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Crossed Products over Arithmetically Graded Rings

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0. INTRODUCTION

In the study of the Brauer group, $\mathrm{Br}(D)$, of a commutative ring D, the subgroup $\mathrm{Br}(S/D)$, consisting of the classes $\alpha \in \mathrm{Br}(D)$ split by the ringextension S of D, play, a fundamental role. If S is a Galois extension of D with Galois group G and $\mathrm{Pic}(S) = 1$, then it is well-known that $\mathrm{Br}(S/D) = H^2(G, U(S))$, where $U(S)$ denotes the multiplicative group of units of S. This states exactly that any Azumaya algebra R over D which represents a class $\alpha \in \mathrm{Br}(S/D)$ is equivalent (in the sense of the Brauer group) to a crossed product algebra $S[u_\sigma, \sigma \in G]$ which is a free S-module generated by elements u_σ, $\sigma \in G$, with multiplication defined by the following relations: $u_\sigma s = \sigma(s) u_\sigma$ for all $s \in S$, $\sigma \in G$; $u_\sigma u_\tau = c_{\sigma,\tau} u_{\sigma,\tau}$ for all $\sigma, \tau \in G$, where $c\colon G \times G \to U(S)$ is a 2-cocycle, i.e., the element of $H^2(G, U(S))$ corresponds to α. In general, $\mathrm{Br}(D)$ is not necessarily described completely by crossed product algebras but this problem can be avoid by introducing étale splitting rings and étale cohomology instead of Galois splitting rings and Galois cohomology. In case D is a local ring, it is known that every Azumaya algebra over D is equivalent to a crossed product algebra.

All of the facts stated are well documented in the litterature, e.g., Auslander and Goldman's fundamental paper [2], or [7, 9].

If D is a commutative $\mathbb{Z}$-graded ring, then one may be interested in the properties of the "graded" Brauer group of D, $\mathrm{Br}^g(D)$, introduced by the author in [16]. The relation between $\mathrm{Br}(D)$ and $\mathrm{Br}^g(D)$ is easily described in case D is a generalised Rees ring, cf. [15]. More generally, if D is a Gr-Dedekind ring, cf. [15], then a result from [16] states that $\mathrm{Br}^g(D)$ is a subgroup of $\mathrm{Br}(D)$. The aim of this paper is twofold. First, we aim to use the exact sequences of cohomology groups for $\mathrm{Br}(D)$ and $\mathrm{Br}^g(D)$ in relating crossed products and graded crossed products over D. Second, we aim to establish explicit at least one good class of Gr-Dedekind rings for which the first-mentioned cohomological methods may be successfully applied. This

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class will be the class of Gr–local Gr–Dedekind rings, i.e., the discrete Gr-valuation rings. Clearly, a ring which is Gr-local is far from being local, as consideration of the Gr–Dedekind ring $k[X]$ shows. Therefore some of the graded techniques developed in [11, 16, 17] will be fundamental in proving that there is a crossed product theory over discrete Gr-valuation rings, which turns out to yield results which are very similar to the local theory.

Since any Gr–Dedekind ring is a regular domain of global-dimension at most two, it follows that we may say that $\mathrm{Br}(D)$, $(\mathrm{Br}^g(D))$, is "determined" by crossed products (graded crossed products). One of the corollaries of our results is that we have solved a problem that arose in [16], i.e., $\mathrm{Br}(k[T, T^{-1}]) = \bigcup_e \mathrm{Br}^g(k[T_{(e)}, T_{(e)}^{-1}])$ where k is a perfect field and $k[T_{(e)}, T_{(e)}^{-1}]$ is the ring $k[T, T^{-1}]$ but with gradation defined by $\deg T = e$, $\deg k = 0$. In other words, it follows that each Azumaya algebra A, graded or not over $k[T, T^{-1}]$, is equivalent to a twisted polynomial ring $\Delta[X, X^{-1}, \varphi]$, where Δ is a finite dimensional skewfield, φ is an automorphism of Δ such that φ^e is inner (and this e is exactly the e for which the class of A is in $\mathrm{Br}^g(k[T_{(e)}, T_{(e)}^{-1}])$.

1. Graded Galois Splitting Rings over Discrete Gr-Valuation Rings

Recall that a *Gr-field* is a graded ring such that each of its homogeneous elements is invertible. Every graded field is of the form $\Delta[X, X^{-1}, \varphi]$ where Δ is a skewfield and φ is an automorphism of Δ, X a variable, and multiplication defined by $X\lambda = \varphi(\lambda)X$ for all $\lambda \in \Delta$, cf. [11]. A commutative Gr-field will be denoted by $k[T, T^{-1}]$, k a field. A *Gr-valuation* ring in a Gr-field is a graded subring such that for each homogeneous element x of the Gr-field, x or x^{-1} is in the subring. If 0_0 is a valuation ring of k, then it is clear that $0_0[T, T^{-1}]$ is a Gr-valuation ring; but there are more complicated examples of Gr-valuation rings, even if they are discrete! A graded domain D is said to be a *Gr-Dedekind* ring if every graded ideal of D is (in essentially a unique way) a product of graded prime ideals of D. The graded analogues of the equivalent characterizations of Dedekind rings hold too, cf. [14]. A Gr-Dedekind ring D such that $DD_1 = D$ is called a *generalized Rees ring* because its structure is easily described as follows: $D = \sum_{n \in \mathbb{Z}} I^n X^n$, where I is a fractional ideal of the Dedekind ring D_0, the part of degree zero of D, and X is a variable, cf. [15]. For a generalized Rees ring D, $\mathrm{Br}^g(D) = \mathrm{Br}(D_0)$ holds, cf. [16], but for a general Gr-Dedekind ring, even if it is Gr-local, i.e., a discrete Gr-valuation ring, $\mathrm{Br}^g(D) \to \mathrm{Br}(D)$ is all that is known.

After these preliminaries we now turn to some basic lemmas, which we will use frequently throughout the paper.

1.1. LEMMA. *A Gr-Dedekind ring D is a regular domain of global dimension at most two.*

Proof. In view of Theorem 2.3 in [12], D is regular if and only if all localizations $Q_P(D)$ are regular, for all graded prime ideals P of D. Since D is a Gr-Dedekind ring, $Q_P^g(D)$ is a discrete Gr-valuation ring (cf. [17] for details about graded localization in abstracto). Now $Q_P(D)$ is obtained from $Q_P^q(D)$ by localization at the Gr-maximal ideal $M = Q_P^g(D)P$. Now recall that there is associated a valuation of K, the field of fractions of D, to any Gr-valuation ring in $K^g = Q_0^g(D)$. Therefore the ring $Q_P(D) = Q_M(Q_P^g(D))$ is a valuation ring of K and since D is Noetherian, $Q_P(D)$ is a discrete valuation ring, hence a regular domain. In this situation we have that global dimension and Krull dimension of D coincide. On the other hand, since the Gr-Krull dimension of D is by definition at most one, and since Gr-K-dim $D \leqslant K$-dim $D \leqslant 1 +$ Gr-K-dim D (Corollary 4.18, [11]) it follows that the Krull dimension if D is at most 2.

