lambda > 0 \),

\[
\omega(f; \lambda \delta) \leq (\lambda + 1) \omega(f; \delta).
\]

The same properties are satisfied by partial continuity moduli.

Definition 2. Let \( f \) be continuous on \( D \) and \((b_n)\) is a sequence of positive numbers such that \( \lim_{n \to \infty} n/b_n = 1 \), then a generalization of multivariate Bernstein polynomials is defined as follows:

\[
B_n(f; x_1, \ldots, x_m) = \sum_{k_1, \ldots, k_m=0}^{n} f \left( \frac{k_1}{b_n}, \ldots, \frac{k_m}{b_n} \right) \prod_{j=1}^{m} \binom{n}{k_j} x_j^{k_j} (1 - x_j)^{n-k_j}
\]

(1)

where \( \sum_{k_1, k_2, \ldots, k_m=0}^{n} = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_m=0}^{n} \).

When \( n \) is taken instead of \( b_n \) in (1), classical Bernstein polynomials are obtained.

2. Convergence analysis

Theorem 3. Let \( L_n \) be a sequence of linear positive operators, acting from \( C_{IR}^m(D) \) to \( C_{IR}^m(D) \) and satisfying \((m + 2)\) conditions

\[
\|L_n(1; x) - 1\|_{C_{IR}^m(D)} = 0, \quad n \to \infty, \tag{2}
\]

\[
\|L_n(t_j; x) - x_j\|_{C_{IR}^m(D)} = 0, \quad n \to \infty, \quad j = 1, 2, \ldots, m, \tag{3}
\]

\[
\|L_n(|t|^2; x) - |x|^2\|_{C_{IR}^m(D)} = 0, \quad n \to \infty \tag{4}
\]

where \( t = (t_1, t_2, \ldots, t_m) \), \( x = (x_1, x_2, \ldots, x_m) \); then for any function \( f \in C_{IR}^m(D) \) the result given below is obtained:

\[
\|L_n(f; x) - f(x)\|_{C_{IR}^m(D)} = 0, \quad n \to \infty.
\]

The proof of this theorem is shown in [7].
Theorem 4. If $f$ is continuous in $D$ and $B_n(f; x_1, \ldots, x_m)$ is the generalization of multivariate Bernstein polynomials of $f$, then $B_n(f; x_1, \ldots, x_m)$ uniformly converges to $f$ as $n \to \infty$.

Proof. It is obvious that $B_n$ is a linear positive operator and the equations and inequality given below are easy to show:

$$B_n(1; x_1, x_2, \ldots, x_m) = 1,$$

$$B_n(t_j; x_1, x_2, \ldots, x_m) = \left(\frac{n}{b_n} - 1\right)x_j + x_j, \quad j = 1, 2, \ldots, m,$$

$$B_n(|t|^2; x_1, x_2, \ldots, x_m) = |x|^2 + \sum_{j=1}^{m} \left(\left(\frac{n^2}{b_n^2} - 1\right)x_j^2 + \frac{n^2}{b_n^2}x_j(1 - x_j)\right)$$

$$\leq |x|^2 + m \left[\left(\frac{n^2}{b_n^2} - 1\right) + \frac{n^2}{4b_n^2}\right].$$

By Theorem 3,

$$\|B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)\|_{C_{Lip}(D)} = 0, \quad n \to \infty$$

is obtained. □

Example 5. When $m = 2$ is chosen the multivariate Bernstein polynomials of $f(x_1, x_2)$ is as in the following form:

$$B_n(f; x_1, x_2) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} f\left(\frac{k_1}{b_n}, \frac{k_2}{b_n}\right) \binom{n}{k_1} \binom{n}{k_2} x_1^{k_1}(1 - x_1)^{n-k_1}x_2^{k_2}(1 - x_2)^{n-k_2}. \quad (5)$$

For $n = 1, 2, 3, 5, 10$ and $b_n = 1/(\ln(n+1) - \ln(n))$, the convergence of $B_n(f; x_1, x_2)$ to $f(x_1, x_2) = (x_1^5 + x_2^5)/100$ will be illustrated in Fig. 1.
Example 6. When $m = 3$ is chosen the multivariate Bernstein polynomials of $f(x_1, x_2, x_3)$ is as in the following form:

$$B_n(f; x_1, x_2, x_3) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \sum_{k_3=0}^{n} f\left(\frac{k_1}{b_n}, \frac{k_2}{b_n}, \frac{k_3}{b_n}\right) {n \choose k_1} {n \choose k_2} {n \choose k_3}$$

$$\times x_1^{k_1}(1-x_1)^{n-k_1}x_2^{k_2}(1-x_2)^{n-k_2}x_3^{k_3}(1-x_3)^{n-k_3}. \quad (6)$$