1.2. COROLLARIES. (1) *From* [12] *recall that* $\mathrm{Br}(D) = \bigcap_{\mathrm{ht}\,P=1} \mathrm{Br}(Q_P(D))$, *where the intersection is taken over the prime ideals of height* 1.

(2) *It is a direct consequence of the proof that every non-zero graded prime ideal of D is a minimal prime ideal (if we exclude the positively graded case).*

(3) *The following diagram of injective group homomorphisms is a commutative diagram:*

$$\begin{array}{ccccc} \mathrm{Br}^g(D) & \hookrightarrow & \mathrm{Br}(D) & \hookrightarrow & \mathrm{Br}(K) \\ \downarrow & & \downarrow & & \| \\ \mathrm{Br}^g(K^g) & \hookrightarrow & \mathrm{Br}(K^g) & \hookrightarrow & \mathrm{Br}(K) \end{array}$$

(a) *The injection* $\mathrm{Br}^g(D) \to \mathrm{Br}(D)$ *has been established in* [16].

(b) *The injection* $\mathrm{Br}(D) \to \mathrm{Br}(K)$ *is Corollary* 1.2.1 *above.*

(c) *The injection* $\mathrm{Br}^g(K^g) \to \mathrm{Br}(K^g)$ *follows from* (a) *with* $K^g = D$.

(d) *The injection* $\mathrm{Br}(K^g) \to \mathrm{Br}(K)$ *is well known since* K^g *is a Dedekind ring* ($K^g \cong k[T, T^{-1}]$).

Commutativity of the diagram is easily checked and so injectivity of $\mathrm{Br}^g(D) \to \mathrm{Br}^g(K^g)$ *follows from it (a direct proof is obtained by following the lines of proof for the ungraded analogue).*

(4) *If P is a prime ideal of D, then either* $P_g \neq 0$ *is a graded prime ideal of D or* $P_g = 0$ *in which case all homogeneous elements of D are inver-*

tible in $Q_P(D)$, *i.e.,* $Q_P(D)$ *contains* $k[T, T^{-1}]$. *In combination with the first corollary above it follows that* $\mathrm{Br}(D) = \mathrm{Br}(K^g) \cap (\bigcap_{P \text{ graded ht } P=1} \mathrm{Br}(Q_P(D))$.

Because of the foregoing corollary we deduce:

$$\mathrm{Br}^g(D) = \bigcap_{\substack{P \text{ graded} \\ \text{ht } P=1}} \mathrm{Br}^g(Q_P^g(D)).$$

If P is a graded prime ideal of the Gr-Dedekind ring D such that P does not contain D_1, then $Q_P^g(D)$ contains an invertible element of degree one, i.e., $Q_P^g(D) = (Q_P^g(D))_0 [X, X^{-1}]$ with $\deg X = 1$ and $(Q_P^g(D))_0$ a discrete valuation ring. If P does contain D_1, then the structure of $Q_P^g(D)$ is somewhat more complicated. From [15] recall that if e_P is the ramification of the valuation associated to $Q_P^g(D)$ compared with the valuation of $(Q_P^g(D))_0$, then e_P is exactly the maximal number such tha $D_1 \subset P^e$ and also e_P is exactly the minimal number (positive) such that $(Q_P^g(D))_e$ contains a unit.

1.3. LEMMA. (1) *If R is graded Galois over a* Gr-*local ring then every finitely generated graded projective R-module M of constant rank is* Gr-*free, i.e., has a basis of homogeneous elements.*

(2) *Let R be a Noetherian graded ring and M a finitely generated graded projective module. Let I be any graded ideal of R contained in the graded Jacobson, radical,* $J^g(R)$, *of R (cf.* [11] *for details about* $J^g(R)$), *and assume that M/IM is* Gr-*free as an R/I-module. Then M is a* Gr-*free R-module.*

Proof. Formally similar to the proof of the corresponding ungraded statement.

1.4. COROLLARIES. (1) *If R is as in the lemma and moreover a domain, then the condition about M being of constant rank may be dropped in* 1.3.1.

(2) *If R is* Gr-*semilocal then every invertible graded module of rank one is* Gr-*free, thus* $\mathrm{Pic}^g(R) = 1$.

(3) *If R is a uniformly* Gr-*semilocal graded Krull domain, then* $1 = \mathrm{Pic}^g(R) = \mathrm{Pic}(R)$, *because equality of* $\mathrm{Pic}^g(R)$ *and* $\mathrm{Pic}(R)$ *holds for any graded Krull domain. In particular, if S is the integral closure of a discrete* Gr-*valuation ring D in some separable field extension L of* $K = Q(D)$, *then* $\mathrm{Pic}^g(S) = 1$.

1.5. LEMMA. *Any graded Azumaya algebra over a discrete* Gr-*valuation ring D is* Gr-*equivalent to a graded Azumaya algebra over D which is a domain.*

Proof. Let A be a graded Azumaya algebra over D. The total graded ring of fractions $Q^g(A) \simeq A \otimes_D K^g$ is a Gr-central simple algebra; hence it is isomorphic to $M_n(\Delta[X, X^{-1}, \varphi])_{\mathbf{d}}$, $\mathbf{d} \in \mathbb{Z}^n$ as described in [11]. Let B be a graded maximal D-order in the Gr-field $\Delta[X, X^{-1}, \varphi]$. It is easily verified that $M_n(B)_{\mathbf{d}}$ can be identified with a maximal graded order in $Q^g(A)$. Consider $I_c = \{x \in Q^g(A),\ xM_n(B)_{\mathbf{d}} \subset A\}$, the conductor of $M_n(B)_{\mathbf{d}}$ into A. Clearly, I_c is a graded left ideal of A and a graded right ideal of $M_n(B)_{\mathbf{d}}$. Write M_D for the unique Gr-maximal ideal of D and let $M = AM_D$ be the unique Gr-maximal ideal of A lying over M_D. Let L be any graded left ideal of A, then L/ML is a graded left ideal of $A/AM \cong M_r(\Delta_1[Y, Y^{-1}, \psi])_{\mathbf{d}}$, where Δ_1 is a skewfield, ψ an automorphism of Δ_1. Now from [11] we know that L/ML is generated by one homogeneous element, say $\bar{y}$. Select a homogeneous element $y \in L$ representing $\bar{y}$. Then $L = Ay + ML$ (note that A and L are both Gr-free D-modules and L is a direct summand of A, so $M \cap L = ML$), but the graded version of Nakayama's lemma entails $L = Ay$. Applying this to I_c it follows that $I_c = Au$ for some homogeneous u. From $A \otimes_D K^g = M_n(B) \otimes_D K^g$ it follows that $I_c \cap D \neq 0$, but then au is a unit in $Q^g(A)$ for some homogeneous $a \in A$, hence u is a unit $Q^g(A)$. In this way we obtain that $M_n(B)_{\mathbf{d}} \subset u^{-1}Au$, but as $u^{-1}Au$ is a graded D-order in $Q^g(A)$, the maximality of $M_n(B)_{\mathbf{d}}$ entails $M_n(B)_{\mathbf{d}} \cong u^{-1}Au$. Finally, $M_n(B)_{\mathbf{d}}$ is a graded Azumaya algebra, hence so is B and A is Gr-equivalent to B, moreover $B \subset \Delta[X, X^{-1}, \varphi]$ yields that B is domain.

The idea of the above proof is a mixture of some ideas which have proven succesful in the ungraded theory, e.g., see the proof of 6.32 in [12]. However it is essential that this proof is carried out in $Q^g(A)$ and not in $Q(A)$ because, if $Q(A) = M_n(\Delta)$ for some skewfield Δ over K, then it is not clear how a maximal D-order in Δ may be graded in such way that there exists a $\mathbf{d} \in \mathbb{Z}^n$ such that $M_n(B)_{\mathbf{d}}$ is a maximal graded order in $M_n(\Delta)$ since the gradations of subrings of $M_n(\Delta)$ are fairly arbitrary.

Equivalent to proving Lemma 1.5 using orders in $Q(A)$ instead of in $Q^g(A)$ is the problem of proving Lemma 1.5 for arbitrary Azumaya algebras over D. This could be possible but I do not see how, now.

1.6. Proposition. *A graded Azumaya algebra A over a graded field $k[T, T^{-1}]$ may be split by an extension in degree zero*: $l[T, T^{-1}]$.

Proof. From [11] we know that $A \simeq M_n(\Delta[X, X^{-1}, \varphi])_d$ where Δ is a skewfield, φ an automorphism of Δ such that φ^m is inner in Δ for some $m \in \mathbb{N}$, and $\mathbf{d} \in \mathbb{Z}^n$ describes the gradation on the matrix ring. It will suffice to split $\Delta[X, X^{-1}, \varphi]$. Let $k' = Z(\Delta)$ and l_1 an extension of k' splitting Δ, e.g., a maximal commutative subring of Δ. Consider: $R_1 = \Delta[X, X^{-1}, \varphi] \otimes_{k[T, T^{-1}]} l_1[T, T^{-1}]$, where T is a central element of minimal positive degree. Since $(R_1)_0$ contains zero-divisors it follows that $R_1 \sim$

$M_{n_1}(\Delta_1[X_1, X_1^{-1}, \varphi_1])_{\mathbf{d}_1}$, $\mathbf{d}_1 \in \mathbb{Z}^r$, with $n_1 > 1$. The center of $\Delta_1[X_1, X_1^{-1}, \varphi_1]$ is $l_1[T, T^{-1}]$. Let l_2 be an extension of l_1 splitting Δ_1 and repeat the above argumentation. In each step the degree of the skewfield over its center goes down; since $[A: k[T, T^{-1}]]$ is finite, this proces stops and in the final step we obtain an extension in degree zero $l[T, T^{-1}]$ splitting A. Note that in this proof, the fact that A is graded allows us to avoid the use of Tsen's theorem, what would have forced us to assuming the field k to be perfect. Note also that, at each step, the extension of degree zero, may be chosen to be separable, i.e., l/k may be chosen to be separable.

1.7. PROPOSITION. *Every class $\alpha \in \mathrm{Br}^g(D)$ may be represented by a graded Azumaya algebra over D containing a maximal commutative subring S with the following properties*:

(1) *S is a graded domain and a* Gr-*free D-module of rank $\sqrt{A:D}$.*

(2) *S is of the form $S_0 \otimes_{D_0} D$ where S_0 is a separable extension of D_0.*

Consequently, S is Gr-*semilocal and actually, S is a* Gr-*principal ideal domain.*

Proof. Let B be a graded Azumaya algebra representing α and write $M = BM_D$, $\bar{B} = B/M$ and $m^2 = [B:D]$. Since $\bar{B}$ is a graded Azumaya algebra of rank m^2 over $\bar{D} \cong k[T, T^{-1}]$, it follows that $\bar{B}$ may be split by $l[T, T^{-1}]$ for some separable finite field extension l/k, $l = k(\theta)$, by Proposition 1.6. Note that $\deg\theta = 0$, and let $f \in k[X]$ be the minimal polynomial satisfied by θ over k. Consider a monic polynomial $F \in D[X]$ of the same degree in X as f and such that $F \bmod M_D[X] = f$, obtained by lifting the coefficients of f to D_0. Because D_0 is U.F.D. and F is easily seen to be irreducible, (F) is a prime ideal of $D_0[X]$. It follows that $T_0 = D_0[X]/(F)$ is a separable D_0-algebra which is a domain and free of rank $\deg_X F = \deg_X f$ as a D_0-module. Immediate from this it follows that $T = D[X]/(F)$ is a separable D-algebra and graded free of rank $\deg_X f$ as a D-module. In order to show that T is a domain it will suffice to establish that T has no homogeneous zero divisors. Suppose $u, g, h \in h(D[X])$ (the gradation on $D[X]$ is defined by $\deg X = 0$) such that $uF = gh$. As pointed out before, D contains an invertible element of degree e, say w_e. Pick $0 \neq d_1 \in D_{e\text{-}\deg g}$, $0 \neq d_2 \in D_{e\text{-}\deg h}$. Then $(w_e^{-1} d_1 g)(w_e^{-1} d_2 h) = (w_e^{-2} d_1 d_2 u)F$. However $D[X]F$ is prime in $D_0[X]$, hence $w_e^{-1} d_1 g$ or $w_e^{-1} d_2 h$ is in $D_0[X]F$, say $w_e^{-1} d_1 g = v_0 F$ with $v_0 \in D_0[X]$. Now if $d_1 \notin M_D$, then $g \in D[X]F$ and on the other hand $d_1 \in M_D$ yields $w_e^{-1} d_1 \in M_D$ hence $v_0 F \in M_D[X]$. But $\bar{D}[X] \cong k[X, T, T^{-1}]$ is prime and $F \notin M_D[X]$, hence $v_0 \in M_D[X]_0$. Let π be the homogeneous generator for M_D, then $v_0 = \pi^v v_0^1$, $w_e^{-1} d_1 = \pi^v \lambda$ with $v_0^1 \in D[X] - M_D[X]$ and

$\lambda \in D - M_D$ (same v in both relations!). Therefore $\lambda g = v_0^1 F$ but then $\lambda^{-1} \in D$ yields $g \in D[X]F$.