For $n = 2, 3, 5, 8$ and $b_n = 1/(\ln(n + 1) - \ln(n))$, the convergence of $B_n(f; x_1, x_2, x_3)$ to $f(x_1, x_2, x_3) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 + (x_3 - \frac{1}{2})^2 - \frac{1}{4}$ will be illustrated in Fig. 2.
3. Rate of convergence

Theorem 7. If \( f \in C_{IR^m}(D) \), then for all \((x_1, x_2, \ldots, x_m) \in D\), the following inequalities

\[
|B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)| \leq 2\omega \left( \frac{(n/b_n - 1)^2 + n^4b_n^2}{4\delta_n^2} \right)^{1/2},
\]

\[
|B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)| \leq 2 \sum_{j=1}^{m} \omega^{(j)} \left( \frac{(n/b_n - 1)^2 + n^4b_n^2}{4\delta_n^2} \right)^{1/2}
\]

are obtained.

Proof. When the difference \( B_n f - f \) is represented as follows:

\[
B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m) = \sum_{k_1, k_2, \ldots, k_m=0}^{n} \left\{ f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_m}{b_n} \right) - f(x_1, x_2, \ldots, x_m) \right\}
\]

\[
\times \prod_{j=1}^{m} \left( \left( \frac{n}{k_j} \right) x_j^{k_j} (1 - x_j)^{n-k_j} \right),
\]

the following inequality is obtained.

\[
|B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)| \leq \sum_{k_1, k_2, \ldots, k_m=0}^{n} \left| f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_m}{b_n} \right) - f(x_1, x_2, \ldots, x_m) \right|
\]

\[
\times \prod_{j=1}^{m} \left( \left( \frac{n}{k_j} \right) x_j^{k_j} (1 - x_j)^{n-k_j} \right)
\]

\[
\leq \sum_{k_1, k_2, \ldots, k_m=0}^{n} \omega \left( \frac{\left( \sum_{j=1}^{m} \left( \frac{k_j}{b_n} - x_j \right)^2 \right)^{1/2}}{\delta_n} \right)
\]

\[
\times \prod_{j=1}^{m} \left( \left( \frac{n}{k_j} \right) x_j^{k_j} (1 - x_j)^{n-k_j} \right),
\]

By using well-known properties of the modulus of continuity and applying the Cauchy–Schwartz inequality, we obtain the formula

\[
|B_n(f; x_1, \ldots, x_m) - f(x_1, \ldots, x_m)| \leq \omega(f; \delta_n) \left\{ \frac{1}{\delta_n} \left( \sum_{k_1, k_2, \ldots, k_m=0}^{n} \sum_{j=1}^{m} \left( \frac{k_j}{b_n} - x_j \right)^2 \right)^{2} \right\}^{1/2}
\]

\[
\times \prod_{j=1}^{m} \left( \left( \frac{n}{k_j} \right) x_j^{k_j} (1 - x_j)^{n-k_j} \right) + 1 \right\},
\]

where \( \delta_n \) is the some sequence which tends to zero as \( n \to \infty \).

By taking

\[
\delta_n = \sqrt{m} \left( \left( \frac{n}{b_n} - 1 \right)^2 + \frac{n}{4b_n^2} \right)^{1/2}
\]
Table 1