Next consider the commutative diagram:

$$(*) \qquad \begin{array}{ccccccc} D & \hookrightarrow & T & \hookrightarrow & D^{\mathrm{cl}} & \hookrightarrow & Q(T) \\ \downarrow & & \downarrow & & & & \\ \bar{D} \cong k[T, T^{-1}] & \to & l[T, T^{-1}] & & & & \end{array}$$

where D^{cl} is the integral closure of D in the field of fractions $Q(T)$ of T. Clearly D^{cl} is the integral closure of T in $Q(T)$ and since T is graded, it follows that D^{cl} is a graded ring. That D^{cl} is also a Gr-Dedekind ring follows from the graded version of the Krull–Akizuki theorem for which we refer to [14]. We have established that $[Q(T): Q(D)] = [T:D] = [l:k]$, so $D^{\mathrm{cl}}/D^{\mathrm{cl}}M_D = l[T, T^{-1}]$ while $D^{\mathrm{cl}}M_D \subset J^g(D^{\mathrm{cl}})$. Separability of $Q(T)$ over $Q(D)$ entails that D^{cl} is a finite D-module, so we can apply the graded version of Nakayama's lemma and obtain $D^{\mathrm{cl}} = T$. By construction, T is Gr-local, and from $T = D^{\mathrm{cl}}$ it then follows that T is a discrete graded valuation ring. We have reached the situation where $B \otimes_D T$ is a graded Azumaya algebra over the discrete graded valuation ring T, such that $B \otimes_D T/(B \otimes_D T)M_D = \bar{B} \otimes_{\bar{D}} \bar{T}$ is a graded matrix ring over $\bar{T}$. So we may change notation, i.e., assume that B is such that $\bar{B} = M_m(k[T, T^{-1}]_{\mathbf{d}}$ where $\mathbf{d} \in \mathbb{Z}^m$ describes the gradation on the matrices as in [11]. By Lemma 1.5 we may find a graded Azumaya algebra A over D, Gr-equivalent to B, and such that A is a domain. Then $B \cong M_s(A)_{\mathbf{e}}$ (see construction in Lemma 1.5) yields $\bar{B} \cong M_s(\bar{A})_{\mathbf{e}}$ and $M_s(\bar{A})_{\mathbf{e}} = M_m(k[T, T^{-1}])_{\mathbf{d}}$ yields $\bar{A} \sim_{\mathrm{Gr}} 1$, say $\bar{A} \cong M_t(k[T, T^{-1}])_{\mathbf{f}}$ for some $f \in \mathbb{Z}^t$.

In $M_t(k)$ we find a comutative subring $k[\alpha]$ such that $[k[\alpha]:k] = t$, e.g., let α be a diagonal matrix with different entries. Consider $k[T, T^{-1}][\alpha]$ and lift α to $\beta \in A_0$. Clearly $D[\beta]$ is a commutative subring of A, hence also a domain; but $Q(D)[\beta]$ is a commutative subring in the skewfield $Q(A)$, hence it follows from $[Q(D)[\beta]: Q(D)] \leqslant t = \sqrt{A:D}$ and the fact that $1, \beta, \ldots, \beta^{t-1}$ are D-independent, that $\beta^t = \sum_{i=0}^{t-1} k_i \beta^i$ with $k_i \in Q(D)$, i.e., $d\beta^t = \sum_{i=0}^{t-1} d_i \beta^i$, with $d, d_i \in D$ and utilizing the graded structure of B it follows that we may assume that d, d_i are homogeneous. Moreover since $\deg \beta = 0$, $\deg d = \deg d_i$ for all i. If $d \in M_D$, then $0 = \sum_{i=0}^{t-1} \bar{d}_i \bar{\beta}^i = \sum_{i=0}^{t-1} \bar{d}_i \alpha^i$ implies $d_i \in M_D$ for all i. Therefore the assumption $d \in M_D$ leads to simplification by π (the homogeneous generator of M_D), and after a finite number of repetitions of this we obtain a relation:

$$d'\beta^t = \sum_{i=0}^{t-1} d_i' \beta^i \qquad \text{with } d_1' d_i' \in h(D) \text{ and } d' \notin M_D.$$

Then d' is a unit of D, hence $\beta^t \in D + D\beta + \cdots + D\beta^{t-1}$, stating that the D-module S generated by the D-independent elements $1, \beta, ..., \beta^{t-1}$ is a subring of A, hence also a domain. Put $F = X^t - \sum_{i=0}^{t-1} (d')^{-1} d_i x^i$. Obviously F is a lifting of the minimal polynomial of α over k, and it is clear that $S = D[X]/(F) = D_0[X]/(F) \otimes_{D_0} D$ and the properties of S listed in the statement of the proposition are easily checked. That S splits A is a direct consequence of the fact that $Q(S)$ splits $Q(A)$ because it is a field of dimension t within a skewfield of dimension t^2 over $Q(D)$, and from the injectivity of $\mathrm{Br}^g(D) \to \mathrm{Br}(Q(D))$. Finally, by Corollary 1.4.3 it follows that S is Gr-semilocal, and moreover a Gr-Dedekind ring, i.e., a Gr-principal ideal domain.

1.8. *Remark.* The above proof owes part of its ideas to a proof given by Auslander and Goldman in [2], for which they credit Serre. It is exactly the difference between D and a generalized Rees ring, $D_0[T, T^{-1}]$ with $\deg T = 1$, which makes it impossible to reduce the whole proof to a construction in degree zero. The fact that we consider Gr-Dedekind rings which need not be generalized Rees rings is very essential; see, for example, Section 2.

1.9. THEOREM. *Let D be a discrete* Gr-*valuation ring. Every $\alpha \in \mathrm{Br}^g(D)$ may be represented by a graded Azumaya algebra A over D which contains a maximal commutative subring S with the following properties*:

(1) *S is a graded Galois extension of D.*

(2) *S is a* Gr-*principal ideal domain, hence we have*:

$$\mathrm{Pic}\, S = \mathrm{Pic}^g\, S = 1.$$

(3) $S = S_0 \otimes_{D_0} D$.

(4) *The units of S are homogeneous elements.*

Proof. Let A_1 and S_1 be as A and S in Proposition 1.7. Since S_1 does not contain idempotent elements different from 0, 1, it follows, cf. [7], that the normalization S of S_1 does not contain idempotents different from 0, 1. Let G be the Galois group of S over D. The trace $t: S \to D$ is given by $t(x) = \sum_{\sigma \in G} \sigma(x)$, and it is a free generator of the right S-module $\mathrm{Hom}_D(S, D)$. If $x \in S$ is nilpotent, then $t(xs) = \sum_{\sigma \in G} \sigma(xs)$ is nilpotent and in D, for all $s \in S$; consequently $t(xs) = 0$ for all $s \in S$. However this establishes a relation $tx = 0$ in $\mathrm{Hom}_D(S, D)$ and since t is a free generator, $x = 0$ follows. Thus S is semiprime and also Noetherian and hence we obtain an imbedding $S \to S_1 \oplus \cdots \oplus S_r$ where $S_i = S/P$, $i = 1, ..., r$, P_i being a graded minimal prime ideal. Hence $U(S) \to U(S_1) \oplus \cdots \oplus U(S)$ but $U(S_i) \subset h(S_i)$ because S is a graded domain, so $U(S) \subset h(S)$ follows. In Proposition 1.5 we have shown that $S_1 = (S_1)_0 \otimes_{D_0} D$ and so it is clear that S can be obtained by constructing a D_0-normalization of $(S_1)_0$ and then tensoring by D over D_0. In view of the above remarks we obtain that S_0 has no other idempotents than 0 and 1. The total ring of fractions L of S_0 is a Galois extension of the field of fractions K of D_0, hence semisimple, say $L = L_1 \oplus \cdots \oplus L_r$. We have the commutative diagram:

$$\begin{array}{ccccl} D_0 & \hookrightarrow & S_0 & \hookrightarrow & D_0^{\mathrm{cl}} = (D_0^{\mathrm{cl}})_1 \oplus \cdots \oplus (D_0^{\mathrm{cl}})_r \\ \downarrow & & \downarrow & & \\ k = D_0/(M_D)_0 & \hookrightarrow & S_0/S_0(M_D)_0 = \bar{S}_0 & & \end{array}$$

where D_0^{cl} is the integral closure of D_0 in L, which obviously splits as a direct sum of the integral closures of D_0 in the fields $L_1,..., L_r$; and where $\bar{S}_0$ has Galois group G over k. From $[L:K] = |G| = [\bar{S}_0 : k]$ it follows that $D_0^{\mathrm{cl}} = S_0 + D_0^{\mathrm{cl}}(M_D)_0$ and again from the graded version of Nakayama's lemma we find $S_0 = D_0^{\mathrm{cl}}$. However the only idempotents of S_0 are 0 and 1, so $D_0^{\mathrm{cl}} = (D_0^{\mathrm{cl}})_1$, i.e., $r = 1$ and S_0 is a domain. By the (ungraded) Krull–Akizuki theorem, D_0^{cl} is a Dedekind ring and a finite module over the valuation ring D_0, hence a principal ideal domain. If S has zero-divisors, then it has homogeneous zero-divisors too, say $u_m v_t = 0$ with $0 \neq u_m \in S_m$, $0 \neq v_t \in S_t$. If e is the least positive integer such that D_e contains a unit w_e, then $(w_e^{-m} u_m^e)(w_e^{-t} v_t^e) = 0$ yields zero-divisors of degree zero. Hence $w_e^{-m} u_m^e = 0$ and $u_m^e = 0$, but as S does not contain nonzero nilpotent elements, $u_m = 0$.

Now knowing that S is a domain we may repeat the argument of Proposition 1.7 concerning the diagram (*) and we obtain that $S = D^{\mathrm{cl}}$ is a Gr-semilocal Gr-Dedekind ring, i.e., a Gr-principal ideal domain. Thus $\mathrm{Pic}^g\, S = \mathrm{Pic}\, S = 1$. Let A be Gr-equivalent to A_1 and such that A contains S as a maximal commutative subring, then A and S satisfy all the requirements of the theorem.

1.10. COROLLARY. *Every graded Azumaya algebra over a discrete* Gr-*valuation ring is equivalent to a crossed product* $S[u_\sigma, \sigma \in G]$ *where* S *is a* Gr-*principal ideal domain, and a Galois extension of* D.

1.11. *Remark.* In the proof of 1.9 it has been used implicitly that a normal closure S_1 of a graded ring can be made into a graded ring containing S_1 as a graded subring. That this is indeed true is easily verified if one runs along the lines of proof of Theorem 2.9 in [7] (or the implication $1 \Rightarrow 2$ on p. 89 of [9]). In connection to this, note that a graded separable extension S of D has a separability idempotent $e \in S \otimes_D S$ which may be chosen in $(S \otimes_D S)_0$ (not in $S_0 \otimes_{D_0} S_0$ in general). Indeed $e = s(1)$ for some section of p: $S \otimes_D S \to S$, since p is graded of degree zero, the section s may be chosen to be a graded morphism of degree zero, i.e., $e = s(1) \in (S \otimes_D S)_0$.

2. Cohomological Interpretations

If S is a Galois extension of D with Galois group G, then we have the following well-known exact sequence:

$$1 \to \mathrm{Pic}(D) \to (\mathrm{Pic}(S))^G \xrightarrow{S} H^2(G, U(S)) \xrightarrow{\lambda} \mathrm{Br}(S/D) \to H^1(G, \mathrm{Pic}(S))$$

In case S and D are graded rings there is an analog for this sequence in the graded setting but we need not go into this here because for Gr-Dedekind rings both sequences are easily related, as we will show in the sequel.

Since $U(S)$ consists of homogeneous elements, because S is a graded domain there exists a group homomorphism $\deg^2\colon H^2(G, U(S)) \to H^2(G, \mathbb{Z})$, defined by $(\deg^2(c))_{\sigma,\tau} = \deg c_{\sigma,\tau}$ (G acting trivially on $\mathbb{Z}$). Let us define: $H_g^2(G. U(S)) = \mathrm{Ker}\,\deg^2$.