<table>
<thead>
<tr>
<th>n</th>
<th>Error bound for full continuity modulus of function $f(x_1, x_2)$</th>
<th>Error bound for partial continuity modulus of function $f(x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.414137355</td>
<td>0.579395338</td>
</tr>
<tr>
<td>20</td>
<td>0.299955643</td>
<td>0.410987842</td>
</tr>
<tr>
<td>30</td>
<td>0.247397360</td>
<td>0.335834786</td>
</tr>
<tr>
<td>40</td>
<td>0.215519847</td>
<td>0.290939670</td>
</tr>
<tr>
<td>50</td>
<td>0.193536061</td>
<td>0.260272095</td>
</tr>
<tr>
<td>100</td>
<td>0.138193034</td>
<td>0.184099416</td>
</tr>
<tr>
<td>200</td>
<td>0.098385939</td>
<td>0.130195857</td>
</tr>
<tr>
<td>300</td>
<td>0.080573487</td>
<td>0.106308967</td>
</tr>
<tr>
<td>400</td>
<td>0.069903463</td>
<td>0.092068150</td>
</tr>
<tr>
<td>500</td>
<td>0.062599726</td>
<td>0.082349249</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>n</th>
<th>Error bound for full continuity modulus of function $f(x_1, x_2, x_3)$</th>
<th>Error bound for partial continuity modulus of function $f(x_1, x_2, x_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.7975078486</td>
<td>0.7975078488</td>
</tr>
<tr>
<td>20</td>
<td>0.5955678580</td>
<td>0.5955678576</td>
</tr>
<tr>
<td>30</td>
<td>0.4976237940</td>
<td>0.4976237940</td>
</tr>
<tr>
<td>40</td>
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</tr>
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<tr>
<td>100</td>
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</tr>
<tr>
<td>200</td>
<td>0.2046310352</td>
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<tr>
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</tr>
<tr>
<td>400</td>
<td>0.1462498834</td>
<td>0.1462498834</td>
</tr>
<tr>
<td>500</td>
<td>0.1311639507</td>
<td>0.1311639506</td>
</tr>
</tbody>
</table>

in (6), we obtain the following formula:

$$|B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)| \leq 2 \omega \left( f ; \sqrt{\frac{m}{b_n}} \left( \left( \frac{n}{b_n} - 1 \right)^2 + \frac{n^2}{4b_n^2} \right)^{1/2} \right).$$

In the same way, the second inequality

$$|B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)| \leq 2 \sum_{j=1}^{m} \omega_j \left( f ; \left( \left( \frac{n}{b_n} - 1 \right)^2 + \frac{n}{4b_n^2} \right)^{1/2} \right)$$

is proved. □

**Example 8.** The error of the approximation of $f(x_1, x_2) = x_1 \sin(x_2)$ by using polynomial (5), where $b_n = 1/(\ln(n + 1) - \ln(n))$, are listed in Table 1.

**Example 9.** The error of the approximation of $f(x_1, x_2, x_3) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 + (x_3 - \frac{1}{2})^2 - \frac{1}{4}$ by using polynomial (6), where $b_n = 1/(\ln(n + 1) - \ln(n))$, are listed in Table 2.

The values in the tables above by using Maple 9 program in computer have been obtained.

**Corollary 10.** If $f$ satisfies the Lipschitz conditions,

(i) $|f(x_1, x_2, \ldots, x_m) - f(y_1, y_2, \ldots, y_m)| \leq M \left( \sum_{j=1}^{m} (x_j - y_j)^2 \right)^{1/2},$
(ii) \( |f(x_1, x_2, \ldots, x_j, \ldots, x_m) - f(x_1, x_2, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_m)| \leq M_j |x_j - y_j|^2 \), \( j = 1, 2, \ldots, m \), then the inequalities

\[
|B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)| \leq 2M \left( \frac{n}{b_n} - 1 \right)^2 + \frac{n^2}{4b_n^2} \]

are obtained.

4. Inverse theorems

Theorem 11. If \( f \in CIR^m(D) \) is such that

\[
|B_n(f; x_1, x_2, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)| \leq M \left( \frac{1}{n} \right)^x, \quad (0 < x < 1)
\]

for some positive constant \( M \), then \( f \in \text{Lip}_x \).

Proof. The partial derivative of Eq. (1) with respect to \( x_i \) may be written as follows:

\[
\frac{\partial B_n(f; x_1, \ldots, x_i, \ldots, x_m)}{\partial x_i} = \frac{\partial}{\partial x_i} \left[ \sum_{k_1, k_2, \ldots, k_m=0}^{n} f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_i}{b_n}, \frac{k_m}{b_n} \right) \prod_{j=1}^{m} \binom{n}{k_j} x_j^{k_j} (1 - x_j)^{n-k_j} \right]
\]

\[
= \sum_{k_1, k_2, \ldots, k_m=0}^{n} f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_i}{b_n}, \frac{k_m}{b_n} \right) \binom{n}{k_i} x_i^{k_i} (1 - x_i)^{n-k_i}
\]