2.1. Theorem. *Let D be a* Gr-*Dedekind domain and S a* Gr-*Dedekind domain which is a Galois extension of D with Galois group G, then the following diagram* (**) *is commutative and exact.*

$$\begin{tikzcd}
 & 1 \arrow[r] & 1 \arrow[r] & H^2(G, U(S))/H_g^2(G, U(S)) \\
1 \arrow[r] & \mathrm{Pic}(D) \arrow[r] \arrow[u] & (\mathrm{Pic}(S))^G \arrow[r, "\delta"] \arrow[u] & H^2(G, U(S)) \arrow[u] \\
1 \arrow[r] & \mathrm{Pic}^g(D) \arrow[r] \arrow[u] & (\mathrm{Pic}^g(S)^G \arrow[r, "\delta_g"] \arrow[u] & H^2(G, U(S)) \arrow[u] \\
 & 1 \arrow[u] & 1 \arrow[u] & 1 \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Br}(S/D)/\mathrm{Br}^g(S/D) & \\
{} \arrow[r, "\lambda"] & \mathrm{Br}(S/D) \arrow[r] \arrow[u] & H^1(G, \mathrm{Pic}(S)) \\
{} \arrow[r, "\lambda_g"] & \mathrm{Br}^g(S/D) \arrow[r] \arrow[u] & H^1(G, \mathrm{Pic}^g(S)) \\
 & 1 \arrow[u] &
\end{tikzcd} \qquad (**)$$

where λ_g is the restriction of λ to $H_g^2(G, U(S))$ and $\delta_g = \delta$.

Proof. In view of Corollary 1.2.3 we only have to check that $\lambda(H^2(G, U(S)) \cap \mathrm{Br}^g(S/D) = \lambda(H_g^2(G, U(S))$. By definition, $\alpha \in H^2(G, U(S))$ is in $H_g^2(G, U(S))$ if and only if the 2-cocycle $\alpha_{\sigma,\tau}$ defines a crossed product $A = S[u_\sigma]$ with multiplication defined by $su_\sigma = \sigma(s)u_\sigma$ and $u_\sigma u_\tau = \alpha_{\sigma,\tau} u_{\sigma\tau}$ which is graded over D.

Since τ is an automorphism of S over D which is graded of degree zero, we may deduce from $u_\sigma s = \sigma(s)\, u_\sigma$ taking $s \in h(S)$, that $(u_\sigma)_h s = \sigma(s)(u_\sigma)_h$, where $(u_\sigma)_h$ is the homogeneous component of highest degree appearing in the decomposition of u_σ in A. Then $(u_\sigma)_h u_\sigma^{-1}$ commutes with S and hence $(u_\sigma)_h u_\sigma^{-1} \in S$. But $S[u_\sigma]$ is a free S-module and S is a domain, therefore the elements of S are regular in $S[u_\sigma]$ and consequently $(u_\sigma)_h$ is then also a regular element in $S[u_\sigma] = A$. Writing $(u_\sigma)_h = su_\sigma$ for some $s \in S$ and decomposing s as $s_{i_h} + \cdots + s_{i_m}$ with s_{i_j} of degree i_j and $i_h > \cdots > i_m$, yields the relation $(u_\sigma)_h = (s_{i_h} + \cdots + s_{i_m})\,(u_\sigma)_h$ + lower degree terms. Therefore, the fact that $s_{i_h}(u_\sigma)_h \neq 0$ ($(u_\sigma)_h$ is regular in A!) implies $(u_\sigma)_h = (s_{i_h})(u_\sigma)_h$, i.e., $i_h = 0$. However we see that the homogeneous component $(u_\sigma)_l$ of lowest degree in the decomposition of u_σ is also regular in A (repeat the above argument). But then $s_{i_m}(u_\sigma)_l = 0$ yields that $s_{i_m} = 0$, a contradiction unless $(u_\sigma)_h = (u_\sigma)_m$, i.e., u_σ is homogeneous. Taking degrees in $u_\sigma u_\tau = \alpha_{\sigma,\tau} u_{\sigma\tau}$ yields that $\deg^2\alpha$ is trivial in $H^2(G, \mathbb{Z})$.

Conversely if α is such that $\deg^2 \alpha$ is trivial in $H^2(G, Z)$, then the crossed product $S[u_\sigma]$ defined by α may be graded by putting $\deg u_\sigma = d_\sigma$, where the $d_\sigma \in \mathbb{Z}$ are obtained from $\deg \alpha_{\sigma\tau} = d_\sigma + d_\tau - d_{\sigma\tau}$. All this establishes that we have an injective group morphism

$$H^2(G, U(S))/H_g^2(G, U(S)) \to G^* \cong H^2(G, Z).$$

From $H_g^2(G, U(S)) = \mathrm{Ker}(\deg^2)$ one easily deduces that $\lambda(H^2(G, U(S)) \cap \mathrm{Br}^g(S/D) = \lambda_g(H_g^2(G, U(S))$ and exactness of the diagram follows easily.

2.2. *Note.* Since $1 = \delta((\mathrm{Pic}(S))^G) = \delta((\mathrm{Pic}^g(S))^G)$, it is clear that $\delta^g = \delta$ maps $(\mathrm{Pic}^g(S))^G$ to a graded matrix ring. Let us recall for completeness sake how δ acts and give the graded version of it. If $|V| \in (\mathrm{Pic}^g(S))^G = (\mathrm{Pic}(S))^G$, where V is graded invertible S module of rank one, then $|\sigma \cdot V| = |V|$ implies the existence of a D-isomorphism $\psi_\sigma\colon V \to V$ such that $\psi_\sigma(sx) = \sigma(s)\,\psi_\sigma(x)$ for all $s \in S$, $x \in V$. The map $\psi_{\sigma\tau}\psi_\tau\psi_\sigma^{-1}$ is a D-isomorphism $V \to V$ and it is moreover S-linear, i.e., $\psi_{\sigma\tau}\psi_\tau^{-1}\psi_\sigma^{-1} \in \mathrm{Hom}_S(V, V) \cong S$. It follows that $\psi_{\sigma\tau}\psi_\tau^{-1}\psi_\sigma^{-1}$ is just multiplication by an element $\alpha_{\sigma,\tau}$ in $U(S)$, hence $\alpha_{\sigma,\tau}$ is homogeneous. Again from the relations $\psi_\tau\psi_\sigma = \alpha_{\sigma,\tau}\psi_{\tau,\sigma}$ and $\psi_\sigma s = \sigma(s)\psi_\sigma$, within the graded ring $\mathrm{Hom}_D(V, V)$ (graded because V is graded of finite type over D) one may deduce as in the proof of the theorem that ψ_σ is homogeneous, i.e., a graded morphism, for each $\sigma \in G$. Therefore $\delta(|V|)$ is a graded matrix ring.