\[
\times \prod_{j=1 \atop j \neq i}^{m} \binom{n}{k_j} x_j^{k_j} (1 - x_j)^{n-k_j}
\]

\[
- \sum_{k_i=0}^{n} \sum_{k_1, k_2, \ldots, k_{i-1}, k_{i+1}, \ldots, k_m=0}^{n} f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_i}{b_n}, \frac{k_m}{b_n} \right) \frac{1}{k_i!(n-k_i)!} x_i^{k_i-1} (1 - x_i)^{n-k_i}
\]

\[
\times \binom{n}{k_i} x_i^{k_i} (1 - x_i)^{n-k_i}
\]

\[
\times \prod_{j=1 \atop j \neq i}^{m} \binom{n}{k_j} x_j^{k_j} (1 - x_j)^{n-k_j}
\]

\[
\times \prod_{j=1 \atop j \neq i}^{m} \binom{n}{k_j} x_j^{k_j} (1 - x_j)^{n-k_j}
\]

\[
\times \prod_{j=1 \atop j \neq i}^{m} \binom{n}{k_j} x_j^{k_j} (1 - x_j)^{n-k_j}
\]

\[
\times \prod_{j=1 \atop j \neq i}^{m} \binom{n}{k_j} x_j^{k_j} (1 - x_j)^{n-k_j}
\]
By taking absolute value of both sides, we get

\[
\begin{align*}
&= \sum_{k_i=0}^{n-1} \sum_{k_{i+1}, k_{i+1}, \ldots, k_m=0}^{n} f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_i+1}{b_n}, \ldots, \frac{k_m}{b_n} \right) \frac{n(n-1)!}{k_i!(n-k_i-1)!} \\
&\times x_i^k (1-x_i)^{n-k_i-1} \prod_{j=1 \atop j \neq i}^{m} \left( \binom{n}{k_j} x_j^k (1-x_j) \right)^{n-k_j}
\end{align*}
\]

By using the properties of the modulus of continuity, we obtain

\[
\begin{align*}
&\leq n \sum_{k_i=0}^{n-1} \sum_{k_{i+1}, k_{i+1}, \ldots, k_m=0}^{n} \left| f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_i+1}{b_n}, \ldots, \frac{k_m}{b_n} \right) - f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_i}{b_n}, \ldots, \frac{k_m}{b_n} \right) \right| \\
&\times \frac{(n-1)!}{k_i!(n-k_i-1)!} x_i^k (1-x_i)^{n-k_i-1} \prod_{j=1 \atop j \neq i}^{m} \left( \binom{n}{k_j} x_j^k (1-x_j) \right)^{n-k_j}
\end{align*}
\]

By taking absolute value of both sides, we get

\[
\begin{align*}
&\leq n \sum_{k_i=0}^{n-1} \sum_{k_{i+1}, k_{i+1}, \ldots, k_m=0}^{n} \omega^{(i)} \left( f, \frac{1}{b_n} \right) \frac{(n-1)!}{k_i!(n-k_i-1)!} x_i^k (1-x_i)^{n-k_i-1} \\
&\times \prod_{j=1 \atop j \neq i}^{m} \left( \binom{n}{k_j} x_j^k (1-x_j) \right)^{n-k_j}
\end{align*}
\]

By using the properties of the modulus of continuity, we obtain

\[
\begin{align*}
&\leq n \sum_{k_i=0}^{n-1} \sum_{k_{i+1}, k_{i+1}, \ldots, k_m=0}^{n} \omega^{(i)} \left( f, \frac{1}{b_n} \right) \frac{(n-1)!}{k_i!(n-k_i-1)!} x_i^k (1-x_i)^{n-k_i-1} \\
&\times \prod_{j=1 \atop j \neq i}^{m} \left( \binom{n}{k_j} x_j^k (1-x_j) \right)^{n-k_j}
\end{align*}
\]