2.3. PROPOSITION. *Let D be a* Gr-*Dedekind ring and let $e \in \mathbb{N}$ be the smallest number such that D_e contains a unit, say T. If S is a graded domain which is a Galois extension of D with Galois group G, then the following sequence is exact*:

$$1 \to H_g^2(G, U(S)) \to H^2(G, U(S)) \xrightarrow{\deg^2} G^* \to M \to 1$$

where M is an e-torsion group.

Proof. Pick $d \in H^2(G, \mathbb{Z})$; then ed is a 2-cocycle such that $(ed)_{\sigma,\tau} \in e\mathbb{Z}$. Put $c_{\sigma,\tau} = T^{d_{\sigma\tau}}$. Then it is clear that $c: G \times G \to U(S)$ defined by $c(\sigma, \tau) = c_{\sigma,\tau}$ is an element of $H^2(G, U(S))$ with $\deg^2 c = ed$. Consequently $G^*/\mathrm{Im}(\deg^2)$ is e-torsion.

2.4. *Note.* The 2-cocycle c constructed above lies in $H^2_{\mathrm{sym}}(G, U(D))$; this is especially meaningful in case G is an abelian group.

2.5. THEOREM. *Let D be a* Gr-*Dedekind ring and let $e \in \mathbb{N}$ be the smallest number such that D_e contains a unit, say T. Let S be a domain and a Galois extension of D with group G, assume furthermore that $S = S_0 \otimes_{D_0} D$ where S_0 is a Galois extension of D_0. Then we have*:

(a) $\mathrm{Br}(S/D)/\mathrm{Br}(S_0/D_0) \times G^*$ *is e-torsion.*

(b) $\mathrm{Br}^g(S/D)/\mathrm{Br}(S_0/D_0)$ *is e-torsion.*

Proof. We identify $\mathrm{Br}(S/D) = H^2(G, U(S))$ and $\mathrm{Br}^g(S/D) = H_g^2(G, U(S))$. If $c \in H^2(G, U(S))$, then $c_0 = c^e T^{-d_{\sigma,\tau}}$, where $d = \deg^2(c)$, represents an element of $H^2(G, U(S_0)$. Consequently $c^e = c_0 T^{d_{\sigma,\tau}}$ may be viewed as an element of $H^2(G, U(S_0)) \times H^2(G, \mathbb{Z})$.

Note. $\mathrm{Pic}(S_0) = \mathrm{Pic}(S) = 1$. Hence $\mathrm{Br}(S/D)/\mathrm{Br}(S_0/D_0) \times G^*$ is an e-torsion group. Second, if $c \in H_g^2(G, U(S))$, then $c \in \mathrm{Ker}(\deg^2)$, i.e., $d_{\sigma,\tau} = d_\sigma + d_\tau - d_{\sigma,\tau}$ for certain $d_\sigma \in \mathbb{Z}$. Now $c^e = T^d c_0$, as above, entails that c^e and c_0 represent the same element of $\mathrm{Br}^g(S/D)$ because T^d is trivial (indeed $T^{d_{\sigma,\tau}} = (T^{d_\sigma}) \cdot (T^{d_\tau}) \cdot (T^{-d_{\sigma,\tau}})$). Therefore, $\mathrm{Br}^g(S/D)/\mathrm{Br}(S_0/D_0)$ is e-torsion too.

2.6. COROLLARIES. (1) *If D is a generalised Rees ring, i.e., $e = 1$, then* $\mathrm{Br}(S/D) = \mathrm{Br}(S_0/D_0) \times G^*$ *and* $\mathrm{Br}^g(S/D) = \mathrm{Br}(S_0/D_0)$. *Moreover, since every $\alpha \in \mathrm{Br}(D)$ can be split by a Galois domain extension S of D as required in* 2.5, *it follows that* $\mathrm{Br}(D) = \mathrm{Br}(D_0) \times G^*$, $\mathrm{Br}^g(D) = \mathrm{Br}(D_0)$, *where G^* is the direct limit over splitting Galois groups of $H^2(G, \mathbb{Z})$.*

(2) *If D is of the form $D_0[X, X^{-1}]$ with* $\deg X = e$, *then we may change the gradation of D by giving* $\deg X$ *any other value in $\mathbb{Z}$. Hence*

$\mathrm{Br}(S/D)/\mathrm{Br}(S_0/D_0) \times G^*$ *is e-torsion for every* $e \in \mathbb{Z}$, *hence* $\mathrm{Br}(S/D) = \mathrm{Br}(S_0/D_0) \times G^*$. *It will be established hereafter that, although* $\mathrm{Br}^g(S/D)$ *is e-torsion over* $\mathrm{Br}(S_0/D_0)$ (*e varying with the gradation considered but* $\mathrm{Br}^g(S/D)$ *varying too*!), *the groups* $\mathrm{Br}^g(S/D)$ *may be large in* $\mathrm{Br}(S/D)$ *if the gradation is suitably defined.*

(3) *If D is a discrete* Gr-*valuation ring, then the results of Section* 1 *imply the existence of the domain S reguired in the theorem, for every graded Azumaya algebra over D. So we have established so far that*

(a) $\mathrm{Br}^g(D) = \varinjlim_{G=\mathrm{Gal}(S/D)} H^2_g(G, U(S))$.

(b) $\varinjlim_{G=\mathrm{Gal}(S/D)} H^2(G, U(S))/H^2_g(G, U(S))$ *is e-torsion.*

(c) $\mathrm{Br}^g(D)/\mathrm{Br}(D_0)$ *is e-torsion.*

2.7. THEOREM. *Let D be a* Gr-*Dedekind ring and let S be a graded domain and a Galois extension of D with group G. Let* $S_{(n)}$ *be the graded ring, isomorphic to S as an ungraded ring but with gradation defined by* $(S_{(n)})_{np} = S_p$. *We identify* $H^2_g(G, U(S_{(n)}))$ *with the subgroup of* $H^2(G, U(S))$ *isomorphic to it (by forgetting gradation). Then* $H^2(G, U(S)) = \bigcup_{n\in\mathbb{Z}} H^2_g(G, U(S_{(n)})$.

Proof. Take $c \in H^2(G, U(S))$. Then $\deg^2(c) = d \in H^2(G, \mathbb{Z})$ has finite order, n say, because $H^2(G, \mathbb{Z})$ is a torsion group.