We get

\[
\begin{align*}
\frac{\partial B_n(f; x_1, \ldots, x_i, \ldots, x_m)}{\partial x_i} = n \sum_{k_i=0}^{n-1} \sum_{k_{i+1}, k_{i+1}, \ldots, k_m=0}^{n} \left\{ f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_i+1}{b_n}, \ldots, \frac{k_m}{b_n} \right) - f \left( \frac{k_1}{b_n}, \frac{k_2}{b_n}, \ldots, \frac{k_i}{b_n}, \ldots, \frac{k_m}{b_n} \right) \right\} \\
\times \frac{(n-1)!}{k_i!(n-k_i-1)!} x_i^k (1-x_i)^{n-k_i-1} \prod_{j=1 \atop j \neq i}^{m} \left( \binom{n}{k_j} x_j^k (1-x_j) \right)^{n-k_j}
\end{align*}
\]
On the other hand, we write

\[
\frac{\partial B_n(f; x_1, \ldots, x_i, \ldots, x_m)}{\partial x_i} \leq n \omega^{(i)}(f; \delta) \left\{ 1 + \frac{1}{\delta b_n} \right\} \\
= \omega^{(i)}(f; \delta) \left\{ n + \frac{n}{\delta b_n} \right\}.
\]

For any pair \( x, y \) of points in \([0, 1]\), we write as follows:

\[
\int_x^y \left| \frac{\partial B_n(f; x_1, \ldots, x_i, \ldots, x_m)}{\partial x_i} \right| \, dx_i \leq \omega^{(i)}(f; \delta) \left\{ n + \frac{n}{\delta b_n} \right\} \int_x^y \, dx_i
\]

\[
\int_x^y \left| \frac{\partial B_n(f; x_1, \ldots, x_i, \ldots, x_m)}{\partial x_i} \right| \, dx_i \leq \omega^{(i)}(f; \delta) |x - y| \left\{ n + \frac{n}{\delta b_n} \right\}
\]

by taking \( \delta_n = 1/n \), we obtain

\[
\int_x^y \left| \frac{\partial B_n(f; x_1, \ldots, x_i, \ldots, x_m)}{\partial x_i} \right| \, dx_i \leq \omega^{(i)}(f; \delta) |x - y| \left\{ \frac{1}{\delta_n} + \frac{n}{\delta b_n} \right\}.
\] (8)

The sequence \( \delta_n \) decreases to zero as \( n \to \infty \). Also \( \delta_{n-1} \leq 2\delta_n \) for \( n = 2, 3, \ldots \). Hence for a given \( 0 < \delta \leq 1 \) there exists a natural number \( n \) such that

\[
\delta_n \leq \delta \leq \delta_{n-1} \leq 2\delta_n.
\] (9)

By using (9) in (8), we get

\[
\int_x^y \left| \frac{\partial B_n(f; x_1, \ldots, x_i, \ldots, x_m)}{\partial x_i} \right| \, dx_i \leq \omega^{(i)}(f; \delta) |x - y| \left\{ 2 + \frac{n}{b_n} \right\} \frac{1}{\delta} \omega^{(i)}(f; \delta)|x - y|.
\] (10)

On the other hand, we write

\[
|f(x_1, \ldots, x_m) - f(y_1, \ldots, y_m)| \leq |f(x_1, x_2, \ldots, x_m) - f(y_1, x_2, \ldots, x_m)|
+ |f(y_1, x_2, \ldots, x_m) - f(y_1, y_2, \ldots, x_m)|
+ \cdots + |f(y_1, \ldots, y_{m-1}, x_m) - f(y_1, \ldots, y_m)|
\]

and

\[
|f(x_1, x_2, \ldots, x_m) - f(y_1, x_2, \ldots, x_m)| \leq |f(x_1, x_2, \ldots, x_m) - B_n(f; x_1, x_2, \ldots, x_m)|
+ |f(y_1, x_2, \ldots, x_m) - B_n(f; y_1, x_2, \ldots, x_m)|
+ \int_{x_1}^{y_1} \left| \frac{\partial}{\partial x} B_n(f; x, x_2, \ldots, x_m) \right| \, dx,
\]

\[
|f(y_1, x_2, \ldots, x_m) - f(y_1, y_2, \ldots, x_m)| \leq |f(y_1, x_2, \ldots, x_m) - B_n(f; y_1, x_2, \ldots, x_m)|
+ |f(y_1, y_2, \ldots, x_m) - B_n(f; y_1, y_2, \ldots, x_m)|
+ \int_{x_2}^{y_2} \left| \frac{\partial}{\partial x} B_n(f; y_1, x, \ldots, x_m) \right| \, dx
\]

\[\vdots\]

\[
|f(y_1, \ldots, y_{m-1}, x_m) - f(y_1, \ldots, y_m)| \leq |f(y_1, \ldots, y_{m-1}, x_m) - B_n(f; y_1, \ldots, y_{m-1}, x_m)|
+ |f(y_1, y_2, \ldots, y_m) - B_n(f; y_1, y_2, \ldots, y_m)|
+ \int_{x_m}^{y_m} \left| \frac{\partial}{\partial x} B_n(f; y_1, y_2, \ldots, x) \right| \, dx.
\]
For all fixed natural numbers $n$, we write as follows:

$$|f(x_1, \ldots, x_m) - f(y_1, \ldots, y_m)| \leq |f(x_1, x_2, \ldots, x_m) - B_n(f; x_1, x_2, \ldots, x_m)|$$

$$+ |f(y_1, x_2, \ldots, x_m) - B_n(f; y_1, x_2, \ldots, x_m)|$$

$$+ |f(y_1, x_2, \ldots, x_m) - B_n(f; y_1, y_2, \ldots, x_m)|$$

$$+ \cdots + |f(y_1, \ldots, y_{m-1}, x_m) - B_n(f; y_1, \ldots, y_{m-1}, x_m)|$$

$$+ |f(y_1, y_2, \ldots, y_m) - B_n(f; y_1, y_2, \ldots, y_m)|$$

$$+ \int_{x_1}^{y_1} \frac{\partial}{\partial x} B_n(f; x, x_2, \ldots, x_m) \, dx$$

$$+ \int_{x_2}^{y_2} \frac{\partial}{\partial x} B_n(f; y_1, x, \ldots, x_m) \, dx$$

$$+ \cdots + \int_{x_m}^{y_m} \frac{\partial}{\partial x} B_n(f; y_1, y_2, \ldots, x) \, dx.$$  \hspace{1cm} (11)

By using (9) and (10) in (11), we obtain

$$|f(x_1, \ldots, x_m) - f(y_1, \ldots, y_m)| \leq 2mM \delta^2 + \sum_{i=1}^{m} \left( 2 + \frac{n}{b_n} \right) \frac{1}{\delta} \omega^{(i)}(f; \delta) \left( \sum_{j=1}^{m} (x_j - y_j)^2 \right)^{1/2},$$

$$|f(x_1, \ldots, x_m) - f(y_1, \ldots, y_m)| \leq 2mM \delta^2 + \left( 2 + \frac{n}{b_n} \right) \frac{m}{\delta} \omega(f; \delta) \left( \sum_{j=1}^{m} (x_j - y_j)^2 \right)^{1/2},$$

$$|f(x_1, \ldots, x_m) - f(y_1, \ldots, y_m)| \leq C_1 \left( \delta^2 + \frac{\omega(f; \delta)}{\delta} \left( \sum_{j=1}^{m} (x_j - y_j)^2 \right)^{1/2} \right).$$  \hspace{1cm} (12)

where $C_1 = \max \{2mM, m(2 + n/b_n)\}$.

For $(\sum_{j=1}^{m} (x_j - y_j)^2)^{1/2} \leq h$ where $0 < h \leq 1$ and by the definition of $\omega$, we write

$$\omega(f; h) \leq C_1 \left( \delta^2 + \frac{h}{\delta} \omega(f; \delta) \right), \hspace{0.5cm} 0 < h, \delta \leq 1.$$  \hspace{1cm} (13)

On the other hand, from the inequality $\omega(f; h) \leq C_1 (\delta^2 + (h/\delta) \omega(f; \delta))$ we get the inequality $\omega(f; h) \leq C_2 h^2$ (proof see [1]). Consequently, we write

$$\omega(f; h) \leq C_2 h^2 \hspace{1cm} (0 < \alpha < 1).$$

By the inequality (13), we obtain $f \in \operatorname{Lip} \alpha$. This completes the proof.

**Corollary 12.** If the following inequalities are obtained:

$$|B_n(f; x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) - f(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_m)| \leq M_j \left( \frac{1}{n} \right)^{\alpha_j},$$

where $0 < \alpha_j < 1$, $j = 1, 2, \ldots, m$; and $M_j$ are some positive constants, then

$$f \in \bigcap_{j=1}^{m} \operatorname{Lip}_{x_j} \alpha_j$$

is obtained.
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References