Consequently $nd_{\sigma,\tau} = d_\sigma + d_\tau - d_{\sigma,\tau}$ for all $\sigma, \tau \in G$, some $d_\sigma \in \mathbb{Z}$. In the fradation of S, $c_{\sigma,\tau}$ has degree $d_{\sigma,\tau}$ but in the gradation of $S_{(n)}$, $\deg c_{\sigma,\tau} = nd_{\sigma,\tau}$; in other words $\deg^2 c_{(n)} = nd$, where $c_{(n)}$ is the 2-cocycle c but with the degrees from the gradation on $S_{(n)}$. From $\deg^2 c_{(n)} = nd$ which is trivial in $H^2(G, \mathbb{Z})$ it follows that $c_{(n)}$ represents an element of $H^2_g(G, U(S_{(n)}))$. In other words, the crossed product $S[u_\sigma, \sigma \in G]$ with $u_\sigma u_\tau = c_{\sigma,\tau} u_{\sigma,\tau}$, $u_\sigma s = \sigma(s) u_\sigma$ becomes a graded Azumaya algebra over D, if we change the gradation of S and of D and consider $S_{(n)}$ over $D_{(n)}$, by putting $\deg u_\sigma = d_\sigma$, where d_σ is as above.

2.8. COROLLARIES. (1) *If D is a discrete* Gr-*valuation ring then*

$$\varinjlim_{\mathrm{Gal}(S/D)=G} H^2(G, U(S)) = \bigcup_n \mathrm{Br}^g(D_{(n)}),$$

the limit being taken over the Galois extensions of D which are Gr-*principal ideal domains of the form* $S_0 \otimes_{D_0} D$.

(2) *If* $D = D_0[X, X^{-1}]$ *where* D_0 *is a valuation ring, then*

$$\mathrm{Br}(D) = \bigcup_n \mathrm{Br}^g(D_{(n)}).$$

Proof. Section 1 and Theorem 2.7.

2.9. THEOREM. *Let D be a graded field $k[T, T^{-1}]$ with* $\deg T = e$ *and with k a perfect field. Then* $\mathrm{Br}(D) = \bigcup_n \mathrm{Br}^g(D_{(n)})$.

Consequently, every Azumaya algebra, graded or not, over $k[T, T^{-1}]$ is equivalent to a skew polynomial Ore domain $\Delta[X, X^{-1}, \varphi]$, where Δ is a finite-dimensional skewfield, φ an automorphism of Δ such that φ^e is inner, $\deg X = 1$ and $T = \lambda X^e$, $\lambda \in U(\Delta)$.

Proof. Any Azumaya algebra $k[T, T^{-1}]$ can be split by some $l[T, T^{-1}]$ where l/k is a Galois field extension (Tsen's theorem). Therefore we may take direct limits in Theorem 2.7 and we obtain $\mathrm{Br}(k[T, T^{-1}]) = \bigcup_n \mathrm{Br}^g(k[T_{(n)}, T_{(n)}^{-1}])$, where $T_{(n)} = T$ but with $\deg T_{(n)} = n$. Any graded Azumaya algebra A over $k[T, T^{-1}]$ is of the form $M_n(\Delta[X, X^{-1}, \varphi])_{\mathbf{d}}$, $\mathbf{d} \in \mathbb{Z}^n$, cf. [11], hence A is equivalent to $\Delta[X, X^{-1}, \varphi]$.

Since any Azumaya algebra over $k[T, T^{-1}]$ may be regarded as a graded algebra if we put $\deg T = n$ for some suitable n, the theorem follows.

2.10. *Remark.* Theorem 2.9 is just an easy corollary to 2.7 but *I* guess it is interesting to see how the graded techniques have led to structural results about the elements of the Brauer group. Since no such result seems to be available in the literature *I* stated the property as a theorem in its own right.

2.11. PROPOSITION. *Let D be a* Gr*-Dedekind ring and let S be a graded domain which is a Galois extension of D with abelian Galois group G, then we have*

$$H^2(G, U(S))/H_g^2(G, U(S)) = H_{\mathrm{sym}}^2(G, U(S))/H_g^2(G, U(S)) \cap H_{\mathrm{sym}}^2(G, U(S)).$$

Proof. Consider $d = \deg^2 f$, $f \in H^2(G, U(S))$. By definition, a symmetric 2-cocycle for the abelian group G is given by the system $c_{\sigma,\tau}$ satisfying relations:

$$c_{\sigma,\tau} c_{\sigma\tau,\rho} = c_{\tau,\rho} c_{\sigma,\tau\rho}, \qquad \sigma, \tau, \rho \in G. \tag{*}$$

A result of [10] states that we may fix $n = |G|$ couples $(\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n)$ such that the relations (*) can be solved in the free multiplicative group generated by the elements $c_{\sigma_1,\tau_1}, \ldots, c_{\sigma_n,\tau_n}$. Since $d_{\sigma,\tau}$ satisfies (*) (additively notated), it follows that $d_{\sigma_1,\tau_1}, \ldots, d_{\sigma_n,\tau_n}$ determine d. Now $f_{\sigma_1,\tau_1}, \ldots, f_{\sigma_n,\tau_n}$ are units of S of degree $d_{\sigma_1,\tau_1}, \ldots, d_{\sigma_n,\tau_n}$. Calculate the symmetric 2-cocycle f' satisfying (*) by putting $c_{\sigma_i,\tau_i} = f_{\sigma_i,\tau_i}$, $i = 1, \ldots, n$, (i.e., specialize the free group on the elements c_{σ_i,τ_i}, $i = 1, \ldots, n$, into $U(S)$ by sending c_{σ_i,τ_i} onto f_{σ_i,τ_i}). Since the $f'_{\sigma,\tau}$ are expressions in the f_{σ_i,τ_i}, $i = 1, \ldots, n$, and the $d_{\sigma,\tau}$ are given by exactly the same (but additively written) expressions in d_{σ_i,τ_i}, $i = 1, \ldots, n$, it follows that $\deg^2 f' = d$. So we have lifted any $\mathrm{d} \in \mathrm{Im}(\deg^2)$ to some $f' \in H_{\mathrm{sym}}^2(G, U(S))$; hence the statement of the proposition follows.

2.12. COROLLARY. *If the group G of a* Gr-*Dedekind extension S of D is abelian, then* $\mathrm{Im}(\deg^2) = \mathrm{Im}(\deg^2 \mid H^2_{\mathrm{sym}}(g, U(S))$, this simplifies calculation of $\mathrm{Im}(\deg^2)$ in some examples.

